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ON THE AVERAGE OPERATORS, OSCILLATORY INTEGRALS, SINGULAR INTEGRALS AND THEIR APPLICATIONS*

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Abstract This paper is a survey on three types of integral operators and their applications based on the research work of the authors and their cooperators in the recent decade. The first type is the average operator, including Hardy operators, Hausdorff operators and Hardy-Littlewood maximal operators. The second is the oscillatory type integral operator, such as one-sided oscillatory integral operators, Fourier transforms, fractional Fourier transforms and linear canonical transforms. The third type is the singular integral operators, including Hilbert transform, Riesz transforms, Cauchy type operators, etc. We mainly investigate their norm estimations, boundedness, weighted estimations, compactness characterizations and their properties in various function spaces.

Keywords Average operator, oscillatory integral, singular integral, function space.

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Contents

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1. Introduction

In [68], Stein mentioned that there are three important types of operators in harmonic analysis, namely average operators, oscillatory integrals and singular integrals. The average operators, including Hardy operators, Hausdorff operators, Hardy-Littlewood maximal operators, etc. Properties of these operators in Euclidean space have been well studied, and a large number of results have been obtained. In this paper, we also focus on the boundedness of such operators and their commutators in the setting of homogeneous groups and local fields.

For a locally integrable function b and an integral operator T, the commutator operator for a proper function f can be defined by $[b,T]f := T_b(f) = bT(f) - T(bf)$. During the past decade, the theory of commutator has developed in a variety of directions, such as the role in studying the regularity of solutions to PDEs, see e.g. [42, 49, 53–55], the characterizations of function spaces see e.g. [15, 56–59]. On the Euclidean space, we also obtained some different characterizations of central Campanato spaces via the boundedness and compactness of commutators of Hardy type operators. More precisely, we proved in [52] that commutators of Hardy operators, including the fractional Hardy operator, are compact operators on $L^p(\mathbb{R}^n)(1 < p < \infty)$ spaces if and only if the symbol functions of commutators belong to $CVMO(\mathbb{R}^n)$ spaces (the central $BMO(\mathbb{R}^n)$ closure of $C_c^\infty(\mathbb{R}^n)$). In [27], we addressed two characterizations of $BMO(\mathbb{R}^n)$ type space via the commutators of Hardy operators with homogeneous kernels on Lebesgue spaces by exploiting the center symmetry of Hardy operator deeply and by a more explicit decomposition of the operator and the kernel function. In [29], we obtained characterizations of commutators of several versions of maximal functions on spaces of homogeneous type. In addition, we provided weighted version of the commutator theorems by establishing new characterizations of the weighted BMO space.

On the p-adic field, we established sharp estimates for the p-adic Hardy and Hardy-Littlewood-Pólya operators on power-weighted Lebesgue spaces. Also, we proved that the commutators generated by the p-adic Hardy operators, the fractional p-adic Hardy operators, Hardy-Littlewood-Pólya operators and the central BMO functions are bounded on $L^q(|x|_p^\alpha dx)$, more generally, on Herz spaces [24, 30]. For the weighted version, in [71], we established necessary and sufficient conditions for boundedness of weighted p-adic Hardy operators on p-adic Morrey spaces, p-adic central Morrey spaces and p-adic λ-central BMO spaces, respectively, and obtained their sharp bounds. We also gave the characterization of weight functions for which the commutators generated by weighted p-adic Hardy operators and λ-central BMO functions are bounded on the p-adic central Morrey spaces. This result is different from that on Euclidean spaces due to the special structure of p-adic integers.

In the setting of the Heisenberg group, we obtained the sharp $(p,p)(1 < p \leq \infty)$ estimate for the Hardy operator, and got the best constant in the weak type $(1,1)$ inequality for the Hardy operator. We also established the boundedness for the Hardy operator from H^1 to L^1. Moreover, for $1 \leq p < \infty$, we described the difference between M_p weights and A_p weights and obtained the characterization of such weights using the weighted Hardy inequalities [73]. For the weighted version, we characterized the weights w for which the weighted Hardy operator H_w is bounded on $L^p(\mathbb{H}^n)$, $1 \leq p \leq \infty$, and on $BMO(\mathbb{H}^n)$. Meanwhile, the corresponding operator norm in each case was derived. Furthermore, we introduced a type of weighted multilinear Hardy operators and obtained the characterization of their weights for which the weighted multilinear Hardy operators are bounded on the product of Lebesgue spaces on the Heisenberg group. In addition, the corresponding norms were worked out [14].

In [46, 47], we studied the Hausdorff operator, defined via a general linear mapping A, on weighted Herz spaces and weighted Morrey spaces in the setting of the Heisenberg group. Under some assumptions on the mapping A, we established its sharp boundedness on power-weighted Herz spaces, power-weighted Lebesgue spaces and power-weighted Morrey spaces on the Heisenberg group. In addition, we obtained the boundedness of commutators of such Hausdorff operators on power-weighted Morrey spaces on the Heisenberg group.

We defined weighted Hardy spaces by means of their atomic characterization on the Heisenberg group in [72], and established the sharp boundedness of Hausdorff operators on power-weighted Hardy spaces. Moreover, we obtained sufficient and necessary conditions for the boundedness of Hausdorff operators on local Hardy spaces on the Heisenberg group. In [70], we obtained the boundedness from Lebesgue spaces to Hardy spaces for fractional Hausdorff operators and their compositions with Riesz transforms. We also established the boundedness for two kinds of special Hausdorff operators, the Hausdorff-Poisson operator and the Hausdorff-Gauss operator, on Hardy spaces.

The theory of singular integrals and function spaces has a central role in modern harmonic analysis with extensive applications to other fields such as PDEs, capacity theory and potential theory, see [9, 16, 39, 50, 60–62, 64, 66, 67, 75]. Using the properties for the Hilbert transform and Clifford analytic techniques, Gu et. al [32, 33, 38] established the Riemann-Hilbert problems in Clifford value Hölder spaces and Lebesgue p-integrable spaces. Based on the Newton embedding method, Gu et. al [34–37] also obtained the existence and uniqueness for the nonlinear Riemann-Hilbert problem and also gave the error estimation for the approximate solutions in the Newton embedding procedure.

In section three, we focus on oscillatory type integrals related to the one defined by Ricci and Stein [45], which have been an essential part of harmonic analysis; three sections are devoted to them in the book [68]. The Fourier transform (one of the most important and powerful tools in theoretical and applied mathematics), the Bochner-Riesz means and the Radon transform are some versions of oscillatory integrals. In this section, we considered one-sided oscillatory integral operators, fractional Fourier transforms, and linear canonical transforms. The one-sided weighted classes of Muckenhoupt type were used to study the weighted weak type (1,1) norm inequalities for the one-sided oscillatory singular integrals. Furthermore, we gave the weighted norm inequalities for commutators of the one-sided oscillatory integral operators.

Motivated by analyzing non-stationary signals, there are many results about the Fourier transform of fractional order. Nowadays, the fractional Fourier transform (FRFT for short) has found various applications in scientific research and engineering technology, such as swept filter, artificial neural network, wavelet transform, time-frequency analysis, time-varying filtering, complex transmission, PDEs and so on (see, e.g., [5, 22, 76–80]). We studied the L^p theory of the FRFT and FRFT properties of L^1 functions via the introduction of a suitable chirp operator. We solved the problems of convergence and studied the FRFT inversion problem via approximation by the fractional Gauss and Abel means. Moreover, the regularity of fractional convolution and results on pointwise convergence of FRFT means and the L^p multiplier results and a Littlewood-Paley theorem associated with FRFT were also considered.

In section four, we collect our works for some singular integral operators in complex analysis, including Hilbert transforms, Riesz transforms, Cauchy type operators, Cauchy-Szegö projection operators, etc. Further study on singular integrals and related PDEs leads to commutators of these operators. There are quite a number of recent results on the characterizations of commutators in the above forms for singular integrals in different settings. Inspired by these classical results above, it is natural to ask whether these results hold on stratified Lie groups, especially on Heisenberg groups. Note that in several complex variables, the Heisenberg group $\mathbb{H}^n$ is the boundary of the Siegel upper half space, which is holomorphically equivalent to the unit sphere in $\mathbb{C}^n$. And hence, the role of the Riesz transform on $\mathbb{H}^n$ is similar to the role of the Hilbert transform on the real line. There are also some other works about singular integrals, see e.g. [2, 21, 63] for multilinear integrals and [43] for Marcinkiewicz integrals.

Duong-Li-Li-Wick [18] established the characterization of the BMO space on stratified nilpotent Lie groups via the boundedness of the commutator of the Riesz transforms. This extends the well-known Coifman, Rochberg, Weiss theorem [15] on Euclidean space to the setting of stratified Lie groups. In [7], we obtained the characterization of compactness of the commutators of R_j with respect to VMO, the space of functions with vanishing mean oscillation on stratified Lie group, which extends the well-known result of Uchiyama [69] on Euclidean spaces.

The quaternionic Heisenberg group $\mathcal{H}^{n-1}$ plays a fundamental role in quaternionic analysis and geometry. Its analytic and geometric behaviors are different from the usual Heisenberg group in many aspects, e.g., there is no nontrivial quasiconformal mapping between the quaternionic Heisenberg group while quasiconformal mappings between Heisenberg groups are abundant. The quaternionic Siegel upper half space can be identified with the quaternionic Heisenberg group $\mathcal{H}^{n-1}$. In [6], Chang, Markina and Wang determined the kernel of the Cauchy-Szegő projection on quaternionic Siegel upper half space. We further obtained its explicit formula [4], and then based on this, we proved that the Cauchy–Szegő projection on quaternionic Heisenberg group is a Calderón–Zygmund operator. We also obtained a suitable version of pointwise lower bound for the kernel, which further implies the characterization of the boundedness and compactness of commutators of the Cauchy–Szegő operator via the BMO and VMO spaces on quaternionic Heisenberg group, respectively.

Recently, Lanzani and Stein [40] studied the Cauchy–Szegő projection operator in a bounded strongly pseudoconvex domain D in $\mathbb{C}^n$, whose boundary bD satisfies the minimum regularity condition of class C^2. They obtained the $L^p(bD)$ boundedness $(1<p<\infty)$ of a family of Cauchy integrals $\{\mathcal{C}_\epsilon\}_\epsilon$, We studied the commutator of these Cauchy type integrals, and showed that the commutator $[b,\mathcal{C}]$ is bounded on $L^p(bD)$ $(1<p<\infty)$ or on weighted Morrey space $L^{p,\kappa}_v(bD)$ $(v\in A_p, 1<p<\infty)$ if and only if b is in the BMO space on bD. Moreover, the commutator $[b,\mathcal{C}]$ is compact on $L^p(bD)$ $(1<p<\infty)$ or on weighted Morrey space $L^{p,\kappa}_v(bD)$ $(v\in A_p, 1<p<\infty)$ if and only if b is in the VMO space on bD. Our method can also be applied to the commutator of Cauchy–Leray integral in a bounded, strongly $\mathbb{C}$-linearly convex domain D in $\mathbb{C}^n$ with the boundary bD satisfying the minimum regularity $C^{1,1}$.

2. Average operators

We begin with some function spaces which are needed in this section. Let $1\le p<\infty$. Then for any ball $B\subset\mathbb{R}^n$, the classical Campanato space and Morrey space are defined by the following norms (see e.g. [56, 57])

$$\|f\|_{\mathcal{C}^{p,\lambda}(\mathbb{R}^n)}=\sup_B\frac{1}{|B|^\lambda}\left(\frac{1}{|B|}\int_B|f-f_B|^pdx\right)^{1/p},\quad -1/p<\lambda<1/n$$

and

$$\|f\|_{M^{p,\lambda}(\mathbb{R}^n)}=\sup_B\frac{1}{|B|^\lambda}\left(\frac{1}{|B|}\int_B|f(x)|^pdx\right)^{1/p},\quad -1/p<\lambda<0,$$

respectively. The excellent structures of the spaces $\mathcal{C}^{p,\lambda}(\mathbb{R}^n)$ and $M^{p,\lambda}(\mathbb{R}^n)$ render them useful in the study of PDEs and the Sobolev embedding theorems. The central version of these spaces are defined by the following norm [58]

$$\|f\|_{\dot{\mathcal{C}}^{p,\lambda}(\mathbb{R}^n)}=:\sup_{r>0}\frac{1}{|B(0,r)|^\lambda}\left(\frac{1}{|B(0,r)|}\int_{B(0,r)}|f-f_{B(0,r)}|^pdx\right)^{1/p}$$

and

$$\|f\|_{\dot{M}^{p,\lambda}(\mathbb{R}^n)}=\sup_{r>0}\frac{1}{|B(0,r)|^\lambda}\left(\frac{1}{|B(0,r)|}\int_{B(0,r)}|f|^pdx\right)^{1/p}.$$

If $\lambda=0$, then $\dot{\mathcal{C}}^{p,0}(\mathbb{R}^n)=C\dot{M}O^p(\mathbb{R}^n)$ (the bounded central mean oscillation function space) with the equivalent norm

$$\|f\|_{C\dot{M}O^p(\mathbb{R}^n)}=\sup_{r>0}\inf_{c\in\mathbb{C}}\left(\frac{1}{|B(0,r)|}|f(y)-c|^pdy\right)^{1/p}.$$

If $0<\lambda<\frac{1}{p}$, then $\dot{\mathcal{C}}^{p,0}(\mathbb{R}^n)=C\dot{M}O^{p,\lambda}(\mathbb{R}^n)$(the λ-central bounded mean oscillation space) [24].

2.1. Average operators on Euclidean fields

In this subsection, we focus on Hardy operators in $\mathbb{R}^n$. The $n-$dimensional Hardy operator and its dual operator on $\mathbb{R}^n$ are defined by [13] as

$$Hf(x)=\frac{1}{|x|^n}\int_{|y|<|x|}f(y)dy,\ x\in\mathbb{R}^n\setminus\{0\}$$

and

$$H^*f(x) = \int_{|y|\geq|x|} \frac{f(y)}{|y|^n} dy,\ x \in \mathbb{R}^n \setminus \{0\},$$

respectively. In [24], we gave some characterizations of $C\dot{M}O^p(\mathbb{R}^n)$ for $1 < p < \infty$ by the boundedness of H_b and H_b^*, on Lebesgue spaces as:

$$b \in C\dot{M}O^{\max\{p,p'\}}(\mathbb{R}^n) \Longleftrightarrow H_b(H_b^*) : L^p(\mathbb{R}^n) \longrightarrow L^p(\mathbb{R}^n),$$

where $\frac{1}{p} + \frac{1}{p'} = 1$. Zhao and Lu [84] characterized $C\dot{M}O^{p,\lambda}(\mathbb{R}^n)$ with more restrictions on λ as

$$b \in C\dot{M}O^{\max\{p,p'\},\lambda}(\mathbb{R}^n) \Longleftrightarrow H_b(H_b^*) : L^p(\mathbb{R}^n) \longrightarrow L^q(\mathbb{R}^n),$$
$$0 \leq \lambda = \frac{1}{p} - \frac{1}{q},\ 1 < p < \infty.$$

We settled the characterization for the case $-1/p < \lambda < 0$ under the assumption that b satisfies the following mean value inequality [56]

$$\sup_{B \ni x} |f(x) - f_B| \leq \frac{C}{|B|} \int_B |f(x) - f_B| dx.$$

In [58], we obtained the boundedness characterization of $\dot{\mathcal{C}}^{p,\lambda}(\mathbb{R}^n)$ as follows.

Theorem 2.1. *Let $1 < p < \infty, -1/p < \lambda < 0$, $-1/p_i < \lambda_i < 0, i = 1, 2$, $1/p = \sum_{i=1}^{2} 1/p_i$, $\lambda = \sum_{i=1}^{2} \lambda_i$ and let b satisfy the mean value inequality. Then the following statements are equivalent:*
(a) $b \in \dot{\mathcal{C}}^{p_1,\lambda_1}(\mathbb{R}^n)$;
(b) Both H_b and H_b^* are bounded operators from $\dot{M}^{p_2,\lambda_2}(\mathbb{R}^n)$ to $\dot{M}^{p,\lambda}(\mathbb{R}^n)$.

The mean value inequality is very important in the proof of the Theorem 2.1. If we drop this assumption, the following result can be deduced under some stronger condition.

Theorem 2.2. *Let $2 < p < \infty$ and $-1/(2p) < \lambda < 0$. Then the following statements are equivalent:*
(a) $b \in \dot{\mathcal{C}}^{p,\lambda}(\mathbb{R}^n)$;
(b) Both H_b and H_b^* are bounded operators from $\dot{M}^{p,\lambda}(\mathbb{R}^n)$ to $\dot{M}^{p,2\lambda}(\mathbb{R}^n)$.

For $0 < \alpha < n$, the $n-$dimensional fractional Hardy operator and its dual operator are defined by (see e.g. [24])

$$H_\alpha f(x) = \frac{1}{|x|^{n-\alpha}} \int_{|y|<|x|} f(y) dy,\ x \in \mathbb{R}^n \setminus \{0\},$$

and

$$H_\alpha^* f(x) = \int_{|y|\geq|x|} \frac{f(y)}{|y|^{n-\alpha}} dy,\ x \in \mathbb{R}^n \setminus \{0\},$$

respectively. In [24], we gave some characterizations of $C\dot{M}O^p(\mathbb{R}^n)(1 < p < \infty)$ via the boundedness of $H_{\alpha,b}$ on both Lebesgue spaces and Herz spaces. For $\lambda < 0$, we gave some characterizations of $\dot{\mathcal{C}}^{p,\lambda}(\mathbb{R}^n)$ in [58] as follows.

Theorem 2.3. *Let p, λ, $p_i, \lambda_i, i = 1, 2$, b as in Theorem 2.1, $0 < \alpha < \min\{n(1 - 1/p), n(\lambda_2 + 1/p_2)\}$ and let $\beta = \lambda_2 - \alpha/n$. Then the following statements are equivalent:*
(a) $b \in \dot{\mathcal{C}}^{p_1,\lambda_1}(\mathbb{R}^n)$;
(b) *Both $H_{\alpha,b}$ and $H^*_{\alpha,b}$ are bounded operators from $\dot{M}^{p_2,\beta}(\mathbb{R}^n)$ to $\dot{M}^{p,\lambda}(\mathbb{R}^n)$.*

Theorem 2.4. *Let $2 < p < \infty, -1/(2p) < \lambda < 0$, $0 < \alpha < \min\{n(1 - 1/p), n(\lambda + 1/p)\}$ and let $\beta = \lambda - \alpha/n$. Then the following statements are equivalent:*
(a) $b \in \dot{\mathcal{C}}^{p,\lambda}(\mathbb{R}^n)$;
(b) *Both $H_{\alpha,b}$ and $H^*_{\alpha,b}$ are bounded operators from $\dot{M}^{p,\beta}(\mathbb{R}^n)$ to $\dot{M}^{p,2\lambda}(\mathbb{R}^n)$.*

The study for the compactness of operators can be traced to Uchiyama [69] where the characterization of L^p-compactness of T_b was obtained when T is the Calderón-Zygmund singular integral and $b \in VMO(\mathbb{R}^n)$. Since then, many results were obtained for the compactness of commutators on different function spaces, see e.g. [1, 10, 11]. In [52], we explored the compactness of H_b and H_b^* on $L^p(\mathbb{R}^n)$ space as

$$b \in CVMO(\mathbb{R}^n) \Longleftrightarrow \text{Both } [b, H] \text{ and } [b, H^*] \text{ are compact on } L^p(\mathbb{R}^n),$$

where $CVMO(\mathbb{R}^n)$ denotes the $CBMO(\mathbb{R}^n)$ closure of $C_c^\infty(\mathbb{R}^n)$.

In [27], we considered the following Hardy operator with homogeneous kernel, which was introduced by Fu, Lu and Zhao in [28].

$$H_\Omega f(x) = \frac{1}{|x|^n} \int_{|y|<|x|} \Omega(x - y) f(y) dy, \;\; x \in \mathbb{R}^n \setminus \{0\},$$

where Ω satisfies

$$\Omega(tx) = \Omega(x) \quad \forall\, t > 0 \;\; \& \; x \in \mathbb{R}^n; \tag{2.1}$$

$$\int_{\mathbb{S}^{n-1}} \Omega(x') d\sigma(x') = 0; \tag{2.2}$$

$$\Omega \in L^q(\mathbb{S}^{n-1}) \quad \forall\, q \geq 1. \tag{2.3}$$

Similarly, we can define the dual operator of H_Ω as

$$H^*_\Omega f(x) = \int_{|y|\geq|x|} \frac{\Omega(x - y) f(y)}{|y|^n} dy.$$

So far, there is little information on the function characterizations by the boundedness and compactness of commutators for H_Ω and H^*_Ω, this is because known results for the inverse function characterizations highly depend on the smoothness of the kernels, see e.g. [56] for $\Omega \in C^\infty(\mathbb{S}^{n-1})$. Since then, much work was being spent on weakening the conditions of Ω, for example the following so-called Hölder condition of the log type

$$|\Omega(x') - \Omega(y')| \leq \frac{A}{\left(\log \frac{2}{|x'-y'|}\right)^\gamma} \quad \text{with} \;\; A > 0, \;\; \gamma > 1 \;\; \text{and} \;\; x', y' \in \mathbb{S}^{n-1}. \tag{2.4}$$

If Ω satisfies (2.4), then $\Omega \in L^q(\mathbb{S}^{n-1})$ and also satisfies

$$\int_0^1 \frac{w_q(\delta)}{\delta}(1 + |\log \delta|) d\delta < \infty \quad \text{with} \;\; q \geq 1.$$

If Ω satisfies (2.1) and (2.3), Fu, Lu and Zhao [28] obtained the boundedness of $[b, H_\Omega]$ on $L^p(\mathbb{R}^n)$ with $b \in CBMO(\mathbb{R}^n)$. In [27], we gave some characterizations of $CBMO(\mathbb{R}^n)$ when Ω satisfies (2.1), (2.2) and (2.4).

Theorem 2.5. *Suppose that Ω satisfies (2.1), (2.2) and (2.4). Then*
(a) $b \in CBMO^{\max\{p,s\}}(\mathbb{R}^n)$ with $\frac{1}{p}+\frac{1}{q}+\frac{1}{s}=1$ for $q>1 \Rightarrow$ both $H_{\Omega,b}$ and $H^*_{\Omega,b}$ are bounded on $L^p(\mathbb{R}^n)$.
(b) Both $H_{\Omega,b}$ and $H^*_{\Omega,b}$ are bounded on $L^p(\mathbb{R}^n) \Rightarrow b \in CBMO(\mathbb{R}^n)$.

Theorem 2.6. *Let Ω satisfy (2.1), (2.2), (2.4) and $b \in BMO(\mathbb{R}^n)$. Then*

$$b \in CVMO(\mathbb{R}^n) \Longleftrightarrow \text{Both } H_{\Omega,b} \text{ and } H^*_{\Omega,b} \text{ are compact on } L^p(\mathbb{R}^n).$$

2.2. Average operators on p-adic field

In the past two decades, there is an increasing interest in the study of harmonic analysis on p-adic field and their various generalizations and the related theory of operators and spaces. For a prime number p, let $\mathbb{Q}_p$ be the field of p-adic numbers. It is defined as the completion of the field of rational numbers $\mathbb{Q}$ with respect to the non-Archimedean p-adic norm $|\cdot|_p$. This norm is defined as follows: $|0|_p=0$; if any non-zero rational number x is represented as $x=p^\gamma \frac{m}{n}$, where γ is an integer and integers m, n are indivisible by p, then $|x|_p=p^{-\gamma}$. It is easy to see that the norm satisfies the following properties:$|xy|_p=|x|_p|y|_p$, $\quad |x+y|_p \le \max\{|x|_p,|y|_p\}$. Moreover, if $|x|_p \ne |y|_p$, then $|x+y|_p=\max\{|x|_p,|y|_p\}$. It is well-known that $\mathbb{Q}_p$ is a typical model of non-Archimedean local fields.

The space $\mathbb{Q}_p^n$ consists of points $x=(x_1,x_2,\cdots,x_n)$, where $x_j \in \mathbb{Q}_p$, $j=1,2,\cdots,n$. The p-adic norm on $\mathbb{Q}_p^n$ is

$$|x|_p := \max_{1\le j\le n} |x_j|_p, \quad x \in \mathbb{Q}_p^n.$$

Denote by $B_\gamma(a)=\{x \in \mathbb{Q}_p^n : |x-a|_p \le p^\gamma\}$, the ball with center at $a \in \mathbb{Q}_p^n$ and radius p^γ, and by $S_\gamma(a) := \{x \in \mathbb{Q}_p^n : |x-a|_p = p^\gamma\}$ the sphere with center at $a \in \mathbb{Q}_p^n$ and radius p^γ, $\gamma \in \mathbb{Z}$. It is clear that $S_\gamma(a)=B_\gamma(a)\setminus B_{\gamma-1}(a)$.

Since $\mathbb{Q}_p^n$ is a locally compact commutative group under addition, there exists a Haar measure dx on $\mathbb{Q}_p^n$, which is unique up to positive constant multiple and is translation invariant. We normalize the measure dx by the equality

$$\int_{B_0(0)} dx = |B_0(0)|_H = 1,$$

where $|E|_H$ denotes the Haar measure of a measurable subset E of $\mathbb{Q}_p^n$. Then $|B_\gamma(a)|_H=p^{\gamma n}$, $|S_\gamma(a)|_H=p^{\gamma n}(1-p^{-n})$ for any $a \in \mathbb{Q}_p^n$.

The following definitions of function spaces can be found in [30, 71, 74].

Let $1 \le q < \infty$. A function $f \in L^q_{loc}(\mathbb{Q}_p^n)$ is said to be in $CBMO^q(\mathbb{Q}_p^n)$, if

$$\|f\|_{CBMO^q(\mathbb{Q}_p^n)} := \sup_{\gamma\in\mathbb{Z}} \left(\frac{1}{|B_\gamma(0)|_H} \int_{B_{\gamma(0)}} |f(x)-f_{B_{\gamma(0)}}|^q dx \right)^{\frac{1}{q}} < \infty,$$

where

$$f_{B_{\gamma(0)}} = \frac{1}{|B_\gamma(0)|_H} \int_{B_{\gamma(0)}} f(x)dx.$$

It is obvious that $L^\infty(\mathbb{Q}_p^n) \subset BMO(\mathbb{Q}_p^n) \subset CBMO^q(\mathbb{Q}_p^n)$.

Let $B_k = B_k(0)$, $S_k = B_k \setminus B_{k-1}$ and χ_k is the characteristic function of S_k. Suppose that $\alpha \in \mathbb{R}$, $0 < q < \infty$ and $0 < r < \infty$. *The homogeneous p-adic Herz space* $K_r^{\alpha,q}(\mathbb{Q}_p^n)$ is defined by

$$K_r^{\alpha,q}(\mathbb{Q}_p^n) = \left\{ f \in L_{loc}^r(\mathbb{Q}_p^n) : \|f\|_{K_r^{\alpha,q}(\mathbb{Q}_p^n)} < \infty \right\},$$

where

$$\|f\|_{K_r^{\alpha,q}(\mathbb{Q}_p^n)} = \left(\sum_{k=-\infty}^{+\infty} p^{k\alpha q} \|f\chi_k\|_{L^r(\mathbb{Q}_p^n)}^q \right)^{\frac{1}{q}}$$

with the usual modifications made when $q = \infty$ or $r = \infty$. It's easy to see that $K_q^{\frac{\alpha}{q},q}(\mathbb{Q}_p^n) = L^q(|x|_p^\alpha dx)$, $K_q^{0,q}(\mathbb{Q}_p^n) = L^q(\mathbb{Q}_p^n)$ for all $0 < q \leq \infty$ and $\alpha \in \mathbb{R}$.

Let $1 \leq q < \infty$ and $\lambda \geq -\frac{1}{q}$. *The p-adic Morrey space* $L^{q,\lambda}(\mathbb{Q}_p^n)$ is defined by

$$L^{q,\lambda}(\mathbb{Q}_p^n) = \left\{ f \in L_{loc}^q(\mathbb{Q}_p^n) : \|f\|_{L^{q,\lambda}(\mathbb{Q}_p^n)} < \infty \right\},$$

where

$$\|f\|_{L^{q,\lambda}(\mathbb{Q}_p^n)} = \sup_{a\in\mathbb{Q}_p^n, \gamma\in\mathbb{Z}} \left(\frac{1}{|B_\gamma(a)|_H^{1+\lambda q}} \int_{B_\gamma(a)} |f(x)|^q dx \right)^{\frac{1}{q}}.$$

Clearly, $L^{q,-1/q}(\mathbb{Q}_p^n) = L^q(\mathbb{Q}_p^n)$, $L^{q,0}(\mathbb{Q}_p^n) = L^\infty(\mathbb{Q}_p^n)$.

Let $\lambda \in \mathbb{R}$ and $1 < q < \infty$. *The non-homogeneous p-adic central Morrey space* $B^{q,\lambda}(\mathbb{Q}_p^n)$ is defined by

$$\|f\|_{B^{q,\lambda}(\mathbb{Q}_p^n)} := \sup_{\gamma\in\mathbb{Z}^+} \left(\frac{1}{|B_\gamma|_H^{1+\lambda q}} \int_{B_\gamma} |f(x)|^q dx \right)^{\frac{1}{q}} < \infty.$$

The homogeneous p-adic central Morrey space $\dot{B}^{q,\lambda}(\mathbb{Q}_p^n)$ is defined by

$$\|f\|_{\dot{B}^{q,\lambda}(\mathbb{Q}_p^n)} := \sup_{\gamma\in\mathbb{Z}} \left(\frac{1}{|B_\gamma|_H^{1+\lambda q}} \int_{B_\gamma} |f(x)|^q dx \right)^{\frac{1}{q}} < \infty.$$

Obviously, $\dot{B}^{q,\lambda}(\mathbb{Q}_p^n) \subset B^{q,\lambda}(\mathbb{Q}_p^n)$. If $1 \leq q_1 < q_2 < \infty$, by Hölder's inequality

$$\dot{B}^{q_2,\lambda}(\mathbb{Q}_p^n) \subset \dot{B}^{q_1,\lambda}(\mathbb{Q}_p^n), \quad B^{q_2,\lambda}(\mathbb{Q}_p^n) \subset B^{q_1,\lambda}(\mathbb{Q}_p^n),$$

for $\lambda \in \mathbb{R}$. When $\lambda < -1/q$, the spaces $\dot{B}^{q,\lambda}(\mathbb{Q}_p^n)$ and $B^{q,\lambda}(\mathbb{Q}_p^n)$ reduce to $\{0\}$, and

$$\dot{B}^{q,-\frac{1}{q}}(\mathbb{Q}_p^n) = B^{q,-\frac{1}{q}}(\mathbb{Q}_p^n) = L^q(\mathbb{Q}_p^n).$$

Let $\lambda < \frac{1}{n}$ and $1 < q < \infty$. The space $CBMO^{q,\lambda}(\mathbb{Q}_p^n)$ is defined by the condition

$$\|f\|_{CBMO^{q,\lambda}(\mathbb{Q}_p^n)} := \sup_{\gamma\in\mathbb{Z}} \left(\frac{1}{|B_\gamma|_H^{1+\lambda q}} \int_{B_\gamma} |f(x) - f_{B_\gamma}|^q dx \right)^{\frac{1}{q}} < \infty,$$

where $f_{B_\gamma} = \frac{1}{|B_\gamma|_H}\int_{B_\gamma} f(x)dx$. When $\lambda = 0$, the space $CBMO^{q,\lambda}(\mathbb{Q}_p^n)$ is just $CBMO^q(\mathbb{Q}_p^n)$. If $1 \le q_1 < q_2 < \infty$, by Hölder's inequality, we have

$$CBMO^{q_2,\lambda}(\mathbb{Q}_p^n) \subset CBMO^{q_1,\lambda}(\mathbb{Q}_p^n)$$

for $\lambda \in \mathbb{R}$. By the standard proof as that in $\mathbb{R}^n$, we can see that

$$\|f\|_{CBMO^{q,\lambda}(\mathbb{Q}_p^n)} \sim \sup_{\gamma\in\mathbb{Z}} \inf_{c\in\mathbb{C}} \left(\frac{1}{|B_\gamma|_H^{1+\lambda q}} \int_{B_\gamma} |f(x)-c|^q dx \right)^{\frac{1}{q}}.$$

2.2.1. Hardy operators on p-adic field

For a function f on $\mathbb{Q}_p^n$, the *p-adic Hardy operators* are defined as follows

$$\mathcal{H}^p f(x) = \frac{1}{|x|_p^n}\int_{B(0,|x|_p)} f(t)dt, \quad x \in \mathbb{Q}_p^n \setminus \{0\},$$
$$\mathcal{H}^{p,*} f(x) = \int_{\mathbb{Q}_p^n\setminus B(0,|x|_p)} \frac{f(t)}{|t|_p^n}dt, \quad x \in \mathbb{Q}_p^n \setminus \{0\},$$

where $B(0,|x|_p)$ is a ball in $\mathbb{Q}_p^n$ with center at $0 \in \mathbb{Q}_p^n$ and radius $|x|_p$. Let $b \in L_{loc}(\mathbb{Q}_p^n)$. The commutators of $\mathcal{H}^p, \mathcal{H}^{p,*}$ with b are denoted by $\mathcal{H}_b^p, \mathcal{H}_b^{p,*}$, respectively. In [30], we obtained the sharp estimates for $\mathcal{H}^p$ and $\mathcal{H}^{p,*}$ on $L^q(|x|_p^\alpha dx)$.

Theorem 2.7. *Let $1 < q < \infty$ and $\alpha < n(q-1)$. Then*

$$\|\mathcal{H}^p\|_{L^q(|x|_p^\alpha dx)\to L^q(|x|_p^\alpha dx)} = \|\mathcal{H}^{p,*}\|_{L^q(|x|_p^\alpha dx)\to L^q(|x|_p^\alpha dx)} = \frac{1-p^{-n}}{1-p^{\frac{\alpha}{q}-\frac{n}{q'}}},$$

where $\frac{1}{q}+\frac{1}{q'}=1$.

In particular, when $\alpha = 0$, we have

$$\|\mathcal{H}^p\|_{L^q(\mathbb{Q}_p^n)\to L^q(\mathbb{Q}_p^n)} = \|\mathcal{H}^{p,*}\|_{L^q(\mathbb{Q}_p^n)\to L^q(\mathbb{Q}_p^n)} = \frac{1-p^{-n}}{1-p^{-\frac{n}{q'}}},$$

where $\frac{1}{q}+\frac{1}{q'}=1$. Obviously, the L^q norm of $\mathcal{H}^p$ on $\mathbb{Q}_p^n$ depends on n, however, the L^q norm of $\mathcal{H}$ on $\mathbb{R}^n$ is independent of the dimension n.

In [30], we also got the boundedness of commutators of Hardy operators on homogeneous p-adic Herz spaces.

Theorem 2.8. *Let $0 < q_1 \le q_2 < \infty$, $1 < r < \infty$ and $b \in CBMO^{\max\{r',r\}}(\mathbb{Q}_p^n)$. Then*
(1) If $\alpha < \frac{n}{r'}$, then $\mathcal{H}_b^p$ is bounded from $K_r^{\alpha,q_1}(\mathbb{Q}_p^n)$ to $K_r^{\alpha,q_2}(\mathbb{Q}_p^n)$;
(2) If $\alpha > -\frac{n}{r}$, then $\mathcal{H}_b^{p,*}$ is bounded from $K_r^{\alpha,q_1}(\mathbb{Q}_p^n)$ to $K_r^{\alpha,q_2}(\mathbb{Q}_p^n)$.

Corollary 2.1. *Suppose that $1 < q < \infty$ and $b \in CBMO^{\max\{q',q\}}(\mathbb{Q}_p^n)$. Then*
(1) If $\alpha < \frac{nq}{q'}$, then $\mathcal{H}_b^p$ is bounded from $L^q(|x|_p^\alpha dx)$ to $L^q(|x|_p^\alpha dx)$;
(2) If $\alpha > -n$, then $\mathcal{H}_b^{p,*}$ is bounded from $L^q(|x|_p^\alpha dx)$ to $L^q(|x|_p^\alpha dx)$.

2.2.2. The Hardy-Littlewood-Pólya operator on p-adic field

The Hardy-Littlewood-Pólya operator on $\mathbb{R}$ is defined by

$$Tf(x) = \int_0^{+\infty} \frac{f(y)}{\max(x,y)} dy.$$

And its norm on $L^q(\mathbb{R}^+)$, $1 < q < \infty$, is $\|T\|_{L^q(\mathbb{R}^+)\to L^q(\mathbb{R}^+)} = \frac{q^2}{q-1}$.

In [30], we considered the *p-adic of Hardy-Littlewood-Pólya operator* which is defined as

$$T^p(x) = \int_{\mathbb{Q}_p} \frac{f(y)}{\max(|x|_p, |y|_p)} dy, \qquad x \in \mathbb{Q}_p^*.$$

We obtained the sharp estimates of T^p on $L^q(|x|_p^\alpha dx)$.

Theorem 2.9. *Let $1 < q < \infty$ and $-1 < \alpha < q-1$. Then*

$$\|T^p\|_{L^q(|x|_p^\alpha dx)\to L^q(|x|_p^\alpha dx)} = \left(1 - \frac{1}{p}\right)\left(\frac{1}{1 - p^{\frac{\alpha}{q} - \frac{1}{q'}}} + \frac{p^{-\frac{\alpha+1}{q}}}{1 - p^{-\frac{\alpha+1}{q}}}\right),$$

where $\frac{1}{q} + \frac{1}{q'} = 1$.

When $\alpha = 0$, we can see

$$\|T^p\|_{L^q(\mathbb{Q}_p)\to L^q(\mathbb{Q}_p)} = \left(1 - \frac{1}{p}\right)\left(\frac{1}{1 - p^{-\frac{1}{q'}}} + \frac{p^{-\frac{1}{q}}}{1 - p^{-\frac{1}{q}}}\right).$$

2.2.3. Weighted Hardy operators on p-adic field

On p-adic field, the weighted p-adic Hardy operator $\mathcal{H}_\psi^p$ is defined by

$$\mathcal{H}_\psi^p f(x) = \int_{\mathbb{Z}_p^*} f(tx)\psi(t)dt,$$

where ψ is a non-negative function defined on $\mathbb{Z}_p^*$. Obviously, if $\psi \equiv 1$ and $n = 1$, then $\mathcal{H}_\psi^p$ is just reduced to the p-adic Hardy operator $\mathcal{H}^p$ on $\mathbb{Q}_p$.

In [71], we got the following sufficient and necessary conditions of weight functions, under which the weighted p-adic Hardy operators are bounded on p-adic Morrey, central Morrey and λ-central BMO spaces.

Theorem 2.10. *Let $1 < q < \infty$, $\Theta_n = \int_{\mathbb{Z}_p^*} |t|_p^{n\lambda}\psi(t)dt$, we have*

(i) *when $-1/q < \lambda \le 0$, then $\mathcal{H}_\psi^p$ is bounded on $L^{q,\lambda}(\mathbb{Q}_p^n)$ if and only if $\Theta_n < \infty$. Moreover, $\|\mathcal{H}_\psi^p\|_{L^{q,\lambda}(\mathbb{Q}_p^n)\to L^{q,\lambda}(\mathbb{Q}_p^n)} = \Theta_n$;*

(ii) *when $-1/q < \lambda \le 0$, then $\mathcal{H}_\psi^p$ is bounded on $\dot{B}^{q,\lambda}(\mathbb{Q}_p^n)$ if and only if $\Theta_n < \infty$. Moreover, $\|\mathcal{H}_\psi^p\|_{\dot{B}^{q,\lambda}(\mathbb{Q}_p^n)\to \dot{B}^{q,\lambda}(\mathbb{Q}_p^n))} = \Theta_n$;*

(iii) *when $0 \le \lambda < 1/n$, then $\mathcal{H}_\psi^p$ is bounded on $\mathrm{CMO}^{q,\lambda}(\mathbb{Q}_p^n)$ if and only if $\Theta_n < \infty$. Moreover, $\|\mathcal{H}_\psi^p\|_{\mathrm{CMO}^{q,\lambda}(\mathbb{Q}_p^n)\to \mathrm{CMO}^{q,\lambda}(\mathbb{Q}_p^n))} = \Theta_n$*

Corollary 2.2. *Let $1 < q < \infty$, then*

(i) *when $-1/q < \lambda \le 0$, then $\|\mathcal{H}^p\|_{L^{q,\lambda}(\mathbb{Q}_p)\to L^{q,\lambda}(\mathbb{Q}_p)} = \frac{1-p^{-1}}{1-p^{-(1+\lambda)}}$;*

(ii) *when $-1/q < \lambda \le 0$, then $\|\mathcal{H}^p\|_{\dot{B}^{q,\lambda}(\mathbb{Q}_p)\to \dot{B}^{q,\lambda}(\mathbb{Q}_p)} = \frac{1-p^{-1}}{1-p^{-(1+\lambda)}}$;*

(iii) *when $0 \le \lambda < 1$, then $\|\mathcal{H}^p\|_{\mathrm{CMO}^{q,\lambda}(\mathbb{Q}_p)\to \mathrm{CMO}^{q,\lambda}(\mathbb{Q}_p)} = \frac{1-p^{-1}}{1-p^{-(1+\lambda)}}$.*

Denote by $\mathcal{H}^p_{\psi,b}$ the commutator of the weighted p-adic Hardy operator $\mathcal{H}^p_{\psi}$ and locally integrable function b. We established the following sufficient and necessary condition for weight functions to ensure that $\mathcal{H}^p_{\psi,b}$ are bounded on p-adic central Morrey spaces.

Theorem 2.11. *Let $1 < q < q_1 < \infty$, $1/q = 1/q_1 + 1/q_2$ and $-1/q_1 \le \lambda < 0$. Assume that ψ is a positive integrable function on $\mathbb{Z}_p^*$. Then for any $b \in \mathrm{CMO}^{q_2}(\mathbb{Q}_p^n)$ the commutator $\mathcal{H}^p_{\psi,b}$ is bounded from $\dot{B}^{q_1,\lambda}(\mathbb{Q}_p^n)$ to $\dot{B}^{q,\lambda}(\mathbb{Q}_p^n)$ if and only if*

$$\int_{\mathbb{Z}_p^*} \psi(t)|t|_p^{n\lambda} \log_p \frac{1}{|t|_p} dt < \infty.$$

When $b \in \mathrm{CMO}^{q,\lambda}(\mathbb{Q}_p^n)$ with $\lambda \neq 0$, we have the following conclusion.

Theorem 2.12. *Let $1 < q < q_1 < \infty$, $1/q = 1/q_1 + 1/q_2$, $-1/q < \lambda < 0$, $-1/q_1 < \lambda_1 < 0$, $0 < \lambda_2 < \frac{1}{n}$ and $\lambda = \lambda_1 + \lambda_2$. If*

$$\int_{\mathbb{Z}_p^*} \psi(t)|t|_p^{n\lambda_1} dt < \infty,$$

then for any $b \in \mathrm{CMO}^{q_2,\lambda_2}(\mathbb{Q}_p^n)$, the corresponding commutator $\mathcal{H}^p_{\psi,b}$ is bounded from $\dot{B}^{q_1,\lambda_1}(\mathbb{Q}_p^n)$ to $\dot{B}^{q,\lambda}(\mathbb{Q}_p^n)$.

2.3. Average operators on the Heisenberg group

The Heisenberg group $\mathbb{H}^n$ is a non-commutative nilpotent Lie group, with the underlying manifold $\mathbb{R}^{2n} \times \mathbb{R}$ and the group law [70]

$$\begin{aligned}&(x_1, x_2, \cdots, x_{2n}, x_{2n+1})(x_1', x_2', \cdots, x_{2n}', x_{2n+1}')\\ =&\Big(x_1 + x_1', x_2 + x_2', \cdots, x_{2n} + x_{2n}', x_{2n+1} + x_{2n+1}' + 2\sum_{j=1}^{n}(x_j' x_{n+j} - x_j x_{n+j}')\Big).\end{aligned}$$

The identity element on $\mathbb{H}^n$ is $0 \in \mathbb{R}^{2n+1}$, while the element x^{-1} inverse to x is $-x$. $\mathbb{H}^n$ is a homogeneous group with dilations

$$\delta_r(x_1, x_2, \cdots, x_{2n}, x_{2n+1}) = (rx_1, rx_2, \cdots, rx_{2n}, r^2 x_{2n+1}), \quad r > 0.$$

The Haar measure on $\mathbb{H}^n$ coincides with the usual Lebesgue measure on $\mathbb{R}^{2n} \times \mathbb{R}$. We denote the measure of any measurable set $E \subset \mathbb{H}^n$ by $|E|$. Then

$$|\delta_r(E)| = r^Q |E|, \quad d(\delta_r x) = r^Q dx,$$

where $Q = 2n + 2$ is called the homogeneous dimension of $\mathbb{H}^n$. The Heisenberg distance derived from the norm

$$|x|_h = \left[\left(\sum_{i=1}^{2n} x_i^2\right)^2 + x_{2n+1}^2\right]^{\frac{1}{4}},$$

where $x = (x_1, x_2, \cdots, x_{2n}, x_{2n+1})$, is given by $d(p, q) = d(q^{-1}p, 0) = |q^{-1}p|_h$.

For $r>0$ and $x\in\mathbb{H}^n$, the ball and sphere with center x and radius r on $\mathbb{H}^n$ are given by $B(x,r)=\{y\in\mathbb{H}^n : d(x,y)<r\}$ and $S(x,r)=\{y\in\mathbb{H}^n : d(x,y)=r\}$, respectively. And we have

$$|B(x,r)|=|B(0,r)|=\Omega_Q r^Q,$$

where

$$\Omega_Q=\frac{2\pi^{n+\frac{1}{2}}\Gamma(\frac{n}{2})}{(n+1)\Gamma(n)\Gamma(\frac{n+1}{2})}.$$

The area of $S(0,1)$ on $\mathbb{H}^n$ is $\omega_Q=Q\Omega_Q$.

Let $1<p<\infty$. We say that a weight $w\in A_p(\mathbb{H}^n)$ if there exists a constant C such that for all balls B,

$$\left(\frac{1}{|B|}\int_B w(x)dx\right)\left(\frac{1}{|B|}\int_B w\left(x\right)^{-1/(p-1)}dx\right)^{p-1}\leq C.$$

We say that a weight $w\in A_1(\mathbb{H}^n)$ if there is a constant C such that for all balls B,

$$\frac{1}{|B|}\int_B w\left(x\right)dx\leq C\operatorname*{essinf}_{x\in B} w\left(x\right).$$

We define

$$A_\infty(\mathbb{H}^n)=\bigcup_{1\leq p<\infty}A_p(\mathbb{H}^n).$$

A close relation to $A_\infty(\mathbb{H}^n)$ is the reverse Hölder condition. If there exist $r>1$ and a fixed constant C such that

$$\left(\frac{1}{|B|}\int_B w(x)^r dx\right)^{1/r}\leq\frac{C}{|B|}\int_B w(x)dx$$

for all balls $B\subset\mathbb{H}^n$, we then say that w satisfies the *reverse Hölder condition of order* r and write $w\in RH_r(\mathbb{H}^n)$ (cf. [46] and references therein).

2.3.1. Hardy operator on the Heisenberg group

Let f be a locally integrable function on $\mathbb{H}^n$. *The Hardy operator* on $\mathbb{H}^n$ [73] is defined by

$$\mathsf{H}f(x):=\frac{1}{|B(0,|x|_h)|}\int_{B(0,|x|_h)}f(y)dy,\quad x\in\mathbb{H}^n\setminus\{0\}.$$

The weak $L^1(\mathbb{H}^n)$ space $L^{1,\infty}(\mathbb{H}^n)$ is defined as the set of all measurable functions f on $\mathbb{H}^n$ satisfying

$$\|f\|_{L^{1,\infty}(\mathbb{H}^n)}:=\sup_{\lambda>0}\lambda\left|\{x\in\mathbb{H}^n : |f(x)|>\lambda\}\right|<\infty.$$

A $(1,\infty,0)$-*atom* is a compactly supported $L^\infty(\mathbb{H}^n)$ function f such that (i) there is a ball B whose closure contains supp(f) and satisfying $\|f\|_\infty\leq|B|^{-1}$; (ii) $\int f(x)dx=0$.

The Hardy space $H^1(\mathbb{H}^n)$ can be defined by

$$H^1(\mathbb{H}^n) := \Big\{ f \in L^1(\mathbb{H}^n) : f(x) = \sum_{k=1}^{\infty} \lambda_k f_k(x), \ \sum_{k=1}^{\infty} |\lambda_k| < \infty \Big\},$$

where each f_k is a $(1,\infty,0)$-atom, and the H^1 norm of f can be defined by $\|f\|_{H^1(\mathbb{H}^n)} := \inf \sum_{k=1}^{\infty} |\lambda_k|$, here the infimum is taken over all the decompositions of $f = \sum_k \lambda_k f_k$ as above [19].

In [73], we obtained the following boundedness of H on the Heisenberg group.

Theorem 2.13. *Let $1 < p \le \infty$, then*
(i) H *is bounded from $L^p(\mathbb{H}^n)$ to $L^p(\mathbb{H}^n)$. Moreover,*

$$\|\mathsf{H}\|_{L^p(\mathbb{H}^n)\to L^p(\mathbb{H}^n)} = \frac{p}{p-1}, \quad 1 < p < \infty,$$
$$\|\mathsf{H}\|_{L^\infty(\mathbb{H}^n)\to L^\infty(\mathbb{H}^n)} = 1.$$

(ii) H *is bounded from $L^1(\mathbb{H}^n)$ to $L^{1,\infty}(\mathbb{H}^n)$. Moreover, $\|\mathsf{H}\|_{L^1(\mathbb{H}^n)\to L^{1,\infty}(\mathbb{H}^n)} = 1$.*
(iii) H *is bounded from $H^1(\mathbb{H}^n)$ to $L^1(\mathbb{H}^n)$.*

Remark 2.1. (i) H is not bounded from $L^1(\mathbb{H}^n)$ to $L^1(\mathbb{H}^n)$. For example, we can take

$$f_0(x) = |x|_h^\alpha \chi_{B(0,R)}(x), \quad \alpha > -Q.$$

It is easy to see that $f_0 \in L^1(\mathbb{H}^n)$, and $\|f_0\|_{L^1(\mathbb{H}^n)} = \omega_Q R^{\alpha+Q}/(\alpha+Q)$. But

$$\mathsf{H} f_0(x) = \begin{cases} Q|x|_h^\alpha/(Q+\alpha), & |x| \le R, \\ QR^{Q+\alpha}|x|_h^{-Q}/(Q+\alpha), & |x| > R, \end{cases}$$

does not belong to $L^1(\mathbb{H}^n)$.

(ii) H is not bounded from $H^1(\mathbb{H}^n)$ to $H^1(\mathbb{H}^n)$. To illustrate this, we take

$$f_0(x) = \frac{1-2^Q}{\Omega_Q 2^{Q+1}} \chi_{\{|x|_h \le 1\}}(x) + \frac{1}{\Omega_Q 2^{Q+1}} \chi_{\{1<|x|_h\le 2\}}(x).$$

Then f_0 is a $(1,\infty,0)$-atom of $H^1(\mathbb{H}^n)$, and

$$\mathsf{H} f_0(x) = \frac{1-2^Q}{\Omega_Q 2^{Q+1}} \chi_{\{|x|_h \le 1\}}(x) + \frac{1-2^Q|x|_h^{-Q}}{\Omega_Q 2^{Q+1}} \chi_{\{1<|x|_h\le 2\}}(x).$$

It is clear that

$$\int \mathsf{H} f_0(x)dx = -\frac{Q}{2}\ln 2 \ne 0.$$

Thus $\mathsf{H} f_0 \notin H^1(\mathbb{H}^n)$.

It is known that the Hardy-Littlewood maximal operator on $\mathbb{R}^n$ is bounded from $L^p(v)$ to $L^p(w)$ if and only if $(w,v) \in A^p(\mathbb{R}^n)$. On the Heisenberg group, a pair of nonnegative weights (w,v) belongs to the class $A_p(\mathbb{H}^n)$, $1 \le p < \infty$, if when $1 < p < \infty$

$$\sup_B \left(\frac{1}{|B|} \int_B w(x)dx \right) \left(\frac{1}{|B|} \int_B v(x)^{1-p'} dx \right)^{p-1} < \infty,$$

and when $p = 1$,

$$\frac{1}{|B|}\int_B w(x)dx \le C \inf_{x\in B} v(x).$$

In the Euclidean space, the characterization of the weights (w, v) for the n-dimensional Hardy inequality

$$\int_{\mathbb{R}^n} (\mathcal{H}f(x))^p w(x)dx \le C\int_{\mathbb{R}^n} (f(x))^p v(x)dx, \quad 1 \le p < \infty,$$

was obtained by Drábek et al. in a skillful method.

The good weights (w, v) for $\mathcal{H}$ were called M_p weights [83], $1 \le p < \infty$. A natural question is that whether M_p weights exist on the Heisenberg group. If they exist, do the corresponding results hold for these M_p weights? We provided affirmative answers to these questions.

Assume that (w, v) is a pair of nonnegative functions. (w, v) is called an $M_1(\mathbb{H}^n)$ *weight* if for almost all $x \in \mathbb{H}^n$,

$$\int_{|x|_h>\alpha} |x|_h^{-Q} w(x)dx \le C \operatorname{essinf}_{|x|_h<\alpha} v(x), \quad \alpha > 0,$$

for some constant C. (w, v) is called an $M_p(\mathbb{H}^n)$ $(1 < p < 1)$ *weight* if

$$\sup_{0<\alpha<\infty}\left(\int_{|x|_h>\alpha} |x|_h^{-Qp} w(x)dx\right)^{\frac{1}{p}}\left(\int_{|x|_h<\alpha} v(x)^{1-p'}dx\right)^{\frac{1}{p'}} < \infty,$$

where $1/p + 1/p' = 1$. If $w = v$, we say that $w \in M_p(\mathbb{H}^n)$, $1 \le p < \infty$.

The weighted Hardy inequalities on the Heisenberg group can also characterize M_p weights [73].

Theorem 2.14. *Let w and v be nonnegative weight functions on $\mathbb{H}^n$. For $1 < p \le q < \infty$, the inequality*

$$\left\{\int_{\mathbb{H}^n} w(x)\left(\mathsf{H}f(x)\right)^q dx\right\}^{\frac{1}{q}} \le C\left\{\int_{\mathbb{H}^n} v(x)f(x)^p dx\right\}^{\frac{1}{p}}, \tag{2.5}$$

holds for $f \ge 0$ if and only if

$$A := \sup_{0<\alpha<\infty}\left(\int_{|x|_h>\alpha} |x|_h^{-Qq} w(x)dx\right)^{\frac{1}{q}}\left(\int_{|x|_h<\alpha} v(x)^{1-p'}dx\right)^{\frac{1}{p'}} < \infty.$$

Moreover, if C is the smallest constant for which (2.5) holds, then

$$A \le \Omega_Q C \le A p'^{\frac{1}{p'}} p^{\frac{1}{q}}.$$

Corollary 2.3. *Let w and v be nonnegative weight functions on $\mathbb{H}^n$. For $1 < p < \infty$, the inequality*

$$\left\{\int_{\mathbb{H}^n} w(x)\left(\mathsf{H}f(x)\right)^p dx\right\}^{\frac{1}{p}} \le C\left\{\int_{\mathbb{H}^n} v(x)f(x)^p dx\right\}^{\frac{1}{p}}, \tag{2.6}$$

holds for $f \ge 0$ if and only if $(w, v) \in M_p$. Moreover, if C is the smallest constant for which (2.6) *holds, then*

$$A \le \Omega_Q C \le A p'^{\frac{1}{p'}} p^{\frac{1}{p}}.$$

2.3.2. Weighted Hardy operators on the Heisenberg group

Let $w : [0, 1] \to [0, \infty)$ be a function, *the weighted Hardy operators* H_w on $\mathbb{H}^n$ [14] is defined as

$$\mathsf{H}_w f(x) := \int_0^1 f(\delta_t x) w(t) dt,$$

for a measurable function f on $\mathbb{H}^n$. The adjoint operator of the weighted Hardy operator, the *weighted Cesàro operator* is defined as

$$\mathsf{C}_\omega f(x) := \int_0^1 f(\delta_{1/t} x) t^{-Q} \omega(t) dt, \quad x \in \mathbb{H}^n,$$

which satifies

$$\int_{\mathbb{H}^n} f(x)(\mathsf{H}_\omega g)(x) dx = \int_{\mathbb{H}^n} g(x)(\mathsf{C}_\omega f)(x) dx.$$

Here $f \in L^p(\mathbb{H}^n)$, $g \in L^q(\mathbb{H}^n)$, $1 < p < \infty$, $q = p/(p-1)$, H_ω is bounded on $L^p(\mathbb{H}^n)$ and C_ω is bounded on $L^q(\mathbb{H}^n)$.

Recall that the space $BMO(\mathbb{H}^n)$ is defined to be the space of all locally integrable functions f on $\mathbb{H}^n$ such that

$$\|f\|_{BMO(\mathbb{H}^n)} := \sup_{B \subset \mathbb{H}^n} \frac{1}{|B|} \int_B |f(x) - f_B| dx < \infty,$$

where the supremum is taken over all balls in $\mathbb{H}^n$.

In [14], we gave the characterization of w when H_w and C_w are bounded.

Theorem 2.15. *Let $w : [0, 1] \to (0, \infty)$ be a function and let $1 \le p \le \infty$. Then*
(i) H_w *is bounded on $L^p(\mathbb{H}^n)$ if and only if*

$$\int_0^1 t^{-\frac{Q}{p}} w(t) dt < \infty. \tag{2.7}$$

Moreover, if (2.7) holds, then

$$\|\mathsf{H}_w\|_{L^p(\mathbb{H}^n) \to L^p(\mathbb{H}^n)} = \int_0^1 t^{-\frac{Q}{p}} w(t) dt.$$

(ii) H_w *is bounded on $BMO(\mathbb{H}^n)$ if and only if*

$$\int_0^1 w(t) dt < \infty. \tag{2.8}$$

Moreover, if (2.8) holds, then

$$\|\mathsf{H}_w\|_{BMO(\mathbb{H}^n) \to BMO(\mathbb{H}^n)} = \int_0^1 w(t) dt.$$

Theorem 2.16. *Let $w : [0, 1] \to (0, \infty)$ be a function and let $1 \le q \le \infty$. Then*
(i) C_w *is bounded on $L^q(\mathbb{H}^n)$ if and only if*

$$\int_0^1 t^{-Q(1-1/q)} w(t) dt < \infty. \tag{2.9}$$

Moreover, if (2.9) *holds, then*

$$\|\mathsf{C}_w\|_{L^q(\mathbb{H}^n)\to L^q(\mathbb{H}^n)} = \int_0^1 t^{-Q(1-1/q)} w(t)dt.$$

(ii) C_w *is bounded on* $BMO(\mathbb{H}^n)$ *if and only if*

$$\int_0^1 t^{-Q} w(t)dt < \infty. \tag{2.10}$$

Moreover, if (2.10) *holds, then*

$$\|\mathsf{C}_w\|_{BMO(\mathbb{H}^n)\to BMO(\mathbb{H}^n)} = \int_0^1 t^{-Q} w(t)dt.$$

2.3.3. Hausdorff operator on the Heisenberg group

Let Φ be a locally integrable function on $\mathbb{H}^n$. The Hausdorff operators on $\mathbb{H}^n$ are defined by

$$T_\Phi f(x) = \int_{\mathbb{H}^n} \frac{\Phi(y)}{|y|_h^Q} f\big(\delta_{|y|_h^{-1}} x\big) dy, \qquad T_{\Phi,A} f(x) = \int_{\mathbb{H}^n} \frac{\Phi(y)}{|y|_h^Q} f\left(A(y)x\right) dy,$$

where $A(y)$ is a matrix-valued function [12] and we assume $\det A(y) \neq 0$ almost everywhere on the support of Φ. In the above definition, we note that $T_{\Phi,A} = T_\Phi$ if we choose a special matrix A. For a matrix M, we will use the norm $\|M\| = \sup_{x\in\mathbb{H}^n,\ x\neq 0} |Mx|_h/|x|_h$.

Suppose $\alpha \in \mathbb{R}$, $0 < p$, $q < \infty$. Let w be a weight on $\mathbb{H}^n$, $\mathcal{B}_k = \{x \in \mathbb{H}^n \mid |x|_h < 2^k\}$, $D_k = \mathcal{B}_k \setminus \mathcal{B}_{k-1}$. The *homogeneous weighted Herz space* $\dot{K}_q^{\alpha,p}(\mathbb{H}^n; w)$ is defined by

$$\dot{K}_q^{\alpha,p}(\mathbb{H}^n; w) = \left\{ f \in L^q_{loc}(\mathbb{H}^n \setminus \{0\}; w) : \|f\|_{\dot{K}_q^{\alpha,p}(\mathbb{H}^n;w)} < \infty \right\},$$

where

$$\|f\|_{\dot{K}_q^{\alpha,p}(\mathbb{H}^n;w)} = \left\{ \sum_{k=-\infty}^{+\infty} w(B_k)^{\alpha p/Q} \|f\|^p_{L^q(D_k,w)} \right\}^{1/p}.$$

In [46], we obtained the following boundedness estimates.

Theorem 2.17. *Let* $1 \le p < \infty$, $1 \le\ q_1,\ q_2 < \infty$, $-\infty < \alpha_1 < 0$, $\alpha_2 \in \mathbb{R}$ *and* $1/q_1 + \alpha_1/Q = 1/q_2 + \alpha_2/Q$. *Suppose that* $w \in A_\gamma$, $1 \le \gamma < \infty$, *with the critical index* r_w *for the reverse Hölder condition and* $q_1 > q_2\gamma r_w/(r_w - 1)$.

(*i*) *If* $1/q_1 + \alpha_1/Q \ge 0$, *then for any* $1 < \delta < r_w$,

$$\|T_{\Phi,A} f\|_{\dot{K}_{q_2}^{\alpha_2,p}(\mathbb{H}^n;w)} \preceq C_3 \|f\|_{\dot{K}_{q_1}^{\alpha_1,p}(\mathbb{H}^n;w)},$$

where

$$C_3 = \int_{\|A(y)\|<1} \frac{|\Phi(y)|}{|y|_h^Q} |\det A^{-1}(y)|^{\gamma/q_1} \|A(y)\|^{-\gamma\alpha_1} dy + \int_{\|A(y)\|\ge 1} \frac{\Phi(y)}{|y|_h^Q} |\det A^{-1}(y)|^{\gamma/q_1} \|A(y)\|^{Q\gamma/q_1 - (Q/q_1+\alpha_1)(\delta-1)/\delta} dy.$$

(*ii*) *If* $1/q_1 + \alpha_1/Q < 0$, *then for any* $1 < \delta < r_w$,

$$\|T_{\Phi,A}f\|_{\dot{K}^{\alpha_2,p}_{q_2}(\mathbb{H}^n;w)} \preceq C_4\|f\|_{\dot{K}^{\alpha_1,p}_{q_1}(\mathbb{H}^n;w)},$$

where

$$C_4 = \int_{\|A(y)\|<1} \frac{|\Phi(y)|}{|y|_h^Q} |\det A^{-1}(y)|^{\gamma/q_1} \|A(y)\|^{Q\gamma/q_1-(Q/q_1+\alpha_1)(\delta-1)/\delta} dy + \int_{\|A(y)\|\geq 1} \frac{\Phi(y)}{|y|_h^Q} |\det A^{-1}(y)|^{\gamma/q_1} \|A(y)\|^{-\gamma\alpha_1} dy.$$

The result for the case $\gamma = 1, p < 1$ was also obtained in [46].

If $\|A^{-1}(y)\|$ and $\|A(y)\|^{-1}$ are comparable, we can obtain the following sharp result.

Theorem 2.18. *Let* $1 \leq p,\ q < \infty,\ -Q < \beta < \infty,\ \alpha \in \mathbb{R}$ *and* Φ *be a nonnegative function. Suppose that there is a constant* C *independent of* y *such that* $\|A^{-1}(y)\| \leq C\|A(y)\|^{-1}$ *for all* $y \in \operatorname{supp}(\Phi)$. *Then* $T_{\Phi,A}$ *is bounded on* $\dot{K}^{\alpha,p}_q(\mathbb{H}^n; |\cdot|^\beta)$ *if and only if*

$$\int_{\mathbb{H}^n} \frac{\Phi(y)}{|y|_h^Q} \|A^{-1}(y)\|^{(Q+\beta)(1/q+\alpha/Q)} dy < \infty.$$

Let Φ be a locally integrable function on $\mathbb{H}^n$. If $b \in L_{loc}(\mathbb{H}^n)$, the commutator of Hausdorff operator is defined by

$$T^b_{\Phi,A}f = bT_{\Phi,A}f - T_{\Phi,A}(bf).$$

Define

$$\left(T^{b,1}_{\Phi,A}f\right)(x) = \int_{\|A(y)\|\leq 1} \frac{\Phi(y)}{|y|_h^Q} f\left(A(y)x\right)\left[b(x) - b\left(A(y)x\right)\right] dy,$$
$$\left(T^{b,2}_{\Phi,A}f\right)(x) = \int_{\|A(y)\|> 1} \frac{\Phi(y)}{|y|_h^Q} f\left(A(y)x\right)\left[b(x) - b\left(A(y)x\right)\right] dy.$$

It is clear that

$$T^b_{\Phi,A}f = T^{b,1}_{\Phi,A}f + T^{b,2}_{\Phi,A}f.$$

Theorem 2.19. *Let* $1 \leq p_1,\ p_2,\ q < \infty$ *and* $-1/p_1 \leq \lambda < 0$. *Suppose that* $w \in A_q$ *with the critical index* r_w *for the reverse Hölder condition. If* $p_1 > p_2 q r_w/(r_w - 1)$, *then we have that, for any* $1 < \delta < r_w$,

$$\|T_{\Phi,A}f\|_{\dot{L}^{p_2,\lambda}(\mathbb{H}^n;w)} \preceq C_1\|f\|_{\dot{L}^{p_1,\lambda}(\mathbb{H}^n;w)},$$

where

$$C_1 = \int_{\|A(y)\|>1} \frac{|\Phi(y)|}{|y|_h^Q} \left(\frac{\|A(y)\|^Q}{|\det A(y)|}\right)^{q/p_1} \|A(y)\|^{Q\lambda(\delta-1)/\delta} dy + \int_{\|A(y)\|\leq 1} \frac{|\Phi(y)|}{|y|_h^Q} \left(\frac{\|A(y)\|^Q}{|\det A(y)|}\right)^{q/p_1} \|A(y)\|^{Q\lambda q} dy.$$

Theorem 2.20. *Let $1 \le p,\ p_1,\ p_2,\ q < \infty$ and $-1/p_1 \le \lambda < 0$. Suppose that $w \in A_q$ with the critical index r_w for the reverse Hölder condition. If $1/p > (1/p_1 + 1/p_2)qr_w/(r_w-1)$ and $q \le p_2$, then we have that, for any $1 < \delta < r_w$,*

$$\|T^b_{\Phi,A}f\|_{\dot{L}^{p,\lambda}(\mathbb{H}^n;w)} \preceq C_2\|f\|_{\dot{L}^{p_1,\lambda}(\mathbb{H}^n;w)}\|b\|_{CMO^{p_2}(\mathbb{H}^n;w)},$$

where

$$\begin{aligned}C_2 = &\int_{\|A(y)\|>1} \frac{|\Phi(y)|}{|y|_h^Q}\left(\frac{\|A(y)\|^Q}{|\det A(y)|}\right)^{q/p_1}\|A(y)\|^{Q\lambda(\delta-1)/\delta}\\ &\times \max\left\{\frac{\|A(y)\|^Q}{|\det A(y)|},\ \log_2\|A(y)\|\right\}dy\\ &+\int_{\|A(y)\|\le 1} \frac{|\Phi(y)|}{|y|_h^Q}\left(\frac{\|A(y)\|^Q}{|\det A(y)|}\right)^{q/p_1}\|A(y)\|^{Q\lambda q}\\ &\times \max\left\{\frac{\|A(y)\|^Q}{|\det A(y)|},\ \log_2\frac{1}{\|A(y)\|}\right\}dy.\end{aligned}$$

Especially, if $\|A^{-1}(y)\|$ and $\|A(y)\|^{-1}$ are comparable, the following sharp results hold.

Theorem 2.21. *Let $1 \le p < \infty$, $-1/p \le \lambda < 0, -Q < \alpha < \infty$ and Φ be a nonnegative function. Suppose that there is a constant C_0 independent of y such that $\|A^{-1}(y)\| \le C_0\|A(y)\|^{-1}$ for all $y \in \text{supp}(\Phi)$. Then $T_{\Phi,A}$ is bounded on $\dot{L}^{p,\ \lambda}(\mathbb{H}^n;|\cdot|_h^\alpha)$ if and only if*

$$\int_{\mathbb{H}^n} \frac{\Phi(y)}{|y|_h^Q}\|A(y)\|^{(Q+\alpha)\lambda}dy < \infty. \tag{2.11}$$

Theorem 2.22. *Let $1 \le p,\ p_1,\ p_2 < \infty$, $1/p = 1/p_1 + 1/p_2$, $-1/p_1 < \lambda < 0$, $-Q < \alpha < \infty$ and $p_2 > (Q+\alpha)/Q$ if $0 < \alpha < \infty$ or $p_2 \ge 1$ if $-Q < \alpha \le 0$. Suppose that Φ is a nonnegative function and there is a constant C_0 independent of y such that $\|A^{-1}(y)\| \le C_0\|A(y)\|^{-1}$ for all $y \in \text{supp}(\Phi)$. If $b \in CMO^{p_2}(\mathbb{H}^n;|\cdot|_h^\alpha)$ and* (2.11) *holds, then we have the following conclusions.*

(i) *$T^{b,1}_{\Phi,A}$ is bounded from $\dot{L}^{p_1,\ \lambda}(\mathbb{H}^n;|\cdot|_h^\alpha)$ to $\dot{L}^{p,\ \lambda}(\mathbb{H}^n;|\cdot|_h^\alpha)$ if and only if*

$$\int_{\|A(y)\|\le 1} \frac{\Phi(y)}{|y|_h^Q}\|A(y)\|^{(Q+\alpha)\lambda}\left|\log_2\|A(y)\|\right|dy < \infty.$$

(ii) *$T^{b,2}_{\Phi,A}$ is bounded from $\dot{L}^{p_1,\ \lambda}(\mathbb{H}^n;|\cdot|_h^\alpha)$ to $\dot{L}^{p,\ \lambda}(\mathbb{H}^n;|\cdot|_h^\alpha)$ if and only if*

$$\int_{\|A(y)\|>1} \frac{\Phi(y)}{|y|_h^Q}\|A(y)\|^{(Q+\alpha)\lambda}\log_2\|A(y)\|dy < \infty.$$

Let $\mathcal{S}(\mathbb{H}^n)$ be the Schwartz class on $\mathbb{H}^n$. Its dual space $\mathcal{S}'(\mathbb{H}^n)$ is the space of tempered distributions on $\mathbb{H}^n$. For $f \in \mathcal{S}'(\mathbb{H}^n)$ and $\Psi \in \mathcal{S}(\mathbb{H}^n)$, the *nontangential maximal function* $M_\Psi f$ of f with respect to Ψ is defined by

$$M_\Psi f(x) = \sup_{|x^{-1}y|_h<r<\infty}|f * \Psi_r(y)|, \tag{2.12}$$

where $\Psi_r(y) = r^{-Q}\Psi(\delta_{r^{-1}}y)$. The *nontangential grand maximal function* $M_{(N)}f$ is defined by

$$M_{(N)}f(x) = \sup_{\Psi\in\mathcal{S},\|\Psi\|_N\le 1} M_\Psi f(x). \tag{2.13}$$

For $0 < p \le 1$, let $N_p = [Q(1/p - 1)] + 1$. The *Hardy space* $H^p(\mathbb{H}^n)$ [19] is defined by

$$H^p(\mathbb{H}^n) = \left\{f \in \mathcal{S}'(\mathbb{H}^n) : M_{(N_p)}f \in L^p(\mathbb{H}^n)\right\}$$

with

$$\|f\|_{H^p(\mathbb{H}^n)} = \|M_{(N_p)}f\|_{L^p(\mathbb{H}^n)}.$$

The ordered triplet (p, q, α) is called *admissible* if $0 < p \le 1 \le q \le \infty$, $p \ne q$, $\alpha \in \mathbb{N}$ and $\alpha \ge [Q(\frac{1}{p} - 1)]$, where $[\cdot]$ is the integer funcion. Let (p, q, α) be an admissible triplet. A function $a \in L^q(\mathbb{H}^n)$ is called a (p, q, α)*-atom* [19] centered at x_0 if it satisfies the following conditions:

(i) There exists a ball $B(x_0, r)$ such that $\mathrm{supp}(f) \subset B(x_0, r)$;

(ii) $\|f\|_{L^q(\mathbb{H}^n)} \le |B|^{\frac{1}{q}-\frac{1}{p}}$;

(iii) $\int_{\mathbb{H}^n} f(x)P(x)dx = 0$, for any polynomial P of homogeneous degree less than or equal to α.

A function $a \in L^q(\mathbb{H}^n)$ is called a *big* (p, q)*-atom* centered at x_0 if there exists a ball $B(x_0, r)$ with $r \ge \frac{1}{2}$ such that it satisfies (i) (ii), see [3].

If (p, q, α) is an admissible triplet, the *atomic Hardy space* $H^p_{q,\alpha}(\mathbb{H}^n)$ is the set of all tempered distributions of the form $\sum_j \lambda_j f_j$ (the sum converging in the topology of $\mathcal{S}'$), where each f_j is a (p, q, α)-atom and $\sum_j |\lambda_j|^p < \infty$.

If $f \in H^p_{q,\alpha}(\mathbb{H}^n)$, the quasi-norm $\|f\|_{H^p_{q,\alpha}(\mathbb{H}^n)}$ (it is a norm when $p = 1$) is defined by

$$\|f\|_{H^p_{q,\alpha}(\mathbb{H}^n)} = \inf\left(\sum_j |\lambda_j|^p\right)^{\frac{1}{p}},$$

where the infimum is taken over all (p, q, α)-atom decompositions of f.

Given a weight $w \in A_\infty$ and an admissible triplet $(p, q, [Q(q_w/p - 1)]$, a w-$(p, q, [Q(q_w/p - 1)]$*-atom* centered at x_0 with respect to w will be a function a satisfying the following three conditions.

(i) There exists a ball $B(x_0, r)$ such that $\mathrm{supp}(a) \subset B(x_0, r)$.

(ii) $\|a\|_{L^q_w(\mathbb{H}^n)} \le w(B(x_0, r))^{\frac{1}{q}-\frac{1}{p}}$, if $q < \infty$ or $\|a\|_{L^\infty(\mathbb{H}^n)} \le w(B(x_0, r))^{-\frac{1}{p}}$, if $q = \infty$.

(iii) $\int_{\mathbb{H}^n} a(x)x^I dx = 0$, for all multi-indices $I = (i_1, i_2, \cdots, i_{2n+1}) \in \mathbb{N}^{2n+1}$ with $|I| = \sum_{k=1}^{2n} i_k + 2i_{2n+1} \le [Q(q_w/p - 1)]$.

Let $w \in A_\infty$ be a weight and let $0 < p \le 1 < q \le \infty$. A tempered distribution $f \in \mathcal{S}'$ belongs to $H^p_w(\mathbb{H}^n)$ if and only if f can be written as a series

$$f = \sum_j \lambda_j a_j, \tag{2.14}$$

(the sum converging in $\mathcal{S}'$), where each a_j is a w-$(p, q, [Q(q_w/p - 1)])$-atom and $\sum_j |\lambda_j|^p < \infty$. Moreover, by setting $\|f\|^p_{H^p_w(\mathbb{H}^n)}$ to be the infimum of the sums over all decompositions (2.14), one obtains the norm for such space.

The local maximal functions are defined by taking supremum over $0 < r \le 1$ instead of $0 < r < \infty$ in (2.12) and (2.13):

$$\widetilde{M}_\Psi f(x) = \sup_{|x^{-1}y|_h < r \le 1} |f * \Psi_r(y)|,$$
$$\widetilde{M}_{(N)} f(x) = \sup_{\Psi \in \mathcal{S}, \|\Psi\|_N \le 1} \widetilde{M}_\Psi f(x).$$

Let $0 < p \le 1$. The *local Hardy space* $h^p(\mathbb{H}^n)$ is defined by

$$h^p(\mathbb{H}^n) = \left\{ f \in \mathcal{S}'(\mathbb{H}^n) : \widetilde{M}_{(N_p)} f \in L^p(\mathbb{H}^n) \right\}$$

with

$$\|f\|_{h^p(\mathbb{H}^n)} = \|\widetilde{M}_{(N_p)} f\|_{L^p(\mathbb{H}^n)}.$$

In [72], we got the following behavior of Hausdorff operators on power-weighted Hardy spaces and local Hardy spaces

Theorem 2.23. *Let Φ be a nonnegative function. Suppose that $A(y) \in \mathrm{Aut}(\mathbb{H}^n)$ almost everywhere and there exists a constant M independent of y such that $\|A^{-1}(y)\| \le M\|A(y)\|^{-1}$.*

(i) Let $-Q < \alpha < Q$ and $\alpha \neq 0$. If all entries of the same row of $A(y)$ are nonnegative uniformly or nonpositive uniformly on $y \in \mathrm{supp}(\Phi)$, then $T_{\Phi,A}$ is bounded on $H^1_{|\cdot|_h^\alpha}(\mathbb{H}^n)$ if and only if

$$\int_{\mathbb{H}^n} \frac{\Phi(y)}{|y|_h^Q} \|A^{-1}(y)\|^\alpha \left|\det A^{-1}(y)\right| dy < \infty.$$

(ii) If there exists at least one row of $A(y)$ such that all entries of such row are nonnegative uniformly or nonpositive uniformly on $y \in \mathrm{supp}(\Phi)$, then $T_{\Phi,A}$ is bounded on $H^1(\mathbb{H}^n)$ if and only if

$$\int_{\mathbb{H}^n} \frac{|\Phi(y)|}{|y|_h^Q} \left|\det A^{-1}(y)\right| dy < \infty.$$

Theorem 2.24. *Let Φ be a nonnegative function. Suppose $A(y) \in \mathrm{Aut}(\mathbb{H}^n)$ almost everywhere. If there exists a constant M independent of y such that $\|A^{-1}(y)\| \le M\|A(y)\|^{-1}$, then $T_{\Phi,A}$ is bounded on $h^1(\mathbb{H}^n)$ if and only if*

$$\begin{aligned}&\int_{\|A^{-1}(y)\|<1} \frac{\Phi(y)}{|y|_h^Q} \left|\det A^{-1}(y)\right| \max\left\{1, \ln\left(\|A^{-1}(y)\|^{-1}\right)\right\} dy \\ &+ \int_{\|A^{-1}(y)\|\ge 1} \frac{\Phi(y)}{|y|_h^Q} \left|\det A^{-1}(y)\right| dy \\ &<\infty.\end{aligned}$$

Let Φ be a locally integrable function on $\mathbb{H}^n$. The *fractional Hausdorff operator* on $\mathbb{H}^n$ [70] is defined by

$$T_{\Phi,\beta} f(x) = \int_{\mathbb{H}^n} \frac{\Phi(\delta_{|y|_h^{-1}} x)}{|y|_h^{Q-\beta}} f(y) dy, \quad 0 \le \beta \le Q.$$

We shall define two kinds of special Hausdorff operators. If Φ is a Poisson kernel associated to sub-Laplacians on the Heisenberg group, the corresponding Hausdorff operator T_Φ, written as $T_\Phi^{\mathfrak{p}}$, is called *the Hausdorff-Poisson operator*. It is known that Poisson kernels have the semigroup property:

$$\Phi_s * \Phi_t(x) = \Phi_{s+t}(x),$$

for $s, t > 0$. If $\Phi \in \mathcal{S}(\mathbb{H}^n)$ satisfies another group relation:

$$\Phi_s * \Phi_t(x) = \Phi_{\sqrt{s^2+t^2}}(x),$$

for $s, t > 0$, then the corresponding Hausdorff operator, written as $T_\Phi^{\mathfrak{g}}$, is called the Hausdorff-Gauss operator.

In [70], we obtained the boundedness of Hausdorff operators on Hardy spaces.

Theorem 2.25. *Suppose Φ is radial, $0 \le \beta < Q$ and*

$$\Psi(x) = \frac{\Phi(|x|_h^{-1})}{|x|_h^{Q-\beta}}.$$

Assume $0 < p < 1$, $\alpha = Q(1/p - 1)$ and $1/p = 1/q + \beta/Q$. If $\Psi \in \Gamma_{\alpha+\varepsilon} \cap \Gamma_{\alpha-\varepsilon}$ for some sufficiently small ε such that $\alpha - \varepsilon > 0$, then

$$\|T_{\Phi,\beta}(f)\|_{L^q(\mathbb{H}^n)} \preceq \|f\|_{H^p(\mathbb{H}^n)},$$

where Γ_α is the Lipschitz space of order α (see Section 3).

Theorem 2.26. *Let $\frac{Q}{Q+1} < p \le 1$. We have*

$$\|T_\Phi^{\mathfrak{p}}(f)\|_{H^p(\mathbb{H}^n)} \preceq \|f\|_{H^p(\mathbb{H}^n)}.$$

Theorem 2.27. *For all $0 < p < \infty$,*

$$\|T_\Phi^{\mathfrak{g}}(f)\|_{H^p(\mathbb{H}^n)} \preceq \|f\|_{H^p(\mathbb{H}^n)}.$$

3. Oscillatory type integral operators

3.1. One-sided oscillatory integral operators

The study for one-sided operators was motivated by their natural appearance in harmonic analysis; for example, in the study of one-sided Hardy-Littlewood maximal operator [48]

$$M^+ f(x) = \sup_{h>0} \frac{1}{h} \int_x^{x+h} |f(y)|\, dy \quad \& \quad M^- f(x) = \sup_{h>0} \frac{1}{h} \int_{x-h}^{x} |f(y)|\, dy$$

arising in the ergodic maximal function. Sawyer introduced the one-sided A_p classes A_p^+, A_p^- by the following conditions in [48]:

$$A_p^+: \quad A_p^+(w) := \sup_{a<b<c} \frac{1}{(c-a)^p} \int_a^b w(x)\, dx \left(\int_b^c w(x)^{1-p'}\, dx \right)^{p-1} < \infty,$$

$$A_p^-: \quad A_p^-(w) := \sup_{a<b<c} \frac{1}{(c-a)^p} \int_b^c w(x)\,dx \left(\int_a^b w(x)^{1-p'}\,dx\right)^{p-1} < \infty,$$

when $1 < p < \infty$; also, for $p = 1$, $A_1^+ : M^-w \le Cw$, $A_1^- : M^+w \le Cw$ for some constant C.

We say a function K is a one-sided Calderón-Zygmund kernel [26] if K satisfies the condition as that of Calderón-Zygmund kernel and

$$\left|\int_{a<|x|<b} K(x)\,dx\right| \le C, \quad 0 < a < b$$

with support in $\mathbb{R}^- = (-\infty, 0)$ or $\mathbb{R}^+ = (0, +\infty)$.

The one-sided oscillatory integral operator T^+ and T^- were defined in [26] as

$$T^+f(x) = \lim_{\varepsilon\to 0^+} \int_{x+\varepsilon}^{\infty} e^{iP(x,y)} K(x-y)f(y)dy = \text{p.v.} \int_x^{\infty} e^{iP(x,y)} K(x-y)f(y)dy$$

and

$$T^-f(x) = \lim_{\varepsilon\to 0^+} \int_{-\infty}^{x-\varepsilon} e^{iP(x,y)} K(x-y)f(y)dy = \text{p.v.} \int_{-\infty}^{x} e^{iP(x,y)} K(x-y)f(y)dy,$$

where $P(x, y)$ is a real polynomial defined on $\mathbb{R} \times \mathbb{R}$, and K are the one-sided Calderón-Zygmund kernel.

We obtained the weak (1,1) boundedness of T^+ in [26] as follows:

Theorem 3.1. *If $w \in A_1^+$, then there exists a constant C depending on the total degree of P, $C(K)$ and $A_1^+(w)$ such that*

$$\sup_{\lambda>0} \lambda w(\{x \in \mathbb{R} : |T^+f(x)| > \lambda\}) \le C\|f\|_{L^1(w)}$$

for $f \in \mathcal{S}(\mathbb{R})$.

One-sided BMO spaces were introduced in [44] and defined as

$$BMO^+ := \{f : f_+^\sharp \in L^\infty, \|f\|_{BMO^+} = \|f_+^\sharp\|_{L^\infty}\},$$

and

$$BMO^- := \{f : f_-^\sharp \in L^\infty, \|f\|_{BMO^-} = \|f_-^\sharp\|_{L^\infty}\},$$

where $f_+^\sharp$ and $f_-^\sharp$ were the one-sided sharp maximal function which were defined as

$$f_+^\sharp = \sup_{h>0} \frac{1}{h} \int_x^{x+h} \left(f(y) - \frac{1}{h}\int_{x+h}^{x+2h} f\right)^+ dy,$$

and

$$f_-^\sharp = \sup_{h>0} \frac{1}{h} \int_{x-h}^{x} \left(f(y) - \frac{1}{h}\int_{x-2h}^{x-h} f\right)^+ dy$$

with $z^+ = \max\{z, 0\}$. The commutators formed by T^+ and $b \in BMO^+$ can be defined as follows.

$$T_b^+f(x) = \text{p.v.} \int_x^{\infty} e^{iP(x,y)} K(x-y)(b(x) - b(y))f(y)dy$$

for $b \in BMO^+(\mathbb{R})$. The high order commutators of $T^+(T^-)$ and $b \in BMO^+(BMO^-)$ are

$$T_b^{k,+}f(x) = \text{p.v.}\int_x^{\infty} e^{iP(x,y)}K(x-y)\left(b(x)-b(y)\right)^k f(y)dy$$

and

$$T_b^{k,-}f(x) = \text{p.v.}\int_{-\infty}^{x} e^{iP(x,y)}K(x-y)\left(b(x)-b(y)\right)^k f(y)dy,$$

where $k \in \mathbb{Z}^+$.

In [51,65], we established the boundedness of commutator for $T_b^{k,+}$ and $T_b^{k,-}$ as follows.

Theorem 3.2. *Let $1 < p < \infty$ and $\widetilde{T}^+$ be the one-sided Calderón-Zygmund singular operator.*

(1) *If $w \in A_p^+$ and $\widetilde{T}^+$ is of type (L^2, L^2), then T_b^+ is bounded on $L^p(w)$ for every $b \in BMO^+$, where the norm independent on the coefficients of P.*

(2) *If $w \in A_p^-$ and $\widetilde{T}^-$ is of type (L^2, L^2), then T_b^- is bounded on $L^p(w)$ for every $b \in BMO^-$, where the norm independent on the coefficients of P.*

Theorem 3.3. *Let $1 < p < \infty$ and $k \in \mathbb{Z}^+$.*

(1) *If $w \in A_p^+$ and $\widetilde{T}^+$ is of type (L^2, L^2), then $T_b^{k,+}$ is bounded on $L^p(w)$ for every $b \in BMO^+$, where the norm independent on the coefficients of P.*

(2) *If $w \in A_p^-$ and $\widetilde{T}^-$ is of type (L^2, L^2), then $T_b^{k,-}$ is bounded on $L^p(w)$ for every $b \in BMO^-$, where the norm independent on the coefficients of P.*

We shall say that f in $\mathcal{S}'(x_{-\infty}, \infty)$ belongs to the *one-sided Hardy space* $H_+^q(w)$ [25] if

$$\|f\|_{H_+^q(w)} = \left(\int_{x_{-\infty}}^{\infty} (M_{+,\psi}^2(x))^q w(x)dx\right)^{1/q} < \infty, \quad 0 < q \le 1$$

and $w \in A_p^+(p \ge 1)$. A function $a(x)$ defined on $\mathbb{R}$ is called a *q-atom* with respect to $w(x)$ if there exists an interval I containing the support of $a(x)$ such that

(a) $I \subset (x_{-\infty}, \infty)$ and $w(I) < \infty$,

(b) $\|a\|_{L^\infty} \le w(I)^{-1/q}$,

(c) $|I| < dist(x_{-\infty}, I)$, and $\int_I a(x)dx = 0$.

It is easy to check that $H_+^1(w) \subset L^1(w)$. In [25], we proved the following results.

Theorem 3.4. *Let $P(x)$ be a polynomial which satisfies $P'(0) = 0$ and $w \in A_1^+$. Then there exists a constant $C > 0$, which depends only on $A_1^+(w)$ and the degree of $P(x)$ (not its coefficients), such that*

$$\|T^+f\|_{L^1(w)} \le C\|f\|_{H_+^1(w)}$$

for all $f \in H_+^1(w)$.

Corollary 3.1. *Let P and w be in the above Theorem. Then T^+ is bounded from $L^\infty(w)$ into $[H_+^q(w)]^*$, where $[H_+^q(w)]^*$ is the dual space of $H_+^q(w)$*

In [25], we also gave a criterion for the weighted L^p-boundednesss of T^+.

Theorem 3.5. *Let $P(x,y)$ be a real polynomial, K be a one-sided Calderón-Zygmund kernel and $b(r)$ be a bounded variation function on $[0,\infty)$. For $1 < p < \infty$ and $w \in A_p^+$, we have*

(a) *The operator*

$$\widetilde{T}^{+,b}f(x) = p.v \int_x^\infty b(y-x)K(x-y)f(y)dy$$

is of type $(L^p(w), L^p(w))$.

(b) *The operator*

$$T^{+,b}f(x) = p.v \int_x^\infty e^{iP(x,y)}b(y-x)K(x-y)f(y)dy$$

is of type $(L^p(w), L^p(w))$. *Here its norm depends only on the total degree of* $P(x,y)$ *and* $A_p^+(w)$, *but not on the coefficients of* $P(x,y)$.

Furthermore, we have

Theorem 3.6. *Let* w, p *and* K *be as in the above theorem. Then the following statements are equivalent:*

(a) *If* $P(x,y)$ *is a nontrivial polynomial* ($P(x,y)$ *does not take the form* $P_0(x)+P_1(y)$, *where* P_0 *and* P_1 *are polynomials defined on* $\mathbb{R}$), *then the operator* T^+ *is of type* $(L^p(w), L^p(w))$.

(b) *If* $P(x,y)$ *satisfies* $P(x,y) = P(x-h, y-h) + P_0(x,h) + P_1(y,h)$ *with* $h \in \mathbb{R}$ *and* P_0 *and* P_1 *are polynomials defined on* $\mathbb{R}$, *then the operator* T^+ *is of type* $(L^p(w), L^p(w))$.

(c) *The truncated operator*

$$\widetilde{T}_0^+ f(x) = p.v. \int_x^{x+1} K(x-y)f(y)dy$$

is of type $(L^p(w), L^p(w))$.

By the above two theorems, we can easily obtain the following result for the maximal operator corresponding to T^+:

Theorem 3.7. *Let* w, p *and* K *be as above. Then the maximal operator*

$$T_*^+ f(x) = \sup_{\varepsilon>0} \left| \int_{x+\varepsilon}^\infty e^{iP(x,y)}K(x-y)f(y)dy \right|$$

is of type $(L^p(w), L^p(w))$, *where its norm depends only on the total degree of* $P(x,y)$, *but not on the coefficients of* $P(x,y)$.

Very recently, the authors [23] further give the characterizations of one-sided Triebel–Lizorkin spaces and the boundedness for commutators. These results complement the missing components in the one-sided singular integral operator and function space theory studied before.

3.2. Fractional Fourier transforms

Theory of the FRFT has been developed on the Schwartz space $S(\mathbb{R})$ and $L^2(\mathbb{R})$, while we focus on the FRFT on $L^1(\mathbb{R})$. On $L^1(\mathbb{R})$, problems of convergence arise when certain manipulations of functions are performed and FRFT inversion is not possible.

For $f \in L^1(\mathbb{R})$ and $\alpha \in \mathbb{R}$, the *fractional Fourier transform* of order α of f is (c.f. [8]) defined by

$$(\mathcal{F}_\alpha f)(x) = \begin{cases} \int_{-\infty}^{+\infty} K_\alpha(x,t) f(t)\,\mathrm{d}t, & \alpha \neq n\pi, \quad n \in \mathbb{N}, \\ f(x), & \alpha = 2n\pi, \\ f(-x), & \alpha = (2n+1)\pi, \end{cases}$$

where

$$K_\alpha(x,t) = A_\alpha \exp\left[2\pi i \left(\frac{t^2}{2}\cot\alpha - xt\csc\alpha + \frac{x^2}{2}\cot\alpha\right)\right]$$

is the kernel of FRFT and

$$A_\alpha = \sqrt{1 - i\cot\alpha}.$$

From [8], we can see that $\mathcal{F}_\alpha$ is a bounded linear operator from $L^1(\mathbb{R})$ to $L^\infty(\mathbb{R})$. For $f \in L^1(\mathbb{R})$, $\mathcal{F}_\alpha f$ is uniformly continuous on $\mathbb{R}$.

Theorem 3.8 (Multiplication formula)**.** *For every $f, g \in L^1(\mathbb{R})$ and $\alpha \in \mathbb{R}$ we have*

$$\int_{-\infty}^{+\infty} (\mathcal{F}_\alpha f)(x) g(x)\mathrm{d}x = \int_{-\infty}^{+\infty} f(x)(\mathcal{F}_\alpha g)(x)\mathrm{d}x.$$

Let $f, g \in L^1(\mathbb{R})$. Define *the fractional convolution of order α* by

$$\left(f \overset{\alpha}{*} g\right)(x) = e^{-\pi i x^2 \cot\alpha} \int_{-\infty}^{+\infty} e^{\pi i t^2 \cot\alpha} f(t) g(x-t)\mathrm{d}t = \mathcal{M}_{-\alpha}\left(\mathcal{M}_\alpha f * g\right)(x).$$

The L^1 dilation of a function ϕ is defined as

$$\phi_\varepsilon(x) := \frac{1}{\varepsilon}\phi\left(\frac{x}{\varepsilon}\right), \quad \forall \varepsilon > 0.$$

The following is a fundamental result concerning fractional convolution and approximate identities [8].

Theorem 3.9. *Let $\phi \in L^1(\mathbb{R})$ and $\int_{-\infty}^{+\infty} \phi(x)\,\mathrm{d}x = 1$. If $f \in L^p(\mathbb{R}), 1 \leq p < \infty$, then*
(i)

$$\lim_{\varepsilon \to 0} \left\| \left(f \overset{\alpha}{*} \phi_\varepsilon\right) - f \right\|_p = 0.$$

(ii) *if in addition $\psi(x) = \sup_{|t| \geq |x|} |\phi(t)| \in L^1(\mathbb{R})$, then*

$$\lim_{\varepsilon \to 0} \left(f \overset{\alpha}{*} \phi_\varepsilon\right)(x) = f(x), \quad \text{a.e.} \quad x \in \mathbb{R},$$

where ψ is the decreasing radial dominant functions of ϕ.

Given $\Phi \in C_0(\mathbb{R})$, $\Phi(0) = 1$ and $\varepsilon > 0$, we define

$$M_{\varepsilon,\Phi_\alpha}(f) := \int_{-\infty}^{+\infty} (\mathcal{F}_\alpha f)(x) K_{-\alpha}(x, \cdot)\Phi_\alpha(\varepsilon x)\mathrm{d}x,$$

where

$$\Phi_\alpha(x) := \Phi(x \csc \alpha).$$

The expressions $M_{\varepsilon,\Phi_\alpha}(f)$ (with varying ε) are called *the Φ_α means* of the fractional Fourier integral of f.

We now address the FRFT inversion problem [8].

Theorem 3.10. *If $\Phi, \varphi := \mathcal{F}\Phi \in L^1(\mathbb{R})$ and $\int_{-\infty}^{+\infty} \varphi(x)\, dx = 1$, then*
(i) *the Φ_α means of the Fourier integral of f are convergent to f in L^1 norm, that is,*

$$\lim_{\varepsilon\to 0} \left\| \int_{-\infty}^{+\infty} (\mathcal{F}_\alpha f)(x) K_{-\alpha}(\cdot, x) \Phi_\alpha(\varepsilon x)\, \mathrm{d}x - f(\cdot) \right\|_1 = 0.$$

(ii) *if in addition $\psi = \sup_{|t|\geq|x|} |\varphi(t)| \in L^1(\mathbb{R})$, then the Φ_α means of the Fourier integral of f are a.e. convergent to f, that is,*

$$\int_{-\infty}^{+\infty} (\mathcal{F}_\alpha f)(x) K_{-\alpha}(t, x) \Phi_\alpha(\varepsilon x)\, \mathrm{d}x \to f(t)$$

as $\varepsilon \to 0$ for almost all $t \in \mathbb{R}$.

Theorem 3.11 (Uniqueness of FRFT on $L^1(\mathbb{R})$)**.** *If $f_1, f_2 \in L^1(\mathbb{R})$ and $(\mathcal{F}_\alpha f_1)(x) = (\mathcal{F}_\alpha f_2)(x)$ for all $x \in \mathbb{R}$, then*

$$f_1(t) = f_2(t), \quad a.e.\ t \in \mathbb{R}.$$

We have the following result concerning the action of the FRFT on $L^p(\mathbb{R})$.

Theorem 3.12 (Hausdorff-Young inequality)**.** *Let $1 < p \leq 2$, $p' = p/(p-1)$. Then $\mathcal{F}_\alpha$ are bounded linear operators from $L^p(\mathbb{R})$ to $L^{p'}(\mathbb{R})$. Moreover,*

$$\|\mathcal{F}_\alpha f\|_{p'} \leq A_\alpha^{\frac{2}{p}-1} \|f\|_p.$$

Let $1 \leq p \leq \infty$ and $m_\alpha \in L^\infty(\mathbb{R})$. Define the operator T_{m_α} as

$$\mathcal{F}_\alpha(T_{m_\alpha} f)(x) = m_\alpha(x)(\mathcal{F}_\alpha f)(x), \quad f \in L^2(\mathbb{R}) \cap L^p(\mathbb{R}).$$

The function m_α is called *the L^p Fourier multiplier of order α*, if there exist a constant $C_{p,\alpha} > 0$ such that

$$\|T_{m_\alpha} f\|_p \leq C_{p,\alpha} \|f\|_p, \quad f \in L^2(\mathbb{R}) \cap L^p(\mathbb{R}).$$

Fourier multipliers play an important role in operator theory, partial differential equations, and harmonic analysis. We got some basic multiplier theory results in the FRFT setting.

Theorem 3.13. *Let m_α be a bounded function. If there exists a constant $B > 0$ such that one of the following conditions hold:*

(a) (Mikhlin's condition)

$$\left| \frac{\mathrm{d}}{\mathrm{d}x} m_\alpha(x) \right| \leq B |x|^{-1};$$

(b) (Hörmander's condition)

$$\sup_{R>0} \frac{1}{R} \int_{R<|x|<2R} \left| \frac{\mathrm{d}}{\mathrm{d}x} m_\alpha(x) \right|^2 \mathrm{d}x \le B^2.$$

Then m_α is a fractional L^p multiplier for $1 < p < \infty$, that is, there exist a constant $C > 0$ such that

$$\|T_{m_\alpha} f\|_p = \|\mathcal{F}_{-\alpha} [m_\alpha (\mathcal{F}_\alpha f)]\|_p \le C \|f\|_p, \quad \forall f \in L^p(\mathbb{R}).$$

Theorem 3.14 (Bernstein multiplier theorem)**.** *Let $m_\alpha \in C^1(\mathbb{R}\backslash\{0\})$ be bounded. If $\|m'_\alpha\| < \infty$, then there exist constants $C_1, C_2 > 0$ such that*

$$\|\mathcal{F}_{-\alpha} [m_\alpha (\mathcal{F}_\alpha f)]\|_p \le C_1 \|f\|_p,$$

for $f \in L^p(\mathbb{R})$ $(1 \le p < \infty)$, and

$$\|m_\alpha\|^2 \le C_2 \|m_\alpha\|_2 \|m'_\alpha\|_2.$$

Theorem 3.15 (Marcinkiewicz multiplier theorem)**.** *Let $m_\alpha \in L^\infty(\mathbb{R}) \cap C^1(\mathbb{R}\backslash\{0\})$. If there exist a constant $B > 0$ such that*

$$\sup_{I \in \mathcal{I}} \int_I \left| \frac{\mathrm{d}}{\mathrm{d}x} m_\alpha(x) \right| \mathrm{d}x \le B,$$

where $\mathcal{I} := \{[2^j, 2^{j+1}], [-2^{j+1}, -2^j]\}_{j \in \mathbb{Z}}$ is the set of binary intervals in $\mathbb{R}$, then, for $f \in L^p(\mathbb{R})$ $(1 < p < \infty)$, there exist a constant $C > 0$ such that

$$\|\mathcal{F}_{-\alpha} [m_\alpha (\mathcal{F}_\alpha f)]\|_p \le C \|f\|_p.$$

The Littlewood-Paley is not only a powerful tool in Fourier analysis, but also plays a very important role in other areas, such as partial differential equations.

Let $j \in \mathbb{Z}$. Define the binary intervals in $\mathbb{R}$ as

$$\begin{cases} I_j^\alpha := [2^j \sin\alpha, 2^{j+1} \sin\alpha], -I_j^\alpha := [-2^{j+1} \sin\alpha, -2^j \sin\alpha], \ \alpha \in (0, \pi), \\ I_j^\alpha := [2^{j+1} \sin\alpha, 2^j \sin\alpha], -I_j^\alpha := [-2^j \sin\alpha, -2^{j+1} \sin\alpha], \alpha \in (\pi, 2\pi). \end{cases}$$

Then those binary intervals internally disjoint and

$$\mathbb{R}\backslash\{0\} = \bigcup_{j \in \mathbb{Z}} (-I_j^\alpha \cup I_j^\alpha).$$

Let $\mathcal{I}_\alpha := \{I_j^\alpha, -I_j^\alpha\}_{j \in \mathbb{Z}}$. Define the partial summation operator S_{ρ_α} corresponding to $\rho_\alpha \in \mathcal{I}_\alpha$ by

$$\mathcal{F}_\alpha (S_{\rho_\alpha} f)(x) = \chi_{\rho_\alpha}(x) (\mathcal{F}_\alpha f)(x), \quad \forall f \in L^2(\mathbb{R}) \cap L^p(\mathbb{R}),$$

where χ_{ρ_α} denote the characteristic function of the interval ρ_α. It is obvious that

$$\sum_{\rho_\alpha \in \mathcal{I}_\alpha} \|S_{\rho_\alpha}(f)\|_2^2 = \|f\|_2^2, \quad \forall f \in L^2(\mathbb{R}).$$

For general $L^p(\mathbb{R})$ functions, we have the following result, which is the Littlewood-Paley theorem in fractional setting.

Theorem 3.16. *Let* $f \in L^p(\mathbb{R})$, $1 < p < \infty$. *Then*

$$\left(\sum_{\rho_\alpha \in \mathcal{I}_\alpha} |S_{\rho_\alpha}(f)|^2 \right)^{1/2} \in L^p(\mathbb{R})$$

and there exists constants $C_1, C_2 > 0$ *independent of* f *such that*

$$C_1 \|f\|_p \le \left\| \left(\sum_{\rho_\alpha \in \mathcal{I}_\alpha} |S_{\rho_\alpha}(f)|^2 \right)^{1/2} \right\|_p \le C_2 \|f\|_p .$$

For $f \in L^1(\mathbb{R}^n)$ and $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, ..., \alpha_n) \in \mathbb{R}^n$, the nd-FRFT of f with order $\boldsymbol{\alpha}$ is defined by

$$\mathcal{F}_{\boldsymbol{\alpha}} f(\boldsymbol{z}) = \int_{\mathbb{R}^n} K_{\boldsymbol{\alpha}}(\boldsymbol{x}, \boldsymbol{z}) f(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x},$$

where $K_{\boldsymbol{\alpha}}(\boldsymbol{x}, \boldsymbol{z}) = \prod_{j=1}^n K_{\alpha_j}(x_j, z_j)$ and here for each $j = 1, 2, \cdots, n$, $K_{\alpha_j}(x_j, z_j)$ is defined as follows

$$K_{\alpha_j}(x_j, z_j) = \begin{cases} \sqrt{1 - i \cot \alpha_j} \; e^{i\pi \cot \alpha_j [x_j^2 + z_j^2 - 2x_j z_j \sec \alpha_j]}, & \alpha_j \notin \pi\mathbb{Z}, \\ \delta(x_j - z_j), & \alpha_j \in 2\pi\mathbb{Z}, \\ \delta(x_j + z_j), & \alpha_j \in 2\pi\mathbb{Z} + \pi, \end{cases}$$

where $\boldsymbol{x} = (x_1, x_2, \cdots, x_n)$.

The *multidimensional fractional convolution of order* $\boldsymbol{\alpha}$ can be defined as

$$\left(f \overset{\boldsymbol{\alpha}}{*} g \right)(\boldsymbol{x}) = e_{-\boldsymbol{\alpha}}(\boldsymbol{x}) \int_{\mathbb{R}^n} e_{\boldsymbol{\alpha}}(\boldsymbol{z}) f(\boldsymbol{z}) g(\boldsymbol{x} - \boldsymbol{z}) \mathrm{d}\boldsymbol{z}$$

for $f, g \in L^1(\mathbb{R}^n)$.

We establish the following two approximation theorems in [82]. Theorem 3.17 is about approximation in L^p norm and Theorem 3.18 is about almost everywhere approximation.

Theorem 3.17. *Let* $\phi \in L^1(\mathbb{R}^n)$, $f \in L^p(\mathbb{R}^n), 1 \le p < \infty$ *and* $\int_{\mathbb{R}^n} \phi(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x} = 1$. *Then*

$$\lim_{y \to 0} \left\| \left(f \overset{\boldsymbol{\alpha}}{*} \phi_y \right) - f \right\|_p = 0$$

where $\phi_y := \frac{1}{y^n} \phi\left(\frac{\cdot}{y} \right)$.

Theorem 3.18. *Let* $\phi \in L^1(\mathbb{R}^n)$, $f \in L^p(\mathbb{R}^n), 1 \le p < \infty$ *and* $\int_{\mathbb{R}^n} \phi(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x} = 1$. *Denote by* $\psi(\boldsymbol{z}) = \sup_{|\boldsymbol{x}| \ge \boldsymbol{z}} |\phi(\boldsymbol{x})|$ *the decreasing radial dominant function of* ϕ. *If* $\psi \in L^1(\mathbb{R}^n)$, *then for almost all* $\boldsymbol{z} \in \mathbb{R}^n$,

$$\lim_{y \to 0} \left(f \overset{\boldsymbol{\alpha}}{*} \phi_y \right)(\boldsymbol{z}) = f(\boldsymbol{z}),$$

where $\phi_y := \frac{1}{y^n} \phi\left(\frac{\cdot}{y} \right)$.

The uncertainty principle is a principle of physics introduced by Heisenberg in 1927. It points out that it is impossible to determine precisely the position and momentum of a microscopic particle simultaneously. It is one of the fundamental results in quantum mechanics. The uncertainty principle is expressed mathematically as the Heisenberg inequality. In [82], we also obtained the general Heisenberg inequality.

Theorem 3.19 (General Heisenberg inequality)**.** *Let* $f \in L^2(\mathbb{R}^n)$ *and* $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, ..., \alpha_n)$*,* $\boldsymbol{\beta} = (\beta_1, \beta_2, ..., \beta_n) \in \mathbb{R}^n$*. For any* $\boldsymbol{y} = (y_1, y_2, ..., y_n), \boldsymbol{v} = (v_1, v_2, ..., v_n) \in \mathbb{R}^n$*, if* $\alpha_1 - \beta_1 = \alpha_2 - \beta_2 = ... = \alpha_n - \beta_n$*, then*

$$\left[\int_{\mathbb{R}^n} |\boldsymbol{x} - \widetilde{\boldsymbol{y}}|^2 \left|(\mathcal{F}_{\boldsymbol{\alpha}} f)(\boldsymbol{x})\right|^2 \mathrm{d}\boldsymbol{x}\right] \left[\int_{\mathbb{R}^n} |\boldsymbol{z} - \widetilde{\boldsymbol{v}}|^2 \left|(\mathcal{F}_{\boldsymbol{\beta}} f)(\boldsymbol{z})\right|^2 \mathrm{d}\boldsymbol{z}\right] \geq \frac{n^2\|f\|_2^4}{16\pi^2} \sin^2(\alpha_1 - \beta_1),$$

where

$$\widetilde{\boldsymbol{y}} = (y_1 \sin\alpha_1 + v_1 \cos\alpha_1, y_2 \sin\alpha_2 + v_2 \cos\alpha_2, ..., y_n \sin\alpha_n + v_n \cos\alpha_n),$$
$$\widetilde{\boldsymbol{v}} = (y_1 \sin\beta_1 + v_1 \cos\beta_1, y_2 \sin\beta_2 + v_2 \cos\beta_2, ..., y_n \sin\beta_n + v_n \cos\beta_n).$$

3.3. Linear canonical transforms

The linear canonical transform (LCT) was proposed by Collins and Moshinsky–Quesne almost simultaneously in the early 1970s. Since LCT has more free parameters than the classical Fourier transform (FT) and the fractional Fourier transform (FRFT), it has become an important tool for time-frequency analysis, especially for non-stationary signals or time-varying signals, and is widely used in many fields such as radar, sonar, communication, information security, digital watermarking, etc.

Let's review the definition of 1D-LCT. Denote by $SL(2,\mathbb{R})$ the set of all 2×2 real matrices of determinant 1. For any matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{R})$, the *1D-LCT* is defined by

$$\mathcal{L}_A f(x) = \begin{cases} \int_{-\infty}^{+\infty} K_A(x,t) f(t)\, \mathrm{d}t, & b \neq 0, \\ \sqrt{d} e^{i\frac{cd}{2}x^2} f(dx), & b = 0, \end{cases}$$

where

$$K_A(x,t) = C_A e_{b,d}(x) e_{b,a}(t) e_b(x,t),$$
$$C_A = \sqrt{\frac{1}{i2\pi b}}, \quad e_{b,d}(x) = e^{i\frac{d}{2b}x^2}, \quad e_{b,a}(x) = e^{i\frac{a}{2b}t^2}, \quad e_b(x,t) = e^{-\frac{i}{b}xt}.$$

For any matrix $A_j = \begin{pmatrix} a_j & b_j \\ c_j & d_j \end{pmatrix} \in SL(2,\mathbb{R})$, $b_j \neq 0$, $j = 1, 2$, $\boldsymbol{u} = (u_1, u_2) \in \mathbb{R}^2$, the *2D-LCT* [81] is defined by

$$\mathcal{L}_{\boldsymbol{A}} f(\boldsymbol{u}) = \int_{\mathbb{R}^2} K_{\boldsymbol{A}}(\boldsymbol{u}, \boldsymbol{x}) f(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x},$$

where

$$K_{\boldsymbol{A}}(\boldsymbol{u},\boldsymbol{x}) = K_{A_1}(u_1,x_1)\,K_{A_2}(u_2,x_2) = C_{\boldsymbol{A}} e_{\boldsymbol{b},\boldsymbol{d}}(\boldsymbol{u})\, e_{\boldsymbol{b},\boldsymbol{a}}(\boldsymbol{x})\, e_{\boldsymbol{b}}(\boldsymbol{u},\boldsymbol{x}),$$
$$C_{\boldsymbol{A}} = C_{A_1}C_{A_2},\ \boldsymbol{A} = (A_1, A_2),$$
$$e_{\boldsymbol{b},\boldsymbol{d}}(\boldsymbol{u}) = e_{b_1,d_1}(u_1)e_{b_2,d_2}(u_2) = e^{i\left(\frac{d_1}{2b_1}u_1^2 + \frac{d_2}{2b_2}u_2^2\right)},$$
$$e_{\boldsymbol{b},\boldsymbol{a}}(\boldsymbol{x}) = e_{b_1,a_1}(x_1)e_{b_2,a_2}(x_2) = e^{i\left(\frac{a_1}{2b_1}x_1^2 + \frac{a_2}{2b_2}x_2^2\right)},$$
$$e_{\boldsymbol{b}}(\boldsymbol{u},\boldsymbol{x}) = e_{b_1}(u_1,x_1)e_{b_2}(u_2,x_2) = e^{-i\left(\frac{u_1x_1}{b_1} + \frac{u_2x_2}{b_2}\right)}.$$

In [81], we obtained that

Theorem 3.20 (General Multiplication formula)**.** *Let*

$$\boldsymbol{A} = (A_1, A_2),\ \boldsymbol{A}' = (A_1',\ A_2'),$$
$$A_j = \begin{pmatrix} a_j & b_j \\ c_j & d_j \end{pmatrix} \in SL(2,\mathbb{R}),\ A_j' = \begin{pmatrix} d_j & b_j \\ c_j & a_j \end{pmatrix} \in SL(2,\mathbb{R}),$$

where $j = 1, 2$. For every $f, g \in L^2(\mathbb{R}^2)$ we have

$$\int_{\mathbb{R}^2} [\mathcal{L}_{\boldsymbol{A}} f(\boldsymbol{u})]\, g(\boldsymbol{u})\, \mathrm{d}\boldsymbol{u} = \int_{\mathbb{R}^2} f(\boldsymbol{u})\, [\mathcal{L}_{\boldsymbol{A}'} g(\boldsymbol{u})]\, \mathrm{d}\boldsymbol{u}.$$

Theorem 3.21 (General Heisenberg inequality)**.** *Let $f \in L^2(\mathbb{R}^2)$ and*

$$A_j^k = \begin{pmatrix} a_j^k & b_j^k \\ c_j^k & d_j^k \end{pmatrix} \in SL(2,\mathbb{R}),\ j,k = 1,2.$$

For any $\boldsymbol{y} = (y_1, y_2), \boldsymbol{v} = (v_1, v_2) \in \mathbb{R}^2$, if $A_1^1 \left(A_2^1\right)^* = A_1^2 \left(A_2^2\right)^*$ then

$$\left[\int_{\mathbb{R}^2} |\boldsymbol{x} - \widetilde{\boldsymbol{y}}|^2 \left|(\mathcal{L}_{\boldsymbol{A_1}} f)(\boldsymbol{x})\right|^2 \mathrm{d}\boldsymbol{x}\right] \times \left[\int_{\mathbb{R}^2} |\boldsymbol{u} - \widetilde{\boldsymbol{v}}|^2 \left|(\mathcal{L}_{\boldsymbol{A_2}} f)(\boldsymbol{u})\right|^2 \mathrm{d}\boldsymbol{u}\right]$$
$$\geq \left|a_1^1 c_2^1 - b_1^1 d_2^1\right|^2 \|f\|_2^4,$$

where

$$\boldsymbol{A_1} = \left(A_1^1, A_1^2\right),\ \boldsymbol{A_2} = \left(A_2^1, A_2^2\right),$$
$$\widetilde{\boldsymbol{y}} = \left(y_1 b_1^1 + v_1 a_1^1, y_2 b_1^2 + v_2 a_1^2\right),$$
$$\widetilde{\boldsymbol{v}} = \left(y_1 b_2^1 + v_1 a_2^1, y_2 b_2^2 + v_2 a_2^2\right).$$

For $f, g \in L^1(\mathbb{R}^2)$, we define *the convolution* $\overset{\boldsymbol{A}}{*}$ by

$$\left(f \overset{\boldsymbol{A}}{*} g\right)(\boldsymbol{u}) = e_{-\boldsymbol{b},\boldsymbol{a}}(\boldsymbol{u}) \int_{\mathbb{R}^2} e_{\boldsymbol{b},\boldsymbol{a}}(\boldsymbol{x})\, f(\boldsymbol{x})\, g(\boldsymbol{u} - \boldsymbol{x})\, \mathrm{d}\boldsymbol{x}.$$

For $\varepsilon > 0$, let $\phi_\varepsilon(\boldsymbol{u}) := \frac{1}{\varepsilon^2}\phi\left(\frac{\boldsymbol{u}}{\varepsilon}\right)$.

Theorem 3.22. *Let $\phi \in L^1(\mathbb{R}^2)$ and $\int_{\mathbb{R}^2} \phi(\boldsymbol{u})\,\mathrm{d}\boldsymbol{u} = 1$. If $f \in L^p(\mathbb{R}^2), 1 \le p < \infty$, then*
(i)

$$\lim_{\varepsilon \to 0} \left\| \left(f \overset{\boldsymbol{A}}{*} \phi_\varepsilon \right) - f \right\|_p = 0.$$

(ii) *If in addition the decreasing radial dominant functions $\psi(\boldsymbol{u}) = \sup_{|\boldsymbol{x}| \ge |\boldsymbol{u}|} |\phi(\boldsymbol{x})| \in L^1(\mathbb{R}^2)$ and $f \in L^p(\mathbb{R}^2), 1 \le p < \infty$, then*

$$\lim_{\varepsilon \to 0} \left(f \overset{\boldsymbol{A}}{*} \phi_\varepsilon \right)(\boldsymbol{u}) = f(\boldsymbol{u}), \quad \text{a.e.} \quad \boldsymbol{u} \in \mathbb{R}^2.$$

Let $\Phi \in L^1(\mathbb{R}^2)$ with $\Phi(0,0) = 1$. For any $\varepsilon > 0$, *the $\Phi_{\boldsymbol{b}}$ means* of the linear canonical integral with respect to $\boldsymbol{A}$ of f is defined by

$$M_{\varepsilon,\Phi_{\boldsymbol{b}}}(f)(\boldsymbol{x}) := \int_{\mathbb{R}^2} \mathcal{L}_{\boldsymbol{A}} f(\boldsymbol{u})\, K_{\boldsymbol{A}^*}(\boldsymbol{x},\boldsymbol{u})\, \Phi_{\boldsymbol{b}}(\varepsilon \boldsymbol{u})\,\mathrm{d}\boldsymbol{u},$$

where

$$\Phi_{\boldsymbol{b}}(\boldsymbol{u}) := \Phi(\boldsymbol{u}_{\boldsymbol{b}}).$$

Theorem 3.23. *Let $\Phi, \varphi := \mathcal{F}\Phi \in L^1(\mathbb{R}^2)$ with $\int_{\mathbb{R}^2} \varphi(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = 1$ and $\psi(\boldsymbol{u}) = \sup_{|\boldsymbol{x}| \ge |\boldsymbol{u}|} |\varphi(\boldsymbol{x})| \in L^1(\mathbb{R}^2)$. Then*

(i) *$\Phi_{\boldsymbol{b}}$ means of the linear canonical integral of f are convergent to f in the sense of L^1 norm:*
$$\lim_{\varepsilon \to 0} \left\| M_{\varepsilon,\Phi_{\boldsymbol{b}}}(f) - f \right\|_1 = 0,$$

(ii) *$\Phi_{\boldsymbol{b}}$ means of the linear canonical integral of f are convergent to f almost everywhere, i.e.,*

$$\lim_{\varepsilon \to 0} M_{\varepsilon,\Phi_{\boldsymbol{b}}}(f)(x) = f(\boldsymbol{x}), \quad \text{a.e. } \boldsymbol{x} \in \mathbb{R}^2.$$

From Theorem 3.23, we deduce the following conclusion.

Corollary 3.2. *Suppose $f, \mathcal{L}_{\boldsymbol{A}} f \in L^1(\mathbb{R}^2)$. Then*

$$f(\boldsymbol{x}) = \int_{\mathbb{R}^2} \mathcal{L}_{\boldsymbol{A}} f(\boldsymbol{u})\, K_{\boldsymbol{A}^*}(\boldsymbol{x},\boldsymbol{u})\,\mathrm{d}\boldsymbol{u}, \quad \text{a.e. } \boldsymbol{x} \in \mathbb{R}^2.$$

Corollary 3.3. *For $f_1, f_2, \mathcal{L}_{\boldsymbol{A}} f_1, \mathcal{L}_{\boldsymbol{A}} f_2 \in L^1(\mathbb{R}^2)$ with*

$$\mathcal{L}_{\boldsymbol{A}} f_1(\boldsymbol{u}) = \mathcal{L}_{\boldsymbol{A}} f_2(\boldsymbol{u}), \quad \boldsymbol{u} \in \mathbb{R}^2,$$

we have

$$f_1(\boldsymbol{x}) = f_2(\boldsymbol{x}), \quad \textit{a.e. } \boldsymbol{x} \in \mathbb{R}^2.$$

The LCT also has the following boundedness result as FRFT.

Theorem 3.24 (Hausdorff-Young inequality)**.** *For $1 < p \le 2$, $p' = p/(p-1)$, we have that $\mathcal{L}_{\boldsymbol{A}}$ is a bounded linear operator from $L^p(\mathbb{R}^2)$ to $L^{p'}(\mathbb{R}^2)$. Moreover,*

$$\|\mathcal{L}_{\boldsymbol{A}} f\|_{p'} = C_{\boldsymbol{A}}^{\frac{2}{p}-1} \|f\|_p.$$

For $1 \le p \le \infty$ and $m_{\boldsymbol{A}} \in L^\infty(\mathbb{R}^2)$. The operator $T_{m_{\boldsymbol{A}}}$ is defined by

$$\mathcal{L}_{\boldsymbol{A}}(T_{m_{\boldsymbol{A}}} f)(\boldsymbol{x}) = m_{\boldsymbol{A}}(\boldsymbol{x}) \mathcal{L}_{\boldsymbol{A}} f(\boldsymbol{x}), \quad f \in L^2(\mathbb{R}^2) \cap L^p(\mathbb{R}^2).$$

If there exists a constant $C_{p,A} > 0$ satisfying

$$\|T_{m_{\boldsymbol{A}}} f\|_p \le C_{p,A} \|f\|_p, \quad f \in L^2(\mathbb{R}^2) \cap L^p(\mathbb{R}^2),$$

then $m_{\boldsymbol{A}}$ is called *the L^p linear canonical multiplier.*

Theorem 3.25. *Let $m_{\boldsymbol{A}} \in L^\infty(\mathbb{R}^2)$. If there exists a constant $B > 0$ satisfying one of the following conditions:*

1. *(Mikhlin's condition)*

$$\left| \frac{\partial^2}{\partial u_1 \partial u_2} m_{\boldsymbol{A}}(\boldsymbol{u}) \right| \le B |\boldsymbol{u}|^{-2};$$

2. *(Hörmander's condition)*

$$\sup_{R>0} \frac{1}{R} \int_{R<|\boldsymbol{u}|<2R} \left| \frac{\partial^2}{\partial u_1 \partial u_2} m_{\boldsymbol{A}}(\boldsymbol{u}) \right|^2 \mathrm{d}\boldsymbol{u} \le B.$$

Then there exists $C > 0$ satisfying

$$\|T_{m_{\boldsymbol{A}}} f\|_p = \|\mathcal{L}_{\boldsymbol{A}^*}(m_{\boldsymbol{A}} \mathcal{L}_{\boldsymbol{A}} f)\|_p \le C \|f\|_p, \quad f \in L^p(\mathbb{R}^2).$$

Corollary 3.4. *Assume that $f \in L^p(\mathbb{R}^2)$ $(1 < p < \infty)$ and $m_{\boldsymbol{A}} \in L^\infty(\mathbb{R}^2) \cap C^1(\mathbb{R}^2 \backslash \{0\})$.*
(i) (Bernstein-type multiplier theorem) *If $\|m'_{\boldsymbol{A}}\|_2 < \infty$, then there exists $C > 0$ satisfying*

$$\|\mathcal{L}_{\boldsymbol{A}^*}(m_{\boldsymbol{A}} \mathcal{L}_A f)\|_p \le C \|m_{\boldsymbol{A}}\|_2^{\frac{1}{2}} \|m'_{\boldsymbol{A}}\|_2^{\frac{1}{2}} \|f\|_p.$$

(ii) (Marcinkiewicz-type multiplier theorem) *If there exists $B > 0$ satisfying*

$$\sup_{I \in \Delta} \int_I \left| \frac{\partial^2}{\partial x_1 \partial x_2} m_{\boldsymbol{A}}(\boldsymbol{x}) \right| \mathrm{d}\boldsymbol{x} \le B,$$

where

$$\Delta = \mathcal{I} \times \mathcal{I},$$

$$\mathcal{I} = \left\{ [2^j, 2^{j+1}], [-2^{j+1}, -2^j] \right\}_{j \in \mathbb{Z}}$$

is the set of dyadic rectangles in $\mathbb{R}^2$, then there exists $C > 0$ satisfying

$$\|\mathcal{L}_{\boldsymbol{A}^*}(m_{\boldsymbol{A}} \mathcal{L}_{\boldsymbol{A}} f)\|_p \le C \|f\|_p.$$

We define the partial summation operator $S_{\rho_{\boldsymbol{b}}}$ associated with $\rho_{\boldsymbol{b}} \in \Delta$ as

$$\mathcal{L}_{\boldsymbol{A}}(S_{\rho_{\boldsymbol{b}}} f)(\boldsymbol{u}) = \chi_{\rho_{\boldsymbol{b}}}(\boldsymbol{u}) \mathcal{L}_{\boldsymbol{A}} f(\boldsymbol{u}), \quad f \in L^2(\mathbb{R}^2) \cap L^p(\mathbb{R}^2),$$

where $\chi_{\rho_{\boldsymbol{b}}}$ is the characteristic function of $\rho_{\boldsymbol{b}}$. It is simple to show that

$$\sum_{\rho_{\boldsymbol{b}} \in \Delta} \|S_{\rho_{\boldsymbol{b}}}(f)\|_2^2 = \|f\|_2^2, \quad \forall f \in L^2(\mathbb{R}^2).$$

Theorem 3.26. *For $f \in L^p(\mathbb{R}^2)$, $1 < p < \infty$, we have*

$$\left[\sum_{\rho_{\mathfrak{b}} \in \Delta} \left|S_{\rho_{\mathfrak{b}}}(f)\right|^2\right]^{1/2} \in L^p(\mathbb{R}^2)$$

and there exist $C_1, C_2 > 0$ satisfying

$$C_1 \|f\|_p \le \left\| \left[\sum_{\rho_{\mathfrak{b}} \in \Delta} \left|S_{\rho_{\mathfrak{b}}}(f)\right|^2\right]^{1/2} \right\|_p \le C_2 \|f\|_p .$$

4. Singular integral operators

4.1. Riesz type operators

Recall that a connected, simply connected nilpotent Lie group $\mathcal{G}$ is said to be stratified if its left-invariant Lie algebra $\mathfrak{g}$ (assumed real and of finite dimension) admits a direct sum decomposition

$$\mathfrak{g} = \bigoplus_{i=1}^{s} V_i, \quad [V_1, V_i] = V_{i+1}, \quad \text{for } i \le s-1 \text{ and } [V_1, V_s] = 0.$$

s is called the step of the group $\mathcal{G}$.

There is a natural family of dilations on $\mathfrak{g}$ defined for $r > 0$ as follows:

$$\delta_r\left(\sum_{i=1}^{s} v_i\right) = \sum_{i=1}^{s} r^i v_i, \quad \text{with } v_i \in V_i.$$

This allows the definition of dilation on $\mathcal{G}$, which we still denote by δ_r. We fix once and for all a (bi-invariant) Haar measure dx on $\mathcal{G}$ (which is just the lift of Lebesgue measure on $\mathfrak{g}$ via exp).

Denote by the sub-Laplacian $\Delta = \sum_{j=1}^{n} \mathrm{X}_j^2$, where $\{\mathrm{X}_1, \cdots, \mathrm{X}_n\}$ is the basis of V_1. Let Q denote the homogeneous dimension of $\mathcal{G}$, namely, $Q = \sum_{i=1}^{s} i \dim V_i$. And let p_h $(h > 0)$ be the heat kernel (that is, the integral kernel of $e^{h\Delta}$) on $\mathcal{G}$.

The kernel of the j^{th} Riesz transform $\mathrm{X}_j(-\Delta)^{-\frac{1}{2}}$ $(1 \le j \le n)$ is written simply as $K_j(g, g') = K_j(g'^{-1} \circ g)$. It is known that (c.f. [7, 18])

$$K_j \in C^\infty(\mathcal{G} \setminus \{0\}), \ K_j(\delta_r(g)) = r^{-Q} K_j(g), \quad \forall g \ne 0, \ r > 0, \ 1 \le j \le n.$$

Let ρ be the homogeneous norm on $\mathcal{G}$, which induces a quasi-distance

$$\rho(g, g') = \rho(g'^{-1} \circ g) = \rho(g^{-1} \circ g'), \quad \forall g, g' \in \mathcal{G}.$$

The bounded mean oscillation space $\mathrm{BMO}(\mathcal{G})$ [7] is defined to be the space of all locally integrable functions f on $\mathcal{G}$ such that

$$\|f\|_{\mathrm{BMO}(\mathcal{G})} := \sup_{B \subset \mathcal{G}} M(f, B) := \sup_{B \subset \mathcal{G}} \frac{1}{|B|} \int_B |f(g) - f_B| \, dg < \infty,$$

where $f_B = \frac{1}{|B|} \int_B f(g) dg$.

We define VMO($\mathcal{G}$) as the closure of the C_0^∞ functions on $\mathcal{G}$ under the norm of the BMO space.

Let $1 < p < \infty$, a weight w is said to be of class $A_p(\mathcal{G})$ if

$$[w]_{A_p} := \sup_{B\subset\mathcal{G}} \left(\frac{1}{|B|}\int_B w(g)dg\right)\left(\frac{1}{|B|}\int_B w(g)^{-1/(p-1)}dg\right)^{p-1} < \infty.$$

A weight w is said to be of class $A_1(\mathcal{G})$ if there exists a constant C such that for all balls $B \subset \mathcal{G}$,

$$\frac{1}{|B|}\int_B w(g)dg \le C \operatorname*{essinf}_{x\in B} w(g).$$

For $p = \infty$, we define

$$A_\infty(\mathcal{G}) = \bigcup_{1\le p<\infty} A_p(\mathcal{G}).$$

In [7], we obtained the following lower bound of the Riesz transform kernel and the characterization of the compactness of Riesz transform commutator.

Theorem 4.1. *Suppose that $\mathcal{G}$ is a stratified Lie group with homogeneous dimension Q and that $j \in \{1,2,\ldots,n\}$. There exist a large positive constant r_o and a positive constant C such that for every $g \in \mathcal{G}$ there exists a "twisted truncated sector" $G_g \subset \mathcal{G}$ satisfying that $\inf_{g'\in G_g} \rho(g,g') = r_o$ and that for every $g_1 \in B_\rho(g,1)$ and $g_2 \in G_g$, we have*

$$|K_j(g_1,g_2)| \ge C\rho(g_1,g_2)^{-Q}, \quad |K_j(g_2,g_1)| \ge C\rho(g_1,g_2)^{-Q},$$

and all $K_j(g_1,g_2)$ as well as all $K_j(g_2,g_1)$ have the same sign.

Moreover, this "twisted truncated sector" G_g is regular, in the sense that $|G_g| = \infty$ and that for any $R_2 > R_1 > 2r_o$,

$$|(B_\rho(g,R_2)\setminus B_\rho(g,R_1))\cap G_g| \approx |B_\rho(g,R_2)\setminus B_\rho(g,R_1)|,$$

where the implicit constants are independent of g and R_1, R_2.

Theorem 4.2. *Let $1 < p < \infty$, $w \in A_p(\mathcal{G})$, $b \in L^1_{loc}(\mathcal{G})$. Then $b \in$ VMO($\mathcal{G}$) if and only if for some $\ell \in \{1,\cdots,n\}$, Riesz transform commutator $[b,\mathcal{R}_\ell]$ is compact on $L^p_w(\mathcal{G})$.*

4.2. Cauchy type operators

4.2.1. Cauchy integrals and Cauchy–Leray integral

Recently, Lanzani and Stein [40] studied the Cauchy–Szegő projection operator on a bounded strongly pseudoconvex domain D in $\mathbb{C}^n$ whose boundary bD satisfies the minimum regularity condition of class C^2. The measure that they used on the boundary bD is the Leray–Levi measure $d\lambda$. They obtained the $L^p(bD)$ boundedness ($1 < p < \infty$) of a family of Cauchy integrals $\{\mathcal{C}_\epsilon\}_\epsilon$. Since the role of the parameter ϵ is of no consequence here, when denoting a member in this family we will simply write $\mathcal{C}$. Here the space $L^p(bD)$ is defined with respect to $d\lambda$. We point out that the kernel of these Cauchy integral operators does not satisfy the standard size or smoothness conditions for Calderón–Zygmund operators. To obtain the $L^p(bD)$

boundedness, they decomposed the Cauchy transform $\mathcal{C}$, which is the restriction of such a Cauchy integral on bD into the essential part $\mathcal{C}^\sharp$ and the remainder $\mathcal{R}$, i.e.,

$$\mathcal{C} = \mathcal{C}^\sharp + \mathcal{R},$$

where the kernel of $\mathcal{C}^\sharp$, denoted by $C^\sharp(w,z)$, satisfies the standard size and smoothness conditions for Calderón–Zygmund operators, i.e. there exists a positive constant A_1 such that for every $w, z \in bD$ with $w \neq z$,

$$\begin{cases} a) \ \ |C^\sharp(w,z)| \leq A_1 \dfrac{1}{\mathtt{d}(w,z)^{2n}}; \\ b) \ \ |C^\sharp(w,z) - C^\sharp(w',z)| \leq A_1 \dfrac{\mathtt{d}(w,w')}{\mathtt{d}(w,z)^{2n+1}}, \quad \text{if } \mathtt{d}(w,z) \geq c\mathtt{d}(w,w'); \\ c) \ \ |C^\sharp(w,z) - C^\sharp(w,z')| \leq A_1 \dfrac{\mathtt{d}(z,z')}{\mathtt{d}(w,z)^{2n+1}}, \quad \text{if } \mathtt{d}(w,z) \geq c\mathtt{d}(z,z') \end{cases}$$

for an appropriate constant $c > 0$ and where $\mathtt{d}(z,w)$ is a quasi-distance suitably adapted to D. However, the kernel $R(w,z)$ of $\mathcal{R}$ satisfies a size condition and a smoothness condition for only one of the variables as follows: there exists a positive constant C_R such that for every $w, z \in bD$ with $w \neq z$,

$$\begin{cases} d) \ \ |R(w,z)| \leq C_R \dfrac{1}{\mathtt{d}(w,z)^{2n-1}}; \\ e) \ \ |R(w,z) - R(w,z')| \leq C_R \dfrac{\mathtt{d}(z,z')}{\mathtt{d}(w,z)^{2n}}, \quad \text{if } \mathtt{d}(w,z) \geq c_R \mathtt{d}(z,z') \end{cases}$$

for an appropriate large constant c_R.

In [17], we obtained the following characterization of the commutator of Cauchy transform $\mathcal{C}$.

Theorem 4.3. *Suppose $D \subset \mathbb{C}^n$, $n \geq 2$, is a bounded domain whose boundary is of class C^2 and is strongly pseudoconvex. Suppose $b \in L^1(bD, d\lambda)$. Then for $1 < p < \infty$,*

(1) *$b \in \mathrm{BMO}(bD, d\lambda)$ if and only if the commutator $[b, \mathcal{C}]$ is bounded on $L^p(bD, d\lambda)$.*

(2) *$b \in \mathrm{VMO}(bD, d\lambda)$ if and only if the commutator $[b, \mathcal{C}]$ is compact on $L^p(bD, d\lambda)$.*

We also considered the Cauchy–Leray integral in the setting of Lanzani–Stein [41], where they studied such integral in a bounded domain D in $\mathbb{C}^n$ which is strongly $\mathbb{C}$-linearly convex and the boundary bD satisfies the minimum regularity $C^{1,1}$. The Cauchy–Leray transform of a suitable function f on bD, denoted $\mathcal{C}(f)$, is formally defined by

$$\mathcal{C}(f)(z) = \int_{bD} \frac{f(w)}{\Delta(w,z)^n} d\lambda(w), \quad z \in bD,$$

where $\Delta(w,z) = \langle \partial\rho(w), w - z \rangle$, ρ is the defining function of D. In [41], they obtained the $L^p(bD)$ boundedness $(1 < p < \infty)$ of $\mathcal{C}$ by showing that the kernel $K(w,z)$ of $\mathcal{C}$ satisfies the standard conditions of Calderón–Zygmund operators. Following a similar approach as in the proof for Theorem 4.3, we arrived at the following result on the commutator of the Cauchy–Leray transform.

Theorem 4.4. *Let D be a bounded domain in $\mathbb{C}^n$ of class $C^{1,1}$ that is strongly $\mathbb{C}$-linearly convex and let $b \in L^1(bD, d\lambda)$. Let $\mathcal{C}$ be the Cauchy–Leray transform. Then for $1 < p < \infty$,*

(1) *$b \in \mathrm{BMO}(bD, d\lambda)$ if and only if the commutator $[b, \mathcal{C}]$ is bounded on $L^p(bD, d\lambda)$.*

(2) *$b \in \mathrm{VMO}(bD, d\lambda)$ if and only if the commutator $[b, \mathcal{C}]$ is compact on $L^p(bD, d\lambda)$.*

In [31], we also characterized the boundedness and compactness of the commutator of Cauchy type integral $\mathfrak{C}$ on the weighted Morrey space $L_v^{p,\kappa}(bD)$, $p \in (1, \infty)$, $\kappa \in (0, 1)$ and $v \in A_p(bD)$, which is defined by

$$L_v^{p,\kappa}(bD) := \left\{ f \in L^p_{loc}(bD) : \ \|f\|_{L_v^{p,\kappa}(bD)} < \infty \right\}$$

with

$$\|f\|_{L_v^{p,\kappa}(bD)} := \sup_B \left\{ \frac{1}{[v(B)]^\kappa} \int_B |f(z)|^p v(z)\, d\lambda(z) \right\}^{1/p},$$

here

$$v(B) = \int_B v(z) d\lambda(z).$$

Theorem 4.5. *Suppose $D \subset \mathbb{C}^n$, $n \geq 2$, is a bounded domain whose boundary is of class C^2 and is strongly pseudoconvex. Suppose $b \in L^1(bD), 1 < p < \infty$, $0 < \kappa < 1$ and $v \in A_p$. Then,*

(1) $b \in \mathrm{BMO}(bD)$ if and only if the commutator $[b, \mathfrak{C}]$ is bounded on $L_v^{p,\kappa}(bD)$.

(2) $b \in \mathrm{VMO}(bD)$ if and only if the commutator $[b, \mathfrak{C}]$ is compact on $L_v^{p,\kappa}(bD)$.

Again following a similar approach as in the proof of Theorem 4.3, we obtained the following results on the Cauchy–Leray transform and its commutator.

Theorem 4.6. *Let D be a bounded domain in $\mathbb{C}^n$ of class $C^{1,1}$ that is strongly $\mathbb{C}$-linearly convex. Let $1 < p < \infty, v \in A_p$. Then there exists a positive constant C such that*

$$\|\mathcal{C}(f)\|_{L_v^p(bD)} \leq C \|f\|_{L_v^p(bD)}$$

holds for every function $f \in L_v^p(bD)$.

Theorem 4.7. *Let D be a bounded domain in $\mathbb{C}^n$ of class $C^{1,1}$ that is strongly $\mathbb{C}$-linearly convex and let $b \in L^1(bD), 1 < p < \infty, 0 < \kappa < 1$ and $v \in A_p$. Let $\mathcal{C}$ be the Cauchy–Leray transform (as in [41]). Then for $1 < p < \infty$,*

(1) *$b \in \mathrm{BMO}(bD)$ if and only if the commutator $[b, \mathcal{C}]$ is bounded on $L_v^{p,\kappa}(bD)$.*

(2) *$b \in \mathrm{VMO}(bD)$ if and only if the commutator $[b, \mathcal{C}]$ is compact on $L_v^{p,\kappa}(bD)$.*

4.2.2. Cauchy-Szegő projection operators

Recall that the space $\mathbf{H}$ of quaternion numbers forms a division algebra with respect to the coordinate addition and the quaternion multiplication

$$\begin{aligned} xx' &= (x_1 + x_2\mathbf{i} + x_3\mathbf{j} + x_4\mathbf{k})(x_1' + x_2'\mathbf{i} + x_3'\mathbf{j} + x_4'\mathbf{k}) \\ &= x_1x_1' - x_2x_2' - x_3x_3' - x_4x_4' + (x_1x_2' + x_2x_1' + x_3x_4' - x_4x_3')\,\mathbf{i} \\ &\quad + (x_1x_3' - x_2x_4' + x_3x_1' + x_4x_2')\,\mathbf{j} + (x_1x_4' + x_2x_3' - x_3x_2' + x_4x_1')\,\mathbf{k}, \end{aligned}$$

for any $x = x_1 + x_2\mathbf{i} + x_3\mathbf{j} + x_4\mathbf{k}$, $x' = x_1' + x_2'\mathbf{i} + x_3'\mathbf{j} + x_4'\mathbf{k} \in \mathbf{H}$. The conjugate $\bar{x}$ is defined by

$$\bar{x} = x_1 - x_2\mathbf{i} - x_3\mathbf{j} - x_4\mathbf{k},$$

and the modulus $|x|$ is defined by $|x|^2 = x\bar{x} = \sum_{j=1}^4 x_j^2$. The conjugation inverses the product of quaternion number in the following sense $\overline{q\sigma} = \bar{\sigma}\bar{q}$ for any $q, \sigma \in \mathbf{H}$. It is clear that

$$\begin{aligned}
\operatorname{Im}(\bar{x}x') &= \operatorname{Im}\{(x_1 - x_2\mathbf{i} - x_3\mathbf{j} - x_4\mathbf{k})(x_1' + x_2'\mathbf{i} + x_3'\mathbf{j} + x_4'\mathbf{k})\} \\
&= (x_1x_2' - x_2x_1' - x_3x_4' + x_4x_3')\,\mathbf{i} + (x_1x_3' + x_2x_4' - x_3x_1' - x_4x_2')\,\mathbf{j} \\
&\quad + (x_1x_4' - x_2x_3' + x_3x_2' - x_4x_1')\,\mathbf{k} \\
&=: \sum_{\alpha=1}^{3} \sum_{k,j=1}^{4} b_{kj}^{\alpha} x_k x_j' \mathbf{i}_\alpha,
\end{aligned}$$

where $\mathbf{i}_1 = \mathbf{i}, \mathbf{i}_2 = \mathbf{j}, \mathbf{i}_3 = \mathbf{k}$, and b_{kj}^{α} is the (k, j)-th entry of the following matrices b^{α}:

$$b^1 := \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad b^2 := \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad b^3 := \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}.$$

The Siegel upper half space

$$\mathcal{U}_n := \left\{ q = (q_1, \cdots, q_n) = (q_1, q') \in \mathbf{H}^n \mid \operatorname{Re} q_1 > |q'|^2 \right\},$$

where $q' = (q_2, \cdots, q_n) \in \mathbf{H}^{n-1}$, whose boundary $\partial\mathcal{U}_n := \{(q_1, q') \in \mathbf{H}^n \mid \operatorname{Re} q_1 = |q'|^2\}$ is a quadratic hypersurface and can be identified with the quaternionic Heisenberg group $\mathcal{H}^{n-1}$, which is $\operatorname{Im}\mathbf{H} \times \mathbf{H}^{n-1}$ endowed with the non-commutative multiplication

$$(t, y) \cdot (t', y') = (t + t' + 2\operatorname{Im}\langle y, y'\rangle, y + y'),$$

where $t = t_1\mathbf{i} + t_2\mathbf{j} + t_3\mathbf{k}$, $t' = t_1'\mathbf{i} + t_2'\mathbf{j} + t_3'\mathbf{k} \in \operatorname{Im}\mathbf{H}$, $y, y' \in \mathbf{H}^{n-1}$, and $\langle \cdot, \cdot \rangle$ is the inner product defined by

$$\langle y, y' \rangle = \sum_{l=1}^{n-1} \overline{y}_l y_l', \quad y = (y_1, \cdots, y_{n-1}), \quad y' = (y_1', \cdots, y_{n-1}') \in \mathbf{H}^{n-1}.$$

The Cauchy–Szegő projection operator $\mathcal{P}$ can be defined via the "vertical translate" from Cauchy–Szegő kernel for $\mathcal{U}_n$ by

$$(\mathcal{P}f)(q) = \lim_{\varepsilon \to 0} \int_{\partial\mathcal{U}_n} S(q + \varepsilon\mathbf{e}, p) f(p) d\beta(p), \quad \forall f \in L^2(\partial\mathcal{U}_n), \quad q \in \partial\mathcal{U}_n,$$

where $\mathbf{e} = (1, 0, 0, \cdots, 0) \in \mathbb{H}^n$, the limit exists in the $L^2(\partial\mathcal{U}_n)$ norm and $\mathcal{P}(f)$ is the boundary limit of some function in holomorphic Hardy space $H^2(\mathcal{U}_n)$ (c.f. [4,6]). In view of the action of the quaternionic Heisenberg group, the operator $\mathcal{P}$ can be explicitly described as a convolution operator on this group:

$$(\mathcal{P}f)(g) = (f * K)(g) = p.v. \int_{\mathcal{H}^{n-1}} K(h^{-1} \cdot g) f(h) dh,$$

where the kernel $K(g)$ is the corresponding Cauchy–Szegő kernel on $\mathcal{H}^{n-1}$. We can write

$$(\mathcal{P}f)(g) = p.v. \int_{\mathcal{H}^{n-1}} K(g,h)f(h)dh, \tag{4.1}$$

where $K(g,h) = K(h^{-1} \cdot g)$ for $g \neq h$. Note that (4.1) holds whenever f is an L^2 function supported in a compact set, for ever g outside the support of f.

Based on [6], in [4], we obtained the explicit formula of Cauchy–Szegő kernel for the quaternionic Siegel upper half-space $\mathcal{U}_n$.

Theorem 4.8. *The explicit formula of Cauchy–Szegő kernel for the quaternionic Siegel upper half-space $\mathcal{U}_n$ is given by*

$$S(q,p) = s\Big(q_1 + \overline{p}_1 - 2\sum_{k=2}^{n} \overline{p}_k q_k\Big),$$

for $p = (p_1, p') = (p_1, \cdots, p_n) \in \mathcal{U}_n$, $q = (q_1, q') = (q_1, \cdots, q_n) \in \mathcal{U}_n$, where

$$\begin{aligned} s(\sigma) = {} & c_{n-1} \frac{4(2n-2)!}{|z|^4 (z-\bar{z})^3} \mathbf{i} \\ & \times \left\{ \mathrm{Im}\left[\frac{\bar{z}^2}{z^{2n-2}} \big(z + (2n-1)\frac{z-\bar{z}}{2}\big) \right] \overline{\sigma} - \mathrm{Im}\left[\frac{\bar{z}^2}{z^{2n-3}} \big(z + (2n-2)\frac{z-\bar{z}}{2}\big) \right] \right\}, \end{aligned}$$

here $\sigma = x_1 + x_2\mathbf{i} + x_3\mathbf{j} + x_4\mathbf{k} \in \mathbf{H}$, $z = x_1 + |\operatorname{Im}\sigma|\mathbf{i}$, and c_{n-1} is the one in Theorem A.

Theorem 4.9. *Suppose $j = 1, \ldots, 4n-4$, and we denote Y_j the left-invariant vector fields on $\mathcal{H}^{n-1}$. Then we have*

$$|Y_j K(g)| \lesssim \frac{1}{\rho(g, \mathbf{0})^{Q+1}}, \quad g \in \mathcal{H}^{n-1} \setminus \{\mathbf{0}\},$$

where $\mathbf{0}$ is the neutral element of $\mathcal{H}^{n-1}$.

Then we further have the Cauchy–Szegő kernel $K(g,h)$ on $\mathcal{H}^{n-1}$ $(g \neq h)$ satisfies the following conditions.

$$\text{(i)} \quad |K(g,h)| \lesssim \frac{1}{\rho(g,h)^Q};$$

$$\text{(ii)} \quad |K(g,h) - K(g_0,h)| \lesssim \frac{\rho(g,g_0)}{\rho(g_0,h)^{Q+1}}, \quad \text{if } \rho(g_0,h) \geq c\rho(g,g_0);$$

$$\text{(iii)} \quad |K(g,h) - K(g,h_0)| \lesssim \frac{\rho(h,h_0)}{\rho(g,h_0)^{Q+1}}, \quad \text{if } \rho(g,h_0) \geq c\rho(h,h_0)$$

for some constant $c > 0$, where $Q = 4n+2$ is the homogeneous dimension of $\mathcal{H}^{n-1}$ and ρ is defined in Section 2.

We also got the boundedness and compactness of the commutator of $\mathcal{P}$.

Corollary 4.1. *Let all the notation be the same as above.*

(1) $\mathcal{P}$ extends to a bounded operator on $L^p(\mathcal{H}^{n-1})$ for $1 < p < \infty$;

(2) $\mathcal{P}$ is of weak type $(1,1)$;

(3) $\mathcal{P}$ is bounded from $H^1(\mathcal{H}^{n-1}) \to L^1(\mathcal{H}^{n-1})$;

(4) $\mathcal{P}$ is bounded from $L^\infty(\mathcal{H}^{n-1}) \to \mathrm{BMO}(\mathcal{H}^{n-1})$;

(5) $[b,\mathcal{P}]$ is bounded on $L^p(\mathcal{H}^{n-1})$ for $1<p<\infty$ if $b \in \mathrm{BMO}(\mathcal{H}^{n-1})$;

(6) $[b,\mathcal{P}]$ is compact on $L^p(\mathcal{H}^{n-1})$ for $1<p<\infty$ if $b \in \mathrm{VMO}(\mathcal{H}^{n-1})$.

However, to obtain reverse arguments of (5) and (6) above, we still need to know more about the pointwise lower bound of the kernel K.

Theorem 4.10. *The Cauchy–Szegő kernel $K(\cdot,\cdot)$ on $\mathcal{H}^{n-1}$ satisfies the following pointwise lower bound: there exist a large positive constant r_0 and a positive constant C such that for every $g \in \mathcal{H}^{n-1}$, there exists a "twisted truncated sector" $S_g \subset \mathcal{H}^{n-1}$ such that*

$$\inf_{g' \in S_g} \rho(g,g') = r_0$$

and that for every $g_1 \in B(g,1)$ and $g_2 \in S_g$ we have

$$|K(g_1,g_2)| \geq \frac{C}{\rho(g_1,g_2)^Q}.$$

Moreover, this sector S_g is regular in the sense that $|S_g| = \infty$ and that for every $R_2 > R_1 > 2r_0$

$$\big|\big(B(g,R_2)\backslash B(g,R_1)\big) \cap S_g\big| \approx \big|B(g,R_2)\backslash B(g,R_1)\big|$$

with the implicit constants independent of g and R_1, R_2.

By using the above Theorem, we established the boundedness and compactness of the commutator of $\mathcal{P}$ with respect to $\mathrm{BMO}(\mathcal{H}^{n-1})$ and $\mathrm{VMO}(\mathcal{H}^{n-1})$ in [20], following the ideas and approaches in [7, 17], respectively.

Theorem 4.11. *Suppose $1<p<\infty$ and $b \in L^1_{loc}(\mathcal{H}^{n-1})$.*

(i) *$b \in \mathrm{BMO}(\mathcal{H}^{n-1})$ if and only if $[b,\mathcal{P}]$ is bounded on $L^p(\mathcal{H}^{n-1})$.*

(ii) *$b \in \mathrm{VMO}(\mathcal{H}^{n-1})$ if and only if $[b,\mathcal{P}]$ is compact on $L^p(\mathcal{H}^{n-1})$.*

Theorem 4.12. *Let $p \in (1,\infty)$, $\kappa \in (0,1)$, $w \in A_p(\mathcal{H}^{n-1})$ and $b \in L^1_{\mathrm{loc}}(\mathcal{H}^{n-1})$. Then the Cauchy–Szegö operator commutator $[b,\mathcal{P}]$ has the following boundedness characterization:*

(i) *If $b \in \mathrm{BMO}(\mathcal{H}^{n-1})$, then $[b,\mathcal{P}]$ is bounded on $L^{p,\kappa}_w(\mathcal{H}^{n-1})$.*

(ii) *If b is real-valued and $[b,\mathcal{P}]$ is bounded on $L^{p,\kappa}_w(\mathcal{H}^{n-1})$, then $b \in \mathrm{BMO}(\mathcal{H}^{n-1})$.*

Based on the above Theorem, we further obtained the compactness characterization of Cauchy–Szegö operator commutator.

Theorem 4.13. *Let $p \in (1,\infty)$, $\kappa \in (0,1)$, $w \in A_p(\mathcal{H}^{n-1})$ and $b \in \mathrm{BMO}(\mathcal{H}^{n-1})$. Then the Cauchy–Szegö operator commutator $[b,\mathcal{P}]$ has the following compactness characterization:*

(i) *If $b \in \mathrm{VMO}(\mathcal{H}^{n-1})$, then $[b,\mathcal{P}]$ is compact on $L^{p,\kappa}_w(\mathcal{H}^{n-1})$.*

(ii) *If b is real-valued and $[b,\mathcal{P}]$ is compact on $L^{p,\kappa}_w(\mathcal{H}^{n-1})$, then $b \in \mathrm{VMO}(\mathcal{H}^{n-1})$.*

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