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NUMBER OF LIMIT CYCLES OF A CASE OF POLYNOMIAL SYSTEM VIA THE STABILITY-CHANGING METHOD

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Abstract In this paper, we study bifurcation of limit cycles bifurcating from a planar polynomial system with degree nine. More limit cycles can be obtained by using the stability-changing method compared to the Melnikov function method. We obtain 24 limit cycles bifurcating from a symmetrical compound loop with five saddles.

Keywords Limit cycle, alien limit cycle, homoclinic loop, heteroclinic loop.

MSC(2010) 34C07.

1. Introduction

The following system is widely studied, called near-Hamiltonian system

$$\begin{cases} \dot{x} = H_y(x,y) + \varepsilon f(x,y,\delta), \\ \dot{y} = -H_x(x,y) + \varepsilon g(x,y,\delta), \end{cases} \tag{1.1}$$

where ε is a small parameter, $H(x,y)$, $f(x,y,\delta)$ and $g(x,y,\delta)$ are C^∞ functions and $\delta = (\delta_1, \dots, \delta_m) \in D \subset \mathbb{R}^m$ with D bounded. The main tools to find limit cycles of (1.1) are known as the Melnikov function method and the averaging method. The authors [13] established the equivalence of the two methods. If the unperturbed system has a center, or a homoclinic loop, the Melnikov function method can be used to study Hopf bifurcation and homoclinic bifurcation. However, if the unperturbed system has a polycycle containing a heteroclinic loop L, it may appear to have alien limit cycles that cannot be detected by the Melnikov function method, see [2, 4, 5, 19–21]. In the case that the unperturbed system has a polycycle with n saddles, where $n \geq 2$ is an integer, the authors [8, 11, 15, 16, 26, 29] developed a new approach to obtain alien limit cycles by changing the stability of a homoclinic loop or a double homoclinic loop. From those references, we notice that one can always find more limit cycles bifurcating from a polycycle by the stability-changing method than the Melnikov function method.

When $H(x,y)$, $f(x,y,\delta)$ and $g(x,y,\delta)$ are polynomials, the number of limit cycles of (1.1) is related to the week Hilbert 16th problem [1]. There are many related results about the limit cycles, on their existence [32] and their number [3,7,12,14,17,24,27,30,31]. We introduce some of them here. When the unperturbed

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system $(1.1)_{\varepsilon=0}$ is a cubic system, 12 limit cycles around a singular point are found in [34]. Recently, the authors of [6] analyzed the bifurcation of limit cycles near double saddle loops in cubic Hamiltonian systems under small perturbations. In [33], the authors studied a cubic planar switching polynomial system with Z_2-symmetry to obtain 18 limit cycles. For higher degree, the authors of [25] found 80 limit cycles in a degree nine polynomial vector field of Z_{10}-symmetry. In [28], a class of piecewise smooth systems with degree n is studied. By perturbing a piecewise cubic polynomial system with a cusp and a nilpotent saddle, the authors found $3n-1$ limit cycles, see [28]. Via the stability-changing method, the authors considered a polynomial system with degree seven, in which four limit cycles were found near a 2-polycycle, including an alien limit cycle, see [29]. In [23], the authors found 16 limit cycles bifurcating from a symmetrical compound polycycle with three saddle points, four of which are alien limit cycles.

In this paper, motivated by those references, we discuss the following planar polynomial system of degree 9

$$\begin{cases} \dot{x} = y, \\ \dot{y} = kx(x^2-a)(x^2-b)(x^2-c)(x^2-d) + \varepsilon(b_0x + b_1x^3 + g_0(x)y), \end{cases} \tag{1.2}$$

where $g_0(x) = \sum_{j=0}^{8} a_{2j}x^{2j} + x^{18}$, and k, a, b, c, d are real coefficients with $a > b > c > d > 0$. When $\varepsilon = 0$, it has the unperturbed system

$$\begin{cases} \dot{x} = y, \\ \dot{y} = kx(x^2-a)(x^2-b)(x^2-c)(x^2-d), \end{cases} \tag{1.3}$$

and the Hamiltonian function

$$H(x,y) = \frac{1}{2}y^2 - \int_0^x ks(s^2-a)(s^2-b)(s^2-c)(s^2-d)ds.$$

We remark that the above system has been discussed in [22]. Via the stability changing method, the authors found 20 limit cycles bifurcating from a compound loop with five saddles. In this paper, by choosing a different family of parameters, we find more limit cycles. Our main result is as follows.

Theorem 1.1. *Consider system* (1.2). *When* $k > 0$, $H(\sqrt{c}, 0) < H(\sqrt{a}, 0) = 0$, *then*

(i) system (1.3) *has a large double heterclinic loop surrounding two double homoclinic loops.*

(ii) system (1.2) *has 24 limit cycles bifurcating from a large double heterclinic loop and two double homoclinic loops.*

2. Preliminary lemmas

Assume that the unperturbed system $(1.1)_{\varepsilon=0}$ has three hyperbolic saddles $S_0, S_0^*, \bar{S}_0$ and a double heteroclinic loop $L_0 = L_{01} \cup L_{02} \cup \bar{L}_{01} \cup \bar{L}_{02}$ shown in Figure 1. The system (1.1) has a saddle S_ε near S_0.

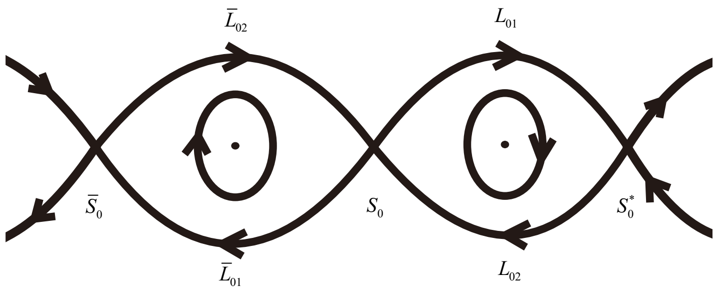

Figure 1. The image of L_0

Denote

$$
\begin{aligned}
&M_0(\delta) = M_{01}(\delta) + M_{02}(\delta),\ M_{0i}(\delta) = \int_{L_{0i}} gdx - fdy,\ i = 1, 2,\\
&\bar{M}_0(\delta) = \bar{M}_{01}(\delta) + \bar{M}_{02}(\delta),\ \bar{M}_{0i}(\delta) = \int_{\bar{L}_{0i}} gdx - fdy,\ i = 1, 2,\\
&c_1(\delta) = (f_x + g_y)(S_0),\ c_1^*(\delta) = (f_x + g_y)(S_0^*),\ \bar{c}_1(\delta) = (f_x + g_y)(\bar{S}_0),\\
&c_2(\delta) = \sum_{i=1}^{2} \int_{L_{0i}} (f_x + g_y)|_{c_1 = c_1^* = 0} dt,\ \bar{c}_2(\delta) = \sum_{i=1}^{2} \int_{\bar{L}_{0i}} (f_x + g_y)|_{c_1 = \bar{c}_1 = 0} dt,\\
&c_{30}(\delta) = \frac{\partial R_1(S_\varepsilon)}{\partial \varepsilon}\Big|_{\varepsilon = 0},
\end{aligned}
\tag{2.1}
$$

where $R_1(S_\varepsilon)$ denotes the first saddle quantity of S_ε. If S_0 is the origin and $H(x, y) = \lambda_0 xy + O(|(x, y)|^3)$ with $\lambda_0 > 0$, near the origin. $c_{30}(\delta)$ has the following form [18],

$$
c_{30}(\delta)|_{c_1=0} = -\tfrac{1}{2\lambda_0} \left\{ (f_{xxy} + g_{xyy}) - \tfrac{1}{\lambda_0} \Big[H_{xyy}(f_{xx} + g_{xy}) + H_{xxy}(f_{xy} + g_{yy}) \Big] \right\} \Bigg|_{x=y=0}.
\tag{2.2}
$$

Lemma 2.1 ([29]). *Suppose that system* $(1.1)_{\varepsilon=0}$ *has a double heteroclinic loop* L_0 *shown in Figure 1. Let* $M_0, \bar{M}_0, M_{01}, \bar{M}_{01}, c_i, c_1^*, \bar{c}_i$ *and* c_{30} *be defined in* (2.1) *and* (2.2)*,* $i = 1, 2$*. If there exists* $\delta_0 \in D$ *such that*

$$
\begin{aligned}
&M_0(\delta_0) = \bar{M}_0(\delta_0) = 0,\ M_{01}(\delta_0)\bar{M}_{01}(\delta_0) > 0,\\
&c_{30}(\delta_0) \neq 0,\ c_i(\delta_0) = \bar{c}_i(\delta_0) = c_1^*(\delta_0) = 0,\ i = 1, 2,\\
&Rank \frac{\partial(M_0, \bar{M}_0, c_1, c_1^*, \bar{c}_1, c_2, \bar{c}_2)}{\partial(\delta_1, \ldots, \delta_m)}(\delta_0) = 7,
\end{aligned}
$$

then system (1.1) *has 10 limit cycles bifurcating from* L_0 *for some* (δ, ε) *near* $(\delta_0, 0)$.

Remark 2.1. From the proof of [29, Theorem 2.2], we can know the distribution of the 10 limit cycles: eight limit cycles are surrounded by two large limit cycles, see Figure 2. Thus, one can see easily that the large limit cycles cannot be found by Melnikov functions, which are called the alien limit cycles.

In order to study the limit cycles bifurcating from a double homoclinic loop by the stability-changing method, we suppose that the unperturbed system $(1.1)_{\varepsilon=0}$

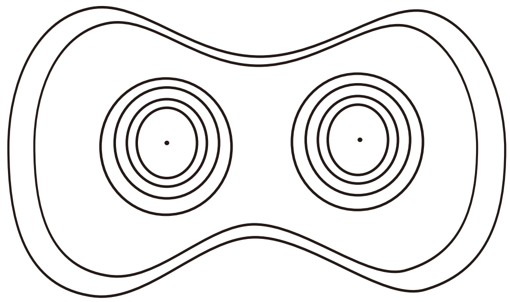

Figure 2. The distribution of 10 limit cycles

has a double homoclinic loop $L_1 = L_{11} \cup L_{12}$ passing through the saddle S_1, see Figure 3.

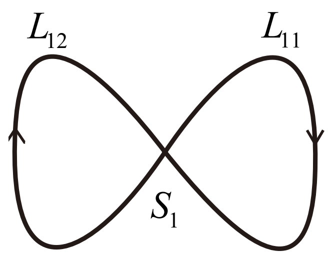

Figure 3. The image of L_1

Denote that

$$\begin{aligned} &M_1(\delta) = M_{11}(\delta) + M_{12}(\delta), \\ &M_{1i}(\delta) = \oint_{L_{1i}} gdx - fdy\Big|_{\varepsilon=0}, \ i = 1, 2. \end{aligned} \tag{2.3}$$

For the existence of a double homoclinc loop of (1.1), we have the following Lemma from [8].

Lemma 2.2 ([8]). *Suppose that system* $(1.1)_{\varepsilon=0}$ *has a double homoclinic loop* L_1 *shown in Figure 3. If there exists* $\delta_0 \in D$ *such that for* $i = 1, 2$, $M_{1i}(\delta_0) = 0$, $\frac{\partial(M_1)}{\partial(\delta_1,\delta_2)}(\delta_0) \neq 0$, *then there exists two functions* $\phi_i = (\varepsilon, \delta_3, \ldots, \delta_m)$ *such that for* $|\delta - \delta_0| + \varepsilon$ *small, system* (1.1) *has a double homoclinic loop* $L_\varepsilon = L_{\varepsilon_1} \cup L_{\varepsilon_2}$ *passing through* S_{ε_1} *if and only if* $\delta_i = \phi_i(\varepsilon, \delta_3, \ldots, \delta_m)$, *where* $S_{\varepsilon_1} \to S_1$ *and* $L_{\varepsilon i} \to L_{1i}$, *as* $\varepsilon \to 0$.

The following lemma illustrates the global stability of a homoclinic loop and a double homoclinic loop.

Lemma 2.3 ([10,11,16]). *Suppose that system* (1.1) *has a double homoclinic loop* $L_\varepsilon = L_{\varepsilon_1} \cup L_{\varepsilon_2}$ *passing through a saddle* S_{ε_1}. *Let*

$$\begin{aligned} &\mu_1 = \varepsilon(f_x + g_y)(S_{\varepsilon_1}), \\ &\mu_2 = \mu_{21} + \mu_{22}, \quad \text{where } \mu_{2i} = \varepsilon \oint_{L_{\varepsilon i}} (f_x + g_y)dt, \\ &\mu_3 = R_1(S_{\varepsilon_1}), \quad i = 1, 2, \end{aligned} \tag{2.4}$$

where $R_1(S_{\varepsilon_1})$ denotes the first saddle quantity of S_{ε_1}. Then the following statements are true.

(1) For $i=1,2$, the homoclinic loop $L_{\varepsilon i}$ is orbitally inside stable(unstable) as $\mu_1 < 0(>0)$, or $\mu_1 = 0, \mu_{2i} < 0(>0)$, or $\mu_1 = \mu_{2i} = 0, \mu_3 > 0(<0)$.

(2) The double homoclinic loop L_ε is orbitally outside stable(unstable) as $\mu_1 < 0(>0)$, or $\mu_1 = 0, \mu_2 < 0(>0)$, or $\mu_1 = \mu_2 = 0, \mu_3 < 0(>0)$.

Now we introduce the following quantities

$$\begin{aligned} c_{11}(\delta) &= (f_x+g_y)(S_1), \\ c_{2i}(\delta) &= \oint_{L_{1i}} (f_x+g_y)|_{c_{11}=0} dt, \ i=1,2, \\ c_{31}(\delta) &= \frac{\partial R_1(S_{\varepsilon_1})}{\partial \varepsilon}\Big|_{\varepsilon=0}. \end{aligned} \tag{2.5}$$

If S_1 is at the origin and $H(x,y) = \lambda_1 xy + O(|(x,y)|^3)$ near the origin with $\lambda_1 > 0$, we further denote that

$$c_{31}(\delta)\Big|_{c_{11}=0} = -\tfrac{1}{2\lambda_1}\Big\{(f_{xxy}+g_{xyy}) - \tfrac{1}{\lambda_1}\Big[H_{xyy}(f_{xx}+g_{xy}) + H_{xxy}(f_{xy}+g_{yy})\Big]\Big\}\Big|_{x=y=0}. \tag{2.6}$$

Following a similar idea of [29], we have

Lemma 2.4. *Suppose that system $(1.1)_{\varepsilon=0}$ has a double homoclinic loop L_1. Let M_1, M_{1i}, and c_{ij} be defined in (2.3), (2.5) and (2.6). If there exists $\delta_0 \in D$ such that*

$$\begin{aligned} & M_1(\delta_0) = 0, \ c_{31}(\delta_0) \neq 0, \ c_{11}(\delta_0) = c_{21}(\delta_0) = c_{22}(\delta_0) = 0, \\ & Rank \frac{\partial(M_{11}, M_{12}, c_{11}, c_{21}, c_{22})}{\partial(\delta_1, \ldots, \delta_m)}(\delta_0) = 5, \end{aligned} \tag{2.7}$$

then system (1.1) has 7 limit cycles bifurcating from L_1 for some (δ, ε) near $(\delta_0, 0)$.

Proof. From Lemma 2.2, we know that under (2.7), system (1.1) has a double homoclinic loop $L_\varepsilon = L_{\varepsilon_1} \cup L_{\varepsilon_2}$ such that $L_{\varepsilon_1} \to L_{11}$, $L_{\varepsilon_2} \to L_{12}$ as $\varepsilon \to 0$. Then we can produce 7 limit cycles by changing the stability of L_ε.

First, let

$$M_{11} = M_{11}(\delta), \ M_{12} = M_{12}(\delta), \ c_{11} = c_{11}(\delta), \ c_{21} = c_{21}(\delta), \ c_{22} = c_{22}(\delta).$$

From (2.7) and the implicit function theory, one can solve δ near δ_0 from the above equations. It implies that $M_{11}, \ M_{12}, \ c_{11}, \ c_{21}, \ c_{22}$ can be taken as free parameters.

Further, when $(f_x+g_y)(S_\varepsilon) = 0$, we have from [29, Lemma 2.4] that

$$\begin{aligned} \oint_{L_{\varepsilon_1}} (f_x+g_y)dt &= \oint_{L_{11}} (f_x+g_y)dt + O(\varepsilon \ln|\varepsilon|), \\ \oint_{L_{\varepsilon_2}} (f_x+g_y)dt &= \oint_{L_{12}} (f_x+g_y)dt + O(\varepsilon \ln|\varepsilon|). \end{aligned}$$

Then from (2.4) and (2.5), we have the following

$$\mu_1 = \varepsilon(c_{11} + O(\varepsilon)),$$

$$\begin{aligned}
&\mu_{21}|_{\mu_1=0} = \varepsilon(c_{21} + O(\varepsilon \ln|\varepsilon|)),\\
&\mu_{22}|_{\mu_1=0} = \varepsilon(c_{22} + O(\varepsilon \ln|\varepsilon|)),\\
&\mu_3 = \varepsilon\left(c_{31}(\delta_0) + O(|\varepsilon| + |c_{11}, c_{21}, c_{22}|)\right).
\end{aligned} \tag{2.8}$$

In the following, we change the signs of μ_1, μ_{21} and μ_{22} by varying c_{11}, c_{21} and c_{22}.

Assuming $c_{33}(\delta_0) > 0$, we have $\mu_3 > 0$ from (2.8). Letting $M_{11} = M_{12} = c_{11} = c_{21} = c_{22} = 0$, Lemma 2.3 derives that L_ε is inside stable and outside unstable.

Keep $M_{11} = M_{12} = c_{11} = c_{21} = 0$. Letting $0 < c_{22} \ll 1$ implies $\mu_{22} > 0$. So that the homoclinic loop L_{ε_2} is inside unstable. Meanwhile, the homoclinic loop L_{ε_1} remains stable and the double homoclinic loop L_ε is still unstable. Then one limit cycle can be found inside L_{ε_2}.

Let $M_{11} = M_{12} = c_{11} = 0$ and $0 < c_{21} \ll 1$, that is $\mu_{21} > 0$. Then from Lemma 2.2 we know that the stability of the homoclinic loop L_{ε_1} has changed from stable to unstable. L_{ε_2} and L_ε are unstable like in the last step. Thus, one limit cycle is produced inside L_{ε_1}.

Now let $M_{11} = M_{12} = 0$. Vary $c_{11} < 0$ and $|c_{11}| \ll \min\{c_{21}, c_{22}\}$, then $\mu_1 < 0$. Hence, L_ε is stable both inside and outside. From last step, we know that the stability of L_{ε_2}, L_{ε_1} and L_ε are changed. Then one can find three more limit cycles, two inside L_{ε_2}, L_{ε_1} and one ouside L_ε, respectively.

At last, letting $M_{11} < 0$ and $M_{12} < 0$, one can find two more limit cycles by breaking L_{ε_2} and L_{ε_1}.

We have found seven limit cycles by varying $M_{11},\ M_{12},\ c_{11},\ c_{21},\ c_{22}$ in the case of $c_{31}(\delta_0) > 0$. Denote (n_1, n_2, n_3) by the distribution of limit cycles near L_ε, such that n_1 limit cycles inside L_{ε_2}, n_2 limit cycles inside L_{ε_1} and n_3 limit cycles outside L_ε. From the above proof, seven limit cycles has the distribution of $(3, 3, 1)$, see Figure 4. In fact, we have considered all the possibility of $M_{11},\ M_{12},\ c_{11},\ c_{21},\ c_{22}$ and find no more than 7 limit cycles. By choosing the parameters as follows, one can also find seven limit cycles.

1. When $c_{33}(\delta_0) > 0$, let $0 < \min\{-M_{11}, -M_{12}\} \ll -c_{11} \ll -c_{22} < c_{21} \ll 1$.
   When $c_{33}(\delta_0) < 0$, let $0 < \min\{M_{11}, M_{12}\} \ll c_{11} \ll c_{22} < -c_{21} \ll 1$.
   Distribution of limit cycles is $(3, 1, 3)$.
2. When $c_{33}(\delta_0) > 0$, let $0 < \min\{-M_{11}, -M_{12}\} \ll -c_{11} \ll \min\{c_{21}, c_{22}\} \ll 1$.
   When $c_{33}(\delta_0) < 0$, let $0 < \min\{M_{11}, M_{12}\} \ll c_{11} \ll \min\{-c_{21}, -c_{22}\} \ll 1$.
   Distribution of limit cycles is $(3, 3, 1)$.

It completes the proof of this theorem. $\square$

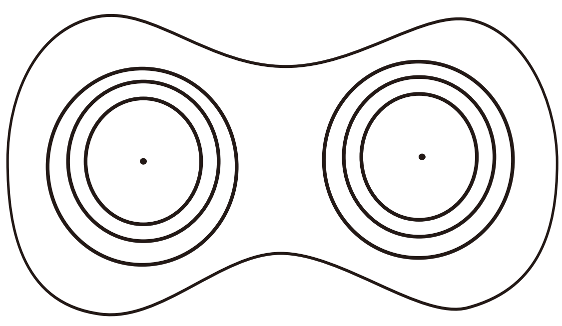

Figure 4. The distribution of 7 limit cycles

Now consider the centrally symmetrical near-Hamiltonian system (1.1), where

$$\begin{aligned}&H(x,y)=H(-x,-y),\\&f(-x,-y,\delta)=-f(x,y,\delta),\\&g(-x,-y,\delta)=-g(x,y,\delta).\end{aligned}$$

Assume that the unperturbed system $(1.1)_{\varepsilon=0}$ has five hyperbolic saddles $S_0(0,0)$, $S_0^*(x_1^s,0)$, $\bar{S}_0(-x_1^s,0)$, $S_1(x_2^s,0)$, $\bar{S}_1(-x_2^s,0)$, and four centers $C_i(x_i^c,0)$, $\bar{C}_i(-x_i^c,0)$ where $x_i^c>0, i=1,2$. By the symmetry of system $(1.1)_{\varepsilon=0}$, we have

$$H(x_i^s,0)=H(-x_i^s,0),\quad H(x_i^c,0)=H(-x_i^c,0),\quad i=1,2.$$

Suppose system $(1.1)_{\varepsilon=0}$ has a double heteroclinic loop L_0 surrounding two homoclinic loops L_1,$\bar{L}_1$,where $L_0=L_{01}\cup L_{02}\cup\bar{L}_{01}\cup\bar{L}_{02}$, $L_1=L_{11}\cup L_{12}$ and $\bar{L}_1=\bar{L}_{11}\cup\bar{L}_{12}$, see Figure 5. We can get that

$$H(\pm x_1^s,0)=H(0,0).$$

Without loss of generality, we further assume that

$$H(0,0)=0,\ H(\pm x_2^s,0)=h_0<0,\ H(\pm x_i^c,0)=h_i<h_0,\ i=1,2,$$

from which we know L_0 is in clockwise orientation.

Then by the symmetry of (1.1), we can obtain the following theorem easily from Lemmas 2.1, 2.4 and Remark 2.1.

Theorem 2.1. *Suppose the symmetrical system $(1.1)_{\varepsilon=0}$ has a double heteroclinic loop L_0 surrounding two double homoclinic loops L_1 and $\bar{L}_1$ like Figure 5. Let M_i, $\bar{M}_i$, M_{ij}, $\bar{M}_{ij}$, c_i, $\bar{c}_i$, c_1^*, c_{3j} be defined in (2.1), (2.2), (2.3), (2.5) and (2.6). If there exist $\delta_0\in D$ such that*

$$\begin{aligned}&M_0(\delta_0)=\bar{M}_0(\delta_0)=M_1(\delta_0)=0, M_{01}(\delta_0)\bar{M}_{01}(\delta_0)>0,\ c_{3j}(\delta_0)\neq 0, j=0,1,\\&c_{11}(\delta_0)=c_{2i}(\delta_0)=c_i(\delta_0)=\ \bar{c}_i(\delta_0)=c_1^*(\delta_0)=0, i=1,2,\\&Rank\frac{\partial(M_{11},M_{12},M_0,\bar{M}_0,c_{11},c_{21},c_{22},c_1,c_1^*,\bar{c}_1,c_2,\bar{c}_2)}{\partial(\delta_1,\ldots,\delta_m)}(\delta_0)=12,\end{aligned}\tag{2.9}$$

then system (1.1) has 24 limit cycles, for some (δ,ε) near $(\delta_0,0)$, fourteen of which are bifurcated from $L_1,\bar{L}_1$ and ten of which are from L_0. Among them, there are two alien limit cycles.

Remark 2.2. We know that by using the Melnikov function, a double heteroclinic loop L_0 can produce six limit cycles, see [22], and a double homoclinic loops L_1 can produce seven limit cycles, see [9]. So twenty limit cycles can be generated from system $(1.1)_{\varepsilon=0}$. In summary, we can find four more limit cycles near L_1, $\bar{L}_1$ and L_0 by the stability-changing method than the Melnikov function method.

3. Proof of the main theorem

First, we prove Theorem 1.1(i).

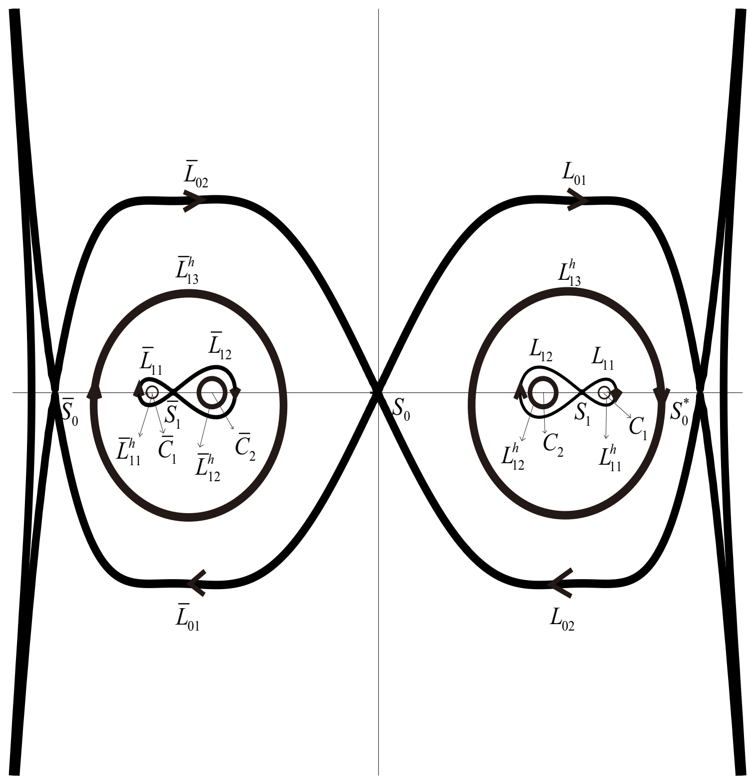

Figure 5. The image of L_0, L_1 and $\bar{L}_1$

Notice that system (1.3) has 9 singular points $(0,0)$, $(\pm\sqrt{a},0)$, $(\pm\sqrt{b},0)$, $(\pm\sqrt{c},0)$ and $(\pm\sqrt{d},0)$. Let $J(x,y)$ be the Jacobi matrix at point (x,y), defined as

$$J(x,y)=\begin{bmatrix} 0 & 1 \\ \frac{-\partial H_x}{\partial x} & 0 \end{bmatrix},$$

where $-H_x = kx(x^2-a)(x^2-b)(x^2-c)(x^2-d)$.

Then one can caculate easily the eigenvalues λ of $J(x,y)$ at points $(0,0)$, $(\sqrt{a},0)$, $(\sqrt{b},0)$, $(\sqrt{c},0)$ and $(\sqrt{d},0)$, which are listed in the following table.

Table 1. Eigenvalues of sigular points

(x,y)	$(0,0)$	$(\sqrt{a},0)$	$(\sqrt{b},0)$	$(\sqrt{c},0)$	$(\sqrt{d},0)$
λ	$\pm\sqrt{abcdk}$	$\pm\sqrt{2ak(a-b)(a-c)(a-d)}$	$\pm\sqrt{2bk(b-a)(b-c)(b-d)}$	$\pm\sqrt{2ck(c-a)(c-b)(c-d)}$	$\pm\sqrt{2dk(d-a)(d-b)(d-c)}$

Under the assumptions $a>b>c>d>0$ and $k>0$, we can conclude that $(0,0)$, $(\sqrt{a},0)$ and $(\sqrt{c},0)$ are saddles, $(\sqrt{b},0)$ and $(\sqrt{d},0)$ are centers. Then by the symmetry of (1.3) and $H(\sqrt{c},0)<H(\sqrt{a},0)=H(0,0)=0$, Theorem 1.1(i) can be proved directly.

Now let $k=1$, $a=4$, $b=\frac{8}{5}$, $c=2$ and $d=1$. Then the phase portrait of system (1.3) is Figure 6.

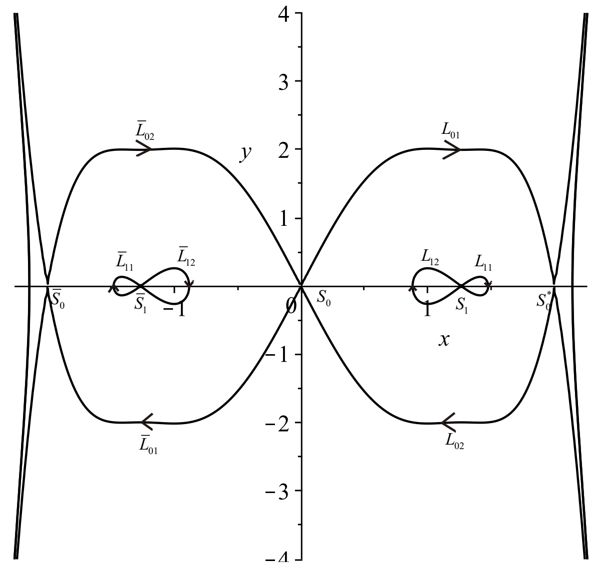

Figure 6. The phase portrait of system (1.3)

Consider the number of limit cycles for the following Liénard system

$$\begin{cases} \dot{x} = y, \\ \dot{y} = -x^9 + \dfrac{43}{5}x^7 - \dfrac{126}{5}x^5 + \dfrac{152}{5}x^3 - \dfrac{64}{5}x + \varepsilon(b_0x + b_1x^3 + g_0(x)y), \end{cases} \tag{3.1}$$

where $g_0(x) = a_0 + a_2x^2 + a_4x^4 + a_6x^6 + a_8x^8 + a_{10}x^{10} + a_{12}x^{12} + a_{14}x^{14} + a_{16}x^{16} + x^{18}, b_0,\ b_1 \neq 0$. The unperturbed system $(1.1)_{\varepsilon=0}$ has five saddles $S_0^* = (2,0),\ \bar{S}_0 = (-2,0),\ S_1 = (\frac{2\sqrt{10}}{5},0),\ \bar{S}_1 = (\frac{-2\sqrt{10}}{5},0),\ S_0 = (0,0)$, and four centers $(\pm1,0)$, $(\pm\sqrt{2},0)$. It has the following Hamiltonian function

$$H(x,y) = \frac{1}{2}y^2 - \frac{1}{40}\Big(x^2(4x^4 - 11x^2 - 16)(x-2)^2(x+2)^2\Big).$$

By (2.1) we have the following integrals along the curves $L_{0i} : y = y_i(x) = (-1)^{i+1}\frac{1}{2\sqrt{5}}x(4-x^2)(4x^4 - 11x^2 + 16)^{\frac{1}{2}},\quad 0 \le x \le 2,\quad i = 1,2,$

$$\begin{aligned} M_0(\delta) &= M_{01}(\delta) + M_{02}(\delta) \\ &= \int_{L_{01}} \big(b_0x + b_1x^3 + g_0(x)y\big)dx + \int_{L_{02}} \big(b_0x + b_1x^3 + g_0(x)y\big)dx \\ &= \int_{L_{01}} g_0(x)y_1(x)dx + \int_{L_{02}} g_0(x)y_2(x)dx \\ &\quad + \int_{L_{01}} (b_0x + b_1x^3)dx + \int_{L_{02}} (b_0x + b_1x^3)dx \\ &= \frac{1}{\sqrt{5}}\int_0^2 g_0(x)\big(x(4-x)(4x^4 - 11x^2 + 16)^{\frac{1}{2}}\big)dx \\ &= \frac{1}{\sqrt{5}}(I_0a_0 + I_2a_2 + I_4a_4 + I_6a_6 + I_8a_8 + I_{10}a_{10} + I_{12}a_{12} + I_{14}a_{14} + I_{16}a_{16} + I_{18}) \end{aligned} \tag{3.2}$$

where

$$L_{0i}: y=(-1)^{i+1}\frac{1}{2\sqrt{5}}x(4-x^2)(4x^4-11x^2+16)^{\frac{1}{2}},\ 0\le x\le 2,\ i=1,2,$$

$$I_i=\int_0^2 x^{2m+1}(4-x^2)(4x^4-11x^2+16)^{\frac{1}{2}},\ 0\le m\le 9,\ i=2m,$$

and

$$\begin{aligned}
I_0&=\frac{2923}{384}+\frac{2835\ln 3}{512},\ I_{10}=\frac{5059743146611}{5637144576}-\frac{30359256435\ln 3}{1073741824},\\
I_2&=\frac{115805}{12288}+\frac{142965\ln 3}{16384},\ I_{12}=\frac{370903396451989}{135291469824}-\frac{440637580935\ln 3}{8589934592},\\
I_4&=\frac{14284121}{491520}+\frac{1390365\ln 3}{131072},\ I_{14}=\frac{13337650412525201}{1546188226560}-\frac{6023334177945\ln 3}{137438953472},\\
I_6&=\frac{752541529}{7864320}+\frac{15698205\ln 3}{2097152},\\
I_{16}&=\frac{26682779747440136519}{952451947560960}+\frac{43631905014405\ln 3}{1099511627776},\\
I_8&=\frac{130791632753}{440401920}-\frac{86187645\ln 3}{16777216},\\
I_{18}&=\frac{94778250349461631 5817}{10159487440650240}+\frac{7808910149157345\ln 3}{35184372088832}.
\end{aligned}$$

Similarly, along the homoclinic loop L_{11}: $y=\pm\frac{1}{250}(500x^6-3775x^4+7640x^2-3888)^{\frac{1}{2}}(5x^2-8)$, $\frac{2\sqrt{10}}{5}\le x\le\sqrt{2}$, we have from (2.3) that

$$\begin{aligned}
M_{11}(\delta)&=\oint_{L_{11}}\big(b_0x+b_1x^3+g_0(x)y\big)dx\\
&=\int_{L_{11}}\big(b_0x+b_1x^3+g_0(x)y_3(x)\big)dx\\
&=\frac{1}{125}\int_{\frac{2\sqrt{10}}{5}}^{\sqrt{2}} g_0(x)(500x^6-3775x^4+7640x^2-3888)^{\frac{1}{2}}(5x^2-8)dx\\
&=\frac{1}{125}(J_0a_0+J_2a_2+J_4a_4+J_6a_6+J_8a_8+J_{10}a_{10}\\
&\quad+J_{12}a_{12}+J_{14}a_{14}+J_{16}a_{16}+J_{18})
\end{aligned}\tag{3.3}$$

where

$$J_i=\int_{\frac{2\sqrt{10}}{5}}^{\sqrt{2}} x^{2m}(500x^6-3775x^4+7640x^2-3888)^{\frac{1}{2}}(5x^2-8)dx,\ 0\le m\le 9,\ i=2m,$$

obtained by Maple using numerical integration

$$\begin{aligned}
J_0&=9.226513336\cdots, & J_2&=17.18815710\cdots,\\
J_4&=32.10371920\cdots, & J_6&=60.11415378\cdots,\\
J_8&=112.8375462\cdots, & J_{10}&=212.2978888\cdots,\\
J_{12}&=400.3246508\cdots, & J_{14}&=756.5074514\cdots,\\
J_{16}&=1432.545119\cdots, & J_{18}&=2718.055189\cdots.
\end{aligned}$$

We can also obtain $M_{12}(\delta)$ from (2.3) that

$$
\begin{aligned}
M_{12}(\delta) &= \oint_{L_{12}} \big(b_0 x + b_1 x^3 + g_0(x) y\big) dx \\
&= \int_{L_{12}} \big(b_0 x + b_1 x^3 + g_0(x) y_3(x)\big) dx \\
&= \frac{1}{125} \int_1^{\frac{2\sqrt{10}}{5}} g_0(x)(500x^6 - 3775x^4 + 7640x^2 - 3888)^{\frac{1}{2}}(5x^2 - 8) dx \\
&= \frac{1}{125}(K_0 a_0 + K_2 a_2 + K_4 a_4 + K_6 a_6 + K_8 a_8 + K_{10} a_{10} \\
&\quad + K_{12} a_{12} + K_{14} a_{14} + K_{16} a_{16} + K_{18})
\end{aligned} \tag{3.4}
$$

where

$$
K_i = \int_1^{\frac{2\sqrt{10}}{5}} x^{2m}(500x^6 - 3775x^4 + 7640x^2 - 3888)^{\frac{1}{2}}(5x^2 - 8) dx, \ 0 \le m \le 9, \ i = 2m,
$$

$$
L_{12}: \ y = \pm \frac{1}{250}(500x^6 - 3775x^4 + 7640x^2 - 3888)^{\frac{1}{2}}(5x^2 - 8), \quad 1 \le x \le \frac{2\sqrt{10}}{5}.
$$

Calculating K_i, we have

$$
\begin{aligned}
&K_0 = -26.64042911\cdots, && K_2 = -31.79318924\cdots, \\
&K_4 = -38.46007978\cdots, && K_6 = -47.17856128\cdots, \\
&K_8 = -58.69889628\cdots, && K_{10} = -74.07464795\cdots, \\
&K_{12} = -94.79365732\cdots, && K_{14} = -122.9682972\cdots, \\
&K_{16} = -161.6127875\cdots, && K_{18} = -215.0487424\cdots.
\end{aligned}
$$

For the expression of $c_1(\delta)$, $c_1^*(\delta)$ and $c_{11}(\delta)$, we can easily obtain from (2.1) and (2.5) that

$$
\begin{aligned}
c_1(\delta) &= a_0, \\
c_1^*(\delta) &= \sum_{j=0}^{8} 4^j a_{2j} + 4^9, \\
c_{11}(\delta) &= \sum_{j=0}^{8} (\frac{8}{5})^j a_{2j} + (\frac{8}{5})^9.
\end{aligned} \tag{3.5}
$$

Let $c_1(\delta) = c_1^*(\delta) = 0$, which yields that $a_2 = -(4a_4 + 4^2 a_6 + 4^3 a_8 + 4^4 a_{10} + 4^5 a_{12} + 4^6 a_{14} + 4^7 a_{16} + 4^8)$. Then, taking a_2 into (2.1), we have

$$
\begin{aligned}
c_2(\delta) &= \int_{L_{01} \cup L_{02}} g_0(x)\, dt = \int_{L_{01} \cup L_{02}} \frac{g_0(x)}{y}\, dx \\
&= 2 \int_0^2 \frac{g_0(x)}{\frac{1}{2\sqrt{5}} x(4 - x^2)(4x^4 - 11x^2 + 16)^{\frac{1}{2}}}\, dx \\
&= 4\sqrt{5} \int_0^2 \frac{g_0(x)}{x(4 - x^2)(4x^4 - 11x^2 + 16)^{\frac{1}{2}}}\, dx \\
&= 4\sqrt{5} \bigg(\sum_{j=2}^{8} P_{2j} a_{2j} + P_{18} \bigg)
\end{aligned} \tag{3.6}
$$

where

$$P_i = \int_0^2 \frac{x^{2m-1} - 2^{2m-2}x}{x(4-x^2)(4x^4 - 11x^2 + 16)^{\frac{1}{2}}} dx,\ 2 \le m \le 9,\ i = 2m,$$

and

$$\begin{aligned}
&P_4 = -\frac{\ln 3}{2}, \quad P_6 = -\frac{1}{4} - \frac{43 \ln 3}{16},\\
&P_8 = -\frac{193}{64} - \frac{2859 \ln 3}{256}, \quad P_{10} = -\frac{30743}{1536} - \frac{89695 \ln 3}{2048},\\
&P_{12} = -\frac{5168267}{49152} - \frac{11260723 \ln 3}{65536}, \quad P_{14} = -\frac{324823563}{655360} - \frac{357451269 \ln 3}{524288},\\
&P_{16} = -\frac{23237669873}{10485760} - \frac{22854210367 \ln 3}{8388608},\\
&P_{18} = \frac{5662048512319}{587202560} - \frac{732140715175 \ln 3}{67108864}.
\end{aligned}$$

Let $c_{11}(\delta) = 0$, from (3.5) we obtain that

$$a_0 = -\Big(\frac{8}{5}a_2 + (\frac{8}{5})^2 a_4 + (\frac{8}{5})^3 a_6 + (\frac{8}{5})^4 a_8 + (\frac{8}{5})^5 a_{10} + (\frac{8}{5})^6 a_{12} + (\frac{8}{5})^7 a_{14} + (\frac{8}{5})^8 a_{16} + (\frac{8}{5})^9\Big).$$

It implies from (2.5) that

$$\begin{aligned}
c_{21}(\delta) &= \oint_{L_{11}} g_0(x)dt = \oint_{L_{11}} \frac{g_0(x)}{y_3(x)} dx\\
&= 500 \int_{\frac{2\sqrt{10}}{5}}^{\sqrt{2}} \frac{g_0(x)}{(500x^6 - 3775x^4 + 7640x^2 - 3888)^{\frac{1}{2}}(5x^2 - 8)} dx \qquad (3.7)\\
&= 500(Q_2 a_2 + Q_4 a_4 + Q_6 a_6 + Q_8 a_8 + Q_{10} a_{10} + Q_{12} a_{12}\\
&\quad + Q_{14} a_{14} + Q_{16} a_{16} + Q_{18})
\end{aligned}$$

where

$$Q_i = \int_{\frac{2\sqrt{10}}{5}}^{\sqrt{2}} \frac{x^{2m} - (\frac{8}{5})^m}{(500x^6 - 3775x^4 + 7640x^2 - 3888)^{\frac{1}{2}}(5x^2 - 8)} dx,\ 1 \le m \le 9,\ i = 2m,$$

and

$$\begin{aligned}
&Q_2 = 0.001316505038\cdots, \ Q_4 = 0.004490334120\cdots, \ Q_6 = 0.01151924117\cdots,\\
&Q_8 = 0.02634464921\cdots, \ Q_{10} = 0.05665708129\cdots, \ Q_{12} = 0.1173417856\cdots,\\
&Q_{14} = 0.2370405044\cdots, \ Q_{16} = 0.4706303950\cdots, \ Q_{18} = 0.9229379649\cdots.
\end{aligned}$$

And

$$\begin{aligned}
c_{22}(\delta) &= \oint_{L_{12}} g_0(x)dt = \oint_{L_{12}} \frac{g_0(x)}{y_3(x)} dx\\
&= 2 \int_1^{\frac{2\sqrt{10}}{5}} \frac{g_0(x)}{\frac{(500x^6 - 3775x^4 + 7640x^2 - 3888)^{\frac{1}{2}}(5x^2 - 8)}{250}} dx \qquad (3.8)\\
&= 500(R_2 a_2 + R_4 a_4 + R_6 a_6 + R_8 a_8 + R_{10} a_{10} + R_{12} a_{12}\\
&\quad + R_{14} a_{14} + R_{16} a_{16} + R_{18})
\end{aligned}$$

where

$$R_i = \int_1^{\frac{2\sqrt{10}}{5}} \frac{x^{2m} - (\frac{8}{5})^m}{(500x^6 - 3775x^4 + 7640x^2 - 3888)^{\frac{1}{2}}(5x^2 - 8)} dx, \ 1 \le m \le 9, \ i = 2m,$$

and

$$\begin{aligned}
&R_2 = 0.002010818873\cdots, \ R_4 = 0.005788571050\cdots, \ R_6 = 0.01261172859\cdots, \\
&R_8 = 0.02462290372\cdots, \ R_{10} = 0.04539362244\cdots, \ R_{12} = 0.08085102069\cdots, \\
&R_{14} = 0.1407954475\cdots, \ R_{16} = 0.2413815522\cdots, \ R_{18} = 0.4091684847\cdots.
\end{aligned}$$

To compute $c_{30}(\delta)$, we make the variable transformation

$$\begin{cases} x = u - v, \\ y = \dfrac{8\sqrt{5}}{5}(u + v), \end{cases}$$

which carries the system (3.1) into

$$\begin{cases} \dot{u} = -\dfrac{\sqrt{5}}{16}\left[H_x\Big(u - v, \dfrac{8\sqrt{5}}{5}(u + v)\Big) - \dfrac{64}{5}(u + v)\right] + \varepsilon \tilde{f}(u, v), \\ \dot{v} = -\dfrac{\sqrt{5}}{16}\left[H_x\Big(u - v, \dfrac{8\sqrt{5}}{5}(u + v)\Big) + \dfrac{64}{5}(u + v)\right] + \varepsilon \tilde{f}(u, v), \end{cases}$$

where $\tilde{f}(u, v) = \frac{\sqrt{5}}{16}\Big(b_0(u-v) + b_1(u-v)^3 + g_0(u-v)\frac{8\sqrt{5}}{5}(u+v)\Big)$. The Hamiltonian function of system (3.1) is

$$\begin{aligned}
\tilde{H}(u, v) &= \frac{\sqrt{5}}{16} H\Big(u - v, \frac{8\sqrt{5}}{5}(u + v)\Big) \\
&= \frac{2\sqrt{5}}{5}(u + v)^2 - \frac{2\sqrt{5}}{640}(u - v)^2 \\
&\quad \times \Big(4(u - v)^4 - 11(u - v)^2 + 16\Big)(u - v - 2)^2(u - v + 2)^2 \\
&= \frac{8\sqrt{5}}{5} uv + O(|(u, v)|^3).
\end{aligned}$$

Then it implies from (2.2) that

$$\begin{aligned}
c_{30}(\delta) &= -\frac{\sqrt{5}}{16}\left\{(\tilde{f}_{uuv} + \tilde{f}_{uvv}) - \frac{\sqrt{5}}{8}\big[\tilde{H}_{uvv}(\tilde{f}_{uu} + \tilde{f}_{uv}) + \tilde{H}_{uuv}(\tilde{f}_{uv} + \tilde{f}_{vv})\big]\right\}\Big|_{u=v=0} \\
&= \frac{\sqrt{5}}{8} a_2.
\end{aligned} \tag{3.9}$$

Similarly, to compute $c_{31}(\delta)$, we make the variable transformation

$$\begin{cases} x = u - v + \dfrac{2\sqrt{10}}{5}, \\ y = \dfrac{24\sqrt{2}}{25}(u + v), \end{cases}$$

under which system (3.1) becomes

$$\begin{cases} \dot{u} = -\dfrac{\sqrt{2}}{2400}\Big[625H_x - 1152(u+v)\Big] + \varepsilon \hat{f}(u,v), \\ \dot{v} = -\dfrac{\sqrt{2}}{2400}\Big[625H_x + 1152(u+v)\Big] + \varepsilon \hat{f}(u,v), \end{cases}$$

where

$$\begin{aligned} \hat{H}(u,v) &= \frac{25\sqrt{2}}{96} H\Big(u - v + \frac{2\sqrt{10}}{5}, \frac{24\sqrt{2}}{25}(u+v)\Big) = \frac{24\sqrt{2}}{25} uv + O(|(u,v)|^3), \\ \hat{f}(u,v) &= \frac{25\sqrt{2}}{96}\Big(b_0(u - v + \frac{2\sqrt{10}}{5}) + b_1(u - v + \frac{2\sqrt{10}}{5}) \\ &\quad + g_0(u - v + \frac{2\sqrt{10}}{5})\frac{24\sqrt{5}}{5}(u+v)\Big). \end{aligned}$$

Then we have from (2.5) that

$$\begin{aligned} c_{31}(\delta) = &\frac{\sqrt{2}}{375000}(1660944384 + 1171875a_2 + 5000000a_4 + 15000000a_6 + 38400000a_8 \\ &+ 89600000a_{10} + 196608000a_{12} + 412876900a_{14} + 838860800a_{16}). \end{aligned} \tag{3.10}$$

Combining (3.2)–(3.8), solving the equations $M_0(\delta) = M_{11}(\delta) = M_{12}(\delta) = c_1(\delta) = c_1^*(\delta) = c_{11}(\delta) = c_2(\delta) = c_{21}(\delta) = c_{22}(\delta) = 0$ gives

$$\begin{aligned} &\hat{a}_0 = 9.226513336, \ \hat{a}_2 = 17.18815710, \ \hat{a}_4 = 32.10371920, \\ &\hat{a}_6 = 60.11415378, \ \hat{a}_8 = 112.8375462, \ \hat{a}_{10} = 212.2978888, \\ &\hat{a}_{12} = 400.3246508, \ \hat{a}_{14} = 756.5074514, \ \hat{a}_{16} = 1432.545119. \end{aligned}$$

Thus, we can take $\delta_0 = (\hat{a}_0, \hat{a}_2, \hat{a}_4, \hat{a}_6, \hat{a}_8, \hat{a}_{10}, \hat{a}_{12}, \hat{a}_{14}, \hat{a}_{16})$. In this case, we have

$$\begin{aligned} &M_0(\delta) = M_{11}(\delta) = M_{12}(\delta) = c_1(\delta) = c_1^*(\delta) = c_{11}(\delta) = c_2(\delta) = c_{21}(\delta) = c_{22}(\delta) = 0, \\ &\det \frac{\partial(M_0, M_{11}, M_{12}, c_1, \ c_1^*, c_{11}, c_2, c_{21}, c_{22},)}{\partial(a_0, a_2, a_4, a_6, a_8, a_{10}, a_{12}, a_{14}, a_{16})} = -2.794147105 \times 10^{44} \neq 0. \end{aligned}$$

Furthermore, from (3.9) and (3.10) we have

$$c_{30}(\delta_0) = 314.4911458\sqrt{5} \neq 0, \quad c_{31}(\delta_0) = -2.406510261 \times 10^6\sqrt{5} \neq 0.$$

Then (2.9) of Theorem 2.1 are satisfied, which implies that Theorem 1.1(ii) is true. It completes the proof of Theorem 1.1.

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