Journal of Applied Analysis and Computation
Volume 14, Number 1, February 2024, 379–391
Website:http://www.jaac-online.com
DOI:10.11948/20230226

EXISTENCE AND ASYMPTOTIC BEHAVIOR OF GROUND STATE SOLUTIONS FOR A CLASS OF MAGNETIC KIRCHHOFF CHOQUARD TYPE EQUATION WITH A STEEP POTENTIAL WELL

Li Zhou[1], Chuanxi Zhu[2,3] and Shufen Liu[4,†]

Abstract In this paper, we consider the following nonlinear magnetic Kirchhoff Choquard type equation

$$\begin{aligned}&[a+b\int_{\mathbb{R}^N}(|\nabla_A u|^2+\lambda V(x)|u|^2)\mathrm{d}x](-\Delta_A u+\lambda V(x)u)\\ =&(I_\alpha * F(|u|))\frac{f(|u|)}{|u|}u,\ \text{in } \mathbb{R}^N,\end{aligned}$$

where $u:\mathbb{R}^N\to\mathbb{C}$, $A:\mathbb{R}^N\to\mathbb{R}^N$ is a vector potential, $N\geq 3$, $a>0$, $b>0$, $\alpha\in(N-2,N]$, $V:\mathbb{R}^N\to\mathbb{R}$ is a scalar potential function and I_α is a Riesz potential of order $\alpha\in(N-2,N]$. Under certain assumptions on $A(x)$, $V(x)$ and $f(t)$, we prove that the equation has at least one ground state solution by variational methods and investigate the asymptotic behavior of solutions.

Keywords Magnetic Laplace operator, ground state solutions, Nehari manifold, asymptotic behavior.

MSC(2010) 35J60, 35J20.

1. Introduction

In this article, we study the following Kirchhoff Choquard type equation

$$\begin{cases} [a+b\int_{\mathbb{R}^N}(|\nabla_A u|^2+\lambda V(x)|u|^2)\mathrm{d}x](-\Delta_A u+\lambda V(x)u)\\ =(I_\alpha * F(|u|))\frac{f(|u|)}{|u|}u,\ \text{in } \mathbb{R}^N,\\ |u|\in H^1(\mathbb{R}^N).\end{cases} \tag{1.1}$$

[†]The corresponding author.
[1]Department of Mathematics, Zhejiang University of Science & Technology, Hangzhou, Zhejiang 310023, China
[2]School of Mathematics, Dalian University of Technology, Dalian, Liaoning 116024, China
[3]Department of Mathematics, Nanchang University, Nanchang, Jiangxi 330031, China
[4]Department of Basic Discipline, Nanchang JiaoTong Institute, Nanchang, Jiangxi 330031, China
[*]The authors were supported by Scientific research start-up fund (Grant No. F701108M03), National Natural Science Foundation of China (11771198, 11901276) and School level teaching research & reform project (Grant No. 2023-jg51).
Email: zhouli8869@163.com(L. Zhou), 2573627764@qq.com(C. Zhu), 397621669@qq.com(S. Liu)

Let us define $\nabla_A u = -i\nabla u - Au$ and the magnetic Laplace operator $\Delta_A u := (\nabla - iA)^2 u = \Delta u - 2iA(x)\cdot\nabla u - |A(x)|^2 u - iu\mathrm{div}(A(x))$. Here i is the imaginary unit, $u : \mathbb{R}^N \to \mathbb{C}$, $A : \mathbb{R}^N \to \mathbb{R}^N$ is a vector magnetic potential, $N \geq 3$, $a > 0$, $b > 0$, $\lambda > 0$, $F(t) = \int_0^t f(s)\mathrm{d}s$, $V : \mathbb{R}^N \to \mathbb{R}$ is a scalar potential function and I_α is a Riesz potential whose order is $\alpha \in (N-2, N]$ defined by $I_\alpha = \frac{\Gamma(\frac{N-\alpha}{2})}{\Gamma(\frac{\alpha}{2})\pi^{\frac{N}{2}}2^\alpha |x|^{N-\alpha}}$, where Γ is the Gamma function. $V(x) : \mathbb{R}^N \to \mathbb{R}$ is a continuous, bounded potential function satisfying:

(V1) $V(x) \in C(\mathbb{R}^N, \mathbb{R})$, and $V(x) \geq 0$ for all $x \in \mathbb{R}^N$,

(V2) There exists $V_0 > 0$ such that $\mathcal{V}_0 := \{x \in \mathbb{R}^N : V(x) \leq V_0\}$ is nonempty and has a finite measure,

(V3) $\Omega := intV^{-1}(0)$ is a nonempty open set which has locally Lipschitz boundary and $\overline{\Omega} = V^{-1}(0)$.

We also suppose A satisfies:

(A1) $\liminf\limits_{|x|\to+\infty} A(x) = A_\infty$,

(A2) $A \in L^\upsilon(\mathbb{R}^N, \mathbb{R}^N), \upsilon > N \geq 3$,

(AV) $|A(y)|^2 + V(y) < |A_\infty|^2 + V_\infty$.

Moreover, we assume that the function $f \in C^1(\mathbb{R}, \mathbb{R})$ verifies:

$(f1)$ $f(t) = o(t^{\frac{\alpha}{N}})$ as $t \to 0$,

$(f2)$ $\lim\limits_{|t|\to+\infty} \frac{f(t)}{t^{\frac{\alpha+2}{N-2}}} = 0$,

$(f3)$ there exists $\theta > 2$ such that $f(t)t > \theta F(t)$,

$(f4)$ $f(t)$ is increasing on $\mathbb{R}$.

These hypotheses of $V(x)$ were first put forward by Bartsch and Wang [8] in the research of the nonlinear Schrödinger equations and have attracted the attention of several researchers, see e.g. [7, 35, 39]. We note that the conditions (V1)-(V3) imply that λV represents a potential well which has the bottom $V^{-1}(0)$ and its steepness is controlled by the positive parameter λ. In consideration of this, λV is often referred as the steep potential well if λ is sufficiently large.

It should be noted that if $b = 0$, problem (1.1) is related to the following equation

$$-\Delta_A u + V(x)u = (I_\alpha * F(|u|))\frac{f(|u|)}{|u|}u, \text{ in } \mathbb{R}^N. \tag{1.2}$$

When $A \equiv 0$ it conduces to the Choquard equation. There is a huge collections of articles on the subject and some good reviews of the Choquard equation can be found in [18,26–32,41]. Recently, Alves et al. [5] studied the existence of multi-bump solutions for the Choquard equation as follows

$$-\Delta u + (\lambda V(x) + 1)u = (\frac{1}{|x|^\mu} * |u|^p)|u|^{p-2}u, \text{ in } \mathbb{R}^3, \tag{1.3}$$

where $\mu \in (0,3)$, $p \in (2, 6-\mu)$, the nonnegative continuous function $V(x)$ has a potential well.

If $b \neq 0$ and $A \equiv 0$, problem (1.1) is related to the Kirchhoff type equation

$$\begin{cases} -(a + b\int_{\mathbb{R}^N} |\nabla u|^2 \mathrm{d}x)\Delta u + V(x)u = g(x,u), \text{ in } \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N), \end{cases} \tag{1.4}$$

where $a > 0$, $b \geq 0$, $V : \mathbb{R}^N \to \mathbb{R}$ is a potential function and $g \in C(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$. This problem has an interesting physical context. Indeed, if we set $V(x) = 0$ and replace

$\mathbb{R}^N$ by a bounded domain $\Omega \subset \mathbb{R}^N$ in (1.4), then we get the following Kirchhoff Dirichlet problem

$$\begin{cases} -(a+b\int_\Omega |\nabla u|^2 \mathrm{d}x)\Delta u = g(x,u), & x \in \Omega, \\ u = 0 & x \in \ \partial\Omega. \end{cases} \tag{1.5}$$

It is related to the stationary analogue of the equation

$$u_{tt} - (a+b\int_\Omega |\nabla u|^2 \mathrm{d}x)\Delta u = g(x,u).$$

Such a hyperbolic equation is a general version of the equation

$$\rho\frac{\partial^2 u}{\partial t^2} - (\frac{\rho_0}{h} + \frac{E}{2L}\int_0^L |\frac{\partial u}{\partial x}| \mathrm{d}x)\frac{\partial^2 u}{\partial x^2} = 0,$$

which was proposed by G. Kirchhoff as an extension of classical D′Alembert′s wave equations for free vibration of elastic strings. Kirchhoff′s model takes into account the changes in length of the string produced by transverse vibrations. Later, J. L. Lions introduced a functional analysis approach. After that, (1.4) has been paid much attention to by many scholars. Readers can see [10,19,34,40] for recent work. Moreover, Kirchhoff type problems with steep potential well have also been studied by many researchers, see [11,23,25] and references therein. It is worth mentioning that for the nonlocal problems with steep potential well, Alves and Figueiredo [4] considered the following Kirchhoff problem

$$\begin{cases} M[\int_{\mathbb{R}^3} |\nabla u|^2 \mathrm{d}x + \int_{\mathbb{R}^3} (\lambda\alpha(x)+1)u^2)\mathrm{d}x](-\Delta u + (\lambda\alpha(x)+1)u) = f(u), \text{ in } \mathbb{R}^3, \\ u \in H^1(\mathbb{R}^3). \end{cases} \tag{1.6}$$

Assuming that the nonnegative function $\alpha(x)$ has a potential well with $int(\alpha^{-1}(0))$ consisting of k disjoint components $\Omega_1,\ \Omega_2,\ \cdots,\ \Omega_k$ and the nonlinearity $f(t)$ has a subcritical growth, they established the existence and multiplicity of positive multi-bump solutions by using variational methods.

On the other hand, there are also many works concerning the following nonlinear Schrödinger equations with a magnetic field recently:

$$-\Delta_A u + V(x)u = |u|^{p-2}u, \text{ in } \Omega \subset R^N,\ N \geq 2. \tag{1.7}$$

Here $u : \Omega \to \mathbb{C}$, $2 < p \leq 2^*$, where $2^* = \frac{2N}{N-2}$ if $N \geq 3$ and $2^* = \infty$ if $N = 1$ or 2. Besides, $A : \Omega \to \mathbb{R}^N$ and $V : \Omega \to \mathbb{R}$ are smooth. It is well known that the first paper in which problem (1.7) has been studied maybe Esteban and Lions [17]. They used the concentration-compactness principle and minimization arguments to obtain solutions for a local equation for $N = 2$ and $N = 3$. More recently, applying constrained minimization and a minimax-type argument, Arioli-Szulkin [6] considered the equation in a magnetic filed. They established the existence of nontrivial solutions both in the critical and in the subcritical case, provided that some technical conditions relating to A and V were assumed. In [2], the authors use the penalization method and Ljusternik-Schnirelmann category theory to prove the multiplicity and concentration results of solutions for the following nonlinear Schrödinger equation with magnetic field:

$$\begin{cases} (\frac{\varepsilon}{i}\nabla - A(x))^2 u + V(x)u = f(|u|^2)u, \text{ in } \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N, \mathbb{C}), \end{cases} \tag{1.8}$$

where $f \in C^1$ has the subcritical growth. We must point out that if f is only continuous, then the arguments developed in [2] fail. In [21], Ji and Rădulescu used the method of the Nehari manifold, the penalization technique and Ljusternik-Schnirelmann category theory to study the multiplicity and concentration results for the above nonlinear magnetic Schrödinger equation in which the subcritical nonlinearity f is only continuous. After that, Ji and Rădulescu [22] continued to study multiplicity and concentration of the solutions for the magnetic Schrödinger equation with critical growth. Recently, Alves et al. [3] obtained the existence of multiple solutions for a nonlinear magnetic Choquard equation by using the penalization method and Ljusternik-Schnirelmann category theory.

Besides, we also refer to [16] for other results related to problem (1.2) in the presence of the magnetic field when the nonlinearity has a subcritical growth. Furthermore, we must mention the works [1, 20] for the critical case and also refer to the recent papers [33, 37] for the study of various classes of PDEs with magnetic potential.

Inspired by the above works, we want to research the the equation (1.1). Our aim of this paper is to prove the existence of the ground state solutions for problem (1.1), that is a nontrivial solution with minimal energy, and investigate the asymptotic behavior of solutions.

Notice that if we define

$$\tilde{f}(t) = \begin{cases} \frac{f(t)}{t}, & t \neq 0, \\ 0, & t = 0, \end{cases}$$

our assumptions assure that $\tilde{f}(t)$ is continuous. Therefore, equation (1.1) can be rewritten in the form

$$[a+b\int_{\mathbb{R}^N}(|\nabla_A u|^2+\lambda V(x)|u|^2)\mathrm{d}x](-\Delta_A u+\lambda V(x)u) = (I_\alpha * F(|u|))\tilde{f}(|u|)u, \quad (1.9)$$

which generalizes the study in [9, 13–15, 38].

Our main result is as follows:

Theorem 1.1. *If $\alpha \in (N-2, N]$, (A1), (A2), (V1), (V2), (AV) are valid, and $f \in C^1(\mathbb{R}, \mathbb{R})$ verifies (f1)-(f4), then problem(1.1) has at least a ground state solution.*

Theorem 1.2. *Assume that u_{λ_n} are solutions for problem (1.1), and Ω is defined by (V3), then $u_{\lambda_n} \to \overline{u}$ in E_λ as $\lambda_n \to \infty$, where $\overline{u} \in H_0^1(\Omega)$ is a nontrivial solution of*

$$\begin{cases} (a+b\int_{\mathbb{R}^N}|\nabla_A u|^2\mathrm{d}x)(-\Delta_A u) = (I_\alpha * F(|u|))\frac{f(|u|)}{|u|}u, \ x \in \Omega, \\ u = 0 \ \text{on } \partial\Omega. \end{cases} \quad (1.10)$$

2. Preliminaries

In this section, we will establish the variational framework for equation (1.1) and give some very important inequalities and lemmas.

For the convenience of expression, from now on, we use the following notations:

- $H^1 = \{u \in L^2(\mathbb{R}^N, \mathbb{C}) : \nabla_A u \in L^2(\mathbb{R}^N, \mathbb{C})\}$ is equipped with an equivalent norm:

$$\|u\|^2 = \int_{\mathbb{R}^N}(|\nabla_A u|^2 + V(x)|u|^2)\mathrm{d}x,$$

- For $\lambda > 0$, we define the space $H^1_\lambda := (H^1, \|u\|_\lambda)$ equipped with scalar product

$$\langle u, v\rangle_\lambda = \Re e \int_{\mathbb{R}^N} (\nabla_A u \cdot \overline{\nabla_A v} + \lambda V(x) u\overline{v}) \mathrm{d}x,$$

  therefore the norm

$$\|u\|^2_\lambda = \int_{\mathbb{R}^N} (|\nabla_A u|^2 + \lambda V(x)|u|^2) \mathrm{d}x,$$

- $L^s(\mathbb{R}^N)(1 \le s \le \infty)$ denotes the Lebesgue space with the norm $|u|_s = (\int_{\mathbb{R}^N} |u|^s \mathrm{d}x)^{1/s}$,
- For any $x \in \mathbb{R}^N$ and $r > 0$, $B_r(x) := \{y \in \mathbb{R}^N : |y - x| < r\}$,
- $C, C_\varepsilon, C_1, C_2, ...$ represent positive constants possibly different in different lines.

Remark 2.1. It is obviously that for $\forall \lambda \ge 1$, $\|u\|_\lambda \ge \|u\|$.

Remark 2.2. $\|u\|_\lambda$ is an equivalent norm to the norm obtained by considering $V \equiv 1$, see [24].

Lemma 2.1. [17] *Assume $u \in H^1_\lambda$, then $|u| \in H^1(\mathbb{R}^N)$ and the diamagnetic inequality holds $\big|\nabla|u|(x)\big| \le |\nabla_A u(x)|$.*

Remark 2.2. It is well known that the embedding $H^1_\lambda \hookrightarrow L^r(\mathbb{R}^N, \mathbb{C})$ is compact for $r \in [1, 2^*)$ and $H^1_\lambda \hookrightarrow L^r(\mathbb{R}^N, \mathbb{C})$ is continuous for $r \in [1, 2^*]$.

Lemma 2.3. *Assume (f1)-(f4) hold, then we have*

1. *for all $\varepsilon > 0$, there is a $C_\varepsilon > 0$ such that $|f(t)| \le \varepsilon|t|^{\frac{\alpha}{N}} + C_\varepsilon|t|^{\frac{\alpha+2}{N-2}}$ and $|F(t)| \le \varepsilon|t|^{\frac{N+\alpha}{N}} + C_\varepsilon|t|^{\frac{N+\alpha}{N-2}}$,*
2. *for all $\varepsilon > 0$, there is a $C_\varepsilon > 0$ such that for every $p \in (2, 2^*)$, $|F(t)| \le \varepsilon(|t|^{\frac{N+\alpha}{N}} + |t|^{\frac{N+\alpha}{N-2}}) + C_\varepsilon|t|^{\frac{p(N+\alpha)}{2N}}$, and $|F(t)|^{\frac{2N}{N+\alpha}} \le \varepsilon(|t|^2 + |t|^{\frac{2N}{N-2}}) + C_\varepsilon|t|^p$,*
3. *for any $s \ne 0$, $F(s) > 0$.*

Proof. One can easily obtain the results by elementary calculation. □

Lemma 2.4. [9] *Let $O \subset \mathbb{R}^N$ be any open set, for $1 < p < \infty$, and $\{f_n\}$ be a bounded sequence in $L^p(O, \mathbb{C})$ such that $f_n(x) \rightharpoonup f(x)$ a.e., then $f_n(x) \rightharpoonup f(x)$.*

Lemma 2.5. [9] *Suppose that $u_n \rightharpoonup u_0$ in $H^1_\lambda(\mathbb{R}^N, \mathbb{C})$, and $u_n(x) \to u_0(x)$ a.e. in $\mathbb{R}^N$, then $I_\alpha * F(|u_n(x)|) \rightharpoonup I_\alpha * F(|u_0(x)|)$ in $L^{\frac{2N}{\alpha}}(\mathbb{R}^N)$.*

Corollary 2.6. *Suppose that $u_n \rightharpoonup u_0$ in $H^1_\lambda(\mathbb{R}^N, \mathbb{C})$, then $\Re e \int_{\mathbb{R}^N} I_\alpha * F(|u_n|) f(|u_n|)u_n\overline{\varphi} \to \Re e \int_{\mathbb{R}^N} I_\alpha * F(|u_0|) f(|u_0|)u_0\overline{\varphi}$ for $\varphi \in C_c^\infty(\mathbb{R}^N, \mathbb{C})$.*

Lemma 2.7. (Hardy-Little-Sobolev inequality [24]). *Let $0 < \alpha < N$, $p, q > 1$ and $1 \le r < s < \infty$ be such that*

$$\frac{1}{p} + \frac{1}{q} = 1 + \frac{\alpha}{N}, \ \frac{1}{r} - \frac{1}{s} = \frac{\alpha}{N}.$$

1. *For any $f \in L^p(\mathbb{R}^N)$ and $g \in L^q(\mathbb{R}^N)$, one has*

$$\left| \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{f(x)g(y)}{|x - y|^{N-\alpha}} \mathrm{d}x\mathrm{d}y \right| \le C(N, \alpha, p)\|f\|_{L^p(\mathbb{R}^N)} \|g\|_{L^q(\mathbb{R}^N)}.$$

2. *For any* $f \in L^r(\mathbb{R}^N)$ *one has*

$$\left\| \frac{1}{|\cdot|^{N-\alpha}} * f \right\|_{L^s(\mathbb{R}^N)} \leq C(N,\alpha,r)\|f\|_{L^r(\mathbb{R}^N)}.$$

Remark 2.8. By Lemma 2.3(1), Lemma 2.7(1) and Sobolev imbedding theorem, we can get

$$\begin{aligned} \left| \int_{\mathbb{R}^N} \big(I_\alpha * F(u)\big)F(u)\mathrm{d}x \right| &\leq C|F(u)|^2_{\frac{2N}{N+\alpha}} \\ &\leq C\Big[\int_{\mathbb{R}^N} \big(|u|^{\frac{N+\alpha}{N}} + |u|^{\frac{N+\alpha}{N-2}}\big)^{\frac{(2N)}{N+\alpha}}\mathrm{d}x\Big]^{\frac{N+\alpha}{N}} \\ &\leq C\Big[\int_{\mathbb{R}^N} \big(|u|^2 + |u|^{\frac{2N}{N-2}}\big)\mathrm{d}x\Big]^{\frac{N+\alpha}{N}} \\ &\leq C(\|u\|_\lambda^{\frac{2N+2\alpha}{N}} + \|u\|_\lambda^{\frac{2N+2\alpha}{N-2}}). \end{aligned} \tag{2.1}$$

3. Variational formulation for problem (1.1)

The enenrgy functional associated to problem (1.1) is given by:

$$J_\lambda(u) = \frac{a}{2}\|u\|_\lambda^2 + \frac{b}{4}\|u\|_\lambda^4 - \frac{1}{2}\int_{\mathbb{R}^N} (I_\alpha * F(|u|))F(|u|)\mathrm{d}x. \tag{3.1}$$

The derivative of the energy functional $J_\lambda(u)$ is given by

$$\langle J'_\lambda(u), \varphi\rangle = a\langle u, \varphi\rangle_\lambda + b\|u\|_\lambda^2\langle u, \varphi\rangle_\lambda - \Re e\int_{\mathbb{R}^N} (I_\alpha * F(|u|))\tilde{f}(|u|)u\overline{\varphi}\mathrm{d}x. \tag{3.2}$$

Thus,

$$\langle J'_\lambda(u), u\rangle = a\|u\|_\lambda^2 + b\|u\|_\lambda^4 - \int_{\mathbb{R}^N} (I_\alpha * F(|u|))f(|u|)|u|\mathrm{d}x. \tag{3.3}$$

Now, we can prove the following results.

Lemma 3.1. *The functional* J_λ *possesses the mountain-pass geometry, that is*

1. *there exists* $\rho, \delta > 0$ *such that* $J_\lambda \geq \delta$ *for all* $\|u\|_\lambda = \rho$;
2. *for any* $u \in H^1_\lambda(\mathbb{R}^N, \mathbb{C})\backslash\{0\}$, *there exists* $\tau \in (0, +\infty))$ *such that* $\|\tau u\|_\lambda > \rho$ *and* $J_\lambda(\tau u) < 0$.

Proof. (1) By Lemma 2.7(1) and Lemma 2.3, one can get

$$J_\lambda(u) \geq \frac{1}{2}\|u\|_\lambda^2 - C(\|u\|_\lambda^{\frac{2N+2\alpha}{N}} + \|u\|_\lambda^{\frac{2N+2\alpha}{N-2}}).$$

Thus there exists $\rho, \delta > 0$ such that $J_\lambda \geq \delta$ for all $\|u\|_\lambda = \rho > 0$ small enough.

(2) For any fixed $u_0 \in H^1_\lambda\backslash\{0\}$, and consider the function $g_{u_0}(t) : (0, +\infty) \to \mathbb{R}$ given by

$$g_{u_0}(t) = \frac{1}{2}\int_{\mathbb{R}^N} \Big(I_\alpha * F(\frac{t|u_0|}{\|u_0\|_\lambda})\Big)F(\frac{t|u_0|}{\|u_0\|_\lambda})\mathrm{d}x, \tag{3.4}$$

then

$$\begin{aligned} g'_{u_0}(t) &= \int_{\mathbb{R}^N} \Big(I_\alpha * F\big(\frac{t|u_0|}{\|u_0\|_\lambda}\big)\Big) f\big(\frac{t|u_0|}{\|u_0\|_\lambda}\big)\frac{|u_0|}{\|u_0\|_\lambda}\mathrm{d}x \\ &= \frac{2\theta}{t}\int_{\mathbb{R}^N} \frac{1}{2}\Big(I_\alpha * F\big(\frac{t|u_0|}{\|u_0\|_\lambda}\big)\Big)\frac{1}{\theta} f\big(\frac{t|u_0|}{\|u_0\|_\lambda}\big)\frac{t|u_0|}{\|u_0\|_\lambda}\mathrm{d}x \\ &\geq \frac{2\theta}{t} g_{u_0}(t) > 0, (t>0). \end{aligned} \tag{3.5}$$

Thus, $\mathrm{ln} g_{u_0}(t)\big|_1^{\tau\|u_0\|_\lambda} \geq 2\theta \mathrm{ln} t\big|_1^{\tau\|u_0\|_\lambda}$. So $\frac{g_{u_0}(\tau\|u_0\|_\lambda)}{g_{u_0}(1)} \geq (\|u_0\|_\lambda)^{2\theta}$ which implies that $g_{u_0}(\tau\|u_0\|_\lambda) \geq M(\|u_0\|_\lambda)^{2\theta}$ for a constant $M>0$. Since $\theta > 2$, we can get

$$J_\lambda(\tau u_0) = \frac{a\tau^2}{2}\|u_0\|_\lambda^2 + \frac{b\tau^4}{4}\|u_0\|_\lambda^4 - g_{u_0}(\tau\|u_0\|_\lambda) \leq C_1\tau^2 + C_2\tau^4 - C_3\tau^{2\theta} \tag{3.6}$$

yields that $J_\lambda(\tau u_0) < 0$ when τ is large enough. □

Hence we can define the mountain-pass level of J_λ:

$$c = \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} J_\lambda(\gamma(t)) > 0,$$

where: $\Gamma = \{\gamma \in C([0,1], X) : \gamma(0) = 0, J_\lambda(\gamma(1)) < 0\}$.

Now we recall the Nehari manifold

$$\mathcal{N}_\alpha := \{u \in H^1_\lambda(\mathbb{R}^N, \mathbb{C})\backslash\{0\} : \langle J'_\lambda(u), u\rangle = 0\}.$$

Let $c_\alpha = \inf\limits_{u\in\mathcal{N}_\alpha} J_\lambda(u)$, Moreover by the similar argument as Chapter 4 [36], we have the following characterization

$$c = \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} J_\lambda(\gamma(t)) = c_\alpha = \inf_{u\in\mathcal{N}_\alpha} J_\lambda(u) = c^* = \inf_{u\in H^1_\lambda(\mathbb{R}^N,\mathbb{C})\backslash\{0\}}\max_{t\geq 0} J_\lambda(tu).$$

4. Ground state solution for equation (1.1)

In this section, we prove the theorem 1.1.

Proof of Theorem 1.1. Let $\{u_n\}$ be minimizing sequence given as a consequence of Lemma 3.1, i.e. $\{u_n\} \subset H^1_\lambda$ such that $J'_\lambda(u_n) \to 0$, $J_\lambda(u_n) \to c$, where

$$c = c_\alpha = \inf_{u\in\mathcal{N}_\alpha} J_\lambda(u) = c^* = \inf_{u\in H^1_\lambda(\mathbb{R}^N,\mathbb{C})\backslash\{0\}}\max_{t\geq 0} J_\lambda(tu).$$

Then we have

$$\begin{aligned} c_\alpha + o(1) &= J_\lambda(u_n) - \frac{1}{4}\langle J'_\lambda(u_n), u_n\rangle \\ &= \frac{a}{4}\int_{\mathbb{R}^N} [|\nabla_A u_n|^2 + \lambda V(x)|u_n|^2]\mathrm{d}x \\ &\quad + \frac{1}{4}\int_{\mathbb{R}^N} (I_\alpha * F(|u_n|))[f(|u_n|)|u_n| - 2F(|u_n|)]\mathrm{d}x \\ &\geq \frac{a}{4}\|u_n\|^2_\lambda. \end{aligned} \tag{4.1}$$

Consequence, $\{u_n\}$ is bounded. Then by standard methods we can get the convergence of $\{u_n\}$.

Next, let $\delta := \limsup_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int_{B_1(y)} |u_n|^2 \mathrm{d}x$. We claim $\delta > 0$. On the contrary, by Lion's concentration compactness principle, we have $u_n \to 0$ in $L^p(\mathbb{R}^N)$ for $2 < p < 2^*$. By Lemma 2.3(2), for any $\varepsilon > 0$ there exists a constant $C_\varepsilon > 0$ such that

$$
\begin{aligned}
&\limsup_{n\to\infty} \int_{\mathbb{R}^N} (I_\alpha * F(|u_n|))f(|u_n|)|u_n|\mathrm{d}x \\
&\le C \limsup_{n\to\infty} \left[\varepsilon\left(\int_{\mathbb{R}^N} |u_n|^2\mathrm{d}x + \int_{\mathbb{R}^N} |u_n|^{\frac{2N}{N-2}}\mathrm{d}x\right) + C_\varepsilon \int_{\mathbb{R}^N} |u_n|^p \mathrm{d}x\right]^{\frac{N+\alpha}{N}} \\
&\le C\left[\varepsilon C_1 + C_\varepsilon \limsup_{n\to\infty} \int_{\mathbb{R}^N} |u_n|^p\mathrm{d}x\right]^{\frac{N+\alpha}{N}} \\
&= C(\varepsilon C_2)^{\frac{N+\alpha}{N}}.
\end{aligned}
$$

Note that ε is arbitrary, we get

$$
\int_{\mathbb{R}^N} (I_\alpha * F(|u_n|))f(|u_n|)|u_n|\mathrm{d}x = o(1).
$$

Combining with $J'_\lambda(u_n) \to 0$, we can get

$$
\begin{aligned}
o(1) &= \langle J'_\lambda(u_n), u_n\rangle \\
&= a\|u_n\|^2_\lambda + b\|u_n\|^4_\lambda - \int_{\mathbb{R}^N} (I_\alpha * F(|u_n|))f(|u_n|)|u_n|\mathrm{d}x,
\end{aligned} \tag{4.2}
$$

which implies that

$$
a\|u_n\|^2_\lambda + b\|u_n\|^4_\lambda = \int_{\mathbb{R}^N} (I_\alpha * F(|u_n|))f(|u_n|)|u_n|\mathrm{d}x + o(1) = 2o(1). \tag{4.3}
$$

Then we have $\|u_n\|^2_\lambda = \int_{\mathbb{R}^N} [|\nabla_A u_n|^2 + \lambda V(x)|u_n|^2]\mathrm{d}x \to 0$, which implies $u_n \to 0$ in H^1_λ. We deduce that $c_\alpha = 0$, which contradicts to the fact that $c_\alpha > 0$. Hence $\delta > 0$ and there exists $\{y_n\} \subset \mathbb{R}^N$ such that $\int_{B_1(y_n)} |u_n|^p\mathrm{d}x \ge \frac{\delta}{2} > 0$. We set $v_n(x) = u_n(x + y_n)$, then $\|u_n\| = \|v_n\|$, $\int_{B_1(0)} |v_n|^p\mathrm{d}x > \frac{\delta}{2}$ and $J_\lambda(v_n) \to c_\alpha = c$, $J'_\lambda(v_n) \to 0$. Thus there exists a $v_0 \neq 0$ such that

$$
\begin{cases}
v_n \rightharpoonup v_0 \text{ in } H^1_\lambda, \\
v_n \to v_0 \text{ in } L^s(\mathbb{R}^N),\ \forall\, s \in [2, 2^*), \\
v_n \to v_0\ a.e. \text{ on } \mathbb{R}^N.
\end{cases}
$$

Then for any $\varphi \in C_0^\infty(\mathbb{R}^N)$ we have $0 = \langle J'_\lambda(v_n), \varphi\rangle + o(1) = \langle J'_\lambda(v_0), \varphi\rangle$, which means v_0 is a solution of equation (1.1).

On the other hand, combining with the Fatou Lemma, we can obtain

$$
\begin{aligned}
c_\alpha &= J_\lambda(v_n) - \frac{1}{4}\langle J'_\lambda(v_n), v_n\rangle + o(1) \\
&= \frac{a}{4}\int_{\mathbb{R}^N} [|\nabla_A v_n|^2 + V(x)|v_n|^2]\mathrm{d}x \\
&\quad + \frac{1}{4}\int_{\mathbb{R}^N} (I_\alpha * F(|v_n|))[f(|v_n|)|v_n| - 2F(|v_n|)]\mathrm{d}x + o(1)
\end{aligned}
$$

$$
\begin{aligned}
&\geq \frac{a}{4}\int_{\mathbb{R}^N}[|\nabla_A v_0|^2+V(x)|v_0|^2]\mathrm{d}x \\
&\quad+\frac{1}{4}\int_{\mathbb{R}^N}(I_\alpha * F(|v_0|))[f(|v_0|)|v_0|-2F(|v_0|)]\mathrm{d}x+o(1) \\
&=J_\lambda(v_0)-\frac{1}{4}\langle J_\lambda'(v_0),v_0\rangle+o(1) \\
&=J_\lambda(v_0)+o(1). \qquad (4.4)
\end{aligned}
$$

At the same time, we know $c_\alpha \leq J_\lambda(v_0)$ by the definition of c_α. Then we can deduce that v_0 is a ground state solution of equation (1.1). □

5. Asymptotic behavior of solutions for equation (1.1)

In this section, we will investigate the asymptotic behavior of solutions for (1.1).

Proof of Theorem 1.2. Let u_λ be ground state solution of (1.1) obtained in Theorem 1.1, we can get that $J_\lambda(u_\lambda)=c_{\alpha_\lambda}$ and $J_\lambda'(u_\lambda)=0$. Define $u_n:=u_{\lambda_n}$, then there exists a sequence $\{u_n\}$ such that $J_{\lambda_n}(u_n)=c_{\alpha_{\lambda_n}}$ and $J_{\lambda_n}'(u_n)=0$. It follows from (4.1) that $\{u_n\}$ is bounded in $H^1_{\lambda_n}$, that is there exist $T>0$ such that

$$\|u_n\|_{\lambda_n}\leq T. \qquad (5.1)$$

Thus, up to a subsequence, we may assume that there exist a u_0 such that

$$
\begin{cases}
u_n \rightharpoonup u_0 \text{ in } H^1_{\lambda_n},\\
u_n \to u_0 \text{ in} L^s_{loc}(\mathbb{R}^N),\ \forall\, s\in[2,2^*),\\
u_n \to u_0\ a.e. \text{ on } \mathbb{R}^N.
\end{cases}
$$

Now we show that $u_n\to u_0$ in $L^s(\mathbb{R}^N)$ for $s\in(2,2^*)$. We define

$$D_R:=\{x\in\mathbb{R}^N\backslash B_R : V(x)\geq V_0\} \qquad (5.2)$$

and

$$A_R:=\{x\in\mathbb{R}^N\backslash B_R : V(x)< V_0\}. \qquad (5.3)$$

Then we have $meas(A_R)\to 0$ as $R\to\infty$ by (V2) and

$$\int_{D_R}u_n^2\mathrm{d}x\leq\frac{1}{\lambda_n V_0}\int_{D_R}\lambda_n V(x)u_n^2\mathrm{d}x\leq\frac{C_8}{\lambda_n V_0}\to 0 \qquad (5.4)$$

as $\lambda_n\to\infty$. Combing with the Hölder and Sobolev inequality, for any $s\in(2,2^*)$ we get

$$\int_{A_R}u_n^2\mathrm{d}x\leq\left(\int_{A_R}u_n^s\mathrm{d}x\right)^{\frac{2}{s}}\left(\int_{A_R}1\mathrm{d}x\right)^{\frac{s-2}{s}}\leq\|u_n\|^2_{\lambda_n}(meas(A_R))^{\frac{s-2}{s}}. \qquad (5.5)$$

Thus, we can obtain

$$\begin{aligned}\int_{B_R^c} u_n^s \mathrm{d}x &= \left(\int_{B_R^c} |u_n|^2 \mathrm{d}x\right)^{\frac{2^*-s}{2^*-2}} \left(\int_{B_R^c} |u_n|^{2^*} \mathrm{d}x\right)^{\frac{s-2}{2^*-2}} \\ &\leq C_9 \left(\int_{D_R} u_n^2 \mathrm{d}x + \int_{A_R} u_n^2 \mathrm{d}x\right)^{\frac{2^*-s}{2^*-2}} \\ &\leq C_9 \left(\frac{C_8}{\lambda_n V_0} + C_{10}(meas(A_R))^{\frac{s-2}{s}}\right)^{\frac{2^*-s}{2^*-2}} \\ &\to 0\end{aligned} \tag{5.6}$$

as $\lambda_n \to \infty$, where $B_R^c := \{x \in \mathbb{R}^N : |x| \geq R\}$. Then,

$$\int_{B_R^c} \big||u_n|^s - |u_0|^s\big| \mathrm{d}x \leq \int_{B_R^c} |u_n|^s \mathrm{d}x + \int_{B_R^c} |u_0|^s \mathrm{d}x \to 0, \tag{5.7}$$

as $R \to \infty$. Since $u_n \to u_0$ in $L^s_{loc}(\mathbb{R}^N)$ with $s \in (2, 2^*)$, we derive

$$\int_{|x|<R} |u_n|^s \mathrm{d}x \to \int_{|x|<R} |u_0|^s \mathrm{d}x. \tag{5.8}$$

Therefore, $u_n \to u_0$ in $L^s(\mathbb{R}^N)$ for $s \in (2, 2^*)$ as $\lambda_n \to \infty$.

Next, we set $z_n := u_n - u_0$ and by standard methods we can get that $z_n \to 0$ in H^1_λ.

Thus together with Fatou's Lemma and (5.1), we have

$$\int_{\mathbb{R}^N} V(x) u_0^2 \mathrm{d}x \leq \liminf_{n\to\infty} \int_{\mathbb{R}^N} V(x) u_n^2 \mathrm{d}x \leq \frac{\liminf_{n\to\infty} \|u_n\|^2_{\lambda_n}}{\lambda_n} \to 0. \tag{5.9}$$

Hence by (V3), we deduce that $u_0 = 0$ a.e. $x \in \mathbb{R}^N \backslash \Omega$ and $u_0 \in H^1_0(\Omega)$. Then we obtain

$$\begin{aligned}&a\Re e \int_\Omega (\nabla_A u_0 \cdot \overline{\nabla_A v}) \mathrm{d}x + b \int_\Omega (|\nabla_A u_0|^2) \mathrm{d}x \Re e \int_\Omega (\nabla_A u_0 \cdot \overline{\nabla_A v}) \mathrm{d}x \\ =&\Re e \int_\Omega (I_\alpha * F(|u_0|)) \tilde{f}(|u_0|) u_0 \overline{v} \mathrm{d}x,\end{aligned} \tag{5.10}$$

for any $v \in H^1_0(\Omega)$. This completes the proof. □

Acknowledgements

This work was supported by Scientific research start-up fund (Grant No. F701108M03), National Natural Science Foundation of China (Grant No. 11771198, 11901276) and School level teaching research & reform project (Grant No. 2023-jg51).

References

[1] C. O. Alves and G. M. Figueiredo, *Multiple solutions for a semilinear elliptic equation with critical growth and magnetic field*, Milan J. Math., 2014, 82, 389–405.

[2] C. O. Alves, G. M. Figueiredo and M. F. Furtado, *Multiple solutions for a nonlinear Schrödinger equation with magnetic fields*, Comm. Partial Differential and Equations, 2011, 36, 1565–1586.

[3] C. O. Alves, G. M. Figueiredo and M. B. Yang, *Multiple semiclassical solutions for a nonlinear Choquard equation with magnetic field*, Asymptot. Anal., 2016, 96, 135–159.

[4] C. O. Alves and G. M. Figueiredo, *Multi-bump solutions for a Kirchhoff-type problem*, Adv. Nonlinear Anal., 2016, 51, 1–26.

[5] C. O. Alves, A. B. Nóbrega and M. B. Yang, *Multi-bump solutions for Choquard equation with deepning potential well*, Calc. Var. Partial Differ. Equ., 2016, 55(3), 1–28.

[6] G. Arioli and A. Szulkin, *A semilinear Schrödinger equation in the presence of a magnetic field*, Arch. Ration. Mech. Anal., 2003, 170, 277–293.

[7] T. Bartsch, A. Pankov and Z. Q. Wang, *Nonlinear Schrödinger equations with steep potential well*, Commun. Contemp. Math., 2001, 3, 549–569.

[8] T. Bartsch and Z. Q. Wang, *Existence and multiplicity results for superlinear elliptic problems on* $\mathbb{R}^N$, Commun. Partial Differ. Equ., 1995, 20, 1725–1741.

[9] H. Bueno, G. G. Mamani and G. A. Pereira, *Ground state of a magnetic nonlinear Choquard equation*, Nonlinear Anal., 2019, 181, 189–199.

[10] E. Cabanillas Lapa, *Global solutions for a nonlinear Kirchhoff type equation with viscosity*, Opuscula Math., 2023, 43(5), 689–701.

[11] C. Chen, Y. Kuo and T. Wu, *The Nehari manifold for a Kirchhoff type problem involving sign-changing weight functions*, J. Diff. Eqs., 2011, 250, 1876–1908.

[12] H. Chen and C. X. Zhu, *Delayed-state-feedback exponential stabilization for uncertain Markovian jump systems with mode-dependent time-varying state delays*, Nonlinear Dyn., 2012, 69, 1023–1039.

[13] S. Cingolani, M. Clapp and S. Secchi, *Multiple solutions to a magnetic nonlinear Choquard equation*, Z. Angew. Math. Phys., 2012, 63, 233–248.

[14] S. Cingolani, M. Clapp and M. Secchi, *Intertwining semiclassical solutions to a Schrödinger-Newton system*, Discrete Contin. Dyn. Syst. Ser. S, 2013, 6(4), 891–908.

[15] S. Cingolani, S. Secchi and M. Squassina, *Semi-classical limit for Schrödinger equations solutions with magnetic field and Hartree-type nonlinearities*, Proc. Roy. Soc. Edinburgh., 2010, 140A, 973–1009.

[16] M. Clapp and A. Szulkin, *Multiple solutions to a nonlinear Schrödinger equation with Aharonov-Bohm magnetic potential*, NoDEA Nonlinear Differential Equations Appl., 2010, 17, 229–248.

[17] M. J. Esteban and P. L. Lions, *Stationary solutions of a nonlinear Schrödinger equations with an external magnetic field*, Partial Differential Equations and the Calculus of Variations, Essays in Honor of Ennio De, 1989, 401–409.

[18] M. Ghimenti and J. Van Schaftingen, *Nodal solutions for the Choquard equation*, J. Funct. Anal., 2016, 271, 107–135.

[19] G. Gu and Z. Yang, *On the singularly perturbation fractional Kirchhoff equations: critical case*, Adv. Nonlinear Anal., 2022, 11(1), 1097–1116.

[20] C. Ji and V. D. Rădulescu, *Multi-bump solutions for the nonlinear magnetic Schrödinger equation with exponential critical growth in* $\mathbb{R}^2$, Manuscripta Math., 2021, 164, 509–542.

[21] C. Ji and V. D. Rădulescu, *Multiplicity and concentration of solutions to the nonlinear magnetic Schrödinger equation*, Calc. Var. Partial Differential Equations, 2020, 59, art 115, 28 pp.

[22] C. Ji and V. D. Rădulescu, *Concentration phenomena for nonlinear magnetic Schrödinger equations with critical growth*, Isr. J. Math., 2021, 241, 465–500.

[23] G. Li and H. Ye, *Existence of positive ground state solutions for the nonlinear Kirchhoff type equations in* $\mathbb{R}^3$, J. Diff. Eqs., 2014, 257, 566–600.

[24] E. H. Lieb and M. Loss, *Analysis*, second ed, Grad. Stud. Math., vol 14, American Mathematical Scoiety, Province, RL, 2001.

[25] D. Lü, *A note on Kirchhoff-type equations with Hartree-type nonlinearities*, Nonlinear Annal, 2014, 99, 35–48.

[26] H. X. Luo, *Ground state solutions of Pohozaev type and Nehari type for a class of nonlinear Choquard equations*, J. Math. Anal. Appl., 2018, 467, 842–862.

[27] L. Ma and Z. Lin, *Classification of positive solitary solutions of the nonlinear Choquard equation*, Arch Ration. Mech. Aral., 2010, 195, 455–467.

[28] V. Moroz and J. Van Schaftingen, *Ground states of nonlinear Choquard equations: Existence, qualitative properties and decay asymptotics*, J. Funct. Anal., 2013, 265, 153–184.

[29] V. Moroz and J. Van Schaftingen, *Existence of ground states for a class of nonlinear Choquard equations*, Trans. Amer. Math. Soc., 2015, 367, 6557–6579.

[30] V. Moroz and J. Van Schaftingen, *Nonexistence and optimal decay of supersolutions to Choquard equations in exterior domains*, J. Diff. Eqs., 2013, 254, 3089–3145.

[31] V. Moroz and J. Van Schaftingen, *Ground states of nonlinear Choquard equations: Hardy-Littlewood-Sobolev critical exponent*, Commun. Contemp. Math., 2015, 17, 1550005.

[32] V. Moroz and J. Van Schaftingen, *A guide to the Choquard equation*, J. Fixed Point Theory Appl., 2017, 19, 773–813.

[33] H. M. Nguyen, A. Pinamonti, M. Squassina and E. Vecchi, *New characterizations of magnetic Soblev spaces*, Adv. Nonlinear Anal., 2018, 7, 227–245.

[34] X. Sun, Y. Song, S. Liang and B. Zhang, *Critical Kirchhoff equations involving the p-sub-Laplacians operators on the Heisenberg group*, Bull. Math. Sci., 2023, 13(2), Paper No. 2250006.

[35] Z. P. Wang and H. S. Zhou, *Positive solutions for nonlinear Schrödinger equations with deepening potential well*, J. Eur. Math. Soc., 2009, 11, 545–573.

[36] M. Willem, *Minimax Theorems, Proress in Nonlinear Differential Equations and Their Applications 24*, Birkhäuser, Boston, MA, 1996.

[37] A. Xia, *Multiplicity and concentration results for magnetic relativistic Schrödinger equations*, Adv. Nonlinear Anal., 2020, 9, 1161–1186.

[38] M. Yang and Y. Wei, *Existence and multiplicity of solutions for nonlinear Schrödinger equations solutions with magnetic field and Hartree type nonlinearities*, J. Math. Anal. Appl., 2013, 403, 680–694.

[39] L. F. Yin and X. P. Wu, *Existence and concentration of ground state solutions for critical Schrödinger equation with steep potential well*, Comput. Math. Appl., 2019, 78, 3862–3871.

[40] L. Zhang, X. Tang and P. Chen, *On the planar Kirchhoff-type problem involving supercritical exponential growth*, Adv. Nonlinear Anal., 2022, 11(1), 1412–1446.

[41] S. Zhou, Z. Liu and J. Zhang, *Ground states for Choquard type equations with weighted potentials and Hardy-Littlewood-Sobolev lower critical exponent*, Adv. Nonlinear Anal., 2022, 11(1), 141–158.