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DELAYED CONSENSUS IN MEAN-SQUARE OF MASS UNDER MARKOV SWITCHING TOPOLOGIES AND BROWN NOISE

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Abstract The delayed consensus in mean-square issue of nonlinear multi-agent systems (NMASs) under uncertain nonhomogeneous Markov switching (UNMS) topologies and Brown noise is investigated in this paper. Firstly, there are two delays $d(t)$ and τ. $d(t)$ represents the time-varying delay among followers. τ stands for the delay between the leader and the followers, which is the delay in delayed consensus in mean-square. When $\tau = 0$, the delayed consensus degenerates to identical consensus. Secondly, the random communication topologies are modeled as nonhomogeneous Markov switching topologies in which the transition rates (TRs) are partially or totally unknown. Further, communication noise is also considered, which is assumed to be Brown noise. Sufficient conditions of delayed consensus in mean-square for the systems are gained on account of qualitative and stability theory, theory of random differetntial equations and distributed control theory. Finally, the correctness of the results is verified through the example given.

Keywords Nonlinear multi-agent systems, delayed consensus in mean-square, Markov switching topologies, Brown noise.

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1. Introduction

The agent is a computing entity that can continuously and autonomously play its role in a particular environment. Multi-agent refers to any independent entity that has intelligence and can interact with the environment in complex systems. Multi-agent systems (MASs) are networked systems consisted of multi-agent with sensing, interaction, calculation and implementation capabilities through mutual interaction and coordination. MASs have applications in many fields, such as UAV

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[23], formation [3], continuous monitoring [27], etc. The primary researches of MASs include the cluster problems [8], tracking problems [5], controllability problems [17], consensus problems [1, 15, 18, 36], etc.

Consensus is the process of exchanging message from each agent in MASs to adjacent agent, which ultimately enables each individual to achieve the same state. The consensus problems are one of the research hotspots for distributed coordination control of MASs. They are usually studied with identical consensus [35], partial component consensus [9, 10, 28, 30] and delayed consensus [19, 24–26]. DeGroot [1] pioneered the consensus problem in management and statistical sciences for solving the acquisition of information with uncertainty by multiple sensors. Identical consensus means that there are N agents in the considered MASs, where the ith agent satisfies $\lim_{t\to\infty} \|x_i(t) - x_j(t)\| = 0$. In some sense, identical consensus is harmful and needs to be avoided, while delayed consensus is beneficial and needs to be promoted. For example, a certain number of vehicles on the highway may cause traffic jams when they arrive at a specific position at the same time, however, if they have a suitable time delay in the arrival time, it will not cause traffic jams. Delayed consensus refers to that the state vectors of j followers reaches the previous state of the leader within a specific time, and there is a time delay τ_j between them. Here, $\tau_j = \tau$ for the convenience of the study. When $\tau = 0$, the delayed consensus degenerates to identical consensus. People first studied the delayed consensus problems of first-order MASs [16, 29]. In recent years, delayed consensus of second-order MASs has received more and more attention (such as [24–26]). In [26], the issue of pinned controlled delayed consensus for NMASs was studied. In [24], a per-follower control protocol with the local information from adjacent agents was introduced for delayed consensus of second-order NMASs. In [25], a cluster-delayed consensus approach was submitted for the congestion problem of second-order NMASs on capacity-constrained paths. In [24–26], the network communication topologies were all deterministic topologies. However, in open communication networks, various reasons, such as noise, cyber attacks and energy depletion, may lead to stochastic changes in the communication topologies. For the above reasons, the issue of delayed consensus in mean-square of NMASs under stochastic communication topologies is investigated in this paper.

Switching systems are made up of several subsystems and a decision of individual subsystems when a particular subsystem is activated [7], and have extensive applications in networked control systems, aircraft systems and power control systems (such as [31–34]). There are many achievements in modeling communication topologies as Markov switching topologies to achieve consensus of MASs [2, 11, 21, 22]. Ding et al. [2] researched consensus issue for NMASs in Markov switching topologies and assumed that the TRs were time-varying. Li et al. [11] researched mean-square consensus issue of NMASs with random switching topologies as well as noise. However, the TRs are time-varying to be more realistic. For this reason, Li et al. [11] further extended the case where switching topologies were semi-Markov switching topologies and the TRs were partially unknown. Wang et al. [22] studied leader-following consensus problem for NMASs subjected to network attacks in UNMS. Building on [22], Wang et al. [21] further studied consensus issue of MASs with time-varying delay. In [22] and [21], the Markov switching considered was uncertain and nonhomogeneous, which meaned that its TRs were partially or even completely unknown. The consensus studied in [2, 11, 21, 22] was all identical consensus.

MASs are inevitably disturbed by environmental noise during operation and noise is usually modeled as standard Brown motion or even as more complex fractional Brown motion (such as [37–39]). In [26] and [24], the authors not only studied delayed consensus issue for MASs but also considered the case of the effects brought by noise. In [4], the authors not only examined mean-square consensus issue of heterogeneous NMASs with communication noise, but also gave sufficient conditions of consensus with a leader and without a leader. Moreover, information exchange in networks is usually not instantaneous, and time-varying delay is prevalent in practical networked control systems. In [13], Ma et al. studied delayed consensus issue of discrete-time MASs under communication delays. Sun et al. [20] studied consensus issue of continuous-time NMASs under random topologies as well as nonuniform time-varying delay.

According to previous analysis, the delayed consensus in mean-square of NMASs with stochastic communication topologies and Brown noise is investigated. The innovations are as follows:

(i) The delayed consensus in mean-square of the NMASs is considered, which generalizes the delayed consensus of deterministic systems and identical consensus. The stochastic communication topologies are modeled as UNMS topologies, and the TRs are partly or even totally unknown. The communication noise is designed as Brown noise.

(ii) The designed controller is advanced and contains two different types of delays (the delay τ between the leader and the followers and the delay $d(t)$ among the followers), Brown noise and UNMS topologies, which brings substantial challenges to the design of the controller.

(iii) In [24,26], the authors considered delayed consensus in a noisy environment and containment control. However, time-varying delays among followers were not considered and the communication topologies were deterministic. Different from identical consensus studied under Markov switching topologies in [6, 21, 22], the delayed consensus in mean-square is studied and the effects brought by two time delays and Brown noise on this basis are considered. The authors studied delayed consensus of discrete time NMASs under the communication delays among followers in [13, 14], but UNMS topologies and Brown noise were not considered.

Structure of this paper is shown as below, preparatory knowledge will be used in Sect.2. Sufficient conditions of delayed consensus in mean-square for NMASs under UMNS topologies and Brown noise are given in Sect.3. In Sect.4, numerical simulation is given. In Sect.5, draw conclusions and give outlook.

2. Preliminary

2.1. UNMS process

The basic content of Markov switching process will be introduced below. $\{\vartheta(t), t \geq 0\}$ denotes a discrete-state UNMS process on the complete probability space $(\mathscr{O}, \mathscr{X}, \mathrm{P})$ and takes the value in the finite set $\mathbf{Z} = \{1, 2, \ldots, z\}$. The transition probabilities are as below

$$\mathrm{P}(\vartheta(t+\Delta) = s \mid \vartheta(t) = r) = \begin{cases} \kappa_{rs}(h)\Delta + o(\Delta), & s \neq r, \\ 1 + \kappa_{rr}(h)\Delta + o(\Delta), & s = r, \end{cases}$$

where $\lim_{\Delta\to 0}\frac{o(\Delta)}{\Delta}=0$, $\kappa_{rs}(h)$ stands for the TRs of state r to state s, which are time-varying, when $r=s$, $\kappa_{rr}(h)=-\sum_{s=1,s\neq r}^{z}\kappa_{rs}(h)$ $(\forall r\in\mathbf{Z})$. $\bar{\kappa}_{rs}$ and $\underline{\kappa}_{rs}$ stand for upper bound and lower bound of $\kappa_{rs}(h)$, respectively. Define the TRs matrix of Markov chain as

$$\Pi=\begin{pmatrix} ? & \kappa_{12}(h) & ? & \ldots & \kappa_{1z}(h) \\ \kappa_{21}(h) & ? & \kappa_{23}(h) & \ldots & ? \\ \kappa_{31}(h) & ? & ? & \ldots & ? \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ ? & \kappa_{z2}(h) & ? & \ldots & ? \end{pmatrix},$$

where "?" represent different unknown elements. Define $\mathbf{Z}_k^r=\{s\mid\kappa_{rs}(h)\text{ is known}\}$, $\mathbf{Z}_{uk}^r=\{s\mid\kappa_{rs}(h)\text{ is unknown}\}$, obviously $\mathbf{Z}=\mathbf{Z}_k^r+\mathbf{Z}_{uk}^r$.

Remark 2.1. Different from the deterministic communication topologies in [24,26], stochastic switching topologies of NMASs are modeled as UNMS topologies and the TRs are partly or totally unknown.

2.2. Uncertain nonhomogeneous Markov switching topologies

Assume that $\mathcal{G}=(\mathcal{V},\varepsilon,\mathcal{W})$ stands for the communication topology graph of the agents, where $\mathcal{V}=\{1,2,\ldots,\bar{n}\}$ is the node set, $\varepsilon\subseteq\mathcal{V}\times\mathcal{V}$ represents the set of edges, and $\mathcal{W}=[w_{ij}]\in\mathbb{R}^{n\times n}$ stands for the adjacency matrix. An edge $\varepsilon_{ij}=(j,i)\in\varepsilon$ denotes that messages are able to be transferred from j to i. If $\varepsilon_{ij}\in\varepsilon$, then $w_{ij}>0$, otherwise, $w_{ij}=0$. The Laplacian matrix $L=[l_{ij}]\in\mathbb{R}^{n\times n}$ is described as $l_{ij}=-w_{ij}$ $(i\neq j)$ and $l_{ii}=\sum_{j=1}^{n}w_{ij}$.

$\mathcal{G}_{\vartheta(t)}=\{\mathcal{V},\varepsilon_{\vartheta(t)},\mathcal{W}_{\vartheta(t)}\}$ are uncertain nonhomogeneous Markov switching topologies among agents at time t, where $\vartheta(t)$ is nonhomogeneous Markov chain and takes the value in the finite set $\mathbf{Z}=\{1,2,\ldots,z\}$.

Remark 2.2. It is inevitable that NMASs are interfered during its operation, which leads to the change of communication topologies and makes deterministic topologies no longer applicable. Therefore, it is appropriate to assume that the communication topologies are random Markov switching topologies.

2.3. Modeling the systems

The dynamics model of the leader is described as

$$\begin{aligned}\dot{\chi}_0(t)&=\nu_0(t),\\ \dot{\nu}_0(t)&=g\left(\chi_0(t),\nu_0(t)\right),\end{aligned}\tag{2.1}$$

where $\chi_0(t)\in\mathbb{R}^m$ ($\mathbb{R}^m$ stands for the m-dimensional real space), $\nu_0(t)\in\mathbb{R}^m$, $\chi_0(0)\in\mathbb{R}^m$ represent position vector, velocity vector and the initial value of the leader, respectively. $g\left(\chi_0(t),\nu_0(t)\right)$ stands for the nonlinear vector function of leader and satisfies the Lipschitz condition, which will be given later.

The MASs are made up of n followers. The dynamics model of the ith follower is described as

$$\begin{aligned}\dot{\chi}_i(t) &= \nu_i(t),\\ \dot{\nu}_i(t) &= g\left(\chi_i(t), \nu_i(t)\right) + u_i(\vartheta(t), t),\end{aligned} \tag{2.2}$$

where $\chi_i(t) \in \mathbb{R}^m$, $\nu_i(t) \in \mathbb{R}^m$, $f\left(\chi_i(t), \nu_i(t)\right)$, $u_i(\vartheta(t), t)$ stand for position vector, velocity vector, the nonlinear vector function and the control input vector of the ith follower, respectively. $\chi_i(0) \in \mathbb{R}^m$ and $\nu_i(0) \in \mathbb{R}^m$ represent the initial value of the ith follower.

The control input vector $u_i(\vartheta(t), t)$ is described as

$$\begin{aligned}u_i(\vartheta(t), t) = &-\alpha(t) \sum_{j=1}^{n} w_{ij(\vartheta(t))} \left[(\chi_i(t) - \chi_j(t)) + (\nu_i(t) - \nu_j(t))\right]\\ &- \sum_{j=1}^{n} w_{ij(\vartheta(t))} \left[(\chi_i(t - d(t)) - \chi_j(t - d(t)))\right]\\ &- \sum_{j=1}^{n} w_{ij(\vartheta(t))} \left[(\nu_i(t - d(t)) - \nu_j(t - d(t)))\right]\\ &- \alpha(t) b_{i(\vartheta(t))} \left[(\chi_i(t) - \chi_0(t - \tau)) + (\nu_i(t) - \nu_0(t - \tau))\right]\\ &- \varsigma \sum_{j=1}^{n} w_{ij(\vartheta(t))} \left[(\chi_i(t) - \chi_j(t)) + (\nu_i(t) - \nu_j(t))\right] \dot{\omega}_i(t),\end{aligned} \tag{2.3}$$

where $\alpha(t)$ is the adaptive control law and will be given later. Feedback gain $b_{i(\vartheta(t))}$ are nonnegative numbers and $b_{i(\vartheta(t))} > 0$ means that messages are exchanged between the leader and the ith follower. $\omega_i(t)$ is the n-dimensional standard Brown motion. $d(t)$ is time-varying delay among the followers and has $0 \leq \dot{d}(t) \leq \bar{d}$, where $\bar{d}$ is the upper bound of $\dot{d}(t)$. ς is the noise intensity. τ is the delay between the leader and followers.

Remark 2.3. In the controller (2.3), the delay among followers, the delay between leader and followers, UNMS topologies and Brown noise are considered, which brings substantial challenges to the design of the controller. Noise between leader and followers is not taken into account because the calculation is too complicated, which may be considered in future work.

Let $\hat{\chi}_i(t) = \chi_i(t) - \chi_0(t - \tau)$, $\hat{\nu}_i(t) = \nu_i(t) - \nu_0(t - \tau)$, $\hat{\aleph}(t) = \left(\hat{\chi}(t), \hat{\nu}(t)\right)^T$ with $\hat{\chi}(t) = \left[\hat{\chi}_1^T(t), \hat{\chi}_2^T(t), \ldots, \hat{\chi}_n^T(t)\right]^T$ and $\hat{\nu}(t) = \left[\hat{\nu}_1^T(t), \hat{\nu}_2^T(t), \ldots, \hat{\nu}_n^T(t)\right]^T$, $\vartheta(t) = r$. Combining (2.1) - (2.3), one obtains

$$\begin{aligned}d\hat{\aleph}(t) = &\left\{ \begin{bmatrix} 0 & I_n \\ -\alpha(t)\left(L_r + B_r\right) & -\alpha(t)\left(L_r + B_r\right) \end{bmatrix} \otimes I_m \hat{\aleph}(t) + \begin{bmatrix} 0 \\ \mathscr{G}(t) \end{bmatrix} \right.\\ &\left. + \begin{bmatrix} 0 & 0 \\ -L_r & -L_r \end{bmatrix} \otimes I_m \hat{\aleph}(t - d(t)) \right\} dt + \begin{bmatrix} 0 & 0 \\ -\varsigma L_r & -\varsigma L_r \end{bmatrix} \otimes I_m \hat{\aleph}(t) d\omega(t),\end{aligned} \tag{2.4}$$

where $G(\chi(t), \nu(t)) = \left[g^T\left(\chi_1(t), \nu_1(t)\right), g^T\left(\chi_2(t), \nu_2(t)\right), \ldots, g^T\left(\chi_n(t), \nu_n(t)\right)\right]^T$, $\mathscr{G}(t) = G(\chi(t), \nu(t)) - 1_n \otimes g\left(\chi_0(t - \tau), \nu_0(t - \tau)\right)$, $B_r = diag\left(b_{1,r}, b_{2,r}, \ldots, b_{n,r}\right)$,

L_r is the Laplacian matrix of $\mathcal{G}_r$. $\otimes$ is the Kronecker product. $\hat{\aleph}(0)$ is the initial value. I_m stands for the m-dimensional identity matrix.

The adaptive control law $\alpha(t)$ is satisfied

$$\dot{\alpha}(t) = \hat{\aleph}^T(t) \begin{bmatrix} \Im_r & \Im_r \\ \Im_r & \Im_r \end{bmatrix} \otimes I_m \hat{\aleph}(t), \tag{2.5}$$

where $\Im_r = P_r(L_r + B_r) + (L_r + B_r)^T P_r$. P_r are unknown matrices. $\alpha(0)$ is the initial value.

Remark 2.4. The adaptive control law can reduce the control input and the energy consumption.

Definition 2.1 ([12, 26]). NMASs (2.1) and (2.2) realize delayed consensus in mean-square, if the following conditions are satisfied

$$\lim_{t\to+\infty} \boldsymbol{E}\left\{\|\chi_i(t) - \chi_0(t-\tau)\|^2\right\} = 0,$$

$$\lim_{t\to+\infty} \boldsymbol{E}\left\{\|\nu_i(t) - \nu_0(t-\tau)\|^2\right\} = 0,$$

where $\tau > 0$, $\boldsymbol{E}$ represents mathematical expectation.

Assumption 2.1 ([26]). *(Lipschitz Conditions) For all* $\chi_1(t),\ \chi_2(t),\ \nu_1(t),\ \nu_2(t) \in \mathbb{R}^m$, *there exist* $\varpi_1 \geq 0$ *and* $\varpi_2 \geq 0$, *such that*

$$\|g(\chi_1(t), \nu_1(t)) - g(\chi_2(t), \nu_2(t))\| \leq \varpi_1 \|\chi_1(t) - \chi_2(t)\| + \varpi_2 \|\nu_1(t) - \nu_2(t)\|.$$

3. Main results

Theorem 3.1. *For given scalars* $\beta > \lambda_{\max}(P_r)/\lambda_{\min}(\Im_r)$, $\ell_1 > 0$, $\ell_2 > 0$, $\varpi_1 \geq 0$, $\varpi_2 \geq 0$, *under Assumption* 2.1, *NMASs* (2.1) *and* (2.2) *can reach delayed consensus in mean-square, if there exist matrices* $\Xi > 0$, $P_r > 0$ *and* $Z_r \in \mathbb{R}^{n\times n}$, *such that*

$$\Theta_1 = \begin{bmatrix} \Upsilon_1 + \eth + \Xi & \Upsilon_2 \\ * & (\bar{d}-1)\Xi \end{bmatrix} < 0,\ \ r \in \mathbf{Z}_k^r, \tag{3.1}$$

$$\Theta_2 = \begin{bmatrix} \Upsilon_1 + \Psi_1 + \Xi & \Upsilon_2 \\ * & (\bar{d}-1)\Xi \end{bmatrix} < 0,\ \ r \in \mathbf{Z}_{uk}^r, \tag{3.2}$$

$$\bar{\Lambda}_s + \bar{Z}_r \leq 0,\ \ \forall s \in \mathbf{Z}_{uk}^r,\ \ s \neq r, \tag{3.3}$$

$$\bar{\Lambda}_s + \bar{Z}_r \geq 0,\ \ \forall s \in \mathbf{Z}_{uk}^r,\ \ s = r, \tag{3.4}$$

where

$$\Lambda_r = \begin{bmatrix} \beta\Im_r & P_r \\ P_r & P_r \end{bmatrix} \otimes I_m,\ \ \bar{\Lambda}_r = \begin{bmatrix} \beta\Im_r & P_r \\ P_r & P_r \end{bmatrix},\ \ \bar{\Lambda}_s = \begin{bmatrix} \beta\Im_s & P_s \\ P_s & P_s \end{bmatrix},$$

$$\Lambda_s = \begin{bmatrix} \beta\Im_s & P_s \\ P_s & P_s \end{bmatrix} \otimes I_m,\ \ \bar{Z}_r = \begin{bmatrix} Z_r & 0 \\ 0 & Z_r \end{bmatrix},\ \ \Upsilon_2 = -\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \otimes (P_r L_r) \otimes I_m,$$

$\Psi_1 = \sum_{s\in\mathbf{Z}_k^r} \kappa_{rs}(h)\left(\bar{\Lambda}_s + \bar{Z}_r\right)\otimes I_m,\ \ \Im_r = P_r\left(L_r + B_r\right) + \left(L_r + B_r\right)^T P_r,$
$\eth = \kappa_{rr}(h)\Lambda_r + \sum_{s=1,s\neq r}^{z} \kappa_{rs}(h)\Lambda_s + \sum_{s\in\mathbf{Z}_k^r} \kappa_{rs}(h)\bar{Z}_r\otimes I_m,$
$\Omega = \varsigma^2\left\|L_r^T\lambda_{\max}\left(P_r\right)L_r\right\|,\ \ \varrho_1 = 3\varpi_1 + \varpi_2,\ \ \varrho_2 = \varpi_1 + 3\varpi_2 + 2,$
$\Upsilon_1 = \begin{bmatrix} \varrho_1 P_r - \beta\Im_r + \Omega I_n & 0 \\ 0 & \varrho_2 P_r - \beta\Im_r + \Omega I_n \end{bmatrix}\otimes I_m.$

Proof. The Lyapunov function is constructed as

$$V(t) = \hat{\aleph}^T(t)\Lambda_r\hat{\aleph}(t) + \int_{t-d(t)}^{t}\hat{\aleph}^T(s)\Xi\hat{\aleph}(s)ds + \frac{1}{2}(\alpha(t)-\beta)^2, \tag{3.5}$$

where

$$\Xi = \begin{bmatrix} \Xi_{11} & \Xi_{12} \\ * & \Xi_{22} \end{bmatrix}\otimes I_m \in \mathbb{R}^{2mn\times 2mn},\ \Lambda_r = \begin{bmatrix} \beta\Im_r & P_r \\ P_r & P_r \end{bmatrix}\otimes I_m \in \mathbb{R}^{2mn\times 2mn},$$

it is easy to know that $\Lambda_r > 0$ iff $\beta > \lambda_{\max}\left(P_r\right)/\lambda_{\min}\left(\Im_r\right)$.

From the It$\hat{o}$ formula, one has

$$\begin{aligned}
&\boldsymbol{E}\left\{\mathcal{L}V(t)\right\} \\
=&2\hat{\aleph}^T(t)\begin{bmatrix} \beta\Im_r & P_r \\ P_r & P_r \end{bmatrix}\begin{bmatrix} 0 & I_n \\ -\alpha(t)\left(L_r+B_r\right) & -\alpha(t)\left(L_r+B_r\right) \end{bmatrix}\otimes I_m\hat{\aleph}(t) \\
&+\varsigma^2\ \text{trace}\ \left(\hat{\aleph}^T(t)\begin{bmatrix} 0 & 0 \\ L_r & L_r \end{bmatrix}^T\begin{bmatrix} \beta\Im_r & P_r \\ P_r & P_r \end{bmatrix}\begin{bmatrix} 0 & 0 \\ L_r & L_r \end{bmatrix}\otimes I_m\hat{\aleph}(t)\right) \\
&+2\hat{\aleph}^T(t)\begin{bmatrix} \beta\Im_r & P_r \\ P_r & P_r \end{bmatrix}\otimes I_m\begin{bmatrix} 0 \\ \mathscr{G}(t) \end{bmatrix} + \hat{\aleph}^T(t)\left[\sum_{s=1}^{z}\kappa_{rs}(h)\Lambda_r\right]\hat{\aleph}(t) \\
&+2\hat{\aleph}^T(t)\begin{bmatrix} \beta\Im_r & P_r \\ P_r & P_r \end{bmatrix}\begin{bmatrix} 0 & 0 \\ -L_r & -L_r \end{bmatrix}\otimes I_m\hat{\aleph}(t-d(t)) \\
&+\hat{\aleph}^T(t)\Xi\hat{\aleph}(t) - (1-\dot{d}(t))\hat{\aleph}^T(t-d(t))\Xi\hat{\aleph}(t-d(t)) + (\alpha(t)-\beta)\dot{\alpha}(t).
\end{aligned} \tag{3.6}$$

Due to $\sum_{s=1}^{z}\kappa_{rs}(h) = 0$, for matrices $\bar{Z}_r, r\in\mathbf{Z}$, one gets

$$\sum_{s=1}^{z}\kappa_{rs}(h)\hat{\aleph}^T(t)\left(\bar{Z}_r\otimes I_m\right)\hat{\aleph}(t) = 0, \tag{3.7}$$

then

$$\begin{aligned}
\hat{\aleph}^T(t)\left[\sum_{s=1}^{z}\kappa_{rs}(h)\Lambda_r\right]\hat{\aleph}(t) =&\hat{\aleph}^T(t)\sum_{s\in\mathbf{Z}_k^r}\kappa_{rs}(h)\left(\bar{\Lambda}_s + \bar{Z}_r\right)\otimes I_m\hat{\aleph}(t) \\
&+\hat{\aleph}^T(t)\sum_{s\in\mathbf{Z}_{uk}^r}\kappa_{rs}(h)\left(\bar{\Lambda}_s + \bar{Z}_r\right)\otimes I_m\hat{\aleph}(t).
\end{aligned} \tag{3.8}$$

Combining (3.6) and (3.8), one obtains

$$
\begin{aligned}
\boldsymbol{E}\{\mathcal{L}V(t)\} \leq & \hat{\aleph}^T(t)\begin{bmatrix} -\beta\Im_r & 0 \\ 0 & 2P_r-\beta\Im_r \end{bmatrix} \otimes I_m \hat{\aleph}(t) \\
& + 2\hat{\aleph}^T(t)\begin{bmatrix} \beta\Im_r & P_r \\ P_r & P_r \end{bmatrix} \otimes I_m \begin{bmatrix} 0 \\ \mathscr{G}(t) \end{bmatrix} \\
& + \varsigma^2 \left\| L_r^T \lambda_{\max}(P_r) L_r \right\| \hat{\aleph}^T(t) \begin{bmatrix} I_n & 0 \\ 0 & I_n \end{bmatrix} \otimes I_m \hat{\aleph}(t) \\
& + \hat{\aleph}^T(t) \sum_{s\in\mathbf{Z}_k^r} \kappa_{rs}(h)\left(\bar{\Lambda}_s+\bar{Z}_r\right) \otimes I_m \hat{\aleph}(t) \\
& - 2\hat{\aleph}^T(t)\begin{bmatrix} P_rL_r & P_rL_r \\ P_rL_r & P_rL_r \end{bmatrix} \otimes I_m \hat{\aleph}(t-d(t)) \\
& - (1-\bar{d})\hat{\aleph}^T(t-d(t))\Xi\hat{\aleph}(t-d(t)) \\
& + \hat{\aleph}^T(t)\Xi\hat{\aleph}(t) + \hat{\aleph}^T(t) \sum_{s\in\mathbf{Z}_{uk}^r} \kappa_{rs}(h)\left(\bar{\Lambda}_s+\bar{Z}_r\right) \otimes I_m \hat{\aleph}(t).
\end{aligned} \tag{3.9}
$$

According to Assumption 2.1, one gets

$$
\begin{aligned}
& 2\hat{\aleph}^T(t)\begin{bmatrix} \beta\Im_r & P_r \\ P_r & P_r \end{bmatrix} \otimes I_m \begin{bmatrix} 0 \\ \mathscr{G}(t) \end{bmatrix} \\
=& 2\left(\hat{x}^T(t)+\hat{v}^T(t)\right)(P_r\otimes I_m)\mathscr{F}(t) \\
=& 2\sum_{i=1}^n p_i\hat{x}_i^T(t)\left[f_i\left(x_i(t),v_i(t)\right)-f\left(x_0(t-\tau(t)),v_0(t-\tau(t))\right)\right] \\
& + 2\sum_{i=1}^n p_i\hat{v}_i^T(t)\left[f_i\left(x_i(t),v_i(t)\right)-f\left(x_0(t-\tau(t)),v_0(t-\tau(t))\right)\right] \\
\leq& 2\varpi_1\sum_{i=1}^n p_i\left\|\hat{\chi}_i(t)\right\|^2 + \varpi_2\sum_{i=1}^n p_i\left(\left\|\hat{\chi}_i(t)\right\|^2+\left\|\hat{\nu}_i(t)\right\|^2\right) \\
& + \varpi_1\sum_{i=1}^n p_i\left(\left\|\hat{\chi}_i(t)\right\|^2+\left\|\hat{\nu}_i(t)\right\|^2\right) + 2\varpi_2\sum_{i=1}^n p_i\left\|\hat{\nu}_i(t)\right\|^2 \\
=& (3\varpi_1+\varpi_2)\sum_{i=1}^n p_i\left\|\hat{x}_i(t)\right\|^2 + (3\varpi_2+\varpi_1)\sum_{i=1}^n p_i\left\|\hat{v}_i(t)\right\|^2 \\
=& \hat{\aleph}^T(t)\begin{bmatrix} (3\varpi_1+\varpi_2)P_r & 0 \\ 0 & (3\varpi_2+\varpi_1)P_r \end{bmatrix} \otimes I_m \hat{\aleph}(t).
\end{aligned} \tag{3.10}
$$

Taking (3.10) into (3.9), a new inequality can be obtained

$$\begin{aligned}\boldsymbol{E}\left\{\mathcal{L}V(t)\right\} \leq & \hat{\aleph}^T(t)\begin{bmatrix}\varrho_1 P_r - \beta\Im_r + \varOmega I_n & 0\\ 0 & \varrho_2 P_r - \beta\Im_r + \varOmega I_n\end{bmatrix}\otimes I_m\hat{\aleph}(t)\\ & + \hat{\aleph}^T(t)\sum_{s\in\mathbf{Z}_k^r}\kappa_{rs}(h)\left(\bar{\varLambda}_s + \bar{Z}_r\right)\otimes I_m\hat{\aleph}(t)\\ & + \hat{\aleph}^T(t)\sum_{s\in\mathbf{Z}_{uk}^r}\kappa_{rs}(h)\left(\bar{\varLambda}_s + \bar{Z}_r\right)\otimes I_m\hat{\aleph}(t)\\ & + \hat{\aleph}^T(t)\varXi\hat{\aleph}(t) - (1-\bar{d})\hat{\aleph}^T(t-d(t))\varXi\hat{\aleph}(t-d(t))\\ & - 2\hat{\aleph}^T(t)\begin{bmatrix}P_rL_r & P_rL_r\\ P_rL_r & P_rL_r\end{bmatrix}\otimes I_m\hat{\aleph}(t-d(t)),\end{aligned}\tag{3.11}$$

where $\varOmega = \varsigma^2\left\|L_r^T\lambda_{\max}\left(P_r\right)L_r\right\|,\ \varrho_1 = 3\varpi_1 + \varpi_2,\ \varrho_2 = \varpi_1 + 3\varpi_2 + 2.$

For any $r \in \mathbf{Z}_k^r$, it gets

$$\begin{aligned}\boldsymbol{E}\left\{\mathcal{L}V(t)\right\} \leq & \hat{\aleph}^T(t-d(t))\left((\bar{d}-1)\varXi\right)\hat{\aleph}(t-d(t))\\ & \hat{\aleph}^T(t)\left(\varUpsilon_1 + \eth + \varXi\right)\hat{\aleph}(t) + 2\hat{\aleph}(t)^T\varUpsilon_2\hat{\aleph}(t-d(t)).\end{aligned}\tag{3.12}$$

Let $\wp(t) = \mathrm{col}(\hat{\aleph}(t), \hat{\aleph}(t-d(t)))$, combining (3.1) and (3.3), it can infer that

$$\boldsymbol{E}\left\{\mathcal{L}V(t)\right\} \leq -\ell_1\boldsymbol{E}\left\{V(t)\right\}.\tag{3.13}$$

For any $r \in \mathbf{Z}_{uk}^r$, there is

$$\begin{aligned}\boldsymbol{E}\left\{\mathcal{L}V(t)\right\} \leq & \hat{\aleph}^T(t-d(t))\left((\bar{d}-1)\varXi\right)\hat{\aleph}(t-d(t))\\ & + \hat{\aleph}^T(t)\left(\varUpsilon_1 + \varPsi_1 + \varXi\right)\hat{\aleph}(t) + 2\hat{\aleph}^T(t)\varUpsilon_2\hat{\aleph}(t-d(t)),\end{aligned}\tag{3.14}$$

combining (3.2), (3.3) and (3.4), one has

$$\boldsymbol{E}\left\{\mathcal{L}V(t)\right\} \leq -\ell_2\boldsymbol{E}\left\{V(t)\right\}.\tag{3.15}$$

Let $\ell = min\left\{\ell_1, \ell_2\right\}$, it gets

$$\boldsymbol{E}\left\{\mathcal{L}V(t)\right\} \leq -\ell\boldsymbol{E}\left\{V(t)\right\},\tag{3.16}$$

integrating (3.16), one has

$$\boldsymbol{E}\left\{V(t)\right\} \leq e^{-\ell t}\boldsymbol{E}\left\{V(0)\right\}.\tag{3.17}$$

When $t \to +\infty$, $\boldsymbol{E}\left\{V(t)\right\} \to 0$. According to Definition 2.1, NMASs (2.1) and (2.2) can reach delayed consensus in mean-square. So the proof is completed. $\square$

If the TRs are completely known, that is to say, $\mathbf{Z}_{uk}^r = 0$ for any $r \in \mathbf{Z}$, one can obtain Corollary 3.1.

Corollary 3.1. *For given scalars $\beta > \lambda_{\max}\left(P_r\right)/\lambda_{\min}\left(\Im_r\right)$, $\ell_1 > 0$, $\varpi_1 \geq 0$, $\varpi_2 \geq 0$, under Assumption* 2.1, *NMASs* (2.1) *and* (2.2) *can reach delayed consensus in mean-square, if there exist matrices $\varXi > 0$ and $P_r > 0$, such that*

$$\varTheta_1 = \begin{bmatrix}\varUpsilon_1 + \eth + \varXi & \varUpsilon_2\\ * & (\bar{d}-1)\varXi\end{bmatrix} < 0,\ r \in \mathbf{Z}_k^r,\tag{3.18}$$

where
$\varOmega = \varsigma^2 \left\| L_r^T \lambda_{\max}(P_r) L_r \right\|$,
$\varrho_1 = 3\varpi_1 + \varpi_2,\ \varrho_2 = \varpi_1 + 3\varpi_2 + 2$,
$\Im_r = P_r (L_r + B_r) + (L_r + B_r)^T P_r$,
$\eth = \hat{\aleph}^T(t) \left[\sum_{s=1}^{z} \kappa_{rs}(h) \varLambda_r\right] \hat{\aleph}(t)$,

$$\varLambda_r = \begin{bmatrix} \beta\Im_r & P_r \\ P_r & P_r \end{bmatrix} \otimes I_m,\quad \varUpsilon_2 = -\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \otimes (P_r L_r) \otimes I_m,$$

$$\varUpsilon_1 = \begin{bmatrix} \varrho_1 P_r - \beta\Im_r + \varOmega I_n & 0 \\ 0 & \varrho_2 P_r - \beta\Im_r + \varOmega I_n \end{bmatrix} \otimes I_m.$$

Remark 3.1. When $\tau = 0$, the delayed consensus in mean-square degenerates to identical consensus in mean-square.

Corollary 3.2. *For given scalars $\beta > \lambda_{\max}(P_r)/\lambda_{\min}(\Im_r)$, $\ell_1 > 0$, $\varpi_1 \geq 0$, $\varpi_2 \geq 0$, under Assumption 2.1, NMASs (2.1) and (2.2) can reach consensus in mean-square, if there exist matrices $\varXi > 0$ and $P_r > 0$, such that*

$$\varTheta_1 = \begin{bmatrix} \varUpsilon_1 + \eth + \varXi & \varUpsilon_2 \\ * & (\bar{d} - 1)\varXi \end{bmatrix} < 0,\ r \in \mathbf{Z}_k^r, \tag{3.19}$$

where
$\varOmega = \varsigma^2 \left\| L_r^T \lambda_{\max}(P_r) L_r \right\|$,
$\varrho_1 = 3\varpi_1 + \varpi_2,\ \varrho_2 = \varpi_1 + 3\varpi_2 + 2$,
$\Im_r = P_r (L_r + B_r) + (L_r + B_r)^T P_r$,
$\eth = \hat{\aleph}^T(t) \left[\sum_{s=1}^{z} \kappa_{rs}(h) \varLambda_r\right] \hat{\aleph}(t)$,

$$\varLambda_r = \begin{bmatrix} \beta\Im_r & P_r \\ P_r & P_r \end{bmatrix} \otimes I_m,\quad \varUpsilon_2 = -\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \otimes (P_r L_r) \otimes I_m,$$

$$\varUpsilon_1 = \begin{bmatrix} \varrho_1 P_r - \beta\Im_r + \varOmega I_n & 0 \\ 0 & \varrho_2 P_r - \beta\Im_r + \varOmega I_n \end{bmatrix} \otimes I_m.$$

Due to the existence of the time-varying term $\kappa_{rs}(h)$, Theorem 3.1 and Corollary 3.1 are not solvable by the LMI Toolbox in MATLAB. The next step is to solve the problem of the time-varying term $\kappa_{rs}(h)$.

Theorem 3.2. *For given scalars $\beta > \lambda_{\max}(P_r)/\lambda_{\min}(\Im_r)$, $\ell_1 > 0$, $\ell_2 > 0$, $\varpi_1 \geq 0$, $\varpi_2 \geq 0$, under Assumption 2.1, NMASs (2.1) and (2.2) can reach delayed consensus in mean-square, if there exist matrices $\varXi > 0$, $P_r > 0$ and $Z_r \in \mathbb{R}^{n \times n}$, such that*

$$\bar{\varTheta}_1 = \begin{bmatrix} \bar{J}_1 & \bar{J}_2 & J_3 & J_3 \\ * & \bar{J}_4 & J_3 & J_3 \\ * & * & J_5 & J_6 \\ * & * & * & J_7 \end{bmatrix} < 0,\ r \in \mathbf{Z}_k^r, \tag{3.20}$$

$$\underline{\Theta}_1 = \begin{bmatrix} \underline{J}_1 & \underline{J}_2 & J_3 & J_3 \\ * & \underline{J}_4 & J_3 & J_3 \\ * & * & J_5 & J_6 \\ * & * & * & J_7 \end{bmatrix} < 0, \ r \in \mathbf{Z}_k^r, \tag{3.21}$$

$$\bar{\Theta}_2 = \begin{bmatrix} \bar{J}_8 & \bar{J}_9 & J_3 & J_3 \\ * & \bar{J}_{10} & J_3 & J_3 \\ * & * & J_5 & J_6 \\ * & * & * & J_7 \end{bmatrix} < 0, \ r \in \mathbf{Z}_{uk}^r, \tag{3.22}$$

$$\underline{\Theta}_2 = \begin{bmatrix} \underline{J}_8 & \underline{J}_9 & J_3 & J_3 \\ * & \underline{J}_{10} & J_3 & J_3 \\ * & * & J_5 & J_6 \\ * & * & * & J_7 \end{bmatrix} < 0, \ r \in \mathbf{Z}_{uk}^r, \tag{3.23}$$

$$\begin{bmatrix} \beta\Im_s + Z_r & P_s \\ * & P_s + Z_r \end{bmatrix} \le 0, \ \forall s \in \mathbf{Z}_{uk}^r, \ s \ne r, \tag{3.24}$$

$$\begin{bmatrix} \beta\Im_s + Z_r & P_s \\ * & P_s + Z_r \end{bmatrix} \ge 0, \ \forall s \in \mathbf{Z}_{uk}^r, \ s = r, \tag{3.25}$$

where
$\Omega = \varsigma^2 \left\| L_r^T \lambda_{\max}(P_r) L_r \right\|$,
$\varrho_1 = 3\varpi_1 + \varpi_2$,
$\varrho_2 = \varpi_1 + 3\varpi_2 + 2$,
$\Im_r = P_r(L_r + B_r) + (L_r + B_r)^T P_r$,
$\bar{J}_1 = \{\varrho_1 P_r + (\bar{\kappa}_{rr} - 1)\beta\Im_r + \Omega I_n + \sum_{s=1,s\ne r}^{z} \bar{\kappa}_{rs}\beta\Im_r + \sum_{s\in\mathbf{Z}_k^r} \bar{\kappa}_{rs} Z_r + \Xi_{11}\} \otimes I_m$,
$\underline{J}_1 = \{\varrho_1 P_r + (\underline{\kappa}_{rr} - 1)\beta\Im_r + \Omega I_n + \sum_{s=1,s\ne r}^{z} \underline{\kappa}_{rs}\beta\Im_r + \sum_{s\in\mathbf{Z}_k^r} \underline{\kappa}_{rs} Z_r + \Xi_{11}\} \otimes I_m$,
$\bar{J}_2 = (\bar{\kappa}_{rr} P_r + \sum_{s=1,s\ne r}^{z} \bar{\kappa}_{rs} P_s + \Xi_{12}) \otimes I_m$,
$\underline{J}_2 = (\underline{\kappa}_{rr} P_r + \sum_{s=1,s\ne r}^{z} \underline{\kappa}_{rs} P_s + \Xi_{12}) \otimes I_m$,
$J_3 = (-P_r L_r) \otimes I_m$,
$\bar{J}_4 = \{\varrho_2 P_r - \beta\Im_r + \Omega I_n + \bar{\kappa}_{rr} P_r + \sum_{s=1,s\ne r}^{z} \bar{\kappa}_{rs} P_s + \sum_{s\in\mathbf{Z}_k^r} \bar{\kappa}_{rs} Z_r + \Xi_{22}\} \otimes I_m$,
$\underline{J}_4 = \{\varrho_2 P_r - \beta\Im_r + \Omega I_n + \underline{\kappa}_{rr} P_r + \sum_{s=1,s\ne r}^{z} \underline{\kappa}_{rs} P_s + \sum_{s\in\mathbf{Z}_k^r} \underline{\kappa}_{rs} Z_r + \Xi_{22}\} \otimes I_m$,
$J_5 = ((\bar{d} - 1)\Xi_{11}) \otimes I_m$,
$J_6 = ((\bar{d} - 1)\Xi_{12}) \otimes I_m$,
$J_7 = ((\bar{d} - 1)\Xi_{22}) \otimes I_m$,
$\bar{J}_8 = \{\varrho_1 P_r - \beta\Im_r + \Omega I_n + \sum_{s\in\mathbf{Z}_k^r} \bar{\kappa}_{rs}(\beta\Im_s + Z_r) + \Xi_{11}\} \otimes I_m$,
$\underline{J}_8 = \{\varrho_1 P_r - \beta\Im_r + \Omega I_n + \sum_{s\in\mathbf{Z}_k^r} \underline{\kappa}_{rs}(\beta\Im_s + Z_r) + \Xi_{11}\} \otimes I_m$,
$\bar{J}_9 = (\sum_{s\in\mathbf{Z}_k^r} \bar{\kappa}_{rs} P_s + \Xi_{12}) \otimes I_m$,
$\underline{J}_9 = (\sum_{s\in\mathbf{Z}_k^r} \underline{\kappa}_{rs} P_s + \Xi_{12}) \otimes I_m$,
$\bar{J}_{10} = \{\varrho_2 P_r - \beta\Im_r + \Omega I_n + \sum_{s\in\mathbf{Z}_k^r} \bar{\kappa}_{rs}(P_s + Z_r) + \Xi_{22}\} \otimes I_m$,

$\underline{J}_{10} = \{\varrho_2 P_r - \beta \Im_r + \Omega I_n + \sum_{s \in \mathbf{Z}_k^r} \underline{\kappa}_{rs}(P_s + Z_r) + \Xi_{22}\} \otimes I_m$.

Proof. For $\kappa_{rs}(h)$, there exist $\gamma_1 > 0, \gamma_2 > 0$ with $\gamma_1 + \gamma_2 = 1$ such that $\kappa_{rs}(h) = \gamma_1 \bar{\kappa}_{rs} + \gamma_2 \underline{\kappa}_{rs}$.

For any $r \in \mathbf{Z}_k^r$, according to (3.12), one has

$$\Theta_1 = \gamma_1 \bar{\Theta}_1 + \gamma_2 \underline{\Theta}_1 < 0. \tag{3.26}$$

Therefore, through calculation, if (3.20) and (3.21) hold, it has $\Theta_1 < 0$. Similarly, for any $r \in \mathbf{Z}_{uk}^r$, if (3.22) and (3.23) hold, it has $\Theta_2 < 0$. Therefore, NMASs (2.1) and (2.2) can reach delayed consensus in mean-square. So the proof is complete. □

4. Numerical simulation

The NMASs (2.1) and (2.2) are composed of a leader and three followers. The random topologies are shown in Fig. 1. The adjacency matrices $\mathcal{W}_r$, pinning

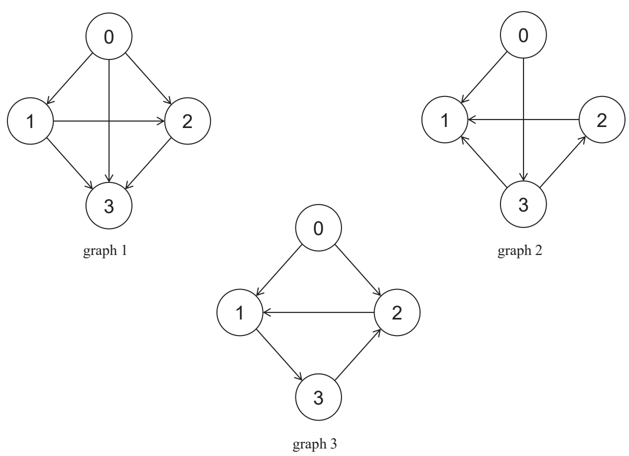

Figure 1. Communication topological diagrams.

matrices B_r and Laplacian matrices L_r are as below

$$\mathcal{W}_1 = \begin{bmatrix} 0\,1\,1 \\ 0\,0\,1 \\ 0\,0\,0 \end{bmatrix}, \ \mathcal{W}_2 = \begin{bmatrix} 0\,0\,0 \\ 1\,0\,0 \\ 1\,1\,0 \end{bmatrix}, \ \mathcal{W}_3 = \begin{bmatrix} 0\,0\,1 \\ 1\,0\,0 \\ 0\,1\,0 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} 1\,0\,0 \\ 0\,1\,0 \\ 0\,0\,1 \end{bmatrix}, \ B_2 = \begin{bmatrix} 1\,0\,0 \\ 0\,0\,0 \\ 0\,0\,1 \end{bmatrix}, \ B_3 = \begin{bmatrix} 1\,0\,0 \\ 0\,1\,0 \\ 0\,0\,0 \end{bmatrix},$$

$$L_1 = \begin{bmatrix} 2 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix},\ L_2 = \begin{bmatrix} 0 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & -1 & 2 \end{bmatrix},\ L_3 = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}.$$

At the same time, the nonlinear vector function $f(\chi_i(t), \nu_i(t))$ is described as

$$f\left(\chi_i(t), \nu_i(t)\right) = \begin{bmatrix} 0.44\sin(\chi_{i1}(t)) \\ 0.44\sin(\chi_{i2}(t)) \\ 0.44\sin(\chi_{i2}(t)) \end{bmatrix}.$$

The TRs of UNMS topologies are partially unknown and described as

$$\begin{pmatrix} \kappa_{11}(h) & ? & ? \\ \kappa_{21}(h) & ? & \kappa_{23}(h) \\ ? & \kappa_{32}(h) & ? \end{pmatrix}.$$

Assume that $d(t) = |sin(t)| \leq 1$. By solving LMI, it can be obtained

$$P_1 = \begin{bmatrix} 0.0638 & 0.0134 & 0.0200 \\ 0.0134 & 0.1025 & 0.0467 \\ 0.0200 & 0.0467 & 0.2611 \end{bmatrix},$$

$$P_2 = \begin{bmatrix} 0.3892 & 0.1410 & 0.0217 \\ 0.1410 & 0.2365 & 0.0169 \\ 0.0217 & 0.0169 & 0.0716 \end{bmatrix},$$

$$P_3 = \begin{bmatrix} 0.1558 & 0.0724 & 0.0981 \\ 0.0724 & 0.1870 & 0.1369 \\ 0.0981 & 0.1369 & 0.3357 \end{bmatrix}.$$

Fig. 2 represents the adaptive control law for different initial values. $\alpha(t)$ converges to a fixed value as time passes. Fig. 3 represents jump modes of Markov chain. From Fig. 4 and Fig. 5, the NMASs under Markov switching topologies and Brown noise can reach delayed consensus in mean-square.

5. Conclusion and outlook

The delayed consensus in mean-square issue for NMASs with stochastic switching topologies and Brown noise has been studied. The time-varying delay among followers and the delay between leader and followers have been considered simultaneously. Meanwhile, the communication topologies of NMASs have been modeled as

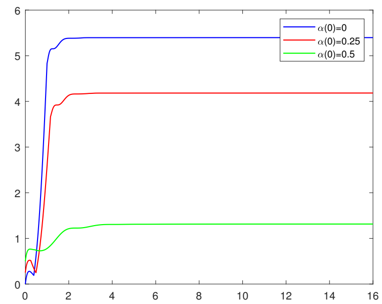

Figure 2. Adaptive control law for different initial values.

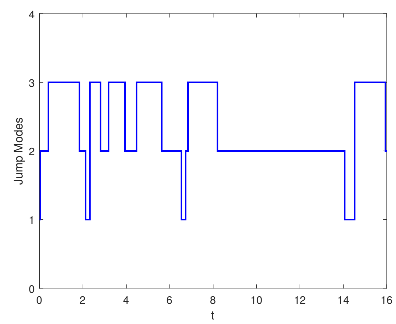

Figure 3. Jump modes of Markov chain.

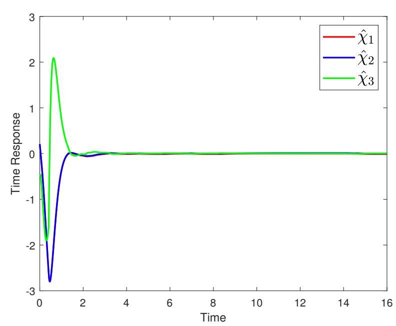

Figure 4. The position delayed error $\hat{\chi}_i(t)$, $i = 1, 2, 3$ with adaptive control under Markov switching topologies.

UNMS topologies, and the TRs have been partly or even totally unknown. Brown noise has been also considered. Sufficient conditions to ensure delayed consensus in mean-square for NMASs have been obtained. The correctness of the results has been verified through the example given. Next, delayed consensus in mean-square for NMASs with Markov switching topologies subjected to network attacks will be further considered. In addition, to reduce unnecessary signal transmission among agents, event-triggered mechanism may also be taken into account to research delayed consensus in mean-square issue for NMASs in the future work.

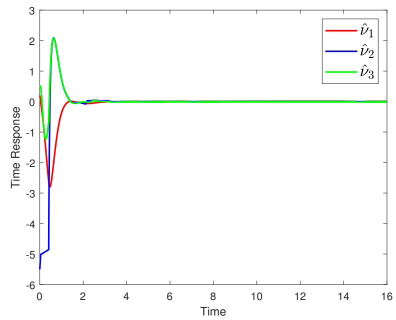

Figure 5. The velocity delayed error $\hat{\nu}_i(t)$, $i = 1, 2, 3$ with adaptive control under Markov switching topologies.

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