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GREEN FUNCTION OF A CLASS OF EIGENPARAMETER DEPENDENT THIRD-ORDER DIFFERENTIAL OPERATORS WITH DISCONTINUITY

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Abstract This paper is concerned with a class of third-order boundary value problems with discontinuity, and the eigenparameter is contained in two of boundary conditions, the transmission conditions are imposed on the discontinuous point. Using operator theoretic formulation, we transfer the considered problem to a new operator T in a modified Hilbert space $\mathcal{H}$. It is proved that T is a self-adjoint operator in $\mathcal{H}$, and we introduce some properties of the spectrum. The Green function and the resolvent operator are obtained. The completeness of eigenfunctions is also proved.

Keywords Third-order differential operators, eigenparameter dependent boundary conditions, transmission conditions, Green function, completeness.

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1. Introduction

In this paper, we consider the following third-order boundary value transmission problems (BVTP)

$$\ell y := \frac{1}{w}\{-i[q_0(q_0y')']' - (p_0y')' + i[q_1y' + (q_1y)'] + p_1y\} = \lambda y,\ x \in I, \tag{1.1}$$

with boundary conditions

$$\mathcal{L}_1 y := (\alpha_1\lambda + \widetilde{\alpha}_1)y(a) - (\alpha_2\lambda + \widetilde{\alpha}_2)y^{[2]}(a) = 0, \tag{1.2}$$

$$\mathcal{L}_2 y := (\beta_1\lambda + \widetilde{\beta}_1)y(b) + (\beta_2\lambda + \widetilde{\beta}_2)y^{[2]}(b) = 0, \tag{1.3}$$

$$\mathcal{L}_3 y := (i + \sin\theta)y^{[1]}(a) + \sqrt{r_1r_2}(1 + i\sin\theta)y^{[1]}(b) = 0, \tag{1.4}$$

and transmission conditions

$$\mathcal{T}_1 y := y(c-) - r_1y(c+) = 0, \tag{1.5}$$

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$$\mathcal{T}_2 y := y^{[1]}(c-) - \sqrt{r_1 r_2}\, y^{[1]}(c+) = 0, \tag{1.6}$$

$$\mathcal{T}_3 y := y^{[2]}(c-) - r_2 y^{[2]}(c+) = 0, \tag{1.7}$$

where $I = [a, c) \cup (c, b]$, $\lambda \in \mathbb{C}$ is the spectral parameter, q_0, q_1, p_0, p_1, w are real-valued continuous functions on each interval $[a, c)$ and $(c, b]$ and satisfy the following conditions

$$q_0^{-1}, q_0^{-2}, p_0, q_1, p_1, w \in L^1(I, \mathbb{R}),\ q_0 > 0,\ w > 0. \tag{1.8}$$

$\theta, \alpha_j,\ \widetilde{\alpha}_j,\ \beta_j,\ \widetilde{\beta}_j,\ r_j\ (j = 1, 2)$ are arbitrary real numbers with

$$\rho_1 = \widetilde{\alpha}_1 \alpha_2 - \alpha_1 \widetilde{\alpha}_2 > 0,\ \rho_2 = \widetilde{\beta}_1 \beta_2 - \beta_1 \widetilde{\beta}_2 > 0,\ r_1 r_2 > 0. \tag{1.9}$$

It is well known that the spectral theory of ordinary differential operators is an important research topic in the differential equation boundary value problems. The research field of differential operators is also very plentiful, such as self-adjoint expansion, the properties of eigenvalues and eigenfunctions, inverse problems, etc. Such problems are well understood for even order differential operators, especially for the so called Sturm-Liouville problems, see for example [14,19,21,22] and references cited therein. However, little is known for odd order differential operators.

Third-order differential equations arise in many physical phenomenons, for example, three-layer beam problem, more backgrounds we refer to [7]. Recently, Hao, Zhang etc in [8] characterized the self-adjoint domain of odd order differential operators by using real parameter solutions, and Niu etc.gave all canonical forms of self-adjoint boundary conditions for regular third-order differential operators [15]. Uğurlu also studied a class of third-order self-adjoint boundary value problems and introduced the dependence of eigenvalues on the problem [17], and generalized these results to boundary value problems with discontinuity [18]. In a very recent paper, Li etc in [10] considered a regular third order differential operators with mixed and eigenparameter dependent boundary conditions and investigated self-adjointness, the properties of eigenvalues, Green function of the operator.

However, there is still lots of work which need to be done for third-order differential operators, namely, differential operators with transmission conditions and eigenparameter dependent boundary conditions which arise in many mathematical physical phenomenon, for example, in electrostatics and magnetostatics, the model problem which describes the heat transfer through an infinitely conductive layer is a transmission problem [5,6,16]. For such problems of second-order Sturm-Liouville operators or Dirac operators, we refer to [1–4, 6, 11–13, 23].

In this paper, we consider third-order boundary value transmission conditions (1.1)-(1.7). By using the classical analysis techniques and spectral theory of linear operator, we transfer the BVTP (1.1)-(1.7) to a self-adjoint operator T in an appropriate Hilbert space $\mathcal{H}$ such that the eigenvalues of the problem (1.1)-(1.7) coincide with those of T. This paper is organized as follows: In Section 2, we investigate some basic notations and preliminaries. In Section 3, we introduce a new Hilbert space and construct an operator T associated with the problem (1.1)-(1.7), the self-adjointness, the properties of eigenvalues of this operator are discussed. The Green function and the resolvent operator are discussed in Section 4. The completeness of eigenfunctions is proved in Section 5.

2. Notations and preliminaries

Let the quasi-derivatives of y be defined as [9]

$$y^{[0]} = y,\ y^{[1]} = -\frac{1+i}{\sqrt{2}} q_0 y',\ y^{[2]} = iq_0(q_0 y')' + p_0 y' - iq_1 y,$$

and $H_w = L^2_w(I) = L^2_w[a,c) \oplus L^2_w(c,b]$ be a weighted Hilbert space consisting of functions y which satisfy $\int_a^c |y_1|^2 w dx + r_1 r_2 \int_c^b |y_2|^2 w dx < \infty$ equipped with the inner product $\langle y, z\rangle_w = \int_a^b y_1 \overline{z}_1 w dx + r_1 r_2 \int_a^b y_2 \overline{z}_2 w dx$. Where $y(x),\ z(x) \in H_w$ and

$$y(x) = \begin{cases} y_1(x),\ x \in [a,c), \\ y_2(x),\ x \in (c,b], \end{cases} \qquad z(x) = \begin{cases} z_1(x),\ x \in [a,c), \\ z_2(x),\ x \in (c,b], \end{cases}$$
$$w(x) = \begin{cases} w_1(x),\ x \in [a,c), \\ w_2(x),\ x \in (c,b]. \end{cases}$$

Denote by $L_{\max}$ the maximal operator with the domain

$$\begin{aligned} D_{\max} = &\{y \in L^2_w(I) \mid y^{[0]},\ y^{[1]},\ y^{[2]} \in AC(I), \\ & y(c\pm), y^{[1]}(c\pm), y^{[2]}(c\pm) < \infty,\ \ell y \in H_w\}, \end{aligned}$$

and the rule

$$L_{\max} y = \ell y,\ y \in D_{\max},\ x \in I.$$

Then for arbitrary $y, z \in D_{\max}$, integration by parts yields Lagrange identity

$$\langle L_{\max} y, z\rangle_w - \langle y, L_{\max} z\rangle_w = [y, \overline{z}]_a^{c-} + r_1 r_2 [y, \overline{z}]_{c+}^b,$$

where

$$\begin{aligned} &[y, \overline{z}]_{\sigma_1}^{\sigma_2} = [y, \overline{z}](\sigma_2) - [y, \overline{z}](\sigma_1), \\ &[y, \overline{z}](x) = y(x)\overline{z^{[2]}(x)} - y^{[2]}(x)\overline{z(x)} + iy^{[1]}(x)\overline{z^{[1]}(x)}. \end{aligned}$$

We can transfer the equation (1.1) to the following first-order system

$$\mathrm{Y}' + \mathrm{QY} = \lambda \mathrm{WY}, \tag{2.1}$$

where

$$\mathrm{Y} = \begin{pmatrix} y^{[0]} \\ y^{[1]} \\ y^{[2]} \end{pmatrix},\ \mathrm{W} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -w & 0 & 0 \end{pmatrix},\ \mathrm{Q} = \begin{pmatrix} 0 & \frac{\sqrt{2}}{(1+i)q_0} & 0 \\ \frac{(1+i)q_1}{\sqrt{2}q_0} & -\frac{ip_0}{q_0^2} & \frac{\sqrt{2}}{(1+i)q_0} \\ -p_1 & \frac{(1+i)q_1}{\sqrt{2}q_0} & 0 \end{pmatrix}. \tag{2.2}$$

Then the following result holds.

Theorem 2.1. [4] *There exists a unique solution for the equation* (1.1) *with initial conditions* $y^{[k]}(d,\lambda) = d_k(\lambda)$, *where* $d \in I,\ k = 0,1,2,\ d_k(\lambda)$ *are arbitrary complex numbers. Moreover,* $y^{[k]}(x,\lambda)$ *are entire functions of* λ.

3. Operator theoretic formulation and self-adjointness

Using the methods of Mukhtarov [4, 12], one can construct a new Hilbert space $\mathcal{H} = H_w \oplus \mathbb{C}^2$ under a suitable inner product by combining the parameters in the boundary and transmission conditions. To this end, the inner product is defined by

$$\langle Y, Z\rangle = \int_a^c y\overline{z}wdx + r_1r_2\int_c^b y\overline{z}wdx + \frac{1}{\rho_1}y_1\overline{z_1} + \frac{r_1r_2}{\rho_2}y_2\overline{z_2}, \tag{3.1}$$

where $Y = (y(x), y_1, y_2)^T, Z = (z(x), z_1, z_2)^T \in \mathcal{H}$.

We shall use the following notations:

$$\begin{aligned} &\mathcal{M}_1(y) = \alpha_1 y(a) - \alpha_2 y^{[2]}(a),\ \mathcal{M}_2(y) = \beta_1 y(b) + \beta_2 y^{[2]}(b),\\ &\mathcal{N}_1(y) = \widetilde{\alpha}_2 y^{[2]}(a) - \widetilde{\alpha}_1 y(a),\ \mathcal{N}_2(y) = -[\widetilde{\beta}_1 y(b) + \widetilde{\beta}_2 y^{[2]}(b)]. \end{aligned} \tag{3.2}$$

Define the operator T in the Hilbert space $\mathcal{H}$ with domain

$$\begin{aligned} D(\mathrm{T}) =\{&Y = (y(x),\ y_1,\ y_2)^T \in \mathcal{H} \mid \mathcal{L}_3 y = \mathcal{T}_1 y = \mathcal{T}_2 y = \mathcal{T}_3 y = 0,\\ &y_1 = \mathcal{M}_1(y),\ y_2 = \mathcal{M}_2(y),\ y \in D_{\max}\}, \end{aligned} \tag{3.3}$$

and

$$Y = (y(x), \mathcal{M}_1(y), \mathcal{M}_2(y))^T \in D(\mathrm{T}),\ \mathrm{T}Y = (\ell y, \mathcal{N}_1(y), \mathcal{N}_2(y))^T. \tag{3.4}$$

Then we get that the eigenvalue problem of BVTP (1.1)-(1.7) is transferred to the spectra problem of the operator T. And it is easily seen that the following results hold.

Lemma 3.1. *BVTP problem* (1.1)-(1.7) *has the same eigenvalues with the operator* T, *and the eigenfunctions of BVTP problem* (1.1)-(1.7) *are the first component of the corresponding eigenvectors of the operator* T.

Proof. For arbitrary $Y = (y(x), y_1, y_2)^T \in D(\mathrm{T})$, by (3.2) and (3.4), we have

$$\mathrm{T}Y = (\ell y, \mathcal{N}_1(y), \mathcal{N}_2(y))^T = \lambda(y(x), \mathcal{M}_1(y), \mathcal{M}_2(y))^T.$$

Comparing this with BVTP problem (1.1)-(1.7) yields the conclusions. □

Lemma 3.2. *The domain* $D(\mathrm{T})$ *is dense in* $\mathcal{H}$.

Proof. Let $F = (f(x), f_1, f_2) \in \mathcal{H},\ F \perp D(\mathrm{T})$, and

$$C_0^\infty(I) = \left\{ \psi(x) = \begin{cases} \psi_1(x),\ x \in [a,c); \\ \psi_2(x),\ x \in (c,b]. \end{cases} \middle| \psi_1(x) \in C_0^\infty[a,c),\ \psi_2(x) \in C_0^\infty(c,b] \right\}.$$

Since $C_0^\infty(I)\oplus 0 \oplus 0 \subset D(\mathrm{T})$ ($0 \in \mathbb{C}$), for arbitrary $G = \{g(x), 0, 0\} \in C_0^\infty(I)\oplus 0\oplus 0$, we have

$$\langle F, G\rangle = \int_a^c f\overline{g}wdx + r_1r_2\int_c^b f\overline{g}wdx = 0.$$

In light of $C_0^\infty(I)$ is dense in H_w, $f(x) = 0$, that is, $F = (0, f_1, f_2)$. For any $U = (u(x), u_1, 0) \in D(\mathrm{T})$, we have

$$\langle F, U\rangle = \frac{1}{\rho_1} f_1\overline{u_1} = 0$$

by the inner product in $\mathcal{H}$. Through the arbitrariness of u_1, then $f_1 = 0$. Moreover, for all $V = (v(x), v_1, v_2) \in D(\mathrm{T})$, we have

$$\langle F, V\rangle = \frac{r_1 r_2}{\rho_2} f_2 \overline{v_2} = 0.$$

By arbitrariness of v_2 and $r_1 r_2 > 0$, we have $f_2 = 0$. Hence $F = (0, 0, 0)$, and the proof is completed. □

Lemma 3.3. *The operator* T *is symmetric.*

Proof. For any $U, V \in D(\mathrm{T})$, integration by parts yields

$$\begin{aligned}\langle \mathrm{T}U, V\rangle - \langle U, \mathrm{T}V\rangle =& [u, \overline{v}]_a^{c-} + r_1 r_2 [u, \overline{v}]_{c+}^b + \frac{1}{\rho_1}[\mathcal{N}_1(u)\overline{\mathcal{M}_1(v)} - \mathcal{M}_1(u)\overline{\mathcal{N}_1(v)}] \\ &+ \frac{r_1 r_2}{\rho_2}[\mathcal{N}_2(u)\overline{\mathcal{M}_2(v)} - \mathcal{M}_2(u)\overline{\mathcal{N}_2(v)}].\end{aligned} \tag{3.5}$$

By boundary and transmission conditions (1.2)-(1.7), we have

$$u^{[2]}(a)\overline{v(a)} - u(a)\overline{v^{[2]}(a)} = \frac{1}{\rho_1}[\mathcal{M}_1(u)\overline{\mathcal{N}_1(v)} - \mathcal{N}_1(u)\overline{\mathcal{M}_1(v)}], \tag{3.6}$$

$$u^{[2]}(b)\overline{v(b)} - u(b)\overline{v^{[2]}(b)} = \frac{1}{\rho_2}[\mathcal{N}_2(u)\overline{\mathcal{M}_2(v)} - \mathcal{M}_2(u)\overline{\mathcal{N}_2(v)}], \tag{3.7}$$

$$u^{[1]}(a)\overline{v^{[1]}(a)} = r_1 r_2 u^{[1]}(b)\overline{v^{[1]}(b)}, \tag{3.8}$$

$$u(c-)\overline{v^{[2]}(c-)} = r_1 r_2 u(c+)\overline{v^{[2]}(c+)}, \tag{3.9}$$

$$u^{[1]}(c-)\overline{v^{[1]}(c-)} = r_1 r_2 u^{[1]}(c+)\overline{v^{[1]}(c+)}, \tag{3.10}$$

$$u^{[2]}(c-)\overline{v(c-)} = r_1 r_2 u^{[2]}(c+)\overline{v(c+)}. \tag{3.11}$$

Inserting (3.6)-(3.11) into (3.5), we have

$$\langle \mathrm{T}U, V\rangle - \langle U, \mathrm{T}V\rangle = 0.$$

Therefore, the operator T is symmetric. □

Theorem 3.4. *The operator* T *is a selfadjoint operator in* $\mathcal{H}$.

Proof. Since T is symmetric, it suffices to prove that for any $U = (u(x), u_1, u_2) \in D(\mathrm{T})$ and some $V \in D(\mathrm{T}^*)$, $Z \in \mathcal{H}$ satisfying $\langle \mathrm{T}U,\ V\rangle = \langle U,\ Z\rangle$, then $V \in D(\mathrm{T})$ and $\mathrm{T}V = Z$, where $V = (v(x), v_1, v_2)$, $Z = (z(x), z_1, z_2)$, i.e.,

(i) $v^{[j]}(x) \in AC(I), j = 0, 1, 2$, and $\ell v \in H_w$;

(ii) $v_1 = \alpha_1 v(a) - \alpha_2 v^{[2]}(a)$, $v_2 = \beta_1 v(b) + \beta_2 v^{[2]}(b)$;

(iii) $\mathcal{L}_3 v = \mathcal{T}_1 v = \mathcal{T}_2 v = \mathcal{T}_3 v = 0$;

(iv) $z(x) = \ell v$;

(v) $z_1 = \widetilde{\alpha}_2 v^{[2]}(a) - \widetilde{\alpha}_1 v(a)$, $z_2 = -[\widetilde{\beta}_1 v(b) + \widetilde{\beta}_2 v^{[2]}(b)]$.

For any $U = \{u(x), 0, 0\} \in \mathbb{C}_0^\infty(I) \oplus 0 \oplus 0 \in D(\mathrm{T})$, by $\langle \mathrm{T}U,\ V\rangle = \langle U,\ Z\rangle$ we have

$$\int_a^c (\ell u)\overline{v} w dx + r_1 r_2 \int_c^b (\ell u)\overline{v} w dx = \int_a^c u\overline{z} w dx + r_1 r_2 \int_c^b u\overline{z} w dx,$$

that is, $\langle \ell u,\ v\rangle_w = \langle u,\ z\rangle_w$. By the classical differential operator theory, we have (i) and (iv) hold. By (3.1), (3.4) and (iv) we get that for all $U = (u(x), u_1, u_2) \in D(\mathrm{T})$,

$\langle \mathrm{T}U, V\rangle = \langle U, Z\rangle$ turns to

$$\langle \ell u,\ v\rangle_w - \langle u,\ \ell v\rangle_w = \frac{1}{\rho_1}[M_1(u)\overline{z_1} - N_1(u)\overline{v_1}] + \frac{r_1 r_2}{\rho_2}[M_2(u)\overline{z_2} - N_2(u)\overline{v_2}].$$

In light of

$$\langle \ell u,\ v\rangle_w = \langle u,\ \ell v\rangle_w + [u, \bar{v}]_a^{c-} + r_1 r_2 [u, \bar{v}]_{c+}^b,$$

hence

$$\frac{1}{\rho_1}[\mathcal{M}_1(u)\overline{z_1} - \mathcal{N}_1(u)\overline{v_1}] + \frac{r_1 r_2}{\rho_2}[\mathcal{M}_2(u)\overline{z_2} - \mathcal{N}_2(u)\overline{v_2}] = [u, \bar{v}]_a^{c-} + r_1 r_2 [u, \bar{v}]_{c+}^b. \quad (3.12)$$

Using Naimark Patching Lemma, there exists a $\widehat{U} = (\widehat{u}(x), \widehat{u}_1, \widehat{u}_2) \in D(\mathrm{T})$ such that

$$\widehat{u}^{[n]}(b) = \widehat{u}^{[n]}(c-) = \widehat{u}^{[n]}(c+) = 0,\ n = 0, 1, 2,$$
$$\widehat{u}(a) = \alpha_2,\ \widehat{u}^{[1]}(a) = 0,\ \widehat{u}^{[2]}(a) = \alpha_1.$$

Substituting this into (3.12) yields $v_1 = \alpha_1 v(a) - \alpha_2 v^{[2]}(a)$. Similarly, there exists a $\widetilde{U} = (\widetilde{u}(x), \widetilde{u}_1, \widetilde{u}_2) \in D(\mathrm{T})$ such that

$$\widetilde{u}^{[n]}(a) = \widetilde{u}^{[n]}(c-) = \widetilde{u}^{[n]}(c+) = 0,\ n = 0, 1, 2,$$
$$\widetilde{u}(b) = \beta_2,\ \widetilde{u}^{[1]}(b) = 0,\ \widetilde{u}^{[2]}(b) = -\beta_1.$$

Then by (3.12), we have $v_2 = \beta_1 v(b) + \beta_2 v^{[2]}(b)$. Therefore, (ii) holds. Using similar methods, one can prove (v) is true.

Choosing $U^* = (u^*(x), u_1^*, u_2^*) \in D(\mathrm{T})$ such that

$$u^*(a) = u^{*[2]}(a) = u^*(b) = u^{*[2]}(b) = u^{*[n]}(c-) = u^{*[n]}(c+) = 0,\ n = 0, 1, 2,$$
$$u^{*[1]}(a) = i - \sin\theta,\ u^{*[1]}(b) = (1 - i\sin\theta)/\sqrt{r_1 r_2}.$$

Then by (3.12), we have $\mathcal{L}_3 v = 0$. Similarly, there exists a $\ddot{U} = (\ddot{u}(x), \ddot{u}_1, \ddot{u}_2) \in D(\mathrm{T})$ such that

$$\ddot{u}(c-) = \ddot{u}^{[1]}(c-) = \ddot{u}(c+) = \ddot{u}^{[1]}(c+) = \ddot{u}^{[n]}(a) = \ddot{u}^{[n]}(b) = 0,\ n = 0, 1, 2,$$
$$\ddot{u}^{[2]}(c-) = r_2,\ \ddot{u}^{[2]}(c+) = 1.$$

Substituting this into (3.12) yields $\mathcal{T}_1 v = 0$. Using similar methods, we can obtain $\mathcal{T}_2 v = 0$ and $\mathcal{T}_3 v = 0$. Therefore, (iii) holds. Hence, the operator T is selfadjoint. □

By self-adjointness of the operator T, we have the following conclusions.

Corollary 3.5. *The eigenvalues of* T *are real-valued.*

Corollary 3.6. *Let λ_1 and λ_2 be two different eigenvalues of* T, *$Y_1 = (y_1(x), y_{11}, y_{12})$ and $Y_2 = (y_2(x), y_{21}, y_{22})$ be the corresponding eigenfunctions respectively, then $y_1(x)$ and $y_2(x)$ are orthogonal in the sense of*

$$\int_a^c y_1\overline{y_2}w dx + r_1 r_2 \int_c^b y_1\overline{y_2}w dx + \frac{1}{\rho_1} M_1(y_1)\overline{M_1(y_2)} + \frac{r_1 r_2}{\rho_2} M_2(y_1)\overline{M_2(y_2)} = 0.$$

Let

$$A_\lambda = \begin{pmatrix} \alpha_1\lambda + \widetilde{\alpha}_1 & 0 & -(\alpha_2\lambda + \widetilde{\alpha}_2) \\ 0 & 0 & 0 \\ 0 & i + \sin\theta & 0 \end{pmatrix},$$

$$B_\lambda = \begin{pmatrix} 0 & 0 & 0 \\ \beta_1\lambda + \widetilde{\beta}_1 & 0 & \beta_2\lambda + \widetilde{\beta}_2 \\ 0 & 1 + i\sin\theta & 0 \end{pmatrix},$$

$$C = \begin{pmatrix} r_1 & 0 & 0 \\ 0 & \sqrt{r_1 r_2} & 0 \\ 0 & 0 & r_2 \end{pmatrix}.$$

Then the boundary and transmission conditions of (1.1)-(1.7) can be rewritten in the following matrix forms

$$A_\lambda \mathrm{Y}(a) + B_\lambda \mathrm{Y}(b) = 0, \tag{3.13}$$

and

$$\mathrm{Y}(c-) - C\mathrm{Y}(c+) = 0, \tag{3.14}$$

where $\mathrm{Y}(x) = (y(x), y^{[1]}(x), y^{[2]}(x))^T$.

Let $\psi_{11}(x,\lambda), \psi_{12}(x,\lambda), \psi_{13}(x,\lambda)$ be the system of linearly independent fundamental solutions of equation (1.1) on interval $[a,c)$, and

$$\Psi_1(x,\lambda) = (\Psi_{11}(x,\lambda),\ \Psi_{12}(x,\lambda),\ \Psi_{13}(x,\lambda)) \tag{3.15}$$

satisfy initial conditions

$$\Psi_1(a,\lambda) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{3.16}$$

where

$$\Psi_{1k}(x,\lambda) = \Big(\psi_{1k}(x,\lambda),\ \psi_{1k}^{[1]}(x,\lambda),\ \psi_{1k}^{[2]}(x,\lambda)\Big)^T,\ k = 1,2,3.$$

Then their Wronskian $w_1(\lambda)$ is independent of x and is an entire function of λ, and we have

$$w_1(\lambda) = \det \Psi_1(x,\lambda) = \det \Psi_1(a,\lambda) = 1. \tag{3.17}$$

Let $\psi_{21}(x,\lambda), \psi_{22}(x,\lambda), \psi_{23}(x,\lambda)$ be the solutions of equation (1.1) on interval $(c,b]$, and satisfying initial conditions

$$\begin{aligned} \Psi_2(c+,\lambda) &= (\Psi_{21}(c+,\lambda),\ \Psi_{22}(c+,\lambda),\ \Psi_{23}(c+,\lambda)) \\ &= C^{-1}(\Psi_{11}(c-,\lambda),\ \Psi_{12}(c-,\lambda),\ \Psi_{13}(c-,\lambda)) \\ &= C^{-1}\Psi_1(c-,\lambda), \end{aligned} \tag{3.18}$$

where

$$\Psi_{2k}(x,\lambda) = \Big(\psi_{2k}(x,\lambda),\ \psi_{2k}^{[1]}(x,\lambda),\ \psi_{2k}^{[2]}(x,\lambda)\Big)^T,\ k = 1,2,3.$$

Thus, their Wronskian $w_2(\lambda)$ is independent of x and is an entire function of λ, and we have

$$w_2(\lambda) = \det \Psi_2(c+,\lambda) = \det C_\lambda^{-1} \det \Psi_1(c-,\lambda) = \frac{1}{r_1 r_2 \sqrt{r_1 r_2}}, \tag{3.19}$$

therefore, $\psi_{21}(x,\lambda), \psi_{22}(x,\lambda), \psi_{23}(x,\lambda)$ are linearly independent on interval $(c,b]$ and

$$\psi_1(x,\lambda) = \begin{cases} \psi_{11}(x),\ x \in [a,c), \\ \psi_{21}(x),\ x \in (c,b], \end{cases} \qquad \psi_2(x,\lambda) = \begin{cases} \psi_{12}(x),\ x \in [a,c), \\ \psi_{22}(x),\ x \in (c,b], \end{cases}$$

$$\psi_3(x,\lambda) = \begin{cases} \psi_{13}(x),\ x \in [a,c), \\ \psi_{23}(x),\ x \in (c,b], \end{cases}$$

are the solutions of equation (1.1) on interval I satisfying transmission conditions (1.5)-(1.7).

We assume that

$$\Psi(x,\lambda) = \begin{cases} \Psi_1(x,\lambda), & x \in [a,c), \\ \Psi_2(x,\lambda), & x \in (c,b], \end{cases}$$

then

$$\Psi(c-,\lambda) = \Psi_1(c-,\lambda),\ \Psi(c+,\lambda) = \Psi_2(c+,\lambda),$$

and for any $x \in I$, $\Psi(x,\lambda)$ is an entire function of λ.

Lemma 3.7. *Let*

$$y(x) = \begin{cases} y_1(x), x \in [a,c), \\ y_2(x), x \in (c,b], \end{cases}$$

be an arbitrary solution of equation (1.1)*, then $y(x)$ can be expressed by*

$$y(x) = \begin{cases} c_{11}\psi_{11}(x) + c_{12}\psi_{12}(x) + c_{13}\psi_{13}(x), x \in [a,c), \\ c_{21}\psi_{21}(x) + c_{22}\psi_{22}(x) + c_{23}\psi_{23}(x), x \in (c,b], \end{cases}$$

where $c_{jk} \in \mathbb{C}$ $(j = 1,2,\ k = 1,2,3)$. If the solution $y(x)$ satisfy the transmission conditions (1.5)-(1.7)*, then $c_{1k} = c_{2k} (k = 1,2,3)$.*

Proof. (i) The first conclusion is clearly true.

(ii) Assume

$$y(x) = \begin{cases} c_{11}\psi_{11}(x) + c_{12}\psi_{12}(x) + c_{13}\psi_{13}(x), x \in [a,c), \\ c_{21}\psi_{21}(x) + c_{22}\psi_{22}(x) + c_{23}\psi_{23}(x), x \in (c,b], \end{cases}$$

is a solution of equation (1.1) satisfying transmission conditions (1.5)-(1.7).

Inserting $y(x)$ into the transmission conditions matrix form (3.14) yields

$$\begin{pmatrix} \sum_{k=1}^{3} c_{1k}\psi_{1k}(c-) \\ \sum_{k=1}^{3} c_{1k}\psi_{1k}^{[1]}(c-) \\ \sum_{k=1}^{3} c_{1k}\psi_{1k}^{[2]}(c-) \end{pmatrix} - C \begin{pmatrix} \sum_{k=1}^{3} c_{2k}\psi_{2k}(c+) \\ \sum_{k=1}^{3} c_{2k}\psi_{2k}^{[1]}(c+) \\ \sum_{k=1}^{3} c_{2k}\psi_{2k}^{[2]}(c+) \end{pmatrix} = 0.$$

The above equation can be expressed as

$$\Psi_1(c-,\lambda)(c_{11},c_{12},c_{13})^T = C\Psi_2(c+,\lambda)(c_{21},c_{22},c_{23})^T,$$

and by (3.18), we have

$$\Psi_1(c-,\lambda)(c_{11},c_{12},c_{13})^T = \Psi_1(c-,\lambda)(c_{21},c_{22},c_{23})^T,$$

namely,

$$\Psi_1(c-,\lambda)(c_{11}-c_{21},c_{12}-c_{22},c_{13}-c_{23})^T = 0, \tag{3.20}$$

by virtue of

$$\det(\Psi_1(\mathrm{c}-,\lambda)) = 1,$$

hence, the system (3.20) have only zero solution, so $c_{11}=c_{21},\ c_{12}=c_{22},\ c_{13}=c_{23}$. □

Lemma 3.8. *A complex number λ is an eigenvalue of the problem* (1.1)-(1.7) *if and only if*

$$\Delta(\lambda) = \det[A_\lambda + B_\lambda\Psi(b,\lambda)] = 0.$$

Proof. Suppose λ_0 is an eigenvalue of the problem (1.1)-(1.7) and $y_0(x)$ is the corresponding eigenfunction. Then we have

$$y_0(x) = \begin{cases} c_{11}\psi_{11}(x,\lambda_0) + c_{12}\psi_{12}(x,\lambda_0) + c_{13}\psi_{13}(x,\lambda_0), & x\in[a,c), \\ c_{11}\psi_{21}(x,\lambda_0) + c_{12}\psi_{22}(x,\lambda_0) + c_{13}\psi_{23}(x,\lambda_0), & x\in[a,c), \end{cases}$$

where at least one of coefficients c_{1k} $(k=1,2,3)$ is not zero. Inserting $y_0(x)$ into boundary condition (3.13) yields

$$A_\lambda \begin{pmatrix} \sum_{k=1}^{3} c_{1k}\psi_{1k}(a,\lambda_0) \\ \sum_{k=1}^{3} c_{1k}\psi_{1k}^{[1]}(a,\lambda_0) \\ \sum_{k=1}^{3} c_{1k}\psi_{1k}^{[2]}(a,\lambda_0) \end{pmatrix} + B_\lambda \begin{pmatrix} \sum_{k=1}^{3} c_{1k}\psi_{2k}(b,\lambda_0) \\ \sum_{k=1}^{3} c_{1k}\psi_{2k}^{[1]}(b,\lambda_0) \\ \sum_{k=1}^{3} c_{1k}\psi_{2k}^{[2]}(b,\lambda_0) \end{pmatrix} = 0,$$

that is

$$\big(A_\lambda\Psi_1(a,\lambda_0) + B_\lambda\Psi_2(b,\lambda_0)\big)(c_{11},c_{12},c_{13})^T = 0,$$

from (3.16), we can see

$$(A_\lambda + B_\lambda\Psi(b,\lambda_0))(c_{11},c_{12},c_{13})^T = 0. \tag{3.21}$$

Due to c_{11},c_{12},c_{13} are not all zero, hence $\det(A_\lambda + B_\lambda\Psi(b,\lambda_0)) = 0$.

On the contrary, if $\det(A_\lambda + B_\lambda \Psi(b,\lambda_0)) = 0$, then the system of the linear equations (3.21) for the constants c_{1k} $(k = 1, 2, 3)$ has non-zero solution $(\widetilde{c}_{11},\ \widetilde{c}_{12},\ \widetilde{c}_{13})^T$. Let

$$y(x) = \begin{cases} \widetilde{c}_{11}\psi_{11}(x,\lambda_0) + \widetilde{c}_{12}\psi_{12}(x,\lambda_0) + \widetilde{c}_{13}\psi_{13}(x,\lambda_0),\ x \in [a,c), \\ \widetilde{c}_{21}\psi_{21}(x,\lambda_0) + \widetilde{c}_{22}\psi_{22}(x,\lambda_0) + \widetilde{c}_{23}\psi_{23}(x,\lambda_0),\ x \in (c,b], \end{cases}$$

then $y(x)$ is the non-trivial solution of equation (1.1) satisfying the conditions (1.2)-(1.7), which implies λ_0 is an eigenvalue of the problem (1.1)-(1.7). □

Theorem 3.9. *The eigenvalues of* T *are discrete and have no finite point of accumulation. Moreover, the multiplicity of each eigenvalue at most* 3.

Proof. By Lemma 3.8 and the self-adjointness of T, we know that the zeros of $\Delta(\lambda)$ are the eigenvalues of operator T, and all the eigenvalues of T are real, that is to say, for any $\lambda \in \mathbb{C}$ with its imaginary part not vanishing, then $\Delta(\lambda) \neq 0$. Therefore, by the distribution of zeros of entire functions, the first part holds. The second conclusion follows from the fact that there at most 3 linearly independent solutions exist for the equation (1.1). □

4. Green's function

In this section, we discuss the Green's function of BVTP (1.1)-(1.7) when λ is not an eigenvalue of the problem (1.1)-(1.7). To this end, consider the following operator equation

$$(\mathrm{T} - \lambda I)Y = F,\ F = (f(x), f_1, f_2) \in \mathcal{H}. \tag{4.1}$$

By the definition of the operator T, the equation (4.1) can be transferred to the following inhomogeneous boundary value problems

$$-i[q_0(q_0y')']' - (p_0y')' + i[q_1y' + (q_1y)'] + (p_1 - \lambda w)y = fw, \tag{4.2}$$

$$L_1y := (\alpha_1\lambda + \widetilde{\alpha}_1)y(a) - (\alpha_2\lambda + \widetilde{\alpha}_2)y^{[2]}(a) = -f_1, \tag{4.3}$$

$$L_2y := (\beta_1\lambda + \widetilde{\beta}_1)y(b) + (\beta_2\lambda + \widetilde{\beta}_2)y^{[2]}(b) = -f_2, \tag{4.4}$$

$$L_3y := (\sin\theta + i)y^{[1]}(a) + (i\sin\theta + 1)y^{[1]}(b) = 0, \tag{4.5}$$

$$\mathcal{T}_1y := y(c-) - r_1y(c+) = 0, \tag{4.6}$$

$$\mathcal{T}_2y := y^{[1]}(c-) - \sqrt{r_1r_2}y^{[1]}(c+) = 0, \tag{4.7}$$

$$\mathcal{T}_3y := y^{[2]}(c-) - r_2y^{[2]}(c+) = 0. \tag{4.8}$$

The general solution of homogeneous equation (1.1) can be represented as follows

$$y(x,\lambda) = \begin{cases} d_{11}\psi_{11}(x,\lambda) + d_{12}\psi_{12}(x,\lambda) + d_{13}\psi_{13}(x,\lambda),\ x \in [a,c), \\ d_{21}\psi_{21}(x,\lambda) + d_{22}\psi_{22}(x,\lambda) + d_{23}\psi_{23}(x,\lambda),\ x \in (c,b], \end{cases} \tag{4.9}$$

where d_{jk} $(j = 1, 2,\ k = 1, 2, 3)$ are the arbitrary constants. From the method of variation of constant, we can get the general solution of the non-homogeneous differential equation (4.2) as

$$y(x,\lambda) = \begin{cases} d_{11}(x,\lambda)\psi_{11}(x,\lambda) + d_{12}(x,\lambda)\psi_{12}(x,\lambda) \\ +d_{13}(x,\lambda)\psi_{13}(x,\lambda),\ x \in [a,c), \\ d_{21}(x,\lambda)\psi_{21}(x,\lambda) + d_{22}(x,\lambda)\psi_{22}(x,\lambda) \\ +d_{23}(x,\lambda)\psi_{23}(x,\lambda),\ x \in (c,b]. \end{cases} \tag{4.10}$$

When $x \in [a, c)$, functions $d_{1k}(x, \lambda)$ $(k = 1, 2, 3)$ satisfy the following conditions

$$\begin{cases} d'_{11}(x,\lambda)\psi_{11}(x,\lambda) + d'_{12}(x,\lambda)\psi_{12}(x,\lambda) + d'_{13}(x,\lambda)\psi_{13}(x,\lambda) = 0, \\ d'_{11}(x,\lambda)\psi^{[1]}_{11}(x,\lambda) + d'_{12}(x,\lambda)\psi^{[1]}_{12}(x,\lambda) + d'_{13}(x,\lambda)\psi^{[1]}_{13}(x,\lambda) = 0, \\ d'_{11}(x,\lambda)\psi^{[2]}_{11}(x,\lambda) + d'_{12}(x,\lambda)\psi^{[2]}_{12}(x,\lambda) + d'_{13}(x,\lambda)\psi^{[2]}_{13}(x,\lambda) = f(x)w(x). \end{cases} \tag{4.11}$$

When $x \in (c, b]$, functions $d_{2k}(x, \lambda)$ $(k = 1, 2, 3)$ satisfy the following conditions

$$\begin{cases} d'_{21}(x,\lambda)\psi_{21}(x,\lambda) + d'_{22}(x,\lambda)\psi_{22}(x,\lambda) + d'_{23}(x,\lambda)\psi_{23}(x,\lambda) = 0, \\ d'_{21}(x,\lambda)\psi^{[1]}_{21}(x,\lambda) + d'_{22}(x,\lambda)\psi^{[1]}_{22}(x,\lambda) + d'_{23}(x,\lambda)\psi^{[1]}_{23}(x,\lambda) = 0, \\ d'_{21}(x,\lambda)\psi^{[2]}_{21}(x,\lambda) + d'_{22}(x,\lambda)\psi^{[2]}_{22}(x,\lambda) + d'_{23}(x,\lambda)\psi^{[2]}_{23}(x,\lambda) = f(x)w(x). \end{cases} \tag{4.12}$$

Since λ is not an eigenvalue, both linear system (4.11) and (4.12) has a unique solution, and hence

$$\begin{aligned} \omega_1(\lambda) &= W(\psi_{11}(x,\lambda), \psi_{12}(x,\lambda), \psi_{13}(x,\lambda)) \\ &= \begin{vmatrix} \psi_{11}(x,\lambda) & \psi_{12}(x,\lambda) & \psi_{13}(x,\lambda) \\ \psi^{[1]}_{11}(x,\lambda) & \psi^{[1]}_{12}(x,\lambda) & \psi^{[1]}_{13}(x,\lambda) \\ \psi^{[2]}_{11}(x,\lambda) & \psi^{[2]}_{12}(x,\lambda) & \psi^{[2]}_{13}(x,\lambda) \end{vmatrix} \\ &= 1, \\ \omega_2(\lambda) &= W(\psi_{21}(x,\lambda), \psi_{22}(x,\lambda), \psi_{23}(x,\lambda)) \\ &= \begin{vmatrix} \psi_{21}(x,\lambda) & \psi_{22}(x,\lambda) & \psi_{23}(x,\lambda) \\ \psi^{[1]}_{21}(x,\lambda) & \psi^{[1]}_{22}(x,\lambda) & \psi^{[1]}_{23}(x,\lambda) \\ \psi^{[2]}_{21}(x,\lambda) & \psi^{[2]}_{22}(x,\lambda) & \psi^{[2]}_{23}(x,\lambda) \end{vmatrix} \\ &= \frac{1}{r_1 r_2 \sqrt{r_1 r_2}}, \end{aligned}$$

and when $x \in [a, c)$ yields

$$\begin{aligned} d_{11}(x,\lambda) &= \int_a^x f(t)w(t)\nabla_{11}(t,\lambda)dt + d^*_{11}, \\ d_{12}(x,\lambda) &= \int_a^x f(t)w(t)\nabla_{12}(t,\lambda)dt + d^*_{12}, \\ d_{13}(x,\lambda) &= \int_a^x f(t)w(t)\nabla_{13}(t,\lambda)dt + d^*_{13}, \end{aligned} \tag{4.13}$$

where $d^*_{11}, d^*_{12}, d^*_{13}$ are arbitrary constants and

$$\nabla_{11} = \begin{vmatrix} \psi_{12} & \psi_{13} \\ \psi^{[1]}_{12} & \psi^{[1]}_{13} \end{vmatrix}, \ \nabla_{12} = \begin{vmatrix} \psi_{13} & \psi_{11} \\ \psi^{[1]}_{13} & \psi^{[1]}_{11} \end{vmatrix}, \ \nabla_{13} = \begin{vmatrix} \psi_{11} & \psi_{12} \\ \psi^{[1]}_{11} & \psi^{[1]}_{12} \end{vmatrix}.$$

When $x \in (c,b]$, we can obtain

$$\begin{aligned} d_{21}(x,\lambda) &= r_1 r_2 \sqrt{r_1 r_2} \int_c^x f(t)w(t)\nabla_{21}(t,\lambda)dt + d_{21}^*, \\ d_{22}(x,\lambda) &= r_1 r_2 \sqrt{r_1 r_2} \int_c^x f(t)w(t)\nabla_{22}(t,\lambda)dt + d_{22}^*, \\ d_{23}(x,\lambda) &= r_1 r_2 \sqrt{r_1 r_2} \int_c^x f(t)w(t)\nabla_{23}(t,\lambda)dt + d_{23}^*, \end{aligned} \tag{4.14}$$

where $d_{21}^*, d_{22}^*, d_{23}^*$ are arbitrary constants and

$$\nabla_{21} = \begin{vmatrix} \psi_{22} & \psi_{23} \\ \psi_{22}^{[1]} & \psi_{23}^{[1]} \end{vmatrix}, \ \nabla_{22} = \begin{vmatrix} \psi_{23} & \psi_{21} \\ \psi_{23}^{[1]} & \psi_{21}^{[1]} \end{vmatrix}, \ \nabla_{23} = \begin{vmatrix} \psi_{21} & \psi_{22} \\ \psi_{21}^{[1]} & \psi_{22}^{[1]} \end{vmatrix}.$$

Inserting $d_{jk}(x,\lambda)(j=1,2,\ k=1,2,3)$ into (4.10), one gets that the general solution of (4.2) has the following representation

$$\begin{aligned} y_1(x,\lambda) =& \psi_{11}(x,\lambda) \int_a^x f(t)w(t)\nabla_{11}(t,\lambda)dt \\ &+ \psi_{12}(x,\lambda) \int_a^x f(t)w(t)\nabla_{12}(t,\lambda)dt \\ &+ \psi_{13}(x,\lambda) \int_a^x f(t)w(t)\nabla_{13}(t,\lambda)dt \\ &+ d_{11}^*\psi_{11}(x,\lambda) + d_{12}^*\psi_{12}(x,\lambda) + d_{13}^*\psi_{13}(x,\lambda),\ x \in [a,c), \end{aligned} \tag{4.15}$$

$$\begin{aligned} y_2(x,\lambda) =& r_1 r_2 \sqrt{r_1 r_2}\psi_{21}(x,\lambda) \int_c^x f(t)w(t)\nabla_{21}(t,\lambda)dt \\ &+ r_1 r_2 \sqrt{r_1 r_2}\psi_{22}(x,\lambda) \int_c^x f(t)w(t)\nabla_{22}(t,\lambda)dt \\ &+ r_1 r_2 \sqrt{r_1 r_2}\psi_{23}(x,\lambda) \int_c^x f(t)w(t)\nabla_{23}(t,\lambda)dt \\ &+ d_{21}^*\psi_{21}(x,\lambda) + d_{22}^*\psi_{22}(x,\lambda) + d_{23}^*\psi_{23}(x,\lambda),\ x \in (c,b]. \end{aligned} \tag{4.16}$$

Inserting (4.15),(4.16) into transmission condition (3.14), one gets that

$$C\Psi_2(c+,\lambda)(d_{21}^*, d_{22}^*, d_{23}^*)^T = \Upsilon(c-,\lambda), \tag{4.17}$$

from (3.18), we can get

$$\Psi_1(c-,\lambda)(d_{21}^*, d_{22}^*, d_{23}^*)^T = \Upsilon(c-,\lambda),$$

thus,

$$\begin{aligned} d_{21}^* &= \det\big(\Upsilon(\mathrm{c}-,\lambda), \Psi_{12}(\mathrm{c}-,\lambda), \Psi_{13}(\mathrm{c}-,\lambda)\big), \\ d_{22}^* &= \det\big(\Psi_{11}(\mathrm{c}-,\lambda), \Upsilon(\mathrm{c}-,\lambda), \Psi_{13}(\mathrm{c}-,\lambda)\big), \\ d_{23}^* &= \det\big(\Psi_{11}(\mathrm{c}-,\lambda), \Psi_{12}(\mathrm{c}-,\lambda), \Upsilon(\mathrm{c}-,\lambda)\big). \end{aligned}$$

At the same time, we have

$$C\Psi_2(c+,\lambda)\begin{pmatrix} d_{21}^* \\ d_{22}^* \\ d_{23}^* \end{pmatrix} = \Psi_1(c-,\lambda)(\begin{pmatrix} \int_a^c f(t)w(t)\nabla_{11}(t,\lambda)dt \\ \int_a^c f(t)w(t)\nabla_{12}(t,\lambda)dt \\ \int_a^c f(t)w(t)\nabla_{13}(t,\lambda)dt \end{pmatrix} + \begin{pmatrix} d_{11}^* \\ d_{12}^* \\ d_{13}^* \end{pmatrix}),$$

by (3.18) we can get

$$\begin{pmatrix} d_{21}^* \\ d_{22}^* \\ d_{23}^* \end{pmatrix} = \begin{pmatrix} \int_a^c f(t)w(t)\nabla_{11}(t,\lambda)dt \\ \int_a^c f(t)w(t)\nabla_{12}(t,\lambda)dt \\ \int_a^c f(t)w(t)\nabla_{13}(t,\lambda)dt \end{pmatrix} + \begin{pmatrix} d_{11}^* \\ d_{12}^* \\ d_{13}^* \end{pmatrix}. \tag{4.18}$$

Substituting (4.18) into (4.16) we obtain

$$\begin{aligned} y_2(x,\lambda) =& r_1r_2\sqrt{r_1r_2}\psi_{21}(x,\lambda)\int_c^x f(t)w(t)\nabla_{21}(t,\lambda)dt \\ &+ r_1r_2\sqrt{r_1r_2}\psi_{22}(x,\lambda)\int_c^x f(t)w(t)\nabla_{22}(t,\lambda)dt \\ &+ r_1r_2\sqrt{r_1r_2}\psi_{23}(x,\lambda)\int_c^x f(t)w(t)\nabla_{23}(t,\lambda)dt \\ &+ \psi_{21}(x,\lambda)\int_a^c f(t)w(t)\nabla_{11}(t,\lambda)dt \\ &+ \psi_{22}(x,\lambda)\int_a^c f(t)w(t)\nabla_{12}(t,\lambda)dt \\ &+ \psi_{23}(x,\lambda)\int_a^c f(t)w(t)\nabla_{13}(t,\lambda)dt \\ &+ d_{11}^*\psi_{21}(x,\lambda) + d_{12}^*\psi_{22}(x,\lambda) + d_{13}^*\psi_{23}(x,\lambda),\ x \in (c,b]. \end{aligned} \tag{4.19}$$

The equation (4.15) and (4.16) can be represented as follows

$$\begin{aligned} y_1(x,\lambda) =& \int_a^b K_1(x,t,\lambda)f(t)w(t)dt + d_{11}^*\psi_{11}(x,\lambda) + d_{12}^*\psi_{12}(x,\lambda) \\ &+ d_{13}^*\psi_{13}(x,\lambda), \qquad x \in [a,c), \end{aligned} \tag{4.20}$$

$$\begin{aligned} y_2(x,\lambda) =& \int_a^b K_2(x,t,\lambda)f(t)w(t)dt + d_{11}^*\psi_{21}(x,\lambda) + d_{12}^*\psi_{22}(x,\lambda) \\ &+ d_{13}^*\psi_{23}(x,\lambda), \qquad x \in (c,b], \end{aligned} \tag{4.21}$$

where

$$K_1(x,t,\lambda) = \begin{cases} S_1(x,t,\lambda), & a \le t \le x < c, \\ 0, & a \le x \le t < c, \\ 0, & a \le x < c, c < t \le b, \end{cases}$$

$$K_2(x,t,\lambda) = \begin{cases} S_2(x,t,\lambda), & a \le t < c, c < x \le b, \\ (r_1r_2)^{\frac{3}{2}}S_3(x,t,\lambda), & c < t \le x \le b, \\ 0, & a \le x \le t \le c, \end{cases}$$

$$S_1(x,t,\lambda) = \begin{vmatrix} \psi_{11}(t,\lambda) & \psi_{12}(t,\lambda) & \psi_{13}(t,\lambda) \\ \psi_{11}^{[1]}(t,\lambda) & \psi_{12}^{[1]}(t,\lambda) & \psi_{13}^{[1]}(t,\lambda) \\ \psi_{11}(x,\lambda) & \psi_{12}(x,\lambda) & \psi_{13}(x,\lambda) \end{vmatrix},$$

$$S_2(x,t,\lambda) = \begin{vmatrix} \psi_{11}(t,\lambda) & \psi_{12}(t,\lambda) & \psi_{13}(t,\lambda) \\ \psi_{11}^{[1]}(t,\lambda) & \psi_{12}^{[1]}(t,\lambda) & \psi_{13}^{[1]}(t,\lambda) \\ \psi_{21}(x,\lambda) & \psi_{22}(x,\lambda) & \psi_{23}(x,\lambda) \end{vmatrix},$$

$$S_3(x,t,\lambda) = \begin{vmatrix} \psi_{21}(t,\lambda) & \psi_{22}(t,\lambda) & \psi_{23}(t,\lambda) \\ \psi_{21}^{[1]}(t,\lambda) & \psi_{22}^{[1]}(t,\lambda) & \psi_{23}^{[1]}(t,\lambda) \\ \psi_{21}(x,\lambda) & \psi_{22}(x,\lambda) & \psi_{23}(x,\lambda) \end{vmatrix}.$$

Obviously,

$$y(x,\lambda) = \begin{cases} y_1(x,\lambda), & x \in [a,c), \\ y_2(x,\lambda), & x \in (c,b], \end{cases} \tag{4.22}$$

is the solution of inhomogeneous equation (4.2) satisfying transmission conditions (1.5)-(1.7). Let (4.22) be expressed as follows

$$\begin{aligned} y(x,\lambda) = & \int_a^b K(x,t,\lambda) f(t) w(t) dt + d_{11}^* \psi_1(x,\lambda) + d_{12}^* \psi_2(x,\lambda) \\ & + d_{13}^* \psi_3(x,\lambda), \qquad x \in I, \end{aligned}$$

where

$$K(x,t,\lambda) = \begin{cases} K_1(x,t,\lambda), & x \in [a,c), \\ K_2(x,t,\lambda), & x \in (c,b]. \end{cases}$$

Substituting the general solution $y = y(x,\lambda)$ into (4.3)-(4.5), one gets that

$$\begin{aligned} & d_{11}^* L_1(\psi_1(x,\lambda)) + d_{12}^* L_1(\psi_2(x,\lambda)) + d_{13}^* L_1(\psi_3(x,\lambda)) \\ = & -\int_a^b f(t) w(t) L_1(K(x,t,\lambda)) dt - f_1, \end{aligned} \tag{4.23}$$

$$\begin{aligned} & d_{11}^* L_2(\psi_1(x,\lambda)) + d_{12}^* L_2(\psi_2(x,\lambda)) + d_{13}^* L_2(\psi_3(x,\lambda)) \\ = & -\int_a^b f(t) w(t) L_2(K(x,t,\lambda)) dt - f_2, \end{aligned} \tag{4.24}$$

$$\begin{aligned} & d_{11}^* L_3(\psi_1(x,\lambda)) + d_{12}^* L_3(\psi_2(x,\lambda)) + d_{13}^* \mathcal{L}_3(\psi_3(x,\lambda)) \\ = & -\int_a^b f(t) w(t) L_3(K(x,t,\lambda)) dt. \end{aligned} \tag{4.25}$$

Due to λ is not an eigenvalue, hence the determinant of coefficients of $d_{11}^*, d_{12}^*, d_{13}^*$ satisfies

$$\begin{aligned} & \begin{vmatrix} L_1(\psi_1(x,\lambda)) & L_1(\psi_2(x,\lambda)) & L_1(\psi_3(x,\lambda)) \\ L_2(\psi_1(x,\lambda)) & L_2(\psi_2(x,\lambda)) & L_2(\psi_3(x,\lambda)) \\ L_3(\psi_1(x,\lambda)) & L_3(\psi_2(x,\lambda)) & L_3(\psi_3(x,\lambda)) \end{vmatrix} \\ = & \det(A_\lambda + B_\lambda \Psi(b,\lambda)) = \Delta(\lambda) \neq 0. \end{aligned} \tag{4.26}$$

Therefore, $d_{11}^*, d_{12}^*, d_{13}^*$ are uniquely determined, and

$$d_{11}^* = \frac{\Re_{11}(\lambda) + \Im_{11}(\lambda)}{\Delta(\lambda)}, d_{12}^* = \frac{\Re_{12}(\lambda) + \Im_{12}(\lambda)}{\Delta(\lambda)}, d_{13}^* = \frac{\Re_{13}(\lambda) + \Im_{13}(\lambda)}{\Delta(\lambda)},$$

where

$$\Re_{11}(\lambda) = \begin{vmatrix} -\int_a^b f(t)w(t)L_1(K(x,t,\lambda))dt & L_1(\psi_2(x,\lambda)) & L_1(\psi_3(x,\lambda)) \\ -\int_a^b f(t)w(t)L_2(K(x,t,\lambda))dt & L_2(\psi_2(x,\lambda)) & L_2(\psi_3(x,\lambda)) \\ -\int_a^b f(t)w(t)L_3(K(x,t,\lambda))dt & L_3(\psi_2(x,\lambda) & L_3(\psi_3(x,\lambda)) \end{vmatrix},$$

$$\Im_{11}(\lambda) = \begin{vmatrix} -f_1 & L_1(\psi_2(x,\lambda)) & L_1(\psi_3(x,\lambda)) \\ -f_2 & L_2(\psi_2(x,\lambda)) & L_2(\psi_3(x,\lambda)) \\ 0 & L_3(\psi_2(x,\lambda)) & L_3(\psi_3(x,\lambda)) \end{vmatrix}.$$

Similarly, $\Re_{12}(\lambda), \Re_{13}(\lambda)$ and $\Im_{12}(\lambda), \Im_{13}(\lambda)$ can be obtained by using Cramer's Rule. Hence the general solution have the following form

$$\begin{aligned} y(x,\lambda) = & \int_a^b K(x,t,\lambda)f(t)w(t)dt \\ & + \frac{1}{\Delta(\lambda)}(\Re_{11}(\lambda)\psi_1(x,\lambda) + \Re_{12}(\lambda)\psi_2(x,\lambda) + \Re_{13}(\lambda)\psi_3(x,\lambda)) \\ & + \frac{1}{\Delta(\lambda)}(\Im_{11}(\lambda)\psi_1(x,\lambda) + \Im_{12}(\lambda)\psi_2(x,\lambda) + \Im_{13}(\lambda)\psi_3(x,\lambda)) \\ = & \int_a^b G(x,t,\lambda)f(t)w(t)dt + \frac{1}{\Delta(\lambda)}\Theta(x,\lambda), \end{aligned} \tag{4.27}$$

where

$$G(x,t,\lambda) = K(x,t,\lambda) - \frac{1}{\Delta(\lambda)}\widetilde{K}(x,t,\lambda), \tag{4.28}$$

$$\widetilde{K}(x,t,\lambda) = \begin{vmatrix} L_1(\psi_1(x,\lambda)) & L_1(\psi_2(x,\lambda)) & L_1(\psi_3(x,\lambda)) & L_1(K(x,t,\lambda)) \\ L_2(\psi_1(x,\lambda)) & L_2(\psi_2(x,\lambda)) & L_2(\psi_3(x,\lambda)) & L_2(K(x,t,\lambda)) \\ L_3(\psi_1(x,\lambda)) & L_3(\psi_2(x,\lambda)) & L_3(\psi_3(x,\lambda)) & L_3(K(x,t,\lambda)) \\ \psi_1(x,\lambda) & \psi_2(x,\lambda) & \psi_3(x,\lambda) & 0 \end{vmatrix},$$

$$\Theta(x,t,\lambda) = \begin{vmatrix} L_1(\psi_1(x,\lambda)) & L_1(\psi_2(x,\lambda)) & L_1(\psi_3(x,\lambda)) & -f_1 \\ L_2(\psi_1(x,\lambda)) & L_2(\psi_2(x,\lambda)) & L_2(\psi_3(x,\lambda)) & -f_2 \\ L_3(\psi_1(x,\lambda)) & L_3(\psi_2(x,\lambda)) & L_3(\psi_3(x,\lambda)) & 0 \\ \psi_1(x,\lambda) & \psi_2(x,\lambda) & \psi_3(x,\lambda) & 0 \end{vmatrix}.$$

In conclusion, for any $F = (f(x), f_1, f_2) \in \mathcal{H}$, there is an unique $Y \in D(\mathrm{T})$, satisfying $(\mathrm{T} - \lambda I)Y = F$.

By the definition of $D(\mathrm{T})$, the components of Y are determined by the first one, i.e., in order to find Y, we only need to find its first component $y(x)$, and $y(x)$ is determined by (4.27).

Definition 4.1. The integral kernel $G(x,t,\lambda)$ in (4.28) is called Green's function of the operator T.

Theorem 4.2. *If λ is not an eigenvalue of the operator* T*, then for any $F = (f(x), f_1, f_2) \in \mathcal{H}$, there exists an unique solution $Y = (y(x), M_1(y), M_2(y))$ of equation $(\mathrm{T} - \lambda I)Y = F$ satisfying*

$$y(x,\lambda) = \int_a^b G(x,t,\lambda)f(t)w(t)dt + \frac{1}{\Delta(\lambda)}\Theta(x,\lambda).$$

The operator $(\mathrm{T} - \lambda I)^{-1}$ is defined in the whole space by Theorem 4.2. It follows from the facts T is symmetric and Closed Graph Theorem that $(\mathrm{T} - \lambda I)^{-1}$ is bounded. Therefore, λ is a regular point of T provided that it is not an eigenvalue of T.

5. Completeness of eigenfunctions

In this part, we study the completeness of the eigenfunctions system of boundary value problem (1.1)-(1.7).

Theorem 5.1. *The operator* T *has only point spectrum, that is to say, $\sigma(\mathrm{T}) = \sigma_p(\mathrm{T})$.*

Proof. We just need to prove that if λ is not an eigenvalue, then $\lambda \in \rho(\mathrm{T})$. Since T is a self-adjoint operator, thus we only need to consider the case where λ is a real number.

We assume that λ is not an eigenvalue. Let's consider equation $(\mathrm{T} - \lambda)Y = F$, where $F = (f(x), f_1, f_2) \in \mathcal{H},\ \lambda \in \mathbb{R}$. With the definition of operator T, we can divide problem into two parts: initial value problem

$$\ell y - \lambda y = f, \qquad x \in I, \tag{5.1}$$

$$\mathrm{Y}(c-) + C\mathrm{Y}(c+) = 0, \tag{5.2}$$

and system of equations

$$\begin{cases} \lambda\mathcal{M}_1(y) - \mathcal{N}_1(y) = -f_1, \\ \lambda\mathcal{M}_2(y) - \mathcal{N}_2(y) = -f_2, \\ (i + \sin\theta)y^{[1]}(a) + \sqrt{r_1 r_2}(1 + i\sin\theta)y^{[1]}(b) = 0. \end{cases} \tag{5.3}$$

By Lemma 3.7, we know that the general solution of the system of equations

$$\begin{cases} \ell y - \lambda y = 0, \qquad x \in I, \\ \mathrm{Y}(c-) + C\mathrm{Y}(c+) = 0, \end{cases}$$

can be expressed as follows

$$y(x) = \begin{cases} c_{11}\psi_{11}(x) + c_{12}\psi_{12}(x) + c_{13}\psi_{13}(x), x \in [a,c), \\ c_{11}\psi_{21}(x) + c_{12}\psi_{22}(x) + c_{13}\psi_{23}(x), x \in (c,b], \end{cases}$$

where $c_{1k} \in \mathbb{C}$, ψ_{jk} $(j = 1, 2,\ k = 1, 2, 3)$ are defined in Section 3.

Let

$$\phi(x) = \begin{cases} \phi_1(x), & x \in [a, c), \\ \phi_2(x), & x \in (c, b], \end{cases}$$

be a special solution of equation (5.1), then the initial value problem (5.1)-(5.2) has the general solution as follows

$$y(x) = \begin{cases} c_{11}\psi_{11}(x) + c_{12}\psi_{12}(x) + c_{13}\psi_{13}(x) + \phi_1(x), x \in [a, c), \\ c_{11}\psi_{21}(x) + c_{12}\psi_{22}(x) + c_{13}\psi_{23}(x) + \phi_2(x), x \in (c, b]. \end{cases} \tag{5.4}$$

Take the general solution (5.4) into system of equations (5.3) yields

$$\begin{aligned} &(\widetilde{\alpha}_1 + \lambda\alpha_1)c_{11} - (\widetilde{\alpha}_2 + \lambda\alpha_2)c_{13} \\ =& - f_1 - (\widetilde{\alpha}_1 + \lambda\alpha_1)\phi_1(a) + (\widetilde{\alpha}_2 + \lambda\alpha_2)\phi_1^{[2]}(a), \end{aligned} \tag{5.5}$$

$$\begin{aligned} &[(\widetilde{\beta}_1 + \lambda\beta_1)\psi_{21}(b) + (\widetilde{\beta}_2 + \lambda\beta_2)\psi_{21}^{[2]}(b)]c_{11} \\ &+ [(\widetilde{\beta}_1 + \lambda\beta_1)\psi_{22}(b) + (\widetilde{\beta}_2 + \lambda\beta_2)\psi_{22}^{[2]}(b)]c_{12} \\ &+ [(\widetilde{\beta}_1 + \lambda\beta_1)\psi_{23}(b) + (\widetilde{\beta}_2 + \lambda\beta_2)\psi_{23}^{[2]}(b)]c_{13} \\ =& - f_2 - (\widetilde{\beta}_1 + \lambda\beta_1)\phi_2(b) - (\widetilde{\beta}_2 + \lambda\beta_2)\phi_2^{[2]}(b), \end{aligned} \tag{5.6}$$

$$\begin{aligned} &\sqrt{r_1 r_2}(1 + i\sin\beta)\psi_{21}^{[1]}(b)c_{11} \\ &+ [(i + \sin\beta) + \sqrt{r_1 r_2}(1 + i\sin\beta)\psi_{22}^{[1]}(b)]c_{12} \\ &+ \sqrt{r_1 r_2}(1 + i\sin\beta)\psi_2^{[1]}(b)c_{13} \\ =& - (i + \sin\beta)\phi_1^{[1]}(b) - \sqrt{r_1 r_2}(1 + i\sin\beta)\phi_2^{[1]}(b). \end{aligned} \tag{5.7}$$

System of equations (5.5)-(5.7) can be expressed as follows

$$\begin{aligned} &\big(A_\lambda + B_\lambda\Psi(b, \lambda)\big)(c_{11}, c_{12}, c_{13})^T \\ =&(-f_1, -f_2, 0) - A_\lambda\big(\phi_1(a), \phi_1^{[1]}(a), \phi_1^{[2]}(a)\big)^T - B_\lambda\big(\phi_1(b), \phi_1^{[1]}(b), \phi_1^{[2]}(b)\big)^T. \end{aligned}$$

Since λ is not an eigenvalue of operator T, therefore $\big(A_\lambda + B_\lambda\Psi(b, \lambda)\big) \neq 0$, thus c_{11}, c_{12} and c_{13} are uniquely determined. Hence, the general solution of the boundary value problem (5.1)-(5.3) is uniquely determined.

From the above discussion, we can see that $(\mathrm{T} - \lambda)^{-1}$ is defined on the whole space $\mathcal{H}$. By the self-adjointness of operator T and the closed image theorem, we know that $(\mathrm{T} - \lambda)^{-1}$ is bounded. Hence $\lambda \in \rho(\mathrm{T})$, so $\sigma(\mathrm{T}) = \sigma_p(\mathrm{T})$. □

Lemma 5.2. *The operator* T *has compact resolvents, ie., for each* $\delta \in \mathbb{R}/\sigma_p(A)$, $(\mathrm{T} - \delta I)^{-1}$ *is compact on* H (c.f.[[22], Theorem 6.3.3]).

By the above lemmas and the spectral theorem for compact operator, we obtain the following theorem.

Theorem 5.3. *The eigenfunctions of the problem* (1.1)-(1.7)*, expanded to become eigenfunctions of* T*, are complete in* H*, i.e., let* $\{\Phi_n = (\phi_n(x), M_1(\phi_n), M_2(\phi_n)); n \in \mathbb{N}\}$ *be a maximum set of orthonormal eigenfunctions of* T*, where* $\{\phi_n(x); n \in \mathbb{N}\}$ *are eigenfunctions of the problem* (1.1)-(1.7)*. Then for all* $F \in H$, $F = \sum_{n=1}^{\infty}\langle F, \Phi_n\rangle\Phi_n$.

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All the authors have no relevant financial or non-financial competing interests.

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