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BIFURCATION OF LIMIT CYCLE AT THE INFINITY ON A CENTER MANIFOLDS IN SPACE VECTOR FIELD*

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Abstract In this paper, we deal with the problem of large amplitude limit cycles bifurcation at infinity in space vector field. By making two appropriate transformations and making use of singular values methods on a center manifold to compute focal values carefully, we obtain the simplified expressions of the first five focal values at the infinity by using symbolic calculation methods. Further, we show the infinity can bifurcate 5 large limit cycles.

Keywords Space vector field, infinity, limit cycle bifurcation, focal values.

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1. Introduction

The goal of this paper is to investigate the limit cycle bifurcation at infinity on a center manifold for the following polynomial systems in space vector field. Investigated systems are as follows:

$$\begin{cases} \dfrac{dx}{d\tau} = -y(x^2+y^2+u^2)^2+2ux(u^3+ux^2-xy+uy^2)+Mu(x+y)(u^2 \\ \qquad -x^2+2xy+y^2)-(x^2+y^2+u^2)(-Au^2+2Cux+Ax^2+2Bxy \\ \qquad -Ay^2) \equiv X(x,y,u), \\ \dfrac{dy}{d\tau} = x(x^2+y^2+u^2)^2+2uy(u^3+ux^2-xy+uy^2)-Mu(x+y)(u^2 \\ \qquad +x^2+2xy-y^2)+(x^2+y^2+u^2)(Bx^2+Bu^2-By^2+2Cuy \\ \qquad +2Axy) \equiv Y(x,y,u), \\ \dfrac{du}{d\tau} = (u^2-x^2-y^2)(u^3+ux^2-xy+uy^2)-2Mu(x^2-y^2) \\ \qquad -(x^2+y^2+u^2)(Cu^2+2Aux-Cx^2+2Buy-Cy^2) \equiv U(x,y,u), \end{cases} \tag{1.1}$$

where A, B, C, M are four real parameters. For system (1.1), the equator Γ_∞ on the Poincaré closed sphere is a trajectory of the system, which is called the infinity

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or the infinite point of the system. If the infinity can bifurcate one limit cycle, then this limit cycle is called a large limit cycle or a equator cycle.

For the limit cycle bifurcation of polynomial differential system in planar vector field, many good published results focus on 2-dimensional systems because of attracting effect from the "Hilbert 16-th problem", up to now, it is a hot topic, for example some recent works (see [8–10], [11, 12] etc). Perhaps, the limit cycle bifurcation problem of space vector field is not a hot topic, and the research literatures are far less than those of planar system, but in recent years, some researchers have paid more and more attentions to the problem of limit cycle bifurcation for polynomial systems in space vector field, some methods and conclusions are also emerging, For example: [1,13] investigated the limit cycle bifurcation of elementary singularity for space Hopf bifurcation systems and the condition and distribution structure of the existence of limit cycles are given; [15, 17] showed averaging theory of second order was valid to study some 3-dimensional Hopf Bifurcation and considered the limit cycle bifurcation on a class of special polynomial system; [7] gave a concrete example with three limit cycles in Zeeman's class 29 for three dimensional Lotka-Volterra competitive systems; [21] used the formal series method to compute the Lyapunov constant and investigated the Hopf bifurcation for a class of three-dimensional systems. By making use of the method offered by [21], [20, 21] showed the limit cycle bifurcation behavior for two classes of three dimensional Lotka-Volterra systems, [19] obtained some results about limit cycle bifurcation at the elementary singularity, [18] showed a class of space Lorenz system could occur 6 small limit cycles, we gave the result that ten limit cycles could occur from a space quadratic system with quadratic perturbation in [22]. We also gave a kind of method namely singular values method to study three-dimensional Hopf bifurcation in [5,6] and listed some cases, for example: [5] showed a class of cubic Kolmogorov model could bifurcate 5 small limit cycles in space vector field by disturbing its parameters, [6] investigated a special class of 3-Dimensional quadratic systems and showed it could bifurcate 8 small limit cycles at an elementary singularity, [3] studied bifurcation of limit cycle for a degenerate singular point in 3-Dimensional vector fields and showed it could bifurcate 5 limit cycles near the origin.

For the case of limit cycle bifurcation at infinity, only several special systems have been studied, see the results of articles [2, 4, 7, 14, 23, 24] and their references. And in the last five years, in the case of the limit cycle bifurcation at infinity in 3-Dimensional vector field, perhaps it is difficult to calculate Lyapunov constant at infinity in 3-Dimensional vector field, there is hardly any published literatures. Next, we will consider this class of bifurcation problem.

In this paper, we will make some attempts to study the limit cycle bifurcation at infinity of system (1.1). At first, by making two appropriate transformations of system (1.1) namely homeomorphous transformation and time transformation in order to consider the bifurcation behavior at infinity of system (1.1), the infinity of system (1.1) is changed into the origin of system (3.3). For system (3.3), we can carry out investigation on limit cycle bifurcation according to the method offered by articles [5,6]. With the help of a computer, we obtain the expressions of the first five focal values for the origin of system (3.3) by carefully computation, and show that the origin of system (3.3) can be a fine focus of fifth order. Moreover, we give the condition that the origin of system (3.3) can bifurcate 5 small limit cycles. From the relation between the infinity of system (1.1) and the origin of system (3.3), we give the bifurcation behavior of the infinity of system (1.1), namely system (1.1) can bifurcate 5 large limit cycles from the infinity. In the whole process of analysis and calculation, our calculations are both symbolic and algebraic, the simplification of the expression is also accurate.

2. Our method to study the 3-dimensional Hopf bifurcations

For Hopf bifurcations system in space vector field, articles [5, 6] gave the singular values method to study the Hopf bifurcations problems of the elementary focus point. At first, we introduce this kind of method in order to research the Hopf bifurcation problem of system (1.1). For the following space Hopf bifurcations systems:

$$\begin{cases} \dfrac{dx}{dt} =& - y + \delta x + \displaystyle\sum_{k+j+l=2}^{\infty} A_{jkl} x^k y^j u^l = X(x,y,u), \\ \dfrac{dy}{dt} =& x + \delta y + \displaystyle\sum_{k+j+l=2}^{\infty} B_{jkl} x^k y^j u^l = Y(x,y,u), \\ \dfrac{du}{dt} =& - du + \displaystyle\sum_{k+j+l=2}^{\infty} d_{jkl} x^k y^j u^l = U(x,y,u), \end{cases} \tag{2.1}$$

where $x, y, u, d, t, A_{jkl}, B_{jkl}, d_{jkl}, \delta \in \mathbf{R}\,(k, j, l \in \mathbf{N})$, $\delta \to 0$. Our method is dedicated to giving the expressions of focal values by comparing and analyzing the relation between focal values in real system and singular values in corresponding complex system. For system (2.1), there exists a positive center manifold $u = u(x, y)$, which expression form can be the polynomial series about x and y as follows:

$$u = x^2 + y^2 + h.o.t., \tag{2.2}$$

in which $h.o.t.$ stands for higher-order term about x and y. It is clear that u can be expanded only from the beginning of a square item. However, here we will consider the implicit function formal series about u, x and y.

Under the following transformation

$$z = x + iy,\ w = x - iy,\ u = u,\ T = it,\ i = \sqrt{-1}, \tag{2.3}$$

system $(2.1)|_{\delta=0}$ is changed into the following complex system

$$\begin{cases} \dfrac{dz}{dT} =& z + \displaystyle\sum_{k+j+l=2}^{\infty} a_{jkl} z^k w^j u^l = Z(z,w,u), \\ \dfrac{dw}{dT} =& - w - \displaystyle\sum_{k+j+l=2}^{\infty} b_{jkl} z^k w^j u^l = -W(z,w,u), \\ \dfrac{du}{dT} =& idu + \displaystyle\sum_{k+j+l=2}^{\infty} \tilde{d}_{jkl} z^k w^j u^l = \tilde{U}(z,w,u), \end{cases} \tag{2.4}$$

in which $z, w, T, a_{jkl}, b_{jkl}, \tilde{d}_{jkl} \in \mathbf{C}\,(k, j, l \in \mathbf{N})$. It is clear that the coefficients of (2.4) a_{jkl} and b_{jkl} satisfy conjugate condition, i.e. $\overline{a_{jkl}} = b_{jkl}, j \geq 0, k \geq 0, l \geq$

$0, j+k+l \geq 2$. System (2.1) and (2.4) are called as concomitant systems. We might as well write $\tilde{d}_{jkl}, \tilde{U}$ as d_{jkl}, U.

The expressions of the singular values of the origin of system (2.4) are given by the following lemma in [5, 6].

Lemma 2.1 ([5, 6]). *Let $c_{110} = 1$, $c_{101} = c_{011} = c_{200} = c_{020} = 0$, $c_{kk0} = 0, k = 2, 3, \cdots$, then the terms of the following formal series of system (2.4) can be derived successively and uniquely as follows:*

$$F(z, w, u) = zw + \sum_{\alpha+\beta+\gamma=3}^{\infty} c_{\alpha\beta\gamma} z^{\alpha} w^{\beta} u^{\gamma}, \tag{2.5}$$

such that

$$\frac{dF}{dT} = \frac{\partial F}{\partial z} Z - \frac{\partial F}{\partial w} W + \frac{\partial F}{\partial u} U = \sum_{m=1}^{\infty} \mu_m (zw)^{m+1}, \tag{2.6}$$

and if $\alpha \neq \beta$ or $\alpha = \beta, \gamma \neq 0$, $c_{\alpha\beta\gamma}$ is determined by the following recursive formula:

$$\begin{aligned} c_{\alpha\beta\gamma} = {} & \frac{1}{\beta - \alpha - id\gamma} \\ & \times \sum_{k+j+l=3}^{\alpha+\beta+\gamma+2} [(\alpha - k + 1) a_{k,j-1,l} - (\beta - j + 1) b_{j,k-1,l} + (\gamma - l) d_{k-1,j-1,l+1}] \\ & \times c_{\alpha-k+1, \beta-j+1, \gamma-l}, \end{aligned} \tag{2.7}$$

and for any positive integer m, μ_m is determined by the following recursive formula:

$$\mu_m = \sum_{k+j=3}^{2m+2} [(m - k + 1) a_{k,j-1,0} - (m - j + 1) b_{j,k-1,0}] c_{m-k+1, m-j+1, 0}, \tag{2.8}$$

and when $\alpha < 0$ or $\beta < 0$ or $\gamma < 0$ or $\gamma = 0, \alpha = \beta$, we have let $c_{\alpha,\beta,\gamma} = 0$.

Obtained μ_m of (2.8) is called as mth singular values at the origin of (2.4). Articles [18, 20] gave the relation between mth singular values and mth Lyapunov constants (or called mth focal values) in the following lemma.

Lemma 2.2 ([18, 20]). *For any positive integer m, the mth Lyapunov constants of system $(2.1)|_{\delta=0}$ and the mth singular values of system (2.4) have the following relationship:*

$$v_{2k+1}(2\pi) = i\pi (\mu_m + \sum_{k=1}^{m-1} \xi_m^{(k)} \mu_k), \tag{2.9}$$

in which $v_{2k+1}(2\pi), (k = 1, 2, \cdots, m-1)$ are the kth Lyapunov constants (or focal values) at the origin of (2.1), $\mu_k (k = 1, 2, \cdots, m-1)$ are the kth singular values at the origin of system (2.4), $\xi_m^{(k)}, (k = 1, 2, \cdots, m-1)$ are polynomial functions of coefficients' combination of system (2.4).

Definition 2.1. For system (2.1), if $v_1(2\pi) \neq 1$, then the origin is called a rough focus (or strong focus); if $v_1(2\pi) = 1$, and $v_2(2\pi) = v_3(2\pi) = \cdots = v_{2k}(2\pi) = 0, v_{2k+1}(2\pi) \neq 0$, then the origin is called a fine focus (or weak focus) of order k, at the same time, $v_{2k+1}(2\pi), k = 1, 2, \cdots$ is the kth focal value (or Lyapunov constants) at the origin of system (2.1); if $v_1(2\pi) = 1$, and for any positive integer k, $v_{2k+1}(2\pi) = 0$, then the origin is called a center.

According to Lemma 2.1 and Lemma 2.2, we have

Lemma 2.3. *For the mth focal value (or Lyapunov constants) at the origin of system* $(2.1)|_{\delta=0}$ *and the mth singular values at the origin of system* (2.4)*, the following relation holds:*

$$v_{2m+1}(2\pi) = i\pi\mu_m, \tag{2.10}$$

when $\mu_k = 0, k = 1, 2, \cdots, m-1$.

In order to study the Hopf bifurcation behavior, Ref. [5] gave the following two results.

Lemma 2.4 ([5])**.** *For system* (2.1)*, we have the following conclusions:*
(a) System (2.1) *can bifurcate m limit cycles at most in a small enough neighborhood at the origin of* (2.1) *when the following conditions hold:*

$$\begin{aligned} &v_1(2\pi,\epsilon) - 1 = \lambda_0\epsilon^{l_0+N} + o(\epsilon^{l_0+N+1}),\\ &v_{2k+1}(2\pi,\epsilon) = \lambda_k\epsilon^{l_k+N} + o(\epsilon^{l_k+N+1}),\\ &k = 1, 2, \cdots, 0 < |\epsilon| \ll 1, \end{aligned} \tag{2.11}$$

where $l_0, l_1, \cdots, l_m, m, N$ *are positive integers and* $l_m = 0, \lambda_m \neq 0$.
(b) If conditions in (a) hold, and $\lambda_k\lambda_{k-1} < 0, (k = 1, 2, \cdots, m), l_{k-1} - l_k > l_k - l_{k+1}, (k = 1, 2, \cdots, m-1)$*, then* $\sum\limits_{k=0}^{m} \lambda_k\epsilon^{l_k}h^{2k} = 0$ *has m positive solutions, i.e.,*

$$h_k(\epsilon) = \sqrt{(-\frac{\lambda_{k-1}}{\lambda_k})\epsilon^{l_{k-1}-l_k}} + o(\epsilon^{\frac{l_{k-1}-l_k}{2}}). \tag{2.12}$$

At the same time, system (2.1) *can bifurcate m limit cycles which are near circles* $x^2 + y^2 = (-\frac{\lambda_{k-1}}{\lambda_k})\epsilon^{l_{k-1}-l_k}$.

Lemma 2.5 ([5, 6])**.** *If the origin of unperturbed system* $(2.1)|_{\delta=0}$ *is a fine focus of n−th order, then the origin of disturbed system* (2.1) *can bifurcate n limit cycles under a suitable perturbation.*

Obviously, for 3-Dimensional Hopf bifurcations system (2.1), we can obtain the first m-th singular values by using recursive formula offered by Lemma 2.1, moreover we can judge whether the origin of (2.1) will be an m-th fine focus. Next, according to Lemma 2.5, we can obtain the origin of (2.1) can bifurcate m limit cycles if the origin of (2.1) is an m-th fine focus.

3. Limit cycle bifurcation behavior of system (1.1)

Lemma 2.1-Lemma 2.5 offered a kind of valid method to compute Lyapunov constants and investigate the Hopf bifurcation at elementary focus point for system (2.1) in space vector field, while the type of system (1.1) is different from that of system (2.1). In order to study the limit cycle bifurcation at the infinity of system (1.1) in space vector field, we may as well make some appropriate transformations so as to carry out our further investigation.

3.1. The reduction of system (1.1)

Make the following homeomorphous transformation

$$x = \frac{x_1}{x_1^2 + y_1^2 + u_1^2},\ y = \frac{y_1}{x_1^2 + y_1^2 + u_1^2},\ u = \frac{u_1}{x_1^2 + y_1^2 + u_1^2}, \tag{3.1}$$

and the following time transformation

$$dt = (x_1^2 + y_1^2 + u_1^2)^2 d\tau, \tag{3.2}$$

system (1.1) becomes the following form

$$\begin{cases} \dfrac{dx_1}{dt} = -y_1 + A(x_1^2 + y_1^2 + u_1^2) + Mu_1(x_1 + y_1), \\ \dfrac{dy_1}{dt} = x_1 + B(x_1^2 + y_1^2 + u_1^2) - Mu_1(x_1 + y_1), \\ \dfrac{du_1}{dt} = -u_1 + C(x_1^2 + y_1^2 + u_1^2) + x_1 y_1. \end{cases} \tag{3.3}$$

Clearly transformation (3.1) is an invertible transformation whose inverse transformation is the same one as the original transformation. Under transformation (3.1), the infinity of system (1.1) becomes the origin of (3.3) correspondingly, at the same time, the infinity of system (1.1) and the origin of system (3.3) can bifurcate the same number of limit cycles, while the difference between them is that one is big and the other is small. On the other hand, the computation of Lyapunov constants or focal values plays pivotal role for studying the limit cycle bifurcation of system (3.3) according to method offered by [5,6]. In this sense, the focal values at the origin of system (3.3) can be called general Lyapunov Constants or general focal values at infinity of system (1.1). Next it is necessary to compute the general Lyapunov constants or general focal values of system (1.1).

3.2. The general general Lyapunov constants of system (1.1)

Clearly, system (3.3) belongs to the class of system (2.1). Let

$$z = x_1 + iy_1,\ w = x_1 - iy_1,\ u = u_1,\ T = it,\ i = \sqrt{-1}, \tag{3.4}$$

then system (3.3) becomes

$$\begin{cases} \dfrac{dz}{dT} = z + (B - iA)u^2 - Mu(z + iw) + (B - iA)wz, \\ \dfrac{dw}{dT} = -w - (B + iA)u^2 + Mu(w - iz) - (B + iA)wz, \\ \dfrac{du}{dT} = -iu + \dfrac{1}{4}(-4iCu^2 + w^2 - z^2 - 4iCzw). \end{cases} \tag{3.5}$$

According to Lemma 2.1 and using computer algebra system, the recursive formula of the singular values of the origin of system (3.5) can be obtained as follows:

Theorem 3.1. *If $k \neq j$ or $k = j, l \neq 0$, c_{kjl} is determined by the following recursive formula:*

$$
\begin{aligned}
c_{k,j,l} =& \frac{1}{4(j-k-il)}(-c_{-2+k,j,1+l} - l\, c_{-2+k,j,1+l} - 4C\, c_{-1+k,-1+j,1+l} \\
& - 4Cl\, c_{-1+k,-1+j,1+l} - 4Aj\, c_{-1+k,j,l} - 4Bj\, c_{-1+k,j,l} - 4M\, c_{-1+k,1+j,-1+l} \\
& - 4jM\, c_{-1+k,1+j,-1+l} + c_{k,-2+j,1+l} + lc_{k,-2+j,1+l} - 4Ak\, c_{k,-1+j,l} + 4Bk \\
& \times c_{k,-1+j,l} + 4C\, c_{k,j,-1+l} - 4Cl\, c_{k,j,-1+l} + 4jM\, c_{k,j,-1+l} - 4kM\, c_{k,j,-1+l} \\
& - 4A\, c_{k,1+j,-2+l} - 4B\, c_{k,1+j,-2+l} - 4Ajc_{k,1+j,-2+l} - 4Bj\, c_{k,1+j,-2+l} \\
& - 4Mc_{1+k,-1+j,-1+l} - 4kMc_{1+k,-1+j,-1+l} - 4A\, c_{1+k,j,-2+l} \\
& + 4Bc_{1+k,j,-2+l} - 4Ak\, c_{1+k,j,-2+l} + 4Bk\, c_{1+k,j,-2+l}),
\end{aligned}
\tag{3.6}
$$

and for any positive integer j, the expressions of singular values μ_j is determined by the following recursive formula:

$$
\begin{aligned}
\mu_{j-1} =& \frac{1}{4}(-c_{-2+j,j,1} - 4iC\, c_{-1+j,-1+j,1} - 4Aj\, c_{-1+j,j,0} \\
& - 4Bj\, c_{-1+j,j,0} + c_{j,-2+j,1} - 4Aj\, c_{j,-1+j,0} + 4Bj\, c_{j,-1+j,0}),
\end{aligned}
\tag{3.7}
$$

and when $k < 0$ or $j < 0$ or $l < 0$ or $l = 0, k = j$, we have let $c_{k,j,l} = 0$.

According to the recursive formula of Theorem 3.1, the expressions of the singular values at the origin of system (3.5) can be given, namely the following theorem.

Theorem 3.2. *The simplified expressions of the first five singular values at the origin of system* (3.5) *are as follows:*

$$
\begin{aligned}
\mu_1 &= \frac{i}{5}M, \\
\mu_2 &= \frac{i}{20}[(1+40C^2)(A^2+B^2) - 22C(A^2-B^2) - 28ABC], \\
\mu_3 &= -\frac{i}{136000}h_3, \\
\mu_4 &= \frac{i}{901680000}h_4, \\
\mu_5 &= -\frac{i}{884764483200000}h_5,
\end{aligned}
\tag{3.8}
$$

in which h_3, h_4, h_5 are the polynomial expressions about A, B, C (see Appendix).

Further we have the following theorem from the relation between the singular values and Lyapunov constants or focal values.

Theorem 3.3. *The first five Lyapunov constants (or focal values) at the origin of system* (3.3) *(or the first five general Lyapunov constants at the infinity of system* (1.1)*) are as follows:*

$$
\begin{aligned}
v_3 &= -\frac{\pi}{5}M, \\
v_5 &= -\frac{\pi}{20}[(1+40C^2)(A^2+B^2) - 22C(A^2-B^2) - 28ABC], \\
v_7 &= \frac{\pi}{136000}h_3,
\end{aligned}
\tag{3.9}
$$

$$v_9 = -\frac{\pi}{901680000} h_4,$$
$$v_{11} = \frac{\pi}{884764483200000} h_5,$$

where the expressions of h_3, h_4, h_5 are the same as those of Theorem 3.2.

3.3. Limit cycle bifurcation of system (1.1)

Obviously, it is necessary to consider the number of order that origin of system (3.3) become a fine focus when the limit cycles bifurcation at the origin of system (3.3) is investigated. According to the results of Lemma 2.5 and Theorem 3.3, the following theorem holds.

Theorem 3.4. *The origin of system* (3.3) *can become a fine focus of 5th order if and only if the following condition holds:*

$$M = 0,\ B \neq 0,\ C(2C-1)(20C-1)(20C^2-1) \neq 0,\ v_5 = v_7 = v_9 = 0. \tag{3.10}$$

Proof. The sufficiency is clear, next we prove the necessity. By analyzing the expressions of $v_k, k \in \{3, 5, 7, 9, 11\}$ in Theorem 3.3, we will try to find a group of real numbers about M, A, B, C such that $v_3 = v_5 = v_7 = v_9 = 0$, $v_{11} \neq 0$.

Let $v_3 = 0$, then $M = 0$. Next we only need to prove that condition (3.10) will deduce $v_{11} \neq 0$.

If $v_5 = v_7 = 0$, then the resultant of v_5 and v_7 with regard to A is zero. Let the resultant of v_5 and v_7 with regard to A be r_1. With the help of computer, we can obtain

$$r_1 = Resultant[v_5, v_7, A] = \frac{-B^4}{924800000000} r_{11}, \tag{3.11}$$

in which r_{11} is the expression on B and C.

Similarly, $v_5 = v_9 = 0$ will deduce the resultant of v_5 and v_9 with regard to A is zero, and $v_5 = v_{11} = 0$ will deduce the resultant of v_5 and v_{11} with regard to A is zero. We also obtain

$$\begin{aligned} r_2 &= Resultant[v_5, v_9, A] = \frac{B^4 r_{22}}{1000648396800000000000000}, \\ r_3 &= Resultant[v_5, v_{11}, A] = \frac{-B^4 r_{33}}{2408640586868194099200000000000000000}, \end{aligned} \tag{3.12}$$

in which r_{22}, r_{33} are the expressions on B and C.

Obviously, B is the common factor of r_1, r_2 and r_3. Supposed that $B = 0$, then $v_3 = v_5 = v_7 = v_9 = v_{11} = 0$ have real number solutions, which will deduce that the origin of system (3.3) can't become a fine focus of 5th order. Hence, if the origin of system (3.3) is a fine focus of 5th order, then $B \neq 0$.

Moreover, if equation groups $v_3 = v_5 = v_7 = v_9 = v_{11} = 0$ have real number solutions, then $r_{11} = r_{22} = r_{33} = 0$. Let r_{12} be the resultant of r_{11} and r_{22} with regard to B, let r_{13} be the resultant of r_{11} and r_{33} with regard to B, then $r_{12} = r_{13} = 0$ will deduce $v_3 = v_5 = v_7 = v_9 = v_{11} = 0$. Hence, we will find the condition such that $r_{12} = 0$ and $r_{13} = 0$ can't hold at the same time. With help of computer, we have

$$\begin{aligned} r_{12} =& Resultant[r_{11}, r_{22}, B] \\ =& 2386420683693101056 \times 10^{12} C^8 \\ & \times (2C-1)^8 (20C-1)^8 (20C^2-1)^2 n_1 n_2, \\ r_{13} =& Resultant[r_{11}, r_{33}, B] \\ =& 12457265120170718331136 \times 10^{16} C^8 \\ & \times (2C-1)^{12} (20C-1)^{12} (20C^2-1)^2 n_3 n_4, \end{aligned} \tag{3.13}$$

in which n_1, n_2, n_3, n_4 are the expressions on C.

From the expressions of r_{12} and r_{13}, $v_3 = v_5 = v_7 = v_9 = 0, v_{11} \neq 0$ will deduce $C(2C-1)(20C-1)(20C^2-1) \neq 0$, hence $v_3 = v_5 = v_7 = v_9 = 0, v_{11} \neq 0$ will deduce condition (3.10) holds. But condition (3.10) is a necessary but not sufficient condition. If let $v_3 = v_5 = v_7 = v_9 = 0, v_{11} \neq 0$, we need to continue our proof that equations group $n_1 n_2 = 0$ and $n_3 n_4 = 0$ have not real number solutions. By making use of computer algebra system Mathematica, it is easy to obtain the resultant of n_1 and n_3 on C isn't zero and the resultant of n_1 and n_4 on C isn't zero and the resultant of n_2 and n_3 on C isn't zero and the resultant of n_2 and n_4 on C isn't zero, so equations group $n_1 n_2 = 0$ and $n_3 n_4 = 0$ have not real number solutions. Hence, the necessity of theorem 3.4 holds.

From the above analysis, condition (3.10) is the sufficient and necessary condition that the origin of system (3.3) can become a fine focus of 5th order. □

In fact, if condition (3.10) holds, we can find 256 groups of real number solutions about M, A, B, C such that $v_3 = v_5 = v_7 = v_9 = 0$ and $v_{11} \neq 0$, for example

$$(1) M = 0,\ A \approx 5.9578224001,\ B \approx -7.0987862676,\ C \approx -0.37308169555; \tag{3.14}$$

$$(2) M = 0,\ A \approx 7.0987862676,\ B \approx -0.9389406545,\ C \approx 0.373081695551; \tag{3.15}$$

$$(3) M = 0,\ A \approx 0.0138858765,\ B \approx -0.0198146188,\ C \approx -0.05403563645. \tag{3.16}$$

Solutions (3.14) will let $v_3 = v_5 = v_7 = v_9 = 0$ and $v_{11} \approx 5.196790 \times 10^9 \pi \neq 0$, Solutions (3.15) will let $v_3 = v_5 = v_7 = v_9 = 0$ and $v_{11} \approx -118025.057754751\pi \neq 0$, Solutions (3.16) will let $v_3 = v_5 = v_7 = v_9 = 0$ and $v_{11} \approx 1.4277903078282437110\pi \times 10^{-10} \neq 0$.

We can deduce the following theorem according to Theorem 3.3 and Theorem 3.4.

Theorem 3.5. *Suppose that the origin of system* (3.3) *is a fine focus of 5th order, then under a certain parameters' perturbed condition, the origin of system* (3.3) *can bifurcate 5 small limit cycles in which 3 limit cycles can be stable cycles; Accordingly, system* (1.1) *can bifurcate 5 large limit cycles, in which 3 limit cycles can be stable cycles.*

Proof. If condition (3.10) holds, then the origin of unperturbed system $(3.3)|_{\delta=0}$ is a fine focus of 5th order. By computing, everyone can easily obtain the Jacobian of the functions group (v_3, v_5, v_7, v_9) with respect to the variables group (M, A, B, C)

as follows:

$$J=\begin{vmatrix} \frac{\partial v_3}{\partial M} & \frac{\partial v_3}{\partial A} & \frac{\partial v_3}{\partial B} & \frac{\partial v_3}{\partial C} \\ \frac{\partial v_5}{\partial M} & \frac{\partial v_5}{\partial A} & \frac{\partial v_5}{\partial B} & \frac{\partial v_5}{\partial C} \\ \frac{\partial v_7}{\partial M} & \frac{\partial v_7}{\partial A} & \frac{\partial v_7}{\partial B} & \frac{\partial v_7}{\partial C} \\ \frac{\partial v_9}{\partial M} & \frac{\partial v_9}{\partial A} & \frac{\partial v_9}{\partial B} & \frac{\partial v_9}{\partial C} \end{vmatrix} = \frac{\pi^4}{383214000000000} m_1, \tag{3.17}$$

in which m_1 is higher-order polynomial expression about A, B, C. By making use of method on resultant, we can obtain $m_1 \neq 0$ when condition (3.10) holds, hence $J \neq 0$ under condition (3.10). Because $J \neq 0$, then equation groups $v_3 = v_5 = v_7 = v_9 = 0$ have a set of real solutions such that $v_{11} \neq 0$. May as well let $M = 0, A = a, B = b, C = c$ be one of those group of solutions, namely $v_3|_{M=0,A=a,B=b,C=c} = v_5|_{M=0,A=a,B=b,C=c} = v_7|_{M=0,A=a,B=b,C=c} = v_9|_{M=0,A=a,B=b,C=c} = 0$, at the same time $v_{11}|_{M=0,A=a,B=b,C=c} \neq 0$.

Let perturbations about these parameters be as follows:

$$v_3(2\pi,\varepsilon) = \varepsilon_1,\ v_5(2\pi,\varepsilon) = \varepsilon_2,\ v_7(2\pi,\varepsilon) = \varepsilon_3,\ v_9(2\pi,\varepsilon) = \varepsilon_4, \tag{3.18}$$

in which ε_1, ε_2, ε_3, ε_4 are a group of arbitrary real numbers. Because $J \neq 0$, then according to existence theorem of implicit function, equation (3.18) has a group of solutions as follows:

$$\begin{aligned} M &= a_1(\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4), \\ A &= a + a_2(\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4), \\ B &= b + a_3(\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4), \\ C &= c + a_4(\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4). \end{aligned} \tag{3.19}$$

Obviously, given perturbations by (3.19) will let (3.18) hold.

In particular, we can let

$$\epsilon_1 = c_1\pi\epsilon^8 + o(\epsilon^9),\ \epsilon_2 = c_2\pi\epsilon^6 + o(\epsilon^7),\ \epsilon_3 = c_3\pi\epsilon^4 + o(\epsilon^5),\ \epsilon_4 = c_4\pi\epsilon^2 + o(\epsilon^3), \tag{3.20}$$

in which

$$\begin{aligned} c_1 =& 21076 j_0,\ c_2 = -7645 j_0,\ c_3 = 1023 j_0,\ c_4 = -55 j_0, \\ j_0 =& v_{11}|_{\epsilon=0,M=0,A=a,B=b,C=c} \neq 0. \end{aligned} \tag{3.21}$$

When the expressions of (3.20) hold, the Lyapunov constants or focal values at the origin of perturbed system (3.3) (or the general Lyapunov constants at infinity of perturbed system (1.1)) are follows:

$$\begin{aligned} v_1(2\pi,\epsilon) &= 1 + c_0\pi\epsilon^{10} + o(\epsilon^{11}), \\ v_3(2\pi,\epsilon) &= c_1\pi\epsilon^8 + o(\epsilon^9), \\ v_5(2\pi,\epsilon) &= c_2\pi\epsilon^6 + o(\epsilon^7), \\ v_7(2\pi,\epsilon) &= c_3\pi\epsilon^4 + o(\epsilon^5), \\ v_9(2\pi,\epsilon) &= c_4\pi\epsilon^2 + o(\epsilon^3), \\ v_{11}(2\pi,\epsilon) &= v_{11}|_{\epsilon=0} + o(\epsilon), \end{aligned} \tag{3.22}$$

then the Poincaré succession function for the origin of system (3.3) is as follows:

$$\begin{aligned} &d(\epsilon h)\\ =&r(2\pi,\epsilon h)-\epsilon h\\ =&(v_1(2\pi,\epsilon)-1)\epsilon h+v_2(2\pi,\epsilon)(\epsilon h)^2+v_3(2\pi,\epsilon)(\epsilon h)^3+\ldots+v_{11}(2\pi,\epsilon)(\epsilon h)^{11}+\ldots\\ =&\pi\epsilon^{11}h[g(h)+\epsilon hG(h,\epsilon)], \end{aligned} \tag{3.23}$$

in which

$$\begin{aligned} g(h)&=c_0+c_1h^2+c_2h^4+c_3h^6+c_4h^8+j_0h^{10}\\ &=-14400j_0+21076j_0h^2-7645j_0h^4+1023j_0h^6-55j_0h^8+j_0h^{10}\\ &=j_0(h^2-1)(h^2-4)(h^2-9)(h^2-16)(h^2-25), \end{aligned} \tag{3.24}$$

and $G(h,\epsilon)$ is an analytic function about h,ϵ at $(0,0)$.

Obviously, $g(h)=0$ has 5 simple positive roots $1,2,3,4,5$. From implicit function theorem, the number of positive zero points of equation $d(\epsilon h)=0$ is equal to one of $g(h)=0$, and these positive roots are close to $1,2,3,4,5$ when $0<|\epsilon|\ll 1$. The above analysis shows there exists 5 small limit cycles in a small enough neighborhood at the origin of system (3.3), which are close to spheres $x_1^2+y_1^2+u_1^2=k^2\epsilon^2, k=1,2,3,4,5$ under the center manifold $u_1=x_1^2+y_1^2$. At the same time, when $v_{11}<0$, there exists 3 stable limit cycles, which are near to spheres $x_1^2+y_1^2+u_1^2=k^2\epsilon^2$, $k=1,3,5$. Correspondingly, system (1.1) has 5 large limit cycles which are close to spheres $x^2+y^2+u^2=\frac{1}{k^2\epsilon^2}$, $(k\in\{1,2,3,4,5\})$. At the same time, when $v_{11}<0$, there exists 3 stable limit cycles, which are close to spheres $x^2+y^2+u^2=\frac{1}{k^2\epsilon^2}$, $k=1,3,5$. □

4. Discussion on relation between the infinity and the origin in space vector field

Through the above investigation, we know the limit cycle bifurcation at infinity for some special classes of Hopf bifurcation systems in space vector field can also be studied by some reductions. The following homeomorphous transformation can often be used to carry out this kind of reduction:

$$x=\frac{x_1}{x_1^2+y_1^2+u_1^2},\ y=\frac{y_1}{x_1^2+y_1^2+u_1^2},\ u=\frac{u_1}{x_1^2+y_1^2+u_1^2}. \tag{4.1}$$

Transformation (4.1) is invertible, namely (4.1) can deduce the following expressions:

$$x_1=\frac{x}{x^2+y^2+u^2},\ y_1=\frac{y}{x^2+y^2+u^2},\ u_1=\frac{u}{x^2+y^2+u^2}. \tag{4.2}$$

Transformation (4.2) is an inverse transformation of transformation (4.1). The origin and infinity can be transformed into each other by transformation (4.1) or (4.2). The limit cycle bifurcation at the origin of system (2.1) can be studied by using method offered in [5,6], then the limit cycle bifurcation at the infinity of the following two types of systems can also be investigated under transformation (4.1) or (4.2):

$$\begin{cases} \dfrac{dx}{dt} = -y(x^2+y^2+u^2)^n + \sum_{k=0}^{2n} X_k(x,y,u), \\ \dfrac{dy}{dt} = x(x^2+y^2+u^2)^n + \sum_{k=0}^{2n} Y_k(x,y,u), \\ \dfrac{du}{dt} = -du(x^2+y^2+u^2)^n + \sum_{k=0}^{2n} U_k(x,y,u), \end{cases} \tag{4.3}$$

$$\begin{cases} \dfrac{dx}{dt} = -y + \dfrac{1}{(x^2+y^2+u^2)^n} \sum_{k=0}^{2n} X_k(x,y,u), \\ \dfrac{dy}{dt} = x + \dfrac{1}{(x^2+y^2+u^2)^n} \sum_{k=0}^{2n} Y_k(x,y,u), \\ \dfrac{du}{dt} = -du + \dfrac{1}{(x^2+y^2+u^2)^n} \sum_{k=0}^{2n} U_k(x,y,u), \end{cases} \tag{4.4}$$

in which $X_k(x,y,u), Y_k(x,y,u), U_k(x,y,u)$ are homogeneous polynomials of degree k about x,y,u.

The work of this paper is a case of system (4.3), one can further research the limit cycle bifurcation of the above two types of systems. Of course, it is possible to carry out the study on the limit cycles bifurcations behavior by using the polar translation on a center manifold and form series, but it's very difficult to calculate and simply.

5. Conclusion

The work of this paper is concerned that the limit cycle bifurcation problem at infinity for a class of polynomial systems in 3-Dimensional vector field. By making two appropriate transformations and making use of singular values methods on center manifolds to compute and simply the general focal values carefully, we give the expressions of the first five focal values at the infinity and prove the conditions of the fifth fine focuses. Moreover, we obtain the infinity can bifurcate 5 large limit cycles. Similar published results are hardly seen, and the result of the number of large limit cycles is new.

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Appendix: Some expressions of Theorem 3.2

$$
\begin{aligned}
h_3 =& - 187A^2 + 51680A^4 - 259760A^3B - 187B^2 - 65280A^2B^2 + 259760AB^3 \\
&+ 51680B^4 + 10510A^2C - 1683680A^4C + 7580ABC - 1621120A^3BC \\
&- 10510B^2C - 1621120AB^3C + 1683680B^4C - 77520A^2C^2 \\
&+ 3209600A^4C^2 - 77520B^2C^2 + 6419200A^2B^2C^2 + 3209600B^4C^2 \\
&+ 432480A^2C^3 + 886720ABC^3 - 432480B^2C^3 - 1088000A^2C^4 \\
&- 1088000B^2C^4,
\end{aligned}
$$

$$
\begin{aligned}
h_4 =& - 15775898A^4 + 4362310160A^6 + 217260000A^3B - 30640889760A^5B \\
&- 15911796A^2B^2 - 5230398160A^4B^2 - 217260000AB^3 - 15775898B^4 \\
&- 5230398160A^2B^4 + 30640889760AB^5 + 4362310160B^6 - 1108111A^2C \\
&+ 1406664380A^4C - 161766872800A^6C - 367914ABC - 1076865880A^3BC \\
&- 143401419200A^5BC + 1108111B^2C - 161766872800A^4B^2C \\
&- 1076865880AB^3C - 286802838400A^3B^3C - 1406664380B^4C \\
&+ 161766872800A^2B^4C - 143401419200AB^5C + 161766872800B^6C \\
&+ 25558106A^2C^2 - 12000865760A^4C^2 + 312173638400A^6C^2 \\
&+ 24278741760A^3BC^2 + 25558106B^2C^2 + 27683414720A^2B^2C^2 \\
&+ 936520915200A^4B^2C^2 - 24278741760AB^3C^2 - 12000865760B^4C^2 \\
&+ 936520915200A^2B^4C^2 + 312173638400B^6C^2 - 341018080A^2C^3 \\
&+ 69536359360A^4C^3 - 462346560ABC^3 + 61525834240A^3BC^3 \\
&+ 341018080B^2C^3 + 61525834240AB^3C^3 - 69536359360B^4C^3 \\
&+ 2093099840A^2C^4 - 158046470400A^4C^4 + 2093099840B^2C^4 \\
&- 316092940800A^2B^2C^4 158046470400B^4C^4 - 6802273920A^2C^5 \\
&- 21654746880ABC^5 + 6802273920B^2C^5 + 23443680000A^2C^6 \\
&+ 23443680000B^2C^6,
\end{aligned}
$$

$$
\begin{aligned}
h_5 =& - 110889692173A^4 - 373141720120692A^6 + 88496136247748320A^8 \\
&- 7253242183980A^3B + 6502218215673120A^5B \\
&- 635345463339038400A^7B + 2973108945534A^2B^2 \\
&- 2657177111703996A^4B^2 - 18513035685464320A^6B^2 \\
&+ 7253242183980AB^3 - 635345463339038400A^5B^3 - 110889692173B^4 \\
&- 2657177111703996A^2B^4 - 214018343866425280A^4B^4 \\
&- 6502218215673120AB^5 + 635345463339038400A^3B^5 \\
&- 373141720120692B^6 - 18513035685464320A^2B^6 \\
&+ 635345463339038400AB^7 + 88496136247748320B^8 \\
&- 47350488984820A^4C + 33337750591518680A^6C \\
&- 3315720387265965120A^8C + 185812739677160A^3BC
\end{aligned}
$$

$$
\begin{aligned}
&- 56943417583497040A^5BC - 2918803512596974080A^7BC \\
&+ 96913384227019160A^4B^2C - 6631440774531930240A^6B^2C \\
&+ 185812739677160AB^3C + 66448864126332640A^3B^3C \\
&- 8756410537790922240A^5B^3C + 47350488984820B^4C \\
&- 96913384227019160A^2B^4C - 56943417583497040AB^5C \\
&- 8756410537790922240A^3B^5C - 33337750591518680B^6C \\
&+ 6631440774531930240A^2B^6C - 2918803512596974080AB^7C \\
&+ 3315720387265965120B^8C - 336376660698A^2C^2 \\
&+ 1041839913831948A^4C^2 - 315930627289605120A^6C^2 \\
&+ 6405271038832224000A^8C^2 - 4845342818866560A^3BC^2 \\
&+ 843334328795736960A^5BC^2 - 336376660698B^2C^2 \\
&- 4205109004063944A^2B^2C^2 + 644348602204519680A^4B^2C^2 \\
&+ 25621084155328896000A^6B^2C^2 + 4845342818866560AB^3C^2 \\
&+ 1041839913831948B^4C^2 + 644348602204519680A^2B^4C^2 \\
&+ 38431626232993344000A^4B^4C^2 - 843334328795736960AB^5C^2 \\
&- 315930627289605120B^6C^2 + 25621084155328896000A^2B^6C^2 \\
&+ 6405271038832224000B^8C^2 + 11705272825236A^2C^3 \\
&- 13628945568181440A^4C^3 + 1818283730869544640A^6C^3 \\
&+ 11028910851144ABC^3 + 13306344842870400A^3BC^3 \\
&+ 849365191701087360A^5BC^3 - 11705272825236B^2C^3 \\
&+ 1818283730869544640A^4B^2C^3 + 13306344842870400AB^3C^3 \\
&+ 1698730383402174720A^3B^3C^3 + 13628945568181440B^4C^3 \\
&- 1818283730869544640A^2B^4C^3 + 849365191701087360AB^5C^3 \\
&- 1818283730869544640B^6C^3 - 150968879889888A^2C^4 \\
&+ 93972550512896640A^4C^4 - 3816454012316774400A^6C^4 \\
&- 93336802526423040A^3BC^4 - 150968879889888B^2C^4 \\
&- 231599489269774080A^2B^2C^4 - 11449362036950323200A^4B^2C^4 \\
&+ 93336802526423040AB^3C^4 93972550512896640B^4C^4 \\
&- 11449362036950323200A^2B^4C^4 - 3816454012316774400B^6C^4 \\
&+ 1122601944409920A^2C^5 - 329186435517189120A^4C^5 \\
&+ 2480798680064640ABC^5 - 352779138724346880A^3BC^5 \\
&- 1122601944409920B^2C^5 - 352779138724346880AB^3C^5 \\
&+ 329186435517189120B^4C^5 - 6936435579690240A^2C^6 \\
&+ 843178193117644800A^4C^6 - 6936435579690240B^2C^6 \\
&+ 1686356386235289600A^2B^2C^6 + 843178193117644800B^4C^6 \\
&+ 14275851889328640A^2C^7 + 70969436538024960ABC^7
\end{aligned}
$$

$$- 142758518893286 40B^2C^7 - 70781158656000000A^2C^8$$
$$- 70781158656000000B^2C^8.$$

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