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THE BASIS PROPERTY OF WEAK EIGENFUNCTIONS FOR STURM-LIOUVILLE PROBLEM WITH BOUNDARY CONDITIONS DEPENDENT RATIONALLY ON THE EIGENPARAMETER*

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Abstract Using the theory of operator pencils in Hilbert space and suitable integral transformation, the basis property of weak eigenfunctions for the Sturm-Liouville problem with eigenparameter dependent rationally on the boundary conditions is obtained, and the asymptotic behavior of eigenvalues is also involved.

Keywords Sturm-Liouville problem, Riesz basis, eigenparameter-dependent boundary condition, weak eigenfunction.

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1. Introduction

As we all know, many authors have researched the Sturm-Liouville problems for these wide applications in various fields such as finance, mathematical physics, quantum mechanics, and acoustic scattering. Among these results, Sturm-Liouville problems containing eigenparameter on the boundary condition have recently been widely researched (see [2–8, 13–16, 18, 21, 22] and the reference cited therein) since the foundation work of Walter [20]. However, most of these results researched Sturm-Liouville problems with eigenparameter dependent linearly on the boundary conditions.

In [5, 6], Binding et al. researched the following Sturm-Liouville problem firstly

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using Prüfer transformation

$$\begin{cases} -y''+q(x)y=\lambda y,\ x\in[0,1], \\ y(0)\cos\alpha=y'(0)\sin\alpha,\ \alpha\in[0,\pi), \\ \dfrac{y'(1)}{y(1)}=g(\lambda)=e\lambda+f-\displaystyle\sum_{h=1}^{M}\frac{b_h}{\lambda-c_h}, \end{cases}$$

here $g(\lambda)$ is a rational function called Herglotz-Nevanlinna type. They considered the oscillation property of eigenfunctions, the inverse problems, and the existence of eigenvalues.

Meanwhile, operator pencil theory in Hilbert space is widely researched (see [8–12, 19]). In [11], Ladyzhenskaia introduced the concept of weak solutions in Hilbert space. This concept allows the authors to reduce the eigenvalue problem to an operator pencil. Belinskiy and Dauer in [3] investigated the weak eigenfunctions of a regular Sturm-Liouville problem with eigenparameter dependent linearly on the boundary conditions. By reducing the considered Sturm-Liouville problem to an operator pencil, they concluded that the spectrum of the considered problem is discrete, and the system of weak eigenfunctions constructs a Riesz basis in the Hilbert space $\mathbb{H} := H^1 \oplus \mathbb{C}$. Later, Olğar et al. in [15–17] studied the weak eigenfunctions of some discontinuous Sturm-Liouville problems, where the boundary condition is also dependent linearly on the eigenparameter. Meanwhile, the similar results to [3] were obtained.

Inspired and motivated by the above works, we consider the Sturm-Liouville equation

$$l_1 y := -(p(x)y')' + q(x)y = \mu w(x)y,\ x\in(a,b), \tag{1.1}$$

subject to the boundary conditions of the form

$$l_2 y := \frac{(py')(a)}{y(a)} = -f(\mu), \quad l_3 y := \frac{(py')(b)}{y(b)} = F(\mu), \tag{1.2}$$

where $p(x) > 0$, $q(x) > 0$, and $w(x) > 0$ are bounded and Lebesgue integrable on the interval (a,b); $\mu \in \mathbb{C}$ is an eigenparameter; $f(\mu)$ and $F(\mu)$ are rational Herglotz-Nevanlinna type functions of the form

$$f(\mu) = z_0\mu + z - \sum_{i=1}^{n}\frac{\alpha_i}{\mu - z_i}, \quad F(\mu) = Z_0\mu + Z - \sum_{j=1}^{N}\frac{\beta_j}{\mu - Z_j},$$

here $z_0,\ Z_0 \geq 0,\ z\in\mathbb{R},\ Z\in\mathbb{R},\ \alpha_i,\ \beta_j > 0,\ \alpha_1<\alpha_2<\cdots<\alpha_n,\ \beta_1<\beta_2<\cdots<\beta_N,\ n,\ N\in\mathbb{Z}_+$. If $f(\mu)=\infty$, then left boundary condition is interprets as a Dirichlet boundary condition $y(a)=0$. Similarly, hypothesis $y(b)=0$ holds if $F(\lambda)=\infty$. It is worth mentioning that both boundary conditions of the Sturm-Liouville problem (1.1)-(1.2) are dependent rationally on the eigenparameter μ. This paper purports to consider the weak eigenfunctions of the Sturm-Liouville problem (1.1)-(1.2). By defining a suitable Hilbert space and exact operator pencil formulae, we prove that the system of weak eigenfunctions of the Sturm-Liouville problem (1.1)-(1.2) constructs a Riesz basis in the Hilbert space.

The rest of this paper is laid out as follows: after this section, we not only recall some notions, definitions, and lemmas but also introduce a new Hilbert space Ξ in

Sect. 2. In Sect. 3, using suitable integral transformation, we reduce the Sturm-Liouville problem (1.1)-(1.2) to an operator pencil by defining the weak (generalized) solutions and introducing some compact operators in this new Hilbert space Ξ. Finally, the main result is established and proved.

2. Preliminaries and lemmas

Firstly, we define some useful spaces. Let $L^2[a,b]$ be the Hilbert space, its inner product is defined by

$$\langle y,z\rangle_0 := \int_a^b y(x)\overline{z}(x)\mathrm{d}x$$

and $H^1[a,b]$ be the Sobolev space, its inner product is defined by

$$\langle y,z\rangle_1 := \int_a^b \Big(y'(x)\overline{z'}(x) + y(x)\overline{z}(x)\Big)\mathrm{d}x.$$

Since $p(x)>0$ and $q(x)>0$ are bounded in $[a,b]$, then, for $y,z\in H^1[a,b]$,

$$\langle y,z\rangle_2 := \int_a^b \Big(p(x)y'(x)\overline{z'}(x) + q(x)y(x)\overline{z}(x)\Big)\mathrm{d}x$$

define a new inner product in $H^1[a,b]$, and the norms $\|\cdot\|_2$ and $\|\cdot\|_1$ are equivalent.

For $\Phi = (\phi(x),\phi_1,\ldots,\phi_n,\varphi_1,\ldots,\varphi_N)^{\mathrm{T}}$ and $\Psi = (\psi(x),\psi_1,\ldots,\psi_n,\xi_1,\ldots,\xi_N)^{\mathrm{T}}$ $\in \Xi := H^1[a,b]\oplus\mathbb{C}^{n+N}$, we define the inner product as follows:

$$\langle\Phi,\Psi\rangle_\Xi := \langle\phi(x),\psi(x)\rangle_1 + \sum_{i=1}^n \phi_i\overline{\psi}_i + \sum_{j=1}^N \varphi_j\overline{\xi}_j. \tag{2.1}$$

It is easy to see that Ξ is a Hilbert space.

Definition 2.1. (c.f. [16]) For a given Hilbert space H, a family of elements $\{\Phi_m\}_{m=1}^\infty$ in H is called a Riesz basis in H if the series $\sum_{m=0}^\infty a_m\Phi_m$ with real coefficients a_m converges in H if and only if $\sum_{m=0}^\infty a_m^2<\infty$.

Lemma 2.1. (c.f. [3]) *An arbitrary invertible bounded operator $\mathcal{A}$ in a given Hilbert space H transforms the orthonormal basis of H into a Riesz basis.*

Lemma 2.2. (c.f. [4]) *Suppose that $y(x)\in H^1[a,b]$, then*

$$|y(a)|^2 \le \frac{2}{(\theta_1-a)}\|y\|^2_{L^2[a,b]} + (\theta_1-a)\|y'\|^2_{L^2[a,b]}, \tag{2.2}$$

$$|y(b)|^2 \le \frac{2}{(b-\theta_2)}\|y\|^2_{L^2[a,b]} + (b-\theta_2)\|y'\|^2_{L^2[a,b]}, \tag{2.3}$$

here (θ_1-a), $(b-\theta_2)\in(0,b-a]$ are constants.

Proof. Since $y(x)\in H^1[a,b]$, we obtain

$$y(a) = y(x) - \int_a^x y'(t)\mathrm{d}t,\ \ x\in[a,\theta_1], \tag{2.4}$$

using the Cauchy-Schwarz inequality, one has

$$\left|\int_a^x y'(t)\mathrm{d}t\right|^2 \leq (x-a)\|y'\|^2_{L_2[a,b]}.$$

This inequality implies that

$$\left\|\int_a^x y'(t)\mathrm{d}t\right\|^2_{L_2[a,\theta_1]} = \int_a^{\theta_1}\left|\int_a^x y'(t)\mathrm{d}t\right|^2 \mathrm{d}x \leq \|y'\|^2_{L_2[a,b]}\int_a^{\theta_1}(x-a)\mathrm{d}x,$$

i.e.,

$$\left\|\int_a^x y'(t)\mathrm{d}t\right\|^2_{L_2[a,\theta_1]} \leq \frac{(\theta_1-a)^2}{2}\|y'\|^2_{L_2[a,b]}.$$

Then, taking $L_2[a,\theta_1]$-norm in both sides of (2.4) and using the inequality

$$(a+b)^2 \leq 2(a^2+b^2),$$

we have

$$(\theta_1-a)|y(a)|^2 \leq 2\|y\|^2_{L_2[a,b]} + (\theta_1-a)^2\|y'\|^2_{L_2[a,b]},$$

i.e.,

$$|y(a)|^2 \leq \frac{2}{(\theta_1-a)}\|y\|^2_{L_2[a,b]} + (\theta_1-a)\|y'\|^2_{L_2[a,b]}.$$

We can obtain (2.3) in a similar method, the proof is completed. □

By using Lemma 2.2, the following result is given.

Corollary 2.1. *For any $y(x)\in H^1[a,b]$, we have*

$$|y(x)|^2 \leq B(x)\|y\|_1^2 \leq B_1(x)\|y\|_2^2, \tag{2.5}$$

where the constants B, B_1 are independent of the selection of function $y(x)$.

3. Main results

For all $\eta\in H^1[a,b]$, multiplying the equation (1.1) by $\overline{\eta}$ and integrating by parts over the interval $[a,b]$, we get

$$\begin{aligned}&-(py')(b)\overline{\eta}(b)+(py')(a)\overline{\eta}(a)+\int_a^b (py')(x)\overline{\eta'}(x)\mathrm{d}x+\int_a^b q(x)y(x)\overline{\eta}(x)\mathrm{d}x\\ =&\mu\int_a^b w(x)y(x)\overline{\eta}(x)\mathrm{d}x.\end{aligned}\tag{3.1}$$

Next, we use the following notations

$$\begin{aligned}y_i &:= -\frac{\alpha_i}{z_i-\mu}y(a),\ i=1,2,\ldots,n,\\ h_j &:= -\frac{\beta_j}{\mu-Z_j}y(b),\ j=1,2,\ldots,N.\end{aligned}\tag{3.2}$$

For $z_0 > 0$, $Z_0 > 0$, the relation (3.2) implies that

$$\begin{aligned} y(a) + \frac{z_i}{\alpha_i} y_i &= \mu \frac{y_i}{\alpha_i},\ i = 1, 2, \ldots, n, \\ -y(b) + \frac{Z_j}{\beta_j} h_j &= \mu \frac{h_j}{\beta_j},\ j = 1, 2, \ldots, N. \end{aligned} \tag{3.3}$$

With the new parameters y_i and h_j, the boundary conditions (1.2) imply that

$$\begin{aligned} (py')(a) &= -(z_0\mu + z)y(a) + \sum_{i=1}^{n} y_i, \\ (py')(b) &= (Z_0\mu + Z)y(b) + \sum_{j=1}^{N} h_j. \end{aligned} \tag{3.4}$$

Plugging (3.4) into identity (3.1), then

$$\begin{aligned} & -Zy(b)\overline{\eta}(b) - zy(a)\overline{\eta}(a) + \sum_{i=1}^{n} y_i \overline{\eta}(a) - \sum_{j=1}^{N} h_j \overline{\eta}(b) \\ & + \int_a^b (py')(x)\overline{\eta'}(x)\mathrm{d}x + \int_a^b q(x)y(x)\overline{\eta}(x)\mathrm{d}x \\ =& \mu \int_a^b w(x)y(x)\overline{\eta}(x)\mathrm{d}x + \mu Z_0 y(b)\overline{\eta}(b) + \mu z_0 y(a)\overline{\eta}(a). \end{aligned} \tag{3.5}$$

Now, we define weak solutions and weak eigenfunctions of the Sturm-Liouville problem (1.1)-(1.2).

Definition 3.1. The element $\Pi = (y(x), y_1, \ldots, y_n, h_1, \ldots, h_N)^{\mathrm{T}}$ of the Hilbert space $\Xi := H^1[a,b] \oplus \mathbb{C}^{n+N}$ satisfying (3.3), and (3.5) for all $\eta \in H^1[a,b]$ is called a weak solution of the Sturm-Liouville problem (1.1)-(1.2). The first component, $y(x)$ of Π, is called a weak eigenfunction of the corresponding Sturm-Liouville problem (1.1)-(1.2).

Theorem 3.1. *The system of the weak eigenfunctions for the Sturm-Liouville problem* (1.1)-(1.2) *constructs a Riesz basis of Hilbert space* $\Xi := H^1[a,b] \oplus C^{n+N}$. *Meanwhile, the eigenvalues* $\{\mu_m\}$ *of the considered problem* (1.1)-(1.2) *are real and discrete with* $\mu_m \to +\infty$ *as* $m \to +\infty$.

For $\eta \in H^1[a,b]$, we define some linear forms as follows:

$$\sigma_0(y,\eta) := -Zy(b)\overline{\eta}(b) - zy(a)\overline{\eta}(a), \tag{3.6}$$

$$\sigma_1(y,\eta) := \int_a^b w(x)y(x)\overline{\eta}(x)\mathrm{d}x, \tag{3.7}$$

$$\sigma_2(y,\eta) := Z_0 y(b)\overline{\eta}(b) + z_0 y(a)\overline{\eta}(a), \tag{3.8}$$

$$\tau_i(y_i,\eta) := y_i\overline{\eta}(a),\ i = \overline{1,n}, \tag{3.9}$$

$$\delta_j(h_j,\eta) := -h_j\overline{\eta}(b),\ j = \overline{1,N}, \tag{3.10}$$

where $y \in H^1[a,b]$, $y_i \in \mathbb{C}$, and $h_j \in \mathbb{C}$. Like the proof of [3], we obtain that all these linear forms are linear functionals in $H^1[a,b]$. Using the equivalence of $\|\cdot\|_1$ and $\|\cdot\|_2$, noticing the positivity of $w(x)$, the classical Riesz representation theorem implies the following theorem.

Theorem 3.2. *Suppose σ_0, σ_1, σ_2, τ_i, and δ_j is defined as* (3.6)-(3.10). *Then there are bounded linear operators Y_0, W, K: $H^1[a,b] \to H^1[a,b]$, and Y_i: $\mathbb{C} \to H^1[a,b]$, $i = \overline{1,n}$, H_j: $\mathbb{C} \to H^1[a,b]$, $j = \overline{1,N}$, such that the following representations hold:*

$$\langle Y_0 y, \eta\rangle_2 = \sigma_0(y,\eta), \tag{3.11}$$

$$\langle W y, \eta\rangle_2 = \sigma_1(y,\eta), \tag{3.12}$$

$$\langle K y, \eta\rangle_2 = \sigma_2(y,\eta), \tag{3.13}$$

$$\langle Y_i y_i, \eta\rangle_2 = \tau_i(y_i,\eta), \tag{3.14}$$

$$\langle H_j h_j, \eta\rangle_2 = \delta_j(h_j,\eta), \tag{3.15}$$

where the functions y, $\eta \in H^1[a,b]$ are arbitrary.

Lemma 3.1. *The operators Y_0, W, and K are self-adjoint and compact. Moreover, operators W and K are also positive in the Hilbert Space $H^1[a,b]$.*

Proof. Firstly, assume that sequence $\{g_k\}$ converges weakly to an element g in $H^1[a,b]$. The boundedness of operators Y_0, W and K imply the weak convergence of $\{Y_0 g_k\}$ to $Y_0 g$, $\{W g_k\}$ to Wg, and $\{K g_k\}$ to Kg in $H^1[a,b]$, respectively. It follows from the embedding theorems that the sequence $\{g_k\}$ converges strongly in $L^2[a,b]$, and the sequences $\{g_k(d)\}$, $d = a$ or b converge in $\mathbb{C}$.

Then by using Theorem 3.2, there is a constant $C_1 > 0$ such that

$$\begin{aligned}\|Y_0(g_k - g_l)\|_2^2 =& \sigma_0\Big(g_k - g_l, Y_0(g_k - g_l)\Big)\\ \le& C_1\Big(\, |(g_k - g_l)(b)| \cdot |(Y_0(g_k - g_l)(b)|\\ &+ |(g_k - g_l)(a)| \cdot |(Y_0(g_k - g_l)(a)|\Big),\end{aligned}$$

similarly, there are two constants $C_2 > 0$, $C_3 > 0$ such that

$$\begin{aligned}\|K(g_k - g_l)\|_2^2 =& \sigma_2\Big(g_k - g_l, K(g_k - g_l)\Big)\\ \le& C_2\Big(\, |(g_k - g_l)(b)| \cdot |(K(g_k - g_l)(b)|\\ &+ |(g_k - g_l)(a)| \cdot |K(g_k - g_l)(a)|\Big),\end{aligned}$$

and

$$\begin{aligned}\|W(g_k - g_l)\|_2^2 &= \sigma_1\Big(g_k - g_l, W(g_k - g_l)\Big)\\ &\le C_3\|(g_k - g_l)\|_0 \cdot \|W(g_k - g_l)\|_2.\end{aligned}$$

Therefore, the compactness of operators Y_0, W and K is got.

Next, we will prove that operators Y_0, W, and K are self-adjoint. Suppose y, $\eta \in H^1[a,b]$ are arbitrary functions. Then it follows from (3.6) and (3.11) that

$$\langle y, Y_0\eta\rangle_2 = \overline{\langle Y_0\eta, y\rangle_2} = \overline{\sigma_0(\eta, y)} = \sigma_0(y,\eta) = \langle Y_0 y, \eta\rangle_2.$$

Relations (3.7) and (3.12) imply that

$$\langle y, W\eta\rangle_2 = \overline{\langle W\eta, y\rangle_2} = \overline{\sigma_1(\eta, y)} = \sigma_1(y, \eta) = \langle Wy, \eta\rangle_2.$$

Similarly, (3.8) and (3.13) imply that

$$\langle y, K\eta\rangle_2 = \overline{\langle K\eta, y\rangle_2} = \overline{\sigma_2(\eta, y)} = \sigma_2(y, \eta) = \langle Ky, \eta\rangle_2.$$

So, the operators Y_0, W and K are self-adjoint in the Hilbert space $H^1[a,b]$.

Finally, the positiveness of operators W and K is obvious since $z_0 > 0$ and $Z_0 > 0$, function $w(x) > 0$ is bounded. □

Lemma 3.2. *The operators* Y_i*,* $i = \overline{1,n}$*, and* H_j*,* $j = \overline{1,N}$*, are compact.*

Proof. The proof of Lemma 3.2 is similar to the proof of Lemma 3.1. □

Using Theorem 3.2, we observe that the relations (3.3) and (3.5) can be rewritten as follows:

$$\begin{aligned}
&\langle y, \eta\rangle_2 + \langle Y_0 y, \eta\rangle_2 + \sum_{i=1}^{n}\langle Y_i y_i, \eta\rangle_2 + \sum_{j=1}^{N}\langle H_j h_j, \eta\rangle_2 = \mu\langle Wy, \eta\rangle_2 + \mu\langle Ky, \eta\rangle_2,\\
&Y_i^* y + \frac{z_i}{\alpha_i} y_i = \mu \frac{y_i}{\alpha_i},\ i = \overline{1,n},\\
&H_j^* y + \frac{Z_j}{\beta_j} h_j = \mu \frac{h_j}{\beta_j},\ i = \overline{1,N}.
\end{aligned}$$

Then, the arbitrariness of $\eta \in H^1[a,b]$ implies that

$$y + Y_0 y + \sum_{i=1}^{n} Y_i y_i + \sum_{j=1}^{N} H_j h_j = \mu W y + \mu K y. \tag{3.16}$$

The following result is given by introducing two $(n+N+1)$-order square matrices U and V, and an unknown $(n+N+1)$-dimensional vector Π.

Lemma 3.3. *Let* I *be an identity operator in* Ξ*. The weak eigenfunctions of Sturm-Liouville problem* (1.1)-(1.2) *satisfy the following operator pencil equation in* Ξ*.*

$$\mathcal{A}(\mu)\Pi = 0, \quad \mathcal{A}(\mu) = U - \mu V, \tag{3.17}$$

here Π *is defined as above, and*

$$U = \begin{bmatrix}
I + Y_0 & Y_1 & \cdots & Y_n & H_1 & \cdots & H_N\\
Y_1^* & \frac{z_1}{\alpha_1} I & 0 & 0 & 0 & \cdots & 0\\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots\\
Y_n^* & 0 & 0 & \frac{z_n}{\alpha_n} I & 0 & \cdots & 0\\
H_1^* & 0 & 0 & 0 & \frac{Z_1}{\beta_1} I & \cdots & 0\\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots\\
H_N^* & 0 & 0 & 0 & 0 & \cdots & \frac{Z_N}{\beta_N} I
\end{bmatrix},$$

$$V=\begin{bmatrix} W+K & 0 & 0 & 0 & 0 & \cdots & 0\\ 0 & \frac{1}{\alpha_1}I & 0 & 0 & 0 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots\\ 0 & 0 & 0 & \frac{1}{\alpha_n}I & 0 & \cdots & 0\\ 0 & 0 & 0 & 0 & \frac{1}{\beta_1}I & \cdots & 0\\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & 0 & 0 & \cdots & \frac{1}{\beta_N}I \end{bmatrix}.$$

Theorem 3.3. *For $\mu_0>0$ sufficiently large, the operator pencil $\mathcal{A}(-\mu_0)$ is positive definite.*

Proof. It follows from relations (3.11)-(3.15) and (3.17) that

$$\begin{aligned}
&\langle \mathcal{A}(-\mu_0)\Pi,\Pi\rangle_\Xi\\
=&\|y\|_2^2+\langle Y_0y(x),y(x)\rangle_2+\sum_{i=1}^{n}\langle Y_iy_i,y(x)\rangle_2+\sum_{j=1}^{N}\langle H_jh_j,y(x)\rangle_2\\
&+\sum_{i=1}^{n}(Y_i^*y(x))\overline{y}_i+\sum_{j=1}^{N}(H_j^*y(x))\overline{h}_j+\sum_{i=1}^{n}\frac{z_i}{\alpha_i}|y_i|^2+\sum_{j=1}^{N}\frac{Z_j}{\beta_j}|h_j|^2\\
&+\mu_0\Big(\langle Wy(x),y(x)\rangle_2+\langle Ky(x),y(x)\rangle_2+\sum_{i=1}^{n}\frac{|y_i|^2}{\alpha_i}+\sum_{j=1}^{N}\frac{|h_j|^2}{\beta_j}\Big)\\
\geq&\|y\|_2^2+\langle Y_0y(x),y(x)\rangle_2+\sum_{i=1}^{n}\langle Y_iy_i,y(x)\rangle_2+\sum_{j=1}^{N}\langle H_jh_j,y(x)\rangle_2\\
&+\sum_{i=1}^{n}(Y_i^*y(x))\overline{y}_i+\sum_{j=1}^{N}(H_j^*y(x))\overline{h}_j+\sum_{i=1}^{n}\frac{z_i}{\alpha_i}|y_i|^2+\sum_{j=1}^{N}\frac{Z_j}{\beta_j}|h_j|^2\\
&+\mu_0\Big(\langle Wy(x),y(x)\rangle_2+\sum_{i=1}^{n}\frac{|y_i|^2}{\alpha_i}+\sum_{j=1}^{N}\frac{|h_j|^2}{\beta_j}\Big),
\end{aligned}\tag{3.18}$$

where $\langle Ky(x),y(x)\rangle_2=z_0|y(a)|^2+Z_0|y(b)|^2\geq 0$ since $z_0>0$ and $Z_0>0$.

We denote

$$\begin{aligned}
G_1(y)&=\int_a^b p(x)|y''(x)|^2\mathrm{d}x,\\
G_2(y)&=\int_a^b q(x)|y(x)|^2\mathrm{d}x,\\
G_3(y)&=\int_a^b w(x)|y(x)|^2\mathrm{d}x,
\end{aligned}\tag{3.19}$$

one has

$$\|y(x)\|_2^2=G_1(y)+G_2(y).\tag{3.20}$$

Meanwhile, since $q(x) > 0$ and $w(x) > 0$ are bounded, there are positive constants C_{i1}, and C_{i2}, $i = 1, 2$ such that

$$|y(a)|^2 \leq C_{11}\delta_1 G_1(y) + \frac{C_{12}}{\delta_1} G_2(y), \tag{3.21}$$

$$|y(b)|^2 \leq C_{21}\delta_2 G_1(y) + \frac{C_{22}}{\delta_2} G_2(y). \tag{3.22}$$

By using (3.19) and inequalities (3.21)-(3.22), one has

$$\begin{aligned}\langle Y_0 y(x), y(x)\rangle_2 &= -Z|y(b)|^2 - z|y(a)|^2 \\ &\geq -\Big(|z|C_{11}\delta_1 + |Z|C_{21}\delta_2\Big)G_1(y) \\ &\quad - \Big(|z|\frac{C_{12}}{\delta_1} + |Z|\frac{C_{22}}{\delta_2}\Big)G_2(y).\end{aligned} \tag{3.23}$$

Using Theorem 3.2 and the Young inequality, we obtain

$$\begin{aligned}\langle Y_i y_i, y(x)\rangle_2 + (Y_i^* y(x))\overline{y_i} &= 2Re(y_i\overline{y(a)}) \\ &\geq -\frac{1}{\gamma_i}|y(a)|^2 - \gamma_i|y_i|^2 \\ &\geq -\frac{1}{\gamma_i}\Big(C_{11}\delta_1 G_1(y) + \frac{C_{12}}{\delta_1}G_2(y)\Big) - \gamma_i|y_i|^2,\end{aligned} \tag{3.24}$$

here $\gamma_i > 0$, $i = \overline{1, n}$, are arbitrary constants. Similarly,

$$\begin{aligned}\langle H_j h_j, y(x)\rangle_2 + (H_j^* y(x))\overline{h_j} &= -2Re(h_j\overline{y(b)}) \\ &\geq -\frac{1}{\rho_j}|y(b)|^2 - \rho_j|h_j|^2 \\ &\geq -\frac{1}{\rho_j}\Big(C_{21}\delta_2 G_1(y) + \frac{C_{22}}{\delta_2}G_2(y)\Big) - \rho_j|h_j|^2,\end{aligned} \tag{3.25}$$

here $\rho_j > 0$, $j = \overline{1, N}$, are arbitrary constants.

Since $q(x) > 0$ and $w(x) > 0$ are bounded, then

$$\langle W y(x), y(x)\rangle_2 = \sigma_1(y, y) \geq Q G_2(y), \tag{3.26}$$

where $Q > 0$ is a constant. Substituting (3.20), (3.23)-(3.26) into (3.18) yields that

$$\begin{aligned}\langle \mathcal{A}(-\mu_0)\Pi, \Pi\rangle_\Xi \geq\, & S_1 G_1(y) + S_2(\mu_0)G_2(y) \\ & + \sum_{i=1}^{n} E_i(\mu_0)|y_i|^2 + \sum_{j=1}^{N} F_j(\mu_0)|h_j|^2,\end{aligned} \tag{3.27}$$

where

$$S_1 := 1 - \Big(|z| + \sum_{i=1}^{n}\frac{1}{\gamma_i}\Big)C_{11}\delta_1 - \Big(|Z| + \sum_{j=1}^{N}\frac{1}{\rho_j}\Big)C_{21}\delta_2, \tag{3.28}$$

$$S_2(\mu_0) := 1 - \Big(|z| + \sum_{i=1}^{n}\frac{1}{\gamma_i}\Big)\frac{C_{12}}{\delta_1} - \Big(|Z| + \sum_{j=1}^{N}\frac{1}{\rho_j}\Big)\frac{C_{22}}{\delta_2} + \mu_0 Q, \tag{3.29}$$

$$E_i(\mu_0) := -\gamma_i + \frac{z_i}{\alpha_i} + \frac{\mu_0}{\alpha_i}, \quad i = \overline{1, n}, \tag{3.30}$$

$$F_j(\mu_0) := -\rho_j + \frac{Z_j}{\beta_j} + \frac{\mu_0}{\beta_j}, \quad j = \overline{1, N}. \tag{3.31}$$

Since $\alpha_i > 0$, $\beta_j > 0$, we can choose the arbitrary small parameters $\delta_1 > 0$, $\delta_2 > 0$ and the parameter $\mu_0 > 0$ large enough such that S_1, $S_2(\mu_0) > 0$, $E_i(\mu_0) > 0$, $F_j(\mu_0) > 0$. Meanwhile, we know that $\langle Ky(x), y(x)\rangle_2 \geq 0$. Then,

$$\langle \mathcal{A}(-\mu_0)\Pi, \Pi\rangle_\Xi \geq S(\mu_0)\|\Pi\|^2_\Xi, \text{ for } \forall \Pi \in \Xi, \tag{3.32}$$

where

$$S(\mu_0) := \min\Big(S_1, S_2(\mu_0), E_1(\mu_0), \ldots, E_n(\mu_0), F_1(\mu_0), \ldots, F_N(\mu_0)\Big).$$

Thus, for $\mu_0 > 0$ sufficiently large, the operator pencil $\mathcal{A}(-\mu_0)$ is positive definite, the proof is completed. □

To reduce the operator pencil equation to a standard equation of Fredholm type, we substitute a spectral parameter $\lambda = \mu_0 + \mu$ in the operator pencil equation (3.17). Then, the spectral problem has the following form:

$$\Big(\mathcal{A}(-\lambda_0) - \lambda V\Big)\Pi = 0. \tag{3.33}$$

We introduce a new unknown parameter $\Lambda = \mathcal{A}(-\mu_0)^{\frac{1}{2}}\Pi$, here $\mathcal{A}(-\mu_0)^{\frac{1}{2}}$ is a positive square root of $\mathcal{A}(-\mu_0)$.

Lemma 3.4. *The function $\Lambda = \mathcal{A}(-\mu_0)^{\frac{1}{2}}\Pi$ satisfies the following operator equation*

$$\begin{aligned} &\Lambda - \lambda T\Lambda = 0, \\ &T = \mathcal{A}(-\mu_0)^{-\frac{1}{2}} V \mathcal{A}(-\mu_0)^{-\frac{1}{2}}, \end{aligned} \tag{3.34}$$

where operator T is compact self-adjoint. Moreover, it is also positive. The spectrum $\{\lambda_m\}$ of equation (3.34) is positive, discrete, and has a unique accumulation point at $+\infty$. In addition, the system of corresponding eigenfunctions constructs an orthogonal basis of Ξ.

Proof. This result can be proved via the Fredholm theorems for the compact self-adjoint operator. Note that the operator V is positive since the unknown parameters α_i and β_j are positive. □

Then the Theorem 3.1 can be obtained directly from Lemma 2.1.

Remark 3.1. For $n = 0$ and $N > 0$; $n > 0$ and $N = 0$; and $n = N = 0$, using a similar method, we obtain that the system of the weak eigenfunctions of the Sturm-Liouville problem (1.1)-(1.2) constructs Riesz basis in $H^1[a, b] \oplus C^N$, $H^1[a, b] \oplus C^n$, $H^1[a, b]$, respectively.

Remark 3.2. When z_0=0 and $Z_0 > 0$, we define the linear function $\sigma_2(y, \eta) := Z_0 y(b)\overline{\eta}(b)$. Similarly, when Z_0=0 and $z_0 > 0$, we define the linear function $\sigma_2(y, \eta) := z_0 y(a)\overline{\eta}(a)$. When $z_0 = 0$ and $Z_0 = 0$, we don't introduce the linear function $\sigma_2(y, \eta)$, i.e., (3.8). Then, a similar discussion can obtain results similar to Theorem 3.1.

Remark 3.3. It is an interesting topic of spectral theory for the discontinuous Sturm-Liouville problem (1.1)-(1.2) with certain interface conditions (see [1, 14–16, 18] for details), we will study the weak eigenfunctions for this kind of operator later.

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