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BEST PROXIMITY POINTS FOR MULTIVALUED MAPPINGS AND EQUATION OF MOTION

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Abstract In this manuscript, we compute coincidence point, best proximity point, and fixed point results for multivalued proximal contractions in the setup of $b-$metric spaces using an alternating distance function. Moreover, we show the corresponding results for single-valued mappings can also be obtained using generalized proximal contractions. To validate our study, examples are given for both multivalued and single-valued mappings that strengthen our main results based on coincidence points. In the end, we apply the obtained result to show the existence of the solution of a particular type of second-order boundary value problem describing the equation of motion.

Keywords Equation of motion, coincidence best proximity point, F-contraction, alternating distance, b-metric space.

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1. Introduction

Fixed point theory is an essential instrument for solving the equation $\Gamma\hbar = \hbar$ for a mapping Γ specified on a subset of a metric space, a simplified linear space, or a topological vector space. Every mapping does not need to have a fixed point. Numerous mappings lack fixed points, including nonself-mappings with two disjoint sets and translation mappings. Also, the fixed point of mappings with strict conditions cannot be located. However, we can approximate the fixed point given certain conditions. We investigate the approximated fixed points in such cases using the best approximation theory.

It is always challenging to find the global minima of a function. Analytical methods are frequently not applicable in such cases, and the use of numerical solution strategies also becomes hard. Applications of Global minimization problems occurs in fitting model parameters to experimental data in chemistry, physics, finance, and engineering. For example, minimize the energy function in case of structure prediction, minimize the path length in case of traveling salesman problem and elec-

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trical circuit design. Keeping in view the above constraints, it is always interesting to establish the existence and convergence of the best proximity points and fixed points as a solution of various minimization problems in optimization theory, as well as existence solutions to various nonlinear problems, which recently attracted the attention of many authors; see for instance, [9, 10, 22, 30–32].

A nonself-mapping $\Gamma : \mathrm{P} \to \mathrm{Q}$ may not have a fixed point, an element $\hbar$ closer to $\Gamma\hbar$ always proceeds certainly. The best proximity point ($\mathbb{BPP}$) and the best approximation theorem are important from this perspective. The root of a $\mathbb{BPP}$ theorem is a global optimization problem of an empirical component, and indicates the error involved in the solution of the approximated operator $\Gamma\hbar = \hbar$. Since, for a nonself-mapping $\Gamma : \mathrm{P} \to \mathrm{Q}$, $d(\hbar, \Gamma\hbar)$ is at least $d(\mathrm{P}, \mathrm{Q})$ for all $\hbar$ in P, the $\mathbb{BPP}$ theorem establishes a globally optimal solution of error $d(\hbar, \Gamma\hbar)$ by constraining an approximate solution $\hbar$ of the equation $\Gamma\hbar = \hbar$ to the condition that $d(\hbar, \Gamma\hbar) = d(\mathrm{P}, \mathrm{Q})$. In real world, $\mathbb{BPP}$ theorems logically follow from fixed point theorems, because the $\mathbb{BPP}$ is a fixed point implicit mapping that represents a self-mapping.

The $\mathbb{BPP}$ is a point in a metric space that is "closest" to a contraction mapping in a certain sense. It is a point whose distance between its image and the point itself is the shortest among all points in the metric space. This distance is also known as the proximity function. The concept of $\mathbb{BPP}$ is helpful in various mathematical situations, including showing the existence and uniqueness of fixed points for contraction mappings, analyzing iterative technique convergence, and solving various optimization issues. The $\mathbb{BPP}$ is a reference point for determining the behavior and properties of the mapping under discussion. Many fields of mathematics and other disciplines, including computer science, engineering, economics, and physics, have applications for studying $\mathbb{BPP}$ and related issues. It provides a theoretical foundation for analyzing mapping behavior and finding optimal solutions to various problems.

In 1997, Sadiq Basha et al. [7, 8] gave the concept of best proximity pair, Eldred et al. [12] established a method to find a best proximity point for the mappings Γ in setting of uniformly convex Banach space. Kikkawa et al. [19] defined some relationship among Kannan mappings and contractions. Anuradha et al. [3] originated the concept of proximal pointwise contraction. Suzuki et al. [28] proposed the idea of property UC, Abkar et al. [1] verified the convergence and presence of best proximity points for asymptotic cyclic contractions having UC property. In [5, 6] Basha et al. proved $\mathbb{BPP}$ theorems for global optimal approximate solutions, Samet et al. [26] originated the concept of $\alpha-$admissible mapping, Jleli et al. [16] initiated $\alpha-$proximal admissible mapping. Gabeleh et al. [15] recently proposed the existence of an optimal approximate solution, known as a best proximity point, for non-self mappings, which are ordered proximal contractions in a more general scenario with an ordered structure.

A classical best approximation theorem, due to Fan [13], claims that if P is a nonempty compact convex subset of a Hausdorff locally convex topological vector space $\mathfrak{X}$ with a semi-norm p and $\Gamma : \mathrm{P} \to \mathfrak{X}$ is a continuous mapping, then there is an element $\hbar$ in P satisfying the condition that $d(\hbar, \Gamma\hbar) = d(\Gamma\hbar, \mathrm{P})$. Komal et al. [20] studied the results of generalized Geraghty proximal cyclic contractions and established coincidence $\mathbb{BPP}$ results in the framework of complete metric space, Latif et al. [21] established partially ordered metric space and developed coincidence $\mathbb{BPP}$ results for $\digamma_{\mathfrak{g}}-$weak contractive mappings. Many successive modifications and different versions of Fan's theorem have been studied, see eg., [2, 23, 25]. It is worth noting that

Motivated from the work done in preceding analysis, in this article, we obtain some coincidence $\mathbb{BPP}$ results in the $b-$metric environment. Our results are more interesting, because we are taking $F-$type proximal contraction using alternating distance function ϕ in the endowed with multivalued mappings, which is a new development within the current state-of-art. We generate multivalued coincidence points, single valued coincidence points, and novel $\mathbb{BPP}$ results with various corollaries. Furthermore, for each outcome, illustrative examples are supplied, making our findings more transparent and authentic. A physics-related application is also proposed, making our study worthwhile and useful.

Let P and Q are two nonempty subsets of a complete b-metric space $(\mathfrak{X},d)$, defined as

$$\mathrm{P}_0 = \{\hbar \in \mathrm{P} : \mathrm{d}(\hbar, \eth) = \mathrm{d}(\mathrm{P}, \mathrm{Q}) \text{ for some } \eth \in \mathrm{Q}\},$$
$$\mathrm{Q}_0 = \{\eth \in \mathrm{Q} : \mathrm{d}(\hbar, \eth) = \mathrm{d}(\mathrm{P}, \mathrm{Q}) \text{ for some } \hbar \in \mathrm{P}\},$$

where

$$\mathrm{d}(\mathrm{P}, \mathrm{Q}) = \inf\{\mathrm{d}(\hbar, \eth) : \hbar \in \mathrm{P}, \eth \in \mathrm{Q}\}(\text{distance of a set P to a set Q}).$$

2. Preliminaries

We will talk about some key definitions from the literature that are relevant to our study in this part. The concept of distance was axiomatically established by Frechet and Haussdorff in the early nineteenth century under the term of metric. Since then, several authors have worked with it and published numerous results in the literature.

Definition 2.1. [14] A function d: $\mathfrak{X} \times \mathfrak{X} \to \mathbb{R}$ whose range is nonnegative real numbers is said to be metric if it satisfies the following properties:

1. $d(\hbar, \eth) \geq 0$,
2. $d(\hbar, \eth) = 0$ if and only if $\hbar = \eth$,
3. $d(\hbar, \eth) = d(\eth, \hbar)$,
4. $d(\hbar, z) + d(z, \eth) \geq d(\hbar, \eth)$

for all $\hbar, \eth, z \in \mathfrak{X}$. Where d is metric on $\mathfrak{X}$ and the pair $(\mathfrak{X},d)$ is called the metric space. The set $\mathfrak{X}$ is named as ground set. The points $\hbar, \eth, z \in \mathfrak{X}$ are known as the elements of metric space $(\mathfrak{X}, \mathrm{d})$. So $\mathfrak{X}$ will represents the metric space $(\mathfrak{X}, \mathrm{d})$.

We present some examples from the literature [14] to get into the concept of distance or metric.

Example 2.1. Consider functions d_1, d_2, $\mathrm{d}_3 : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined as:

$$\mathrm{d}_1(\hbar, \eth) = |\hbar - \eth|,$$
$$\mathrm{d}_2(\hbar, \eth) = \sqrt{|\hbar - \eth|},$$
$$\mathrm{d}_3(\hbar, \eth) = \left|\frac{1}{\hbar} - \frac{1}{\eth}\right|,$$

then d_1, d_2 are metrics on $\mathbb{R}$ and d_3 is metric on $\mathbb{R} \backslash \{0\}$, where $\mathbb{R}$ is set of real numbers.

Example 2.2. Suppose $\mathfrak{X}_1 = (a, b)$ and $\mathfrak{X}_2 = (c, e)$, for all $a, b, c, e \in \mathbb{R}$, then $d(\mathfrak{X}_1, \mathfrak{X}_2)$ defined as:

$$\mathrm{d}(\mathfrak{X}_1, \mathfrak{X}_2) = |a - c| + |b - e| ,$$

is a metric on $\mathbb{R}^2 = \mathfrak{X}$.

Example 2.3. Let $\mathfrak{X}$ be nonempty set of a metric space $(\mathfrak{X}, d_0)$, where $d_0 : \mathfrak{X} \times \mathfrak{X} \to \mathbb{R}$ is a function defined below:

$$\mathrm{d}_0(\hbar, \eth) = \begin{cases} 0, \text{ if } \hbar = \eth, \\ 1, \text{ if } \hbar \neq \eth, \end{cases}$$

then d_0 is said to be discrete metric space.

In order to extrapolate this concept, metric axioms were modified in a variety of ways. Among these, the concept of b-metric is crucial. Bakhtin [4] (and separately Czerwik [11]) introduced the concept of b-metric spaces and demonstrated alternative conclusions for the existence of fixed points. The definition of b-metric, also known as quasi-metric, is provided here for comprehension.

Definition 2.2. [4] Let $\mathfrak{X}$ be a nonempty set and $\mathrm{s} \geq 1$ be a given real number. A function d: $\mathfrak{X} \times \mathfrak{X} \to \mathbb{R}^+$is a b-metric if the following axioms are contended:

1. $d(\hbar, \eth) \geq 0$,
2. $d(\hbar, \eth) = 0$ if and only if $\hbar = \eth$,
3. $d(\hbar, \eth) = d(\eth, \hbar)$,
4. $\mathrm{s}[\mathrm{d}(\hbar, \eth) + \mathrm{d}(\eth, \mathrm{z})] \geq \mathrm{d}(\hbar, \mathrm{z})$

for all $\hbar, \eth, \mathrm{z} \in \mathfrak{X}$. We call $(\mathfrak{X}, \mathrm{d})$ to be a $b-$metric space. Obviously for $\mathrm{s} \geq 1$, $b-$metric space is a generalization of metric space.

Example 2.4. Let $\mathfrak{X} = \{0, 1, 2, 3\}$ and d is defined as $d(\hbar, \eth) = |\hbar - \eth|^2$, where d: $\mathfrak{X} \times \mathfrak{X} \to \mathbb{R}^+$, then $(\mathfrak{X}, \mathrm{d})$ is $b-$metric space for $\mathrm{s} = 2 \geq 1$.

Definition 2.3. [17] Let $(\mathfrak{X}, \mathrm{d})$ be a nonempty set associated with a $b-$metric space. Suppose $\hbar \in \mathfrak{X}$, and $\mathfrak{X}$ possesses a sequence $\{\hbar_n\}$ then,

(i) A sequence $\{\hbar_n\}$ converges to $\hbar$ iff $\lim_{n\to\infty} \hbar_n = \hbar$.

(ii) A sequence $\{\hbar_n\}$ is a Cauchy sequence iff $\lim\limits_{n,m\to\infty} d(\hbar_n, \mathrm{x}_m) = 0$.

(iii) A Cauchy sequence in $\mathfrak{X}$ is complete $\iff$ $\mathfrak{X}$ is convergent.

(iv) A convergent sequence has a unique limit.

(v) The $b-$metric associated with $(\mathfrak{X}, d)$ is not continuous.

(vi) Every subsequence in a Cauchy sequence is convergent.

Definition 2.4. [27] Consider a metric space with a pair of nonempty subsets (P, Q) $[\mathrm{P}_0 \neq \phi]$, then the pair (P, Q) is said to satisfy $\mathfrak{P}-$property if

$$\mathrm{d}(\hbar_1, \eth_1) = \mathrm{d}(\hbar_2, \eth_2) = \mathrm{d}(\mathrm{P}, \mathrm{Q})$$

implies that

$$\mathrm{d}(\hbar_1, \hbar_2) = \mathrm{d}(\eth_1, \eth_2),$$

where $\hbar_1, \hbar_2 \in \mathrm{P}$ and $\eth_1, \eth_2 \in \mathrm{Q}$.

Definition 2.5. [18] A function $\phi : (0,+\infty) \to (0,+\infty)$ is referred to as an alternating distance function if it meets the following criteria:

(i) ϕ is continuous and monotonically increasing,

(ii) $\phi(\hbar) > 0$ for all $\hbar > 0$.

Definition 2.6. [29] Consider a mapping $F : \mathbb{R}^+ \to \mathbb{R}$ upholding the following criteria:

F1: For all $\hbar, \eth \in \mathbb{R}^+$, F is strictly increasing as well as $\hbar < \eth$, $F(\hbar) < F(\eth)$,

F2: For every positive sequence $\{\hbar_n\}_{n\in\mathbb{N}}$, $\lim\limits_{n\to\infty} \hbar_n = 0$, iff $\lim\limits_{n\to\infty} F(\hbar_n) = -\infty$,

F3: For $\lim\limits_{\hbar\to 0^+} \hbar^k F(\hbar) = 0$, there exists $k \in (0,1)$.

Then the map $\Gamma : \mathfrak{X} \to \mathfrak{X}$ on a metric space $(\mathfrak{X},d)$ is referred to as F-contraction, if for any $\tau \in (0,+\infty)$, the following condition is fulfilled:

$$\mathrm{d}(\Gamma\hbar, \Gamma\eth) > 0, \quad \Rightarrow \tau + F(\mathrm{d}(\Gamma\hbar, \Gamma\eth)) \leq F(\mathrm{d}(\hbar, \eth)).$$

Example 2.5. Let $F : (0,+\infty) \to \mathbb{R}$, be a function defined by

$$F(\hbar) = \log \hbar + \hbar,$$

and $\Gamma : \mathfrak{X} \to \mathfrak{X}$ be a mapping defined on complete metric space with metric

$$\mathrm{d}(\hbar, \eth) = |\hbar - \eth|,$$

by $\Gamma(\hbar) = \frac{\hbar}{3}$. Then, Γ is an F contraction.

Definition 2.7. [24] Consider closed and bounded subsets $\mathcal{CB}(\mathfrak{X})$ of $\mathfrak{X}$, and further suppose that H be a Pompeiu–Hausdroff metric induced by metric d defined as

$$\mathcal{H}(\mathrm{P}, \mathrm{Q}) = \max\{\sup_{a\in \mathrm{P}} \mathcal{D}(a, \mathrm{Q}), \sup_{\mathrm{s}\in \mathrm{Q}} \mathcal{D}(b, \mathrm{P})\},$$

for $\mathrm{P}, \mathrm{Q} \subseteq \mathcal{CB}(\mathfrak{X})$, where

$$\mathcal{D}(a, \mathrm{Q}) = \inf\{\mathrm{d}(a, b) : b \in \mathrm{Q}\},$$

and throughout the manuscript we will denote

$$\mathcal{D}^*(a, b) = \mathcal{D}(a, b) - \mathrm{d}(\mathrm{P}, \mathrm{Q}),$$

for $a \in \mathrm{P}$ and $b \in \mathrm{Q}$.

3. Coincidence best proximity points

This section is devoted to present some new results concerning multivalued coincidence point, best proximity point and fixed points within the structure of complete $b-$metric spaces $(\mathfrak{X},d)$.

Definition 3.1. Let P and Q be nonempty closed subsets of a $b-$metric space $(\mathfrak{X},d)$. Consider a pair of mappings $(\mathfrak{g},\Gamma)$ where $\Gamma:\mathrm{P}\to\mathcal{CB}(\mathrm{Q})$ is a multivalued map and $\mathfrak{g}:\mathrm{P}\to\mathrm{P}$, which satisfies

$$\mathcal{D}(\mathfrak{g}\hbar,\Gamma\hbar)=\mathrm{d}(\mathrm{P},\mathrm{Q}),$$

then $\hbar\in\mathrm{P}$ is called coincidence $\mathbb{BPP}$ of the pair of mappings $(\mathfrak{g},\Gamma)$.

Remark 3.1. The coincidence best proximity point results are the generalization of $\mathbb{BPP}$ results and fixed point results because, if we take $\mathfrak{g}=I_{\mathrm{P}}$ then, every coincidence $\mathbb{BPP}$ become $\mathbb{BPP}$ of the mapping Γ, and if this mapping is a self mapping then the $\mathbb{BPP}$ reduces to fixed point.

Definition 3.2. Given $\Gamma:\mathrm{P}\to\mathcal{CB}(\mathrm{Q})$ and $\mathfrak{g}:\mathrm{P}\to\mathrm{P}$. The pair $(\mathfrak{g},\Gamma)$ is said to be a $\digamma_{C_P}-$proximal contraction if there exists a $\tau\in(0,+\infty)$ such that for all $u,v,\hbar,\eth$ in P

$$\mathcal{D}\left(\mathfrak{g}u,\Gamma\hbar\right)=\mathrm{d}(\mathrm{P},\mathrm{Q}),$$
$$\mathcal{D}\left(\mathfrak{g}v,\Gamma\eth\right)=\mathrm{d}(\mathrm{P},\mathrm{Q})$$

implies that

$$\tau+\digamma(\phi(\mathcal{H}(\Gamma\hbar,\Gamma\eth)))\leq\digamma(\phi(\mathcal{N}(u,v,\hbar,\eth))),$$

where

$$\begin{aligned}\mathcal{N}(u,v,\hbar,\eth)=\max\{&\mathrm{d}(\mathfrak{g}u,\mathfrak{g}v),\frac{\mathcal{D}(\mathfrak{g}u,\Gamma u)-\mathrm{s}\,\mathrm{d}(\mathrm{P},\mathrm{Q})}{\mathrm{s}},\\ &\mathcal{D}^{*}(\mathfrak{g}v,\Gamma u),\frac{\mathcal{D}(\mathfrak{g}\hbar,\Gamma v)-\mathrm{s}\mathcal{D}(\mathfrak{g}u,\Gamma v)}{\mathrm{s}}\}.\end{aligned}$$

Definition 3.3. A mapping $\Gamma:\mathrm{P}\to\mathcal{CB}(\mathrm{Q})$ is said to be $\digamma_{B_P}-$proximal contraction if there exists a $\tau\in(0,+\infty)$ such that for all $u,v,\hbar,\eth$ in P,

$$\mathcal{D}\left(u,\Gamma\hbar\right)=\mathrm{d}(\mathrm{P},\mathrm{Q}),$$
$$\mathcal{D}\left(v,\Gamma\eth\right)=\mathrm{d}(\mathrm{P},\mathrm{Q}),$$

implies that

$$\tau+\digamma\left(\phi(\mathcal{H}(\Gamma\hbar,\Gamma\eth))\right)\leq\digamma\left(\phi(\mathcal{N}(u,v,\hbar,\eth))\right),$$

where

$$\begin{aligned}\mathcal{N}(u,v,\hbar,\eth)=\max\Big\{&\mathrm{d}(u,v),\frac{\mathcal{D}(u,\Gamma u)-\mathrm{s}\,\mathrm{d}(\mathrm{P},\mathrm{Q})}{\mathrm{s}},\\ &\mathcal{D}^{*}(v,\Gamma u),\frac{\mathcal{D}(\hbar,\Gamma v)-\mathrm{s}\mathcal{D}(u,\Gamma v)}{\mathrm{s}}\Big\}.\end{aligned}$$

Remark 3.2. It is worth noting that if we take $\mathfrak{g}=I_{\mathrm{P}}$ ($\mathfrak{g}$ as an identity mapping on P), then every $\digamma_{C_P}-$proximal contraction reduces to a $\digamma_{B_P}-$proximal contraction.

Theorem 3.1. *Let $(\mathfrak{X},d)$ be a complete $b-$metric space with nonempty closed subsets* P *and* Q *satisfying the $\mathfrak{P}-$property, where A_0 is nonempty. Given continuous mappings $\Gamma:\mathrm{P}\to\mathcal{CB}(\mathrm{Q})$, $\mathfrak{g}:\mathrm{P}\to\mathrm{P}$ with $\Gamma(\mathrm{P}_0)\subseteq\mathrm{Q}_0$ and $\mathrm{P}_0\subseteq\mathfrak{g}(\mathrm{P}_0)$, where $\mathfrak{g}$ is one-to-one continuous. If the pair $(\mathfrak{g},\Gamma)$ satisfies $\digamma_{C_P}-$proximal contraction with alternating distance ϕ. Then the pair $(\mathfrak{g},\Gamma)$ concedes a coincidence $\mathbb{BPP}$.*

Proof. Let $\hbar_0$ be an arbitrary element in P_0. Since $\Gamma(\mathrm{P}_0)$ is contained in Q_0 and P_0 is contained in $\mathfrak{g}(\mathrm{P}_0)$, there exists an element $\hbar_1$ in P_0 such that

$$\mathcal{D}(\mathfrak{g}\hbar_1, \Gamma\hbar_0) = \mathrm{d}(\mathrm{P}, \mathrm{Q}).$$

Again, since $\Gamma\hbar_1$ is an element of $\Gamma(\mathrm{P}_0)$ which is contained in Q_0, and P_0 is contained in $\mathfrak{g}(\mathrm{P}_0)$, it follows that there is an element $\hbar_2$ in P_0 such that

$$\mathcal{D}(\mathfrak{g}\hbar_2, \Gamma\hbar_1) = \mathrm{d}(\mathrm{P}, \mathrm{Q}).$$

Similarly, for $\hbar_{n-1} \in \mathrm{P}_0$ with $\Gamma(\mathrm{P}_0) \subseteq \mathrm{Q}_0$, then there exists $\hbar_n \in \mathrm{P}_0$ such that

$$\mathcal{D}(\mathfrak{g}\hbar_n, \Gamma\hbar_{n-1}) = \mathrm{d}(\mathrm{P}, \mathrm{Q}). \tag{3.1}$$

Having selected $\{\hbar_n\}$ satisfying the condition, there exists an element $\hbar_{n+1}$ in P_0 satisfying the condition that

$$\mathcal{D}(\mathfrak{g}\hbar_{n+1}, \Gamma\hbar_n) = \mathrm{d}(\mathrm{P}, \mathrm{Q}), \tag{3.2}$$

for every positive integer n.

Using $\mathfrak{P}-$property, we conclude

$$\mathrm{d}(\mathfrak{g}\hbar_n, \mathfrak{g}\hbar_{n+1}) = H(\Gamma\hbar_n, \Gamma\hbar_{n-1}).$$

Since the pair $(\mathfrak{g}, \Gamma)$ is a $\digamma_{C_P}-$proximal contraction, by using equation (3.1) and (3.2), we infer

$$\digamma(\phi(\mathrm{d}(\mathfrak{g}\hbar_n, \mathfrak{g}\hbar_{n+1}))) \leq \digamma(\phi(\mathcal{N}(\hbar_n, \hbar_{n+1}, \hbar_{n-1}, \hbar_n))) - \tau, \tag{3.3}$$

where

$$\begin{aligned}
&\mathcal{N}(\hbar_n, \hbar_{n+1}, \hbar_{n-1}, \hbar_n) \\
&= \max\left\{\begin{array}{c} \mathrm{d}(\mathfrak{g}\hbar_n, \mathfrak{g}\hbar_{n+1}),\ \frac{\mathcal{D}(\mathfrak{g}\hbar_n, \Gamma\hbar_n) - \mathrm{s}\,\mathrm{d}(\mathrm{P},\mathrm{Q})}{\mathrm{s}} \\ \mathcal{D}^*(\mathfrak{g}\hbar_{n+1}, \Gamma\hbar_n),\ \frac{\mathcal{D}(\mathfrak{g}\hbar_{n-1}, \Gamma\hbar_{n+1}) - \mathrm{s}\mathcal{D}(\mathfrak{g}\hbar_n, \Gamma\hbar_{n+1})}{\mathrm{s}} \end{array}\right\}, \\
&\leq \max\left\{\begin{array}{c} \mathrm{d}(\mathfrak{g}\hbar_n, \mathfrak{g}\hbar_{n+1}),\ \frac{\mathrm{s}[\mathrm{d}(\mathfrak{g}\hbar_n, \mathfrak{g}\hbar_{n+1}) + \mathcal{D}(\mathfrak{g}\hbar_{n+1}, \Gamma\hbar_n)] - \mathrm{s}\,\mathrm{d}(\mathrm{P},\mathrm{Q})}{\mathrm{s}} \\ \mathcal{D}(\mathfrak{g}\hbar_{n+1}, \Gamma\hbar_n) - \mathrm{d}(\mathrm{P},\mathrm{Q}),\ \frac{\mathrm{s}[\mathrm{d}(\mathfrak{g}\hbar_{n-1}, \mathfrak{g}\hbar_n) + \mathcal{D}(\mathfrak{g}\hbar_n, \Gamma\hbar_{n+1})] - \mathrm{s}\mathcal{D}(\mathfrak{g}\hbar_n, \Gamma\hbar_{n+1})}{\mathrm{s}} \end{array}\right\} \\
&\leq \max\left\{\begin{array}{l} \mathrm{d}(\mathfrak{g}\hbar_n, \mathfrak{g}\hbar_{n+1}),\ \frac{\mathrm{s}\,\mathrm{d}(\mathfrak{g}\hbar_n, \mathfrak{g}\hbar_{n+1}) + \mathrm{s}\mathrm{d}(\mathrm{P},\mathrm{Q}) - \mathrm{s}\,\mathrm{d}(\mathrm{P},\mathrm{Q})}{\mathrm{s}} \\ 0,\ \frac{\mathrm{s}\,\mathrm{d}(\mathfrak{g}\hbar_{n-1}, \mathfrak{g}\hbar_n) + \mathrm{s}\mathcal{D}(\mathfrak{g}\hbar_n, \Gamma\hbar_{n+1}) - \mathrm{s}\mathcal{D}(\mathfrak{g}\hbar_n, \Gamma\hbar_{n+1})}{\mathrm{s}} \end{array}\right\} \\
&\leq \max\left\{\mathrm{d}(\mathfrak{g}\hbar_n, \mathfrak{g}\hbar_{n+1}), \mathrm{d}(\mathfrak{g}\hbar_n, \mathfrak{g}\mathrm{x}_{n+1}), 0, \mathrm{d}(\mathfrak{g}\hbar_{n-1}, \mathfrak{g}\hbar_n)\right\},
\end{aligned}$$

which yields

$$\mathcal{N}(\hbar_n, \hbar_{n+1}, \hbar_{n-1}, \hbar_n) \leq \max\left\{\mathrm{d}(\mathfrak{g}\hbar_n, \mathfrak{g}\hbar_{n+1}), \mathrm{d}(\mathfrak{g}\hbar_{n-1}, \mathfrak{g}\hbar_n)\right\}.$$

Further, if we choose $\max\{\mathrm{d}(\mathfrak{g}\hbar_{n-1}, \mathfrak{g}\hbar_n), \mathrm{d}(\mathfrak{g}\hbar_n, \mathfrak{g}\mathrm{x}_{n+1}), \mathrm{d}(\mathfrak{g}\hbar_{n+1}, \mathfrak{g}\hbar_{n+2})\} = d(\mathfrak{g}\hbar_n, \mathfrak{g}\hbar_{n+1})$, then the inequality (3.3) becomes

$$\digamma(\phi(\mathrm{d}(\mathfrak{g}\hbar_n, \mathfrak{g}\hbar_{n+1}))) \leq \digamma(\phi(\mathrm{d}(\mathfrak{g}\hbar_n, \mathfrak{g}\hbar_{n+1}))) - \tau$$

which is absurd.

Choosing $\max\{\mathrm{d}(\mathfrak{g}\hbar_{n-1},\mathfrak{g}\mathrm{x}_n),\mathrm{d}(\mathfrak{g}\hbar_n,\mathfrak{g}\hbar_{n+1}),\mathrm{d}(\mathfrak{g}\hbar_{n+1},\mathfrak{g}\hbar_{n+2})\} = d(\mathfrak{g}\hbar_{n-1},\mathfrak{g}\hbar_n)$, then the equation (3.3) will be

$$F(\phi(\mathrm{d}(\mathfrak{g}\hbar_n,\mathfrak{g}\hbar_{n+1}))) \le F(\phi(\mathrm{d}(\mathfrak{g}\hbar_{n-1},\mathfrak{g}\hbar_n))) - \tau. \tag{3.4}$$

Therefore, we can write

$$\mathrm{d}(\mathfrak{g}\hbar_n,\mathfrak{g}\hbar_{n+1}) \le \mathrm{d}(\mathfrak{g}\hbar_{n-1},\mathfrak{g}\hbar_n).$$

Hence, the sequence is a $\{(\mathfrak{g}\hbar_n,\mathfrak{g}\hbar_{n+1})\}$ monotonic non-increasing and bounded below. Thus, there exists $\lambda \ge 0$ such that

$$\lim_{n\to\infty} \mathrm{d}(\mathfrak{g}\hbar_n,\mathfrak{g}\hbar_{n+1}) = \lambda \ge 0. \tag{3.5}$$

Consider $\lim_{n\to\infty} d(\mathfrak{g}\mathrm{x}_n,\mathfrak{g}\hbar_{n+1}) = \lambda > 0$ and using the equation (3.4),(3.5), we obtain

$$F(\phi(\mathrm{d}(\mathfrak{g}\hbar_n,\mathfrak{g}\hbar_{n+1}))) \le F(\phi(\mathrm{d}(\mathfrak{g}\hbar_{n-2},\mathfrak{g}\hbar_{n-1}))) - 2\tau.$$

A repeated use of the above argument leads us to

$$F(\phi(\mathrm{d}(\mathfrak{g}\hbar_n,\mathfrak{g}\hbar_{n+1}))) \le F(\phi(\mathrm{d}(\mathfrak{g}\hbar_0,\mathfrak{g}\hbar_1))) - n\tau. \tag{3.6}$$

Inducing limit as $n \to \infty$, we arrive at

$$\lim_{n\to\infty} F(\phi(\mathrm{d}(\mathfrak{g}\hbar_n,\mathfrak{g}\hbar_{n+1}))) = -\infty.$$

Using the property (F2) of Definition 2.6, we infer

$$\lim_{n\to\infty} \phi(\mathrm{d}(\mathfrak{g}\hbar_n,\mathfrak{g}\hbar_{n+1})) = 0. \tag{3.7}$$

Now, taking (F3) into account, there exists $k \in (0,1)$, such that

$$\begin{aligned} &\lim_{\mathrm{d}(\mathfrak{g}\hbar_n,\mathfrak{g}\hbar_{n+1})\to 0} (\phi(\mathrm{d}(\mathfrak{g}\hbar_n,\mathfrak{g}\hbar_{n+1})))^k F(\phi(\mathrm{d}(\mathfrak{g}\hbar_n,\mathfrak{g}\hbar_{n+1}))) = 0,\\ &\lim_{n\to\infty} (\phi(\mathrm{d}(\mathfrak{g}\hbar_n,\mathfrak{g}\hbar_{n+1})))^k F(\phi(\mathrm{d}(\mathfrak{g}\hbar_n,\mathfrak{g}\hbar_{n+1}))) = 0. \end{aligned} \tag{3.8}$$

Equation (3.6) leads us to

$$\begin{aligned} &F(\phi(\mathrm{d}(\mathfrak{g}\hbar_n,\mathfrak{g}\hbar_{n+1})))\\ &\le F(\phi(\mathrm{d}(\mathfrak{g}\hbar_0,\mathfrak{g}\hbar_1))) - n\tau\\ &\Rightarrow (\phi(\mathrm{d}(\mathfrak{g}\hbar_n,\mathfrak{g}\hbar_{n+1})))^k F(\phi(\mathrm{d}(\mathfrak{g}\hbar_n,\mathfrak{g}\hbar_{n+1}))) - F(\phi(\mathrm{d}(\mathfrak{g}\hbar_0,\mathfrak{g}\hbar_1)))\\ &\le -(\phi(\mathrm{d}(\mathfrak{g}\hbar_n,\mathfrak{g}\hbar_{n+1})))^k n\tau. \end{aligned}$$

Assume that $\beta_n = \phi(d(\mathfrak{g}\hbar_n,\mathfrak{g}\hbar_{n+1}))$, then above inequality becomes

$$(\beta_n)^k(F(\beta_n) - F(\beta_0)) \le -(\beta_n)^k n\tau.$$

By using equation (3.7) and (3.8) and taking $\lim_{n\to\infty}$, we get

$$\lim_{n\to\infty} (\beta_n)^k(F(\beta_n) - F(\beta_0)) \le \lim_{n\to\infty} -(\beta_n)^k n\tau \le 0.$$

The above inequality can be put as

$$\lim_{n\to\infty} n(\beta_n)^k = 0. \tag{3.9}$$

Utilizing (3.9), for given $\epsilon > 0$, there exists $n_1 \in N$ such that

$$\begin{aligned}
&\left|n(\beta_n)^k - 0\right| < \epsilon, \text{ for all } n \geq n_1,\\
&\left|n(\beta_n)^k\right| < \epsilon,\\
&(\beta_n) < \frac{\epsilon}{n^{\frac{1}{k}}}.
\end{aligned}$$

In order to authenticate $\{\mathfrak{g}\hbar_n\}$ is a Cauchy sequence within the setting of complete $b-$metric space, we have, for all $n, p \in \mathbb{N}$

$$\begin{aligned}
&\phi(\mathrm{d}(\mathfrak{g}\hbar_{n+p}, \mathfrak{g}\mathrm{x}_n))\\
&\leq \mathrm{s}\phi(\mathrm{d}(\mathfrak{g}\hbar_{n+p}, \mathfrak{g}\hbar_{n+1})) + b\beta_n\\
&\leq \mathrm{s}^2\phi(\mathrm{d}(\mathfrak{g}\hbar_{n+p}, \mathfrak{g}\hbar_{n+2})) + \mathrm{s}^2\beta_{n+1} + b\beta_n\\
&\leq \mathrm{s}^3\phi(\mathrm{d}(\mathfrak{g}\hbar_{n+p}, \mathfrak{g}\hbar_{n+3})) + \mathrm{s}^3\beta_{n+2} + \mathrm{s}^2\beta_{n+1} + b\beta_n\\
&\quad\vdots\\
&\leq \mathrm{s}^{p-1}\beta_{n+p-1} + \mathrm{s}^{p-1}\beta_{n+p-2} + \ldots + \mathrm{s}^2\beta_{n+1} + b\beta_n\\
&\leq \mathrm{s}^{p}\beta_{n+p-1} + \mathrm{s}^{p-1}\beta_{n+p-2} + \ldots + \mathrm{s}^2\beta_{n+1} + b\beta_n\\
&= \frac{1}{\mathrm{s}^{n-1}}[\mathrm{s}^{n+p-1}\beta_{n+p-1} + \mathrm{s}^{n+p-2}\beta_{n+p-2} + \ldots + \mathrm{s}^{n+1}\beta_{n+1} + \mathrm{s}^n\beta_n]\\
&= \frac{1}{\mathrm{s}^{n-1}}\sum_{i=n}^{n+p-1}\mathrm{s}^i\beta_i \leq \frac{1}{\mathrm{s}^{n-1}}\sum_{i=n}^{\infty}\mathrm{s}^i\frac{\epsilon}{i^{\frac{1}{k}}}.
\end{aligned}$$

Hence for all $n \geq n_1$, $p \in N$ and $k \in (0, 1)$ the above inequality reduces to

$$\phi(\mathrm{d}(\mathfrak{g}\hbar_{n+p}, \mathfrak{g}\mathrm{x}_n)) \leq \frac{1}{\mathrm{s}^{n-1}}\sum_{i=n}^{\infty}\mathrm{s}^i\frac{\epsilon}{i^{\frac{1}{k}}}.$$

Consequently, by $P-$series test, the series $\sum\limits_{i=n}^{\infty}\mathrm{s}^i\frac{\epsilon}{i^{\frac{1}{k}}}$ is convergent for $\frac{1}{k} > 1$. Hence, the Cauchy sequence $\{\mathfrak{g}\hbar_n\}$ is convergent in a complete $b-$metric space $(\mathfrak{X}$,$d)$.

Furthermore, assume that $\{\mathfrak{g}\hbar_n\}$ converges to $\hbar^*$ in $\mathrm{P}_0 \subseteq \mathrm{P}$ (the fact that the set P is closed), which encourages that the sequence $\{\hbar_n\} \subseteq \mathrm{P}_0$, since $\hbar_n \to \hbar^*$. As $(\mathfrak{g}, \Gamma)$ is a pair of continuous mapping, which shows that

$$\mathcal{D}(\mathfrak{g}\hbar^*, \Gamma\hbar^*) = \mathrm{d}(\mathrm{P}, \mathrm{Q}).$$

Thus, $\hbar^*$ is a coincidence point of the pair of mapping $(\mathfrak{g}, \Gamma)$. $\square$

Example 3.1. Consider $\mathfrak{X} = \{0, 1, 2, 3, 4, 5\}$ and d: $\mathfrak{X} \times \mathfrak{X} \to [0, \infty)$ be defined by the following:

d	0	1	2	3	4	5
0	0	3	4	2	6	7
1	3	0	5	8	2	6
2	4	5	0	7	8	2
3	2	8	7	0	3	4
4	6	2	8	3	0	5
5	7	6	2	4	5	0

Clearly, $(\mathfrak{X},d)$ is a complete $b-$metric space. Assuming that $\mathrm{P} = \{0, 1, 2\}$ and $\mathrm{Q} = \{3, 4, 5\}$ are nonempty closed subsets of $b-$metric space $(\mathfrak{X},d)$. After, simple calculation, $d(\mathrm{P}, \mathrm{Q}) = 2$, and the property $\mathfrak{P}$ is satisfied, where $\mathrm{P}_0 = \mathrm{P}$, $\mathrm{Q}_0 = \mathrm{Q}$ and $\mathrm{s} \geq 9$.

Define $\Gamma : \mathrm{P} \to \mathcal{CB}(\mathrm{Q})$ as follows

$$\Gamma\hbar = \begin{cases} 4, & \text{if } \hbar = \{0, 1\}, \\ \{3, 5\}, & \text{if } \hbar = 2, \end{cases}$$

and $\mathfrak{g} : \mathrm{P} \to \mathrm{P}$ is defined by the following

$$\mathfrak{g}\hbar = \begin{cases} 0, & \text{if } \hbar = 2, \\ 1, & \text{if } \hbar = 1, \\ 2, & \text{if } \hbar = 0. \end{cases}$$

Clearly, $\Gamma(\mathrm{P}_0) \subseteq \mathrm{Q}_0$ and $\mathfrak{g}(\mathrm{P}_0) \subseteq \mathrm{P}_0$. The pair $(\mathfrak{g}, \Gamma)$ satisfies $F_{C_P}-$proximal contraction

$$\tau + F(\phi(\mathcal{H}(\Gamma\hbar, \Gamma\eth))) \leq F(\phi(\mathcal{N}(u, v, \hbar, \eth))), \tag{3.10}$$

for all $u, v, \hbar, \eth \in \mathrm{P}$. Since,

$$\mathcal{D}(\mathfrak{g}1, \Gamma 0) = \mathrm{d}(\mathrm{P}, \mathrm{Q}),$$
$$\mathcal{D}(\mathfrak{g}0, \Gamma 2) = \mathrm{d}(\mathrm{P}, \mathrm{Q}),$$

where $u = 1$, $v = 0$, $\hbar = 0$ and $\eth = 2$. With simple calculations, we arrive at

$$\mathcal{H}(\Gamma 0, \Gamma 2) = \mathcal{H}(4, \{3, 5\}) = 3,$$

and

$$\begin{aligned} \mathcal{N}(1, 0, 0, 2) &= \max \begin{pmatrix} \mathrm{d}(\mathfrak{g}1, \mathfrak{g}0), \frac{\mathcal{D}(\mathfrak{g}1, \Gamma 1) - 9(2)}{9}, \\ \mathcal{D}^*(\mathfrak{g}0, \Gamma 1), \frac{\mathcal{D}(\mathfrak{g}0, \Gamma 0) - 9\mathcal{D}(\mathfrak{g}1, \Gamma 0)}{9} \end{pmatrix} \\ &= \max \begin{pmatrix} \mathrm{d}(1, 2), \frac{\mathcal{D}(1,4) - 18}{9} \\ \mathcal{D}^*(2, 4), \frac{\mathcal{D}(2,4) - 9\mathcal{D}(1,4)}{9}. \end{pmatrix} \end{aligned}$$

$$
\begin{aligned}
&= \max\left(5, -\frac{16}{9}, 6, -\frac{10}{9}\right) \\
&= 6.
\end{aligned}
$$

Taking $\digamma(\beta) = \log\beta + \beta$, $\phi = 2t$ and $\tau = 1$, we obtain

$$\phi(\mathcal{H}(\Gamma\hbar, \Gamma\eth)) = 6, \;\; \phi(\mathcal{N}(u, v, \hbar, \eth)) = 12.$$

Hence the inequality (3.10) becomes

$$\tau + \digamma(6) \leq \digamma(12).$$

This confirms that all the conditions of the Theorem 3.1 are contended, and 1 and 2 are the coincidence points of the pair of mapping $(\mathfrak{g}, \Gamma)$.

Corollary 3.1. *Let* P *and* Q *are the nonempty closed subsets of a complete b-metric space* $(\mathfrak{X}, d)$ *satisfying the* $\mathfrak{P}-$*property, where* A_0 *is nonempty. A continuous mappings* $\Gamma : \mathrm{P} \to \mathcal{CB}(\mathrm{Q})$ *with* $\Gamma(\mathrm{P}_0) \subseteq \mathrm{Q}_0$*. Further, if the mapping* Γ *satisfies* $\digamma_{B_P}-$*proximal contraction with alternating distance* ϕ*. Then* Γ *concedes a* $\mathbb{BPP}$*.*

Proof. If we take identity mapping $\mathfrak{g} = I_{\mathrm{P}}$ ($\mathfrak{g}$) is identity on P, then the rest of the proof lies on the similar lines as in the Theorem 3.1.

Note that, If we take $\mathrm{P} = \mathrm{Q} = \mathfrak{X}$ in Theorem 3.1, we derive the following results. □

Corollary 3.2. *Let* $\Gamma : \mathfrak{X} \to \mathcal{CB}(\mathfrak{X})$ *be a multivalued mapping on a complete b-metric space satisfying* ${}^{\tau}\digamma_{B_p}-$*proximal contraction with alternating distance* ϕ*. Then, there exists a fixed point of the mapping* Γ*.*

Example 3.2. Let $(\mathfrak{X}, d)$ be a complete $b-$metric space ($s = 2$), where $\mathfrak{X} = \{1, 2, 3, 4\}$, and d: $\mathfrak{X} \times \mathfrak{X} \to [0, \infty)$ is defined by the following:

$$\mathrm{d}(\hbar, \eth) = \begin{cases} 0, & \text{if } \hbar = \eth, \\ |\hbar - \eth|^2, & \text{otherwise.} \end{cases}$$

Define a multivalued map $\Gamma : \mathfrak{X} \to \mathcal{CB}(\mathfrak{X})$ as follows:

$$\Gamma\hbar = \begin{cases} 1, & \text{if } \hbar = \{1, 2, 3\}, \\ \{2, 3\}, & \text{if } \hbar = 4. \end{cases}$$

We show that the mapping Γ satisfies $\digamma_{B_p}-$proximal contraction

$$\tau + \digamma(\phi(\mathcal{H}(\Gamma\hbar, \Gamma\eth))) \leq \digamma(\phi(\mathcal{N}(\hbar, \eth))), \tag{3.11}$$

for all $\hbar, \eth \in \mathfrak{X}$.

Taking $\digamma(\beta) = \log\beta + \beta$, $\phi = 2t$ and $\tau = 1$, we have following cases:

Case (i). If $\hbar = 2$ and $\eth = 4$, we obtain

$$\mathcal{H}(\Gamma 2, \Gamma 4) = 1 \text{ and } \mathcal{N}(2, 4) = 9.$$

Now, $\phi(\mathcal{H}(\Gamma\hbar, \Gamma\eth)) = 2$ and $\phi(\mathcal{N}(\hbar, \eth)) = 18$, inequality (3.11) reduces to

$$\tau + \digamma(2) \leq \digamma(18).$$

Case (ii). If $\hbar = 3$ and $\eth = 4$ then, $\mathcal{H}(\Gamma 3, \Gamma 4) = 1$ and $\mathcal{N}(3, 4) = 9$, and we obtain, $\phi(\mathcal{H}(\Gamma\hbar, \Gamma\eth)) = 2$ and $\phi(\mathcal{N}(\hbar, \eth)) = 18$.

Consequently, (3.11) becomes

$$\tau + F(2) \leq F(18).$$

Hence, all the conditions of the Corollary 3.2 are met, and 1 is the fixed point of the multivalued mapping Γ.

Corollary 3.3. *If $b = 1$ with all the conditions utilized in Theorem 3.1, then we can obtain coincidence $\mathbb{BPP}$ in the setting of metric space.*

Corollary 3.4. *If $b = 1$ with all the assumptions incorporated in Theorem 3.1, then we can acquire $\mathbb{BPP}$ in the environment of metric space.*

4. Related results in single valued mappings

This portion is devoted to discuss some coincidence $\mathbb{BPP}$ results using single valued mappings within the $b-$metric framework.

Definition 4.1. A pair of mappings $(\mathfrak{g}, \Gamma)$, where $\Gamma : \mathrm{P} \to \mathrm{Q}$ and $\mathfrak{g} : \mathrm{P} \to \mathrm{P}$ is said to be a ${}^{\tau}F_{C_p}-$proximal contraction, if there exists $\tau \in (0, +\infty)$ such that,

$$\mathrm{d}\,(\mathfrak{g}u, \Gamma\hbar) = \mathrm{d}(\mathrm{P}, \mathrm{Q}),$$
$$\mathrm{d}\,(\mathfrak{g}v, \Gamma\eth) = \mathrm{d}(\mathrm{P}, \mathrm{Q})$$

implies that

$$\tau + F(\phi(\mathrm{d}(\Gamma\hbar, \Gamma\eth))) \leq F(\phi(\mathcal{N}(u, v, \hbar, \eth))),$$

where

$$\mathcal{N}(u, v, \hbar, \eth) = \max\{\mathrm{d}(\mathfrak{g}u, \mathfrak{g}v), \frac{\mathrm{d}(\mathfrak{g}u, \Gamma u) - \mathrm{s}\,\mathrm{d}(\mathrm{P}, \mathrm{Q})}{\mathrm{s}},$$
$$\mathrm{d}^*(\mathfrak{g}v, \Gamma u), \frac{\mathrm{d}(\mathfrak{g}\hbar, \Gamma v) - \mathrm{s}\,\mathrm{d}(\mathfrak{g}u, \Gamma v)}{\mathrm{s}}\}.$$

for all $u, v, \hbar, \eth$ in P.

Definition 4.2. A mapping $\Gamma : \mathrm{P} \to \mathrm{Q}$ is said to be $(\tau - F)_{B_P}-$proximal contraction, if there exists some $\tau \in (0, +\infty)$ such that,

$$\mathrm{d}\,(u, \Gamma\hbar) = \mathrm{d}(\mathrm{P}, \mathrm{Q}),$$
$$\mathrm{d}\,(v, \Gamma\eth) = \mathrm{d}(\mathrm{P}, \mathrm{Q})$$

implies that

$$\tau + F(\phi(\mathrm{d}(\Gamma\hbar, \Gamma\eth))) \leq F(\phi(\mathcal{N}(u, v, \hbar, \eth))),$$

where

$$\mathcal{N}(u, v, \hbar, \eth) = \max\{\mathrm{d}(u, v), \frac{\mathrm{d}(u, \Gamma u) - \mathrm{s}\,\mathrm{d}(\mathrm{P}, \mathrm{Q})}{\mathrm{s}},$$
$$\mathcal{D}^*(v, \Gamma u), \frac{\mathrm{d}(\hbar, \Gamma v) - \mathrm{s}\,\mathrm{d}(u, \Gamma v)}{\mathrm{s}}\},$$

for all $u, v, \hbar, \eth$ in P.

Note that, if we take $\mathfrak{g} = I_{\mathrm{P}}$ ($\mathfrak{g}$ as an identical mapping on P), then all ${}^{\tau}\digamma_{C_p}-$proximal contractions will shrink to ${}^{\tau}\digamma_{\mathrm{Q}_p}-$ proximal contraction.

Theorem 4.1. *Let $\Gamma : \mathrm{P} \to \mathrm{Q}$, $\mathfrak{g} : \mathrm{P} \to \mathrm{P}$ be mappings, where P and Q are nonempty closed subset of a complete $b-$metric space $(\mathfrak{X},d)$ satisfying the $\mathfrak{P}$-property with $\Gamma(\mathrm{P}_0) \subseteq \mathrm{Q}_0$ and $\mathrm{P}_0 \subseteq \mathfrak{g}(\mathrm{P}_0)$. If $(\mathfrak{g}, \Gamma)$ is a pair of continuous mappings, where $\mathfrak{g}$ is one-to-one mapping satisfying ${}^{\tau}\digamma_{C_p}-$proximal contraction with alternating distance ϕ. Then, $(\mathfrak{g}, \Gamma)$ concedes a coincidence point.*

Proof. Because any single valued mapping is a multi valued mapping, the remainder of the proof is identical to Theorem 3.1. $\square$

Example 4.1. Let $\mathfrak{X} = \{11, 12, 13, 14, 15, 16\}$ and consider a complete $b-$metric space $(\mathfrak{X},d)$, where d: $\mathfrak{X} \times \mathfrak{X} \to [0, \infty)$ is defined by the following:

$$\mathrm{d}(\hbar, \eth) = \begin{cases} 0, & \text{if } \hbar = \eth, \\ |\hbar - \eth|^2, & \text{otherwise.} \end{cases}$$

Suppose that $\mathrm{P} = \{11, 13, 15\}$ and $\mathrm{Q} = \{12, 14, 16\}$ are nonempty closed subsets of $b-$metric space $(\mathfrak{X},d)$. After simple calculation $d(\mathrm{P}, \mathrm{Q}) = 1$, satisfies the $\mathfrak{P}-$property, where $\mathrm{P}_0 = \mathrm{P}$, $\mathrm{Q}_0 = \mathrm{Q}$ and $\mathrm{s} \geq 5$. A mapping $\Gamma : \mathrm{P} \to \mathrm{Q}$ such that

$$\Gamma\hbar = \begin{cases} 12, \text{ if } \hbar = \{11, 13\}, \\ 14, \text{ if } \hbar = 15, \end{cases}$$

and $\mathfrak{g} : \mathrm{P} \to \mathrm{P}$

$$\mathfrak{g}\hbar = \begin{cases} 11, & \text{if } \hbar = 11, \\ 13, & \text{if } \hbar = 15, \\ 15, & \text{if } \hbar = 13. \end{cases}$$

Clearly, $\Gamma(\mathrm{P}_0) \subseteq \mathrm{Q}_0$ and $\mathfrak{g}(\mathrm{P}_0) \subseteq \mathrm{P}_0$. Now, the pair $(\mathfrak{g}, \Gamma)$ satisfies ${}^{\tau}\digamma_{C_p}-$proximal contraction

$$\tau + \digamma(\phi(\mathrm{d}(\Gamma\hbar, \Gamma \mathrm{y}))) \leq \digamma(\phi(\mathcal{N}(u, v, \hbar, \eth))), \tag{4.1}$$

for all $u, v, \hbar, \eth \in \mathrm{P}$. Since,

$$\mathrm{d}(\mathfrak{g}11, \Gamma 13) = \mathrm{d}(\mathrm{P}, \mathrm{Q}),$$
$$\mathrm{d}(\mathfrak{g}13, \Gamma 15) = \mathrm{d}(\mathrm{P}, \mathrm{Q}),$$

where $u = 11$, $v = 13$, $\hbar = 13$ and $\eth = 15$. With basic analysis, we obtain

$$\mathrm{d}(\Gamma 13, \Gamma 15) = \mathrm{d}(12, 14) = 4,$$

and

$$M(11, 13, 13, 15) = \max \begin{pmatrix} \mathrm{d}(\mathfrak{g}11, \mathfrak{g}13), \frac{\mathrm{d}(\mathfrak{g}11,\Gamma 11)-5(1)}{5}, \\ \mathrm{d}^*(\mathfrak{g}13, \Gamma 11), \frac{\mathrm{d}(\mathfrak{g}13,\Gamma 13)-5\mathrm{d}(\mathfrak{g}11,\Gamma 13)}{5} \end{pmatrix}$$

$$
\begin{aligned}
&= \max\begin{pmatrix} \mathrm{d}(11,15), \frac{\mathrm{d}(11,12)-5}{5} \\ \mathrm{d}^*(15,12), \frac{\mathrm{d}(15,12)-5\mathrm{d}(11,12)}{5}. \end{pmatrix} \\
&= \max\left(16, -\frac{4}{5}, 8, \frac{4}{5}\right) \\
&= 16.
\end{aligned}
$$

Next, consider a function $\digamma$ as follows

$$\digamma(\beta) = \log\beta + \beta.$$

For $\phi = 2t$ and $\tau = 1$, we conclude $\phi(d(\Gamma\hbar, \Gamma\eth)) = 8$, $\phi(N(u, v, \hbar, \eth)) = 32$.

The inequality (4.1) turn to

$$\tau + \digamma(16) \leq \digamma(32).$$

Hence, all the criteria of the Theorem 4.1 are fulfilled and 11 is the coincidence point of $(\mathfrak{g}, \Gamma)$.

Corollary 4.1. *Let* $\Gamma : \mathrm{P} \to \mathrm{Q}$ *be a mapping, where* P *and* Q *are nonempty closed subsets of a complete* $b-$*metric space* $(\mathfrak{X}$,$d)$ *satisfying the* $\mathfrak{P}-$*property with* $\Gamma(\mathrm{P}_0) \subseteq \mathrm{Q}_0$. *If a continuous mappings* Γ *satisfies* ${}^{\tau}\digamma_{B_p}-$*proximal contraction with alternating distance* ϕ. *Then,* Γ *concedes a* $\mathbb{BPP}$.

Proof. If we take identical mapping $\mathfrak{g} = I_{\mathrm{P}}$ ($\mathfrak{g}$ is identity on P), the rest of the proof is similar as of Theorem 3.1. □

Corollary 4.2. *If* $b = 1$ *with all the conditions utilized in Theorem 4.1, then we can obtain coincidence* $\mathbb{BPP}$ *in the setting of metric space.*

Corollary 4.3. *If* $b = 1$ *with all the assumptions incorporated in Corollary 4.1, then we can acquire* $\mathbb{BPP}$ *in the environment of metric space.*

5. Fixed point results

In this section, we obtain fixed point result, if we take $\mathrm{P} = \mathrm{Q} = \mathfrak{X}$ in Theorem 4.1. The following definition is handy in achieving such a novel convergence result (Theorem 5.1).

Definition 5.1. A mapping $\Gamma : \mathfrak{X} \to \mathfrak{X}$ is said to be $(\tau - \digamma)_{F_P}$ contraction, if there exists a there exists a $\tau \in (0, +\infty)$ such that,

$$\tau + \digamma(\phi(\mathrm{d}(\Gamma\hbar, \Gamma\eth))) \leq \digamma(\phi(\mathcal{N}(\hbar, \eth))),$$

where

$$\mathcal{N}(\hbar, \eth) = \max\{\mathrm{d}(\hbar, \eth), \frac{\mathrm{d}(\hbar, \Gamma\hbar)}{s}, \mathrm{d}(\eth, \Gamma\hbar), -\mathrm{d}(\hbar, \Gamma\eth)\},$$

for all $\hbar, \eth$ in $\mathfrak{X}$.

Theorem 5.1. *Let* $\Gamma : \mathfrak{X} \to \mathfrak{X}$ *be a single valued mapping on a complete* $b-$*metric space satisfying* ${}^{\tau}\digamma_{F_P}-$*proximal contraction with alternating distance* ϕ. *Then,* Γ *concedes a fixed point.*

Proof. Taking P = Q = $\mathfrak{X}$, the proof follows due to Theorem 4.1. $\square$

Example 5.1. If we take all the conditions of the Example 3.2 and define a mapping $\Gamma : \mathfrak{X} \to \mathfrak{X}$ such that

$$\Gamma\hbar = \begin{cases} 1, \text{ if } \hbar = \{1,2,3\}, \\ 2, \text{ if } \hbar = 4. \end{cases}$$

To show that the mapping Γ satisfies ${}^{\tau}F_{F_p}$−proximal contraction

$$\tau + F(\phi(\mathrm{d}(\Gamma\hbar, \Gamma\eth))) \leq F(\phi(\mathcal{N}(\hbar, \eth))), \tag{5.1}$$

for all $\hbar, \eth \in \mathfrak{X}$.

We consider the following cases with $F(\beta) = \log\beta + \beta$

Case (i). If $\hbar = 2$ and $\eth = 4$ then, $d(\Gamma 2, \Gamma 4) = 1$ and $\mathcal{N}(2,4) = 9$.

Case (ii). If $\hbar = 3$ and $\eth = 4$ then, $d(\Gamma 3, \Gamma 4) = 1$ and $\mathcal{N}(3,4) = 9$.

Now, choosing $\phi = 2t$ and $\tau = 1$, we get $\phi(d(\Gamma\hbar, \Gamma\eth)) = 2$, $\phi(\mathcal{N}(\hbar, \eth)) = 18$, consequently (5.1) becomes

$$\tau + F(2) \leq F(18).$$

Hence, all the assumptions of Theorem 5.1 are fulfilled, and Γ concedes 1 as the fixed point.

Corollary 5.1. *If $b = 1$ with all the conditions utilized in Theorem 5.1, then we obtain coincidence $\mathbb{BPP}$ in the setting of metric space.*

6. An application to the equation of motion

Let a class of all real-valued continuous functions on $[0,1]$ be denoted by $\mathfrak{X}$ (that is, $\mathfrak{X} = C[[0,1], \mathbb{R}]$). Consider the b-metric d: $\mathfrak{X} \times \mathfrak{X} \to \mathbb{R}$ defined by the following:

$$\mathrm{d}(\hbar, \eth) = \sup_{\curlywedge \in [0,1]} |\hbar(\curlywedge) - \eth(\curlywedge)|^2,$$

for all $\hbar, \eth \in \mathfrak{X}$ and $\curlywedge \in [a,b]$.

Clearly, ($\mathfrak{X}$,d) is complete $b-$metric space for with s = 2.

Problem Statement: A particle of mass m is at rest at $\hbar = 0$, $\curlywedge = 0$. A force $\mathfrak{L}$ starts activity on it in a particular direction such that its velocity immediately jumps from 0 to 1 after $\curlywedge = 0$. Find the position of the particle at time $\curlywedge$. The corresponding motion of the particle is governed by the following second order differential equation.

$$\begin{cases} m\frac{\mathrm{d}^2\hbar}{\mathrm{d}\curlywedge^2} = \mathfrak{L}(\curlywedge, \hbar(\curlywedge)), \\ \hbar(0) = 0, \ \hbar^{'}(0) = 1, \end{cases} \tag{6.1}$$

for all $\curlywedge \in [0,1]$ and $\mathfrak{L} : [0,1] \times \mathbb{R} \to \mathbb{R}$, is a continuous function. We are interested in finding the solution of the equation (6.1), which tells us the nature of the motion of the particle, i.e., how the particle moves.

The Green's function of the problem defined in (6.1) is equivalent to:

$$\dot{\mathrm{G}}(\curlywedge, s) = \begin{cases} \curlywedge & ; 1 \geq s \geq \curlywedge \geq 0, \\ 2\curlywedge - s & ; 1 \geq \curlywedge \geq s \geq 0. \end{cases}$$

We now prove the existence of a solution of the second order differential equation (6.1).

Theorem 6.1. *Suppose for the mappings* $\mathfrak{L} : [0,1] \times \mathbb{R} \to \mathbb{R}$ *and* $\phi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$*, the following axioms are contended:*

1. $|\mathfrak{L}(\curlywedge, a) - \mathfrak{L}(\curlywedge, b)| \leq \sup_{a,b \in [0,1]} |a - b|$, for all $\curlywedge \in [0,1]$ with $\phi(a,b) \geq 0$.
2. For all $\curlywedge \in [0,1]$ and $\hbar_0 \in \mathfrak{X}$,
$$\phi(\Gamma \hbar_0(\curlywedge), \Gamma \hbar_0(\curlywedge)) \geq 0,$$
where $\Gamma : \mathfrak{X} \to \mathfrak{X}$. Then second order differential equation (6.1) governing the equation of motion of a particle has a solution in $\mathfrak{X}$.

Proof. We note that the solution of (6.1) is equivalent to finding the solution of the following integral equation

$$\hbar(\curlywedge) = \int_0^1 \dot{\mathrm{G}}(\curlywedge, s)\mathfrak{L}(s, \hbar(s))ds, \quad \curlywedge \in [0,1]. \tag{6.2}$$

Integral equation (6.2) is equivalent to the second order IVP (6.1) which is constructed with the help of Green's function $\dot{\mathrm{G}}(\curlywedge, s)$.

We define a self-map $\Gamma : \mathfrak{X} \to \mathfrak{X}$ by the following

$$\Gamma\hbar(\curlywedge) = \int_0^1 \dot{\mathrm{G}}(\curlywedge, s)\mathfrak{L}(s, \hbar(s))ds.$$

We have,

$$\begin{aligned} |\Gamma\hbar(\curlywedge) - \Gamma\eth(\curlywedge)| &= \left| \int_0^1 \dot{\mathrm{G}}(\curlywedge, s)\mathfrak{L}(s, \hbar(s)) - \int_0^1 \dot{\mathrm{G}}(\curlywedge, s)\mathfrak{L}(s, \eth(s)) \right| \\ &= \int_0^1 \dot{\mathrm{G}}(\curlywedge, s) \left| \mathfrak{L}(s, \hbar(s)) - \mathfrak{L}(s, \eth(s)) \right| ds \\ &\leq \int_0^1 \dot{\mathrm{G}}(\curlywedge, s) \sup_{\curlywedge \in [0,1]} |\hbar(s) - \eth(s)| \, ds \\ &\leq \sup_{\curlywedge \in [0,1]} |\hbar(\curlywedge) - \eth(\curlywedge)| \int_0^1 \dot{\mathrm{G}}(\curlywedge, s)ds. \end{aligned}$$

This implies that

$$\sup_{\curlywedge \in [0,1]} |\Gamma\hbar(\curlywedge) - \Gamma\eth(\curlywedge)|^2 \leq \sup_{\curlywedge \in [0,1]} |\hbar(\curlywedge) - \eth(\curlywedge)|^2 \sup_{\curlywedge \in [0,1]} \left\{ \int_0^1 \dot{\mathrm{G}}(\curlywedge, s)ds \right\}^2.$$

Observe that $\int_0^1 \dot{G}(\curlywedge, s)ds = \frac{1-\curlywedge^2}{2}$, for all $\curlywedge \in [0,1]$, we infer

$$\sup_{\curlywedge \in [0,1]} \left\{ \int_0^1 \dot{\mathrm{G}}(\curlywedge, s)ds \right\}^2 = \frac{1}{4}.$$

This corresponds to saying

$$\sup_{\curlywedge \in [0,1]} |\Gamma\hbar(\curlywedge) - \Gamma\eth(\curlywedge)|^2 \leq \frac{1}{4} \sup_{\curlywedge \in [0,1]} |\hbar(\curlywedge) - \eth(\curlywedge)|^2.$$

Inducing logarithm, we obtain

$$\log 4 + \log\left(\mathrm{d}(\Gamma\hbar, \Gamma\eth)\right) \leq \log\left(\mathrm{d}(\hbar, \eth)\right).$$

Now, we consider the function $\digamma : [0, \infty) \to \mathbb{R}$ defined by $\digamma(\hbar) = \log \hbar$, we get

$$\log 4 + \digamma\left(\mathrm{d}(\Gamma\hbar, \Gamma\eth)\right) \leq \digamma\left(\mathrm{d}(\hbar, \eth)\right). \tag{6.3}$$

Taking $\log 4 = \tau > o$, (6.3) can be put as

$$\tau + \digamma\left(\mathrm{d}(\Gamma\hbar, \Gamma\eth)\right) \leq \digamma\left(\mathcal{N}(\hbar, \eth)\right).$$

Noting that $\phi > 0$ is continuous and monotonically increasing, we can write

$$\tau + \digamma\left(\phi\, \mathrm{d}(\Gamma\hbar, \Gamma\eth)\right) \leq \digamma\left(\phi\mathcal{N}(\hbar, \eth)\right).$$

Thus, all the conditions of Theorem 5.1 are satisfied. Hence, Γ concedes a fixed point, endorsing that the second order differential equation (6.1) governing the equation of motion has a solution. □

Conclusion

Using the alternating distance function ϕ, we studied the presence of coincidence point outcomes via the multivalued notion over $b-$metric space. Furthermore, examples for multivalued coincidence points, single-valued coincidence points, and fixed point outcomes are provided to enhance our primary results. Finally, we used the acquired result to demonstrate the existence of a solution to a particular form of second-order boundary value problem expressing the equation of motion. The findings have practical and theoretical implications for academics working on fixed point theory applications and scientists dealing with mechanical and industrial issues.

Open Questions.

1. Under what conditions we can obtain coincidence best proximity results for multivalued mappings within the context of b-metric like space?
2. Can we drive best proximity points under the same constraints without taking into account the altering distance functions?
3. Can we obtain Theorem 3.1 and Theorem 4.1 for Kannan or Reich type mappings with applications?

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Competing interests

The authors declare that they have no competing interests.

Author contributions

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