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MULTIPLE STABLE STATES FOR A CLASS OF PREDATOR-PREY SYSTEMS WITH TWO HARVESTING RATES*

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Abstract In this paper, a class of predator-prey systems with two harvesting rates is studied, multiple limit cycles can be obtained by hopf bifurcation, and the Hopf cyclicity at the origin is 4. Multiple stable states can coexist in the predator-prey systems with two harvesting rates. Then by using Poincare-Bendixson theorem and Dulac discriminant method, existence and non-existence conditions of limit cycles are obtained.

Keywords Predator-prey system, limit cycle, Dulac function.

MSC(2010) 34C07.

1. Introduction

In the recent decades, the relation between predators and preys is one of the dominant themes in theoretical ecology. A useful tool to understand and analyze the dynamic behavior of predator-prey systems is to analysis the Mathematical modeling. Predator functional response on prey population that describes the number of prey consumed per predator per unit time for given quantities of prey and predator is the major element in predator-prey interaction. The most important and useful functional responses are Lotka-Volterra functional responses such as Holling type I functional response, Holling type II functional response, Holling type III functional response and so on, see [7, 16]. Both prey and predator species which are subjected to a certain rate of harvesting have also been studied by many authors, such as [8, 15].

The following predator-prey model

$$\begin{aligned}\frac{dx}{dt} &= (a - bx^{\frac{1}{n}} - cx)x - \frac{\alpha x}{1+\beta x}y - q_1 ex,\\ \frac{dy}{dt} &= -dy + k\frac{\alpha x}{1+\beta x}y - q_2 ey,\end{aligned} \tag{1.1}$$

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was studied in [18] and [5] for $n = 3$ and $n = 4$ respectively. For any positive real number n, there is no general results for its dynamical analysis.

Consider a generalized Liénard system given by

$$\dot{x} = p(y) - F(x, \delta), \quad \dot{y} = -g(x), \tag{1.2}$$

where δ is a vector parameter in $\mathbb{R}^m$, p, F and g are analytic functions satisfying $p(0) = F(0, \delta) = g(0) = 0$. Then the origin O is a singular point of the system (1.2). We further suppose

$$F_x(0, \delta^*) = 0 \text{ for some } \delta^* \in \mathbb{R}^m, \quad p'(0) > 0, \quad g'(0) > 0,$$

which implies that the origin is an elementary center or focus of the system (1.2).

Let $G(x) = \int_0^x g(x)\mathrm{d}x$. Then there exists an involution function $\alpha(x) = -x + O(x^2)$ satisfying $G(\alpha(x)) \equiv G(x)$ for $|x| \ll 1$. To ensure the origin being a center of the system (1.2) Han [6] derived the following theorem.

Theorem 1.1. *For fixed $\delta \in \mathbb{R}^m$ and a sufficiently small constant $\varepsilon > 0$, if and only if $F(\alpha(x), \delta) \equiv F(x, \delta)$ for $0 \le x \le \varepsilon$, then the system* (1.2) *has a center at the origin.*

An equivalent form of the necessary and sufficient condition in Theorem 1.1 was obtained by Cherkas for the system (1.2) with $p(y) = y$ [3]. Gasull and Torregrosa developed the Cherkas method to investigate degenerate centers for (1.2) [4].

Limit cycle can be produced around the origin when we vary the vector parameter δ properly. The maximum number of small-amplitude limit cycles appearing near the origin is called Hopf cyclicity at the origin for the system (1.2). The next result of Li et al. [13] gives a method to determine the Hopf cyclicity at the origin. Some integrable conditions were obtained for cubic Z_2 systems, see [9–12]. For multiple limit cycles problem, some models have been investigated. For example, integrability and multiple limit cycles in a predator-prey system with fear effect was considered in [14]. Bifurcation of multiple limit cycles in an epidemic model on adaptive networks was solved in [20,21]. On the coexistence of multiple Limit cycles in H-bridge wireless power transfer systems with zero current switching control was considered in [1]. In [19], the multiple limit cycles problem of the dual-front axle steering heavy truck based on bisectional road was solved. Furthermore, bifurcation analysis of wheel shimmy with non-smooth effects and time delay in the tyre round contact was studied, and multiple limit cycles were obtained [2].

Multiplicity and Stability of Equilibrium States of Three-Dimensional Nonlinear System were also studied in some special system, such as [17].

Theorem 1.2. *Suppose that $F(\alpha(x), \delta) - F(x, \delta)$ has the following expansion*

$$F(\alpha(x), \delta) - F(x, \delta) = \sum_{i=1}^{+\infty} B_i(\delta)x^i.$$

If there exists $k \ge 1$ such that

$$\begin{aligned} &B_{2j+1}(\delta^*) = 0, \; j = 0, \cdots, k-1, \quad B_{2k+1}(\delta^*) \neq 0, \\ &\operatorname{rank} \frac{\partial(B_1, B_3, \cdots, B_{2k-1})}{\partial(\delta_1, \delta_2, \cdots, \delta_m)}\bigg|_{\delta=\delta^*} = k, \end{aligned}$$

then there are at most k limit cycles near the origin for all $|\delta - \delta^*|$ small, and for any given neighborhood U of the origin (1.2) *can have k limit cycles in U for some δ near δ^*.*

The organization of the paper is as follows, Section 2 is devoted to deal with the hopf bifurcation and center conditions of system (1.1). Existence and non-existence conditions of limit cycle of system (1.1) are discussed in Section 3.

2. The Hopf cyclicity of system (1.1)

Considering the practical significance predator-prey models, we only study the positive singular points of system (1.1). However, the only positive singular point of system (1.1) is

$$(x_1, y_1) = (x_1, \frac{(1+\beta x_1)}{\alpha x_1}(a - bx_1^{\frac{1}{n}} - cx_1 - q_1 e)x_1),$$

where $x_1 = \frac{d+q_2 e}{k\alpha - d\beta - eq_2\beta}$. Obviously, it is difficult to study its stability directly. System (1.1) can be transformed to

$$\begin{aligned} \frac{dx}{dt} &= -\frac{x}{n}(\alpha y + (-a + eq_1 + bx + cx^n)(1+\beta x^n)), \\ \frac{dy}{dt} &= -y(d + eq_2 - \alpha k x^n + d\beta x^n + e\beta q_2 x^n) \end{aligned} \tag{2.1}$$

by transformation

$$x = \tilde{x}^n,\ dt = (1+\beta x)d\tau,$$

we still use x to replace $\tilde{x}$ for simple. Namely

$$\begin{aligned} \frac{dx}{dt} &= \frac{1}{n}((a - eq_1)x - bx^2 + (a\beta - eq_1\beta - c)x^{n+1} - b\beta x^{n+2} - c\beta x^{2n+1} - \alpha xy), \\ \frac{dy}{dt} &= -y(d + eq_2) + (\alpha k - d\beta - e\beta q_2)x^n. \end{aligned} \tag{2.2}$$

Obviously, there is no positive singular point for system (2.2) when $\alpha k - d\beta - e\beta q_2 \le 0$, so we always suppose $\alpha k - d\beta - e\beta q_2 > 0$. Then, let $x = k_1\tilde{x}, y = k_2\tilde{y}, t = k_3\tau$, system can be rewritten as

$$\begin{aligned} \frac{dx}{dt} =& \frac{k_3}{n}(a - eq_1)x - \frac{bk_1k_3}{n}x^2 + \frac{a\beta - eq_1\beta - c}{n}k_1^n k_3 x^{n+1} \\ &- \frac{b\beta k_1^{n+1}k_3}{n}x^{n+2} - \frac{c\beta k_1^{2n}k_3}{n}x^{2n+1} - xy, \\ \frac{dy}{dt} =& -y + x^n y, \end{aligned} \tag{2.3}$$

where

$$k_1 = (\frac{d+eq_2}{k\alpha - d\beta - eq_2\beta})^{\frac{1}{n}},\ k_2 = \frac{n(d+eq_2)}{\alpha},\ k_3 = \frac{1}{d+eq_2}.$$

Furthermore, let $A_1 = \frac{k_3}{n}(a - eq_1), A_2 = -\frac{bk_1k_3}{n}, A_3 = \frac{a\beta - eq_1\beta - c}{n}k_1^n k_3, A_4 = -\frac{b\beta k_1^{n+1}k_3}{n}, A_5 = -\frac{c\beta k_1^{2n}k_3}{n}$, system (2.3) can be rewritten as

$$\begin{aligned} \frac{dx}{dt} &= A_1 x + A_2 x^2 + A_3 x^{n+1} + A_4 x^{n+2} + A_5 x^{2n+1} - xy = P(x, y), \\ \frac{dy}{dt} &= -y + x^n y = Q(x, y). \end{aligned} \tag{2.4}$$

Obviously, A_2, A_4, A_5 are negative. System (2.4) has a positive singular point $(1, A_1 + A_2 + A_3 + A_4 + A_5)$ when $A_1 + A_2 + A_3 + A_4 + A_5 > 0$.

In order to study its positive singular point $(1, A_1 + A_2 + A_3 + A_4 + A_5)$, by transformation $x = e^u, y = y_0 e^v$, where $y_0 = A_1 + A_2 + A_3 + A_4 + A_5$, system (2.4) can be changed to

$$\begin{aligned} \frac{du}{dt} &= \Phi(v) - F(u), \\ \frac{dv}{dt} &= -g(u), \end{aligned} \tag{2.5}$$

where

$$\begin{aligned} &\Phi(v) = y_0(e^v - 1), \\ &F(u) = A_1 + A_2 e^u + A_3 e^{nu} + A_4 e^{(n+1)u} + A_5 e^{2nu} - y_0, \\ &g(u) = e^{nu} - 1. \end{aligned}$$

Theorem 2.1. *System* (2.5) *has a center at the origin if and only if one of the following conditions holds:*

(I) $A_2 = A_3 = A_4 = A_5 = 0$*;*

(II) $A_2 = -A_3,\ A_4 = -A_5,\ n = 1$*;*

(III) $A_2 = -A_5,\ A_3 = A_4 = 0,\ n = 1/2$.

Furthermore, the system (2.5) *has the Hopf cyclicity* 4 *at the origin.*

Proof. It is straightforward to prove the sufficiency of the center conditions. Suppose that one of the conditions (I)-(III) holds. Then $F(u) \equiv A_1 - y_0$. Therefore, by Theorem 1.1 the system (2.5) has a center at the origin. It is easy to see that the system (2.5) becomes a Hamiltonian system when $F(u) \equiv A_1 - y_0$.

Next, we prove the necessity of the center conditions. Let $G(u) = \int_0^u g(u)du$. Then $G(u) = (\mathrm{e}^{nu} - 1)/n - u$. From $G(\alpha) = G(u)$, we derive

$$\begin{aligned} \alpha(u) = &- u - \frac{n}{3}u^2 - \frac{n^2}{9}u^3 - \frac{19n^3}{540}u^4 - \frac{17n^4}{1620}u^5 - \frac{13n^5}{4536}u^6 - \frac{229n^6}{340200}u^7 \\ &- \frac{923n^7}{8164800}u^8 + \frac{13n^8}{14696640}u^9 + \frac{40451n^9}{3464208000}u^{10} + \frac{3298063n^{10}}{509238576000}u^{11} \\ &+ \frac{191174143}{79441217856000}n^{11}u^{12} + \frac{622153}{972749606400}n^{12}u^{13} + \cdots . \end{aligned} \tag{2.6}$$

Substituting (2.6) into $F(\alpha) - F(u)$ yields

$$F(\alpha(u)) - F(u) = B_1 u + B_2 u^2 + B_3 u^3 + \cdots + B_9 u^9 + \cdots , \tag{2.7}$$

where

$$\begin{aligned}
B_1 =& A_2 + nA_3 + (1+n)A_4 + 2nA_5,\\
B_3 =& \frac{1}{6}(-A_2(-1+n) + 4A_5n^3 + A_4(1+n)^2),\\
B_5 =& \frac{1}{360}(-36A_5n^5 - A_4(1+n)^2(-3+n+11n^2) - A_2(-3+10n-7n^3)),\\
B_7 =& \frac{1}{15120}[540A_5n^7 - A_2(-3+21n-119n^3+101n^5)\\
&+ A_4(1+n)^2(3+n(-6+n(-54+n(23+169n))))],\\
B_9 =& \frac{1}{16329600}[-375900A_5n^9 - A_2(-45+540n-11886n^3\\
&+ 81320n^5 - 69929n^7) - A_4(1+n)^2(-45+n(225+n(2295+n(-5361\\
&+ n(-38319+n(17159+118021n))))))].
\end{aligned}$$

Then we can obtain

$$\begin{aligned}
A_2 &= -\frac{1}{2}(n+1)A_3 + (2n^2-n-1)A_5,\\
A_4 &= -\frac{1}{2(n+1)}[(n-1)A_3 + (4n^2+2n-2)A_5]
\end{aligned} \tag{2.8}$$

from $B_1 = B_3 = 0$. With (2.8) holding, for (2.7) we further derive

$$\begin{aligned}
B_5 &= -\frac{n(n-1)}{180}[(n+1)(2n-3)A_3 + 2(2n-1)(7n+3)A_5],\\
B_7 &= -\frac{n(n-1)}{1260}[(n+1)(6n^3-11n^2+3n-1)A_3 + 2(2n-1)(22n^3+5n^2+1)A_5].
\end{aligned}$$

Since $n > 0$, solving $B_5 = B_7 = 0$ we get

$$\{A_3 = A_5 = 0\},\ \text{or } \{n=1\},\ \text{or } \{A_3 = 0, n = 1/2\}, \tag{2.9}$$

$$\text{or } \left\{A_3 = \frac{-13+3\sqrt{33}}{2}A_5,\ n = \frac{-3+\sqrt{33}}{4}\right\}. \tag{2.10}$$

Then the equations in (2.8) and (2.9) yield the center conditions (I)-(III). If (2.8) and (2.10) hold, we obtain

$$B_9 = \frac{2553201\sqrt{33} - 14666751}{4}A_5.$$

If $B_9 = 0$, i.e. $A_5 = 0$, then we get a subcase of the center condition (I). Therefore, all the center conditions for the system (2.5) are obtained from $B_1 = B_3 = B_5 = B_7 = 0$.

For the Hopf cyclicity, we have

$$\det\left(\frac{\partial(B_1,B_3,B_5,B_7)}{\partial(A_2,A_3,A_4,n)}\right)\bigg|_{(2.8),(2.10)}$$
$$= \begin{vmatrix}
-2 & \frac{3-\sqrt{33}}{2} & \frac{-1-\sqrt{33}}{2} & \frac{3(\sqrt{33}-11)}{2}A_5\\
\frac{\sqrt{33}-7}{8} & \frac{(3-\sqrt{33})^3}{576} & \frac{-21-5\sqrt{33}}{48} & \frac{3\sqrt{33}-61}{16}A_5\\
\frac{829\sqrt{33}-4819}{5760} & \frac{17(3-\sqrt{33})^5}{1658880} & \frac{-11510-15\sqrt{33}}{11520} & \frac{1161\sqrt{33}-7591}{3840}A_5\\
\frac{231355\sqrt{33}-1329301}{1612800} & \frac{229(3-\sqrt{33})^7}{5573836800} & \frac{-26393+247\sqrt{33}}{3225600} & \frac{139(3637\sqrt{33}-20987)}{1075200}A_5
\end{vmatrix}$$

$$
\begin{aligned}
&= \frac{208681 - 36327\sqrt{33}}{1612800} A_5 \\
&\neq 0,
\end{aligned}
$$

when $A_5 \neq 0$. Then by Theorem 1.2, the system (2.5) has Hopf cyclicity 4 at the origin. □

Because two adjacent limit cycles have opposite stability, so multiple stable states can coexist in the predator-prey systems with harvesting rates.

3. Existence and non-existence of limit cycle of system (1.1)

In this section, we will discuss the existence and non-existence conditions of limit cycles of system (1.1) by using Poincare-Bendixson theorem and Dulac function method. In order to discuss this problem, we should consider its all singular points firstly. System (1.1) may have a boundary singular point which can be determined by

$$M(x) = (a - bx^{\frac{1}{n}} - cx - q_1 e)x = xh(x) = 0.$$

Theorem 3.1. *For system (1.1), when $a - q_1 E \leq 0$, there is no other singular point of system (1.1) except $(0,0)$. When $a - q_1 E > 0$, there is a boundary singular point $(x^*, 0)$, and it is a saddle.*

Proof. Because

$$\dot{h}(x) = -\frac{b}{n} x^{\frac{1}{n}-1} - c < 0,$$

so when $a - q_1 E \leq 0, h(0) = a - q_1 e < 0, h(x) < 0$ for any positive real number x, and there is no positive singular point.

When $a - q_1 E > 0, h(0) = a - q_1 e > 0$, there is a boundary singular point $(x^*, 0)$ where x^* satisfies $h(x^*) = 0$. Meanwhile, $h(x_1) > 0$, so $x_1 < x^*$.

The Jacobi Matrix at the boundary singular point $(x^*, 0)$ of system (1.1) is

$$\begin{bmatrix} x^* \dot{h}(x^*) & -\frac{\alpha x^*}{1+\beta x^*} \\ 0 & -d - q_2 e + \frac{k\alpha x^*}{1+\beta x^*} \end{bmatrix}.$$

So it is a saddle because $x^* > x_1$ which implies $-d - q_2 e + \frac{k\alpha x^*}{1+\beta x^*} > 0$. □

Theorem 3.2. *If system (1.1) has a positive singular point (x_1, y_1) and one of the following conditions holds:*

$$
\begin{aligned}
B_1 =& A_2 + nA_3 + (1+n)A_4 + 2nA_5 > 0, \\
B_1 =& 0,\ B_3 = \frac{1}{6}(-A_2(-1+n) + 4A_5 n^3 + A_4(1+n)^2) > 0, \\
B_1 =& B_3 = 0, \\
B_5 =& \frac{1}{360}(-36A_5 n^5 - A_4(1+n)^2(-3+n+11n^2) - A_2(-3+10n-7n^3)) > 0, \\
B_1 =& B_3 = 0,\ B_5 = 0, \\
B_7 =& \frac{1}{15120}[540A_5 n^7 - A_2(-3+21n-119n^3+101n^5)
\end{aligned}
$$

$$
\begin{aligned}
&\quad + A_4(1+n)^2(3+n(-6+n(-54+n(23+169n))))] \\
&>0, \\
B_1 =& B_3 = B_5 = B_7 = 0, \\
B_9 =& \frac{1}{16329600}[-375900A_5n^9 - A_2(-45+540n-11886n^3+81320n^5-69929n^7) \\
&- A_4(1+n)^2(-45+n(225+n(2295+n(-5361+n(-38319 \\
&+ n(17159+118021n))))))] \\
&>0,
\end{aligned}
$$

then there exists at least one stable limit cycle in neighborhood of the positive singular point.

Proof. At first, let $l_1 : x = x^*$, then

$$\frac{dl_1}{dt} = (a - b(x^*)^{\frac{1}{n}} - cx^*)x^* - \frac{\alpha x^*}{1+\beta x^*}x^* - q_1ex^* = -\frac{\alpha x^*}{1+\beta x^*}y^* < 0.$$

Furthermore, let $l_2 : y = -kx + D$, then

$$\frac{dl_2}{dt} = \dot{y} + k\dot{x} = -(d+q_2E)D + kx(a+d-Eq_1+Eq_2-cx-bx^{\frac{1}{n}}).$$

So $\frac{dl_2}{dt} < 0$ for sufficient large D. Moreover, $x = 0, y = 0$ are solutions of system (1.1), we can construct the outer boundary of Bendixson ring domain by $l_1, l_2, x = 0, y = 0$, the positive singular point (x_1, y_1) is unstable, namely, the positive singular point of system system (1.1) is unstable, so according to Poincare-Bendixson theorem, there exists at least one stable limit cycle in neighborhood of the positive singular point which implies there exists at least one stable limit cycle in neighborhood of the positive singular point $(1, A_1 + A_2 + A_3 + A_4 + A_5)$. □

Remark 3.1. There is no limit cycle when $a - eq_1 \le 0$. When $a - eq_1 > 0$, if system (1.1) has a positive singular point (x_1, y_1), then $x^* > x_1$, and system has a limit cycle, some numerical simulations are given in Figure 1.

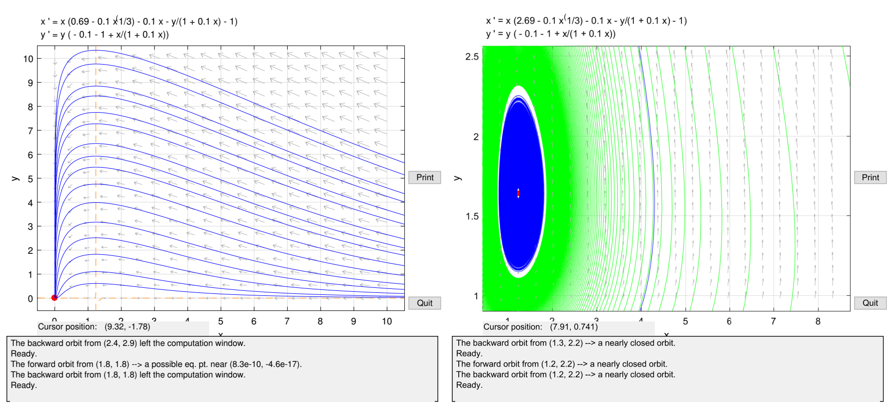

Figure 1. Phase portrait of system (1.1) showing that there is no other singular point and limit cycle when $a - eq_1 \le 0$ and there is three singular points and a limit cycle when $a - eq_1 > 0$.

Theorem 3.3. *For system* (2.5)*, if one of the following conditions holds:*

$$
\begin{aligned}
&(1)\ A_1 \le 1, A_3 \le -1;\\
&(2)\ A_3 \le 0, A_1 + A_3 > 0;\\
&(3)\ A_3 \ge 0, A_1 + A_3 < 0.
\end{aligned}
$$

then there exists no limit cycle in neighborhood of the positive singular point $(1, A_1 + A_2 + A_3 + A_4 + A_5)$.

Proof. Let $F(x, y) = \ln x + \ln y = c (c > 0)$, then

$$\frac{dF}{dt} = (A_1 - 1) + A_2 x + (A_3 + 1)x^n + A_4 x^{n+1} + A_5 x^{2n} - y.$$

When $A_1 \le 1, A_3 \le -1$, $\frac{dF}{dt} < 0$, so there exists no limit cycle in neighborhood of the positive singular point $(1, A_1 + A_2 + A_3 + A_4 + A_5)$.

Let $B(x, y) = x^\alpha y^\beta$, then

$$
\begin{aligned}
\frac{\partial(BP)}{\partial x} + \frac{\partial(BQ)}{\partial y} =&(A_1(\alpha + 1) - (\beta + 1))x^\alpha y^\beta + A_2(\alpha + 2)x^{\alpha+1} y^\beta\\
&- (\alpha + 1)x^\alpha y^{\beta+1} + A_5(1 + 2n + \alpha)x^{2n+\alpha} y^\beta + (A_3(n + 1 + \alpha)\\
&+ (\beta + 1))x^{n+\alpha} y^\beta + A_4(2 + n + \alpha)x^{n+1+\alpha} y^\beta.
\end{aligned}
$$

From

$$A_1(\alpha + 1) - (\beta + 1) = 0,\ A_3(n + 1 + \alpha) + (\beta + 1) = 0,$$

we can get $\alpha = -\frac{A_1+(n+1)A_3}{A_1+A_3}$. If $\alpha \ge -1$, we have

$$
\begin{aligned}
\frac{\partial(BP)}{\partial x} + \frac{\partial(BQ)}{\partial y} =&A_2(\alpha + 2)x^{\alpha+1} y^\beta - (\alpha + 1)x^\alpha y^{\beta+1}\\
&+ A_5(1 + 2n + \alpha)x^{2n+\alpha} y^\beta + A_4(2 + n + \alpha)x^{n+1+\alpha} y^\beta\\
\le&0,
\end{aligned}
$$

because A_2, A_4, A_5 are negative. $\alpha \ge -1$ means that $A_3 \le 0, A_1 + A_3 > 0$ or $A_3 \ge 0, A_1 + A_3 < 0$. So there is no limit cycle according to Dulac discriminant method. □

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