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AVERAGING METHOD FOR MULTI-POINT BOUNDARY VALUE PROBLEMS OF SET-VALUED FUNCTIONAL DIFFERENTIAL EQUATIONS*

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Abstract In this paper, we present a study of the set-valued functional differential equations, of which the right functions are the product of two terms. First, the global averaging method of the equations is considered. Then, by introducing the concept of semi-deviation metric, we consider the averaging method of the above equations for the case in which the limit of a method of an average does not exist. The proof is based on the analysis of support functions and measurable choice sets.

Keywords Set differential equations, multi-point boundary value problems, averaging method, semi-deviation metric, support functions.

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1. Introduction

Due to the difficulty of solving the exact solutions of differential equations, the averaging method has become an effective method to analyze the approximate solutions of ordinary differential equations, the related results can be seen in the references [4–6]. Set-valued differential systems [2,7], as one of the generalized forms of ordinary differential equations, has been widely used in physics, astronomy, biology and engineering. So it is of great significance to study the properties of approximate solutions of set-valued differential equations by using averaging method.

Recently, Plotnikov [11, 14] studied using the average method the asymptotic properties of solutions of different types of set-valued differential equations with Hukuhara derivatives and delay differential equations. Skripnik [19] established a three-step averaging method for set-valued differential equations with generalized derivatives. Wang and Yang considered the averaging method of set-valued impulsive differential equations with initial boundary value conditions. There are numerous results available, including [1, 10, 12].

In addition, functional differential equations are widely used in the real world, it is very important to study the properties of the solutions for such equations.

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Some research results on functional differential equations can be found in literature [3, 9, 16, 17, 20].

Inspired by these results, in this paper, we mainly study the set-valued functional differential equations whose right-hand side are the product of two functions. By introducing the concepts of Hausdorff metric and semi-deviation metric, using the corresponding properties of the support functions and the analysis of measurable choice sets, we discuss two cases respectively when the averaging limit of the function on the right-hand side of the equations exists and does not exist. The asymptotic relationship between the solutions of the original equations and its averaging equations is shown.

2. Preliminaries

In what follows we define the necessary elements for the statements of our main results.

Let $conv(\mathbb{R}^n)$ denote the collection of nonempty, compact and convex subsets of $\mathbb{R}^n$. Given $A,\ B \in conv(\mathbb{R}^n)$ the Hausdorff distance between A and B is defined as

$$H[A,B] = \max\big\{\sup_{a\in A}\inf_{b\in B}\|a-b\|, \sup_{b\in B}\inf_{a\in A}\|a-b\|\big\},$$

where $\|\cdot\|$ denotes the Euclidean norm in $\mathbb{R}^n$ and $\{0\}$ is the zero points set in $conv(\mathbb{R}^n)$.

Given an inteval I in $\mathbb{R}_+$. We say that the set mapping $F : I \to conv(\mathbb{R}^n)$ has a Hukuhara derivative $D_H F(t_0)$ at a point $t_0 \in I$, if

$$\lim_{h\to 0^+}\frac{F(t_0+h)-F(t_0)}{h}, \quad \lim_{h\to 0^+}\frac{F(t_0)-F(t_0-h)}{h}$$

exist in the topology of $conv(\mathbb{R}^n)$ and are equal to $D_H F(t_0)$. By embedding $conv(\mathbb{R}^n)$ as a complete cone in a corresponding Banach space and taking into account result on the differentiation of Bochner integral, we find that if

$$F(t) = X_0 + \epsilon\int_{t_0}^{t}\Phi(s)ds, \quad X_0 \in conv(\mathbb{R}^n),$$

where $\Phi : I \to conv(\mathbb{R}^n)$ is integrable in the sense of Bochner, then $D_H F(t)$ exist and the equality $D_H F(t) = \Phi(t)$ a.e. on I holds.

The Hukuhara integral of F is given by

$$\int_I F(s)ds = cl\Big[\int_I f(s)ds : f \text{ is a continuous selector of } F\Big]$$

for any compact set $I \subset \mathbb{R}_+$, where clA is a closure of set A.

If $F,\ G : I \to conv(\mathbb{R}^n)$ are integrable, then $D[F(\cdot), G(\cdot)] : I \to \mathbb{R}_+$ is integrable and

$$H\Big[\int_{t_0}^{t}F(s)ds, \int_{t_0}^{t}G(s)ds\Big] \le \int_{t_0}^{t}H[F(s),G(s)]ds.$$

The properties of Hausdorff metric and more details in continuity, Hukuhara derivative, Hukuhara integral of the set mapping can be found in the literature [7, 8].

Lemma 2.1 ([14]). *(Gronwall-Bellman lemma) Set $\Theta(t)$ as a real continue function on $\mathbb{R}_+$ and P, Q are positive real numbers. If*

$$\Theta(t) \leq P + Q\int_0^t \Theta(s)ds,\ \ t \in [0,T],$$

then

$$\Theta(t) \leq P\exp\{QT\}.$$

3. Main results

In this part of the paper, we consider multi-point boundary value problem for set-valued functional differential equations

$$\begin{cases} D_H X(t) = \varepsilon\Big(F\big(t, X(t), X(t-\delta(t))\big) \bigotimes G\big(t, X(t), X(t-\delta(t))\big)\Big), \\ \sum\limits_{k=0}^{K} \alpha_k(\varepsilon) X(t_k) = \Phi\big(X(t_0), \cdots, X(t_K), D_H X(t_0), \cdots, D_H X(t_K), \varepsilon\big), \end{cases} \tag{3.1}$$

where $t \in [0,T]$, $D_H X$ is Hukuhara derivative of $X(t)$ on $[0,T]$, $F,\ G : I \times conv(\mathbb{R}^n) \times conv(\mathbb{R}^n) \to conv(\mathbb{R}^n)$, $\bigotimes$ represent cartesian product. $\delta : \mathbb{R}_+ \to \mathbb{R}_+$ is a delay function. Boundary value condition α_k are $n\times n-$dimensional nonsingular matrix, $k = 0, 1, \cdots, K$. $\Phi \in conv(\mathbb{R}^n)$ is a continuous function.

3.1. When the average limit of right-hand side function exists

Suppose the following limits exist

$$\overline{F}(X(t), U(t)) = \lim_{T\to\infty} \frac{1}{T}\int_t^{t+T} F(s, X(s), U(s))ds, \tag{3.2}$$

$$\overline{G}(X(t)) = \lim_{T\to\infty} \frac{1}{T}\int_t^{t+T} G(s, X(s), U(s))ds. \tag{3.3}$$

We associate Eq.(3.1) with the global averaged equation

$$\begin{cases} D_H Y(t) = \varepsilon\Big(\overline{F}\big(Y(t), Y(t-\delta(t))\big) \bigotimes \overline{G}\big(Y(t)\big)\Big), \\ \sum\limits_{k=0}^{K} \alpha_k(\varepsilon) Y(t_k) = \Phi\big(Y(t_0), \cdots, Y(t_K), D_H Y(t_0), \cdots, D_H Y(t_K), \varepsilon\big). \end{cases} \tag{3.4}$$

Theorem 3.1. *Suppose that the following conditions are satisfied in the domain $D = \{(t, X, U)|\ t \geq 0,\ \ X, U \in conv(\mathbb{R}^n)\}$*

$(A_{3.11})$ *The set-value mapping $F, G : D \to conv(\mathbb{R}^n)$ are continuous and bounded, i.e. there exist constants $M_1, \lambda_{11}, \lambda_{12} > 0$, for $X', X'', U', U'' \in conv(\mathbb{R}^n)$ such that*

$$\begin{aligned} &H[F(t, X, U), \{0\}] \bigvee H[G(t, X, U), \{0\}] \leq M_1, \\ &H[F(t, X', U'), F(t, X'', U'')] \bigvee H[G(t, X', U'), G(t, X'', U'')] \\ \leq &\lambda_{11} H[X', X''] + \lambda_{12} H[U', U''] \end{aligned}$$

where $a \bigvee b = \max\{a, b\}$;

$(A_{3.12})$ *The limits (3.2) and (3.3) exist uniformly in* $t \geq 0$*;*

$(A_{3.13})$ *The following inequality*

$$\begin{aligned}
&H\Big[\Phi\big(X(t_0), \cdots, X(t_K), D_H X(t_0), \cdots, D_H X(t_K), \varepsilon\big),\\
&\Phi\big(Y(t_0), \cdots, Y(t_K), D_H Y(t_0), \cdots, D_H Y(t_K), \varepsilon\big)\Big]\\
\leq &\sum_{k=0}^{K} \mu_k(\varepsilon)\Big(H\big[X(t_k), Y(t_k)\big] + H[D_H X(t_k), D_H Y(t_k)]\Big)
\end{aligned}$$

holds, and for any $\varepsilon \in (0, \varepsilon_1]$ *we have*

$$0 < \big\|\big(\sum_{k=0}^{K} \alpha_k(\varepsilon)\big)^{-1}\big\| \times \big(\sum_{k=0}^{K} \mu_k(\varepsilon)\big) < 1,$$

where $\mu_k > 0$ *are continuous functions,* $k = 0, 1, \cdots, K$*.* $\sum\limits_{k=0}^{K} \alpha_k(\varepsilon)$ *is* $n \times n-$*dimensional nonsingular matrix,* A^{-1} *is called the inverse of the matrix* A*;*

$(A_{3.14})$ *The solutions* $Y(t)$ *of the systems (3.4) at* $t \in [0, T]$ *together with its* δ_1*-neighbourhood belong to domain* D*, i.e.* $O(Y(t), \delta_1) \subset D$*, where* $\delta_1 > 0$ *is a constant.*

Then for any $\eta > 0$ *and* $L > 0$*, there exists* $\varepsilon_1(\eta, L) > 0$ *such that for* $\varepsilon \in (0, \varepsilon_1]$ *and* $t \in [0, L\varepsilon^{-1}]$ *the following estimate is correct*

$$H[X(t), Y(t)] \leq \eta,$$

where $X(t)$ *is the set of solutions of equations (3.1),* $Y(t)$ *is the set of solutions of equations (3.4).*

Proof. Due to $(A_{3.11})$ and $(A_{3.12})$ we know, for any $\varepsilon_1 > 0$, there exists T_1, for all $T > T_1$, such that

$$\begin{aligned}
&H\big[\overline{F}(Y, U), \{0\}\big]\\
\leq &H\big[\overline{F}(Y, U), \frac{1}{T}\int_0^T F(s, Y, U)ds\big] + H\big[\frac{1}{T}\int_0^T F(s, Y, U)ds, \{0\}\big]\\
< &\varepsilon_1 + \frac{1}{T}\int_0^T H[F(s, Y, U), \{0\}]ds\\
\leq &\varepsilon_1 + M_1,\\
&H\big[\overline{G}(Y), \{0\}\big]\\
\leq &H\big[\overline{G}(Y), \frac{1}{T}\int_0^T G(s, Y, U)ds\big] + H\big[\frac{1}{T}\int_0^T G(s, Y, U)ds, \{0\}\big]\\
< &\varepsilon_1 + \frac{1}{T}\int_0^T H[G(s, Y, U), \{0\}]ds\\
\leq &\varepsilon_1 + M_1,\\
&H[\overline{F}(Y', U'), \overline{F}(Y'', U'')]
\end{aligned}$$

$$
\begin{aligned}
&\leq H\big[\overline{F}(Y',U'),\frac{1}{T}\int_0^T F(s,Y',U')ds\big]\\
&\quad + H\big[\frac{1}{T}\int_0^T F(s,Y',U')ds,\frac{1}{T}\int_0^T F(s,Y'',U'')ds\big]\\
&\quad + H\big[\frac{1}{T}\int_0^T F(s,Y'',U'')ds,\overline{F}(Y'',U'')\big]\\
&<2\varepsilon_1+\frac{1}{T}\int_0^T H[F(s,Y',U'),F(s,Y'',U'')]ds\\
&\leq 2\varepsilon_1+\lambda_{11}H[Y',Y'']+\lambda_{12}H[U',U''],\\
&\quad H[\overline{G}(Y'),\overline{G}(Y'')]\\
&\leq H\big[\overline{G}(Y'),\frac{1}{T}\int_0^T G(s,Y',U')ds\big]\\
&\quad + H\big[\frac{1}{T}\int_0^T G(s,Y',U')ds,\frac{1}{T}\int_0^T G(s,Y'',U'')ds\big]\\
&\quad + H\big[\frac{1}{T}\int_0^T G(s,Y'',U'')ds,\overline{G}(Y'')\big]\\
&<2\varepsilon_1+\frac{1}{T}\int_0^T H[G(s,Y',U'),G(s,Y'',U'')]ds\\
&\leq 2\varepsilon_1+\lambda_{11}H[Y',Y'']+\lambda_{12}H[U',U''].
\end{aligned}
$$

Thus, for any small $\varepsilon_1 > 0$, the following estimates hold

$$
H[\overline{F}(Y,U),\{0\}]\bigvee H[\overline{G}(Y),\{0\}]\leq M_1, \tag{3.5}
$$

$$
\begin{aligned}
&H[\overline{F}(Y',U'),\overline{F}(Y'',U'')]\bigvee H[\overline{G}(Y'),\overline{G}(Y'')] \\
\leq&\lambda_{11}H[Y',Y'']+\lambda_{12}H[U',U''].
\end{aligned} \tag{3.6}
$$

The solutions of the set-valued differential systems (3.1) satisfy the following equations

$$
\begin{cases}
X(t)=X_0+\varepsilon\displaystyle\int_{t_0}^t\Big(F\big(s,X(s),X(s-\delta(s))\big)\bigotimes G\big(s,X(s),X(s-\delta(s))\big)\Big)ds,\\
\displaystyle\sum_{k=0}^K\alpha_k(\varepsilon)(Z_k)=\Phi\big(Z_0,\cdots,Z_K,D_HX(t_0),\cdots,D_HX(t_K),\varepsilon\big),
\end{cases} \tag{3.7}
$$

where $X_0 = X(t_0)$, $\varrho_k=\int_{t_0}^{t_k}\Big(F\big(s,X(s),X(s-\delta(s))\big)\otimes G\big(s,X(s),X(s-\delta(s))\big)\Big)ds$, $Z_k = X_0 + \varepsilon\varrho_k$.

Similarly, the solutions of the average systems (3.4) are equivalent to integral equations

$$
\begin{cases}
Y(t)=Y_0+\varepsilon\displaystyle\int_{t_0}^t\Big(\overline{F}\big(Y(s),Y(s-\delta(s))\big)\bigotimes\overline{G}\big(Y(s)\big)\Big)ds,\\
\displaystyle\sum_{k=0}^K\alpha_k(\varepsilon)(\bar{Z}_k)=\Phi\big(\bar{Z}_0,\cdots,\bar{Z}_K,D_HY(t_0),\cdots,D_HY(t_K),\varepsilon\big),
\end{cases} \tag{3.8}
$$

where $Y_0 = Y(t_0)$, $\bar{\varrho}_k=\int_{t_0}^{t_k}\Big(\overline{F}\big(Y(s),Y(s-\delta(s))\big)\otimes\overline{G}\big(Y(s)\big)\Big)ds$ and $\bar{Z}_k = Y_0+\varepsilon\bar{\varrho}_k$.

From the integral equations (3.7) and (3.8), it can be obtained

$$
\begin{aligned}
&H[X(t),Y(t)]\\
\leq&\varepsilon H\Big[\int_{t_0}^{t}\Big(F\big(s,X(s),X(s-\delta(s))\big)\bigotimes G\big(s,X(s),X(s-\delta(s))\big)\Big)ds,\\
&\int_{t_0}^{t}\Big(F\big(s,X(s),X(s-\delta(s))\big)\bigotimes G\big(s,Y(s),Y(s-\delta(s))\big)\Big)ds\Big]\\
&+\varepsilon H\Big[\int_{t_0}^{t}\Big(F\big(s,X(s),X(s-\delta(s))\big)\bigotimes G\big(s,Y(s),Y(s-\delta(s))\big)\Big)ds,\\
&\int_{t_0}^{t}\Big(F\big(s,Y(s),Y(s-\delta(s))\big)\bigotimes G\big(s,Y(s),Y(s-\delta(s))\big)\Big)ds\Big]\\
&+\varepsilon H\Big[\int_{t_0}^{t}\Big(F\big(s,Y(s),Y(s-\delta(s))\big)\bigotimes G\big(s,Y(s),Y(s-\delta(s))\big)\Big)ds,\\
&\int_{t_0}^{t}\Big(F\big(s,Y(s),Y(s-\delta(s))\big)\bigotimes \overline{G}\big(Y(s)\big)\Big)ds\Big]\\
&+\varepsilon H\Big[\int_{t_0}^{t}\Big(F\big(s,Y(s),Y(s-\delta(s))\big)\bigotimes \overline{G}\big(Y(s)\big)\Big)ds,\\
&\int_{t_0}^{t}\Big(\overline{F}\big(Y(s),Y(s-\delta(s))\big)\bigotimes \overline{G}\big(Y(s)\big)\Big)ds\Big]+H[X_0,Y_0]\\
\leq&\varepsilon M_1 H\Big[\int_{t_0}^{t}G\big(s,X(s),X(s-\delta(s))\big)ds,\int_{t_0}^{t}G\big(s,Y(s),Y(s-\delta(s))\big)ds\Big]\\
&+\varepsilon M_1 H\Big[\int_{t_0}^{t}F\big(s,X(s),X(s-\delta(s))\big)ds,\int_{t_0}^{t}F\big(s,Y(s),Y(s-\delta(s))\big)ds\Big]\\
&+\varepsilon M_1 H\Big[\int_{t_0}^{t}G\big(s,Y(s),Y(s-\delta(s))\big)ds,\int_{t_0}^{t}\overline{G}\big(Y(s)\big)ds\Big]\\
&+\varepsilon M_1 H\Big[\int_{t_0}^{t}F\big(s,Y(s),Y(s-\delta(s))\big)ds,\int_{t_0}^{t}\overline{F}\big(Y(s),Y(s-\delta(s))\big)ds\Big]\\
&+H[X_0,Y_0]\\
\leq&2\varepsilon M_1\lambda_{11}\int_{t_0}^{t}H[X(s),Y(s)]ds\\
&+2\varepsilon M_1\lambda_{12}\int_{t_0}^{t}H[X(s-\delta(s)),Y(s-\delta(s))]ds\\
&+\varepsilon M_1 H\Big[\int_{t_0}^{t}G\big(s,Y(s),Y(s-\delta(s))\big)ds,\int_{t_0}^{t}\overline{G}\big(Y(s)\big)ds\Big]\\
&+\varepsilon M_1 H\Big[\int_{t_0}^{t}F\big(s,Y(s),Y(s-\delta(s))\big)ds,\int_{t_0}^{t}\overline{F}\big(Y(s),Y(s-\delta(s))\big)ds\Big]\\
&+H[X_0,Y_0],
\end{aligned}
\tag{3.9}
$$

where

$$I_1=2\varepsilon M_1\lambda_{11}\int_{t_0}^{t}H[X(s),Y(s)]ds,$$

$$I_2 = 2\varepsilon M_1 \lambda_{12} \int_{t_0}^{t} H[X(s-\delta(s)), Y(s-\delta(s))]ds,$$
$$I_3 = \varepsilon M_1 H\Big[\int_{t_0}^{t} G\big(s, Y(s), Y(s-\delta(s))\big)ds, \int_{t_0}^{t} \overline{G}\big(Y(s)\big)ds\Big],$$
$$I_4 = \varepsilon M_1 H\Big[\int_{t_0}^{t} F\big(s, Y(s), Y(s-\delta(s))\big)ds, \int_{t_0}^{t} \overline{F}\big(Y(s), Y(s-\delta(s))\big)ds\Big],$$
$$I_5 = H[X_0, Y_0].$$

Supposing that $\delta : [0, T_1] \to \mathbb{R}_+$, then $t-\delta(t) \in [t^* - T_1, T]$. When $t \in [t^* - T_1, 0]$, obviously $X(t-\delta(t)) = Y(t-\delta(t))$. Therefore we can get

$$\begin{aligned} I_2 &\leq 2\varepsilon M_1 \lambda_{12} \int_{t^*-T_1}^{0} H[X(s-\delta(s)), Y(s-\delta(s))]ds \\ &\quad + 2\varepsilon M_1 \lambda_{12} \int_{0}^{t} H[X(s-\delta(s)), Y(s-\delta(s))]ds \\ &= 2\varepsilon M_1 \lambda_{12} \int_{0}^{t} H[X(\tau), Y(\tau)]d\tau, \end{aligned} \tag{3.10}$$

where $\tau = s - \delta(s)$. Since $\delta(t) \in \mathbb{R}_+$, obviously $ds = d\tau$.

From the assumption (3.12) of Theorem 3.1, we know that for any $\xi(t) > 0$, there exists T' such that for all $t > T'$, we have

$$H\Big[\frac{1}{t}\int_0^t F(s, X, U)ds, \overline{F}(X, U)\Big] \leq \xi(t),$$
$$H\Big[\frac{1}{t}\int_0^t G(s, X, U)ds, \overline{G}(X)\Big] \leq \xi(t).$$

Thus, there exists a decreasing function $\xi(t)$, such that

$$H\Big[\int_0^t F(s, X, U)ds, \int_0^t \overline{F}(X, U)ds\Big] \leq t\xi(t), \tag{3.11}$$
$$H\Big[\int_0^t G(s, X, U)ds, \int_0^t \overline{G}(X)ds\Big] \leq t\xi(t), \tag{3.12}$$

where $\lim\limits_{t\to\infty} \xi(t) = 0$.

From the above inequality (3.11), we obtain

$$I_3 \leq \varepsilon M_1 (t - t_0)\xi(t). \tag{3.13}$$

By analogy and (3.12), we get

$$I_4 \leq \varepsilon M_1 (t - t_0)\xi(t). \tag{3.14}$$

Next, to deal with boundary value problems, we give the definition and properties of the support function.

Let A be a nonempty subset of $conv(\mathbb{R}^n)$. The support function of A is defined for all $\psi \in \mathbb{R}^n$ by

$$\mathbb{S}(\psi, A) = \sup\big\{\langle \psi, a\rangle : a \in A\big\}.$$

The properties of the support function

$$\mathbb{S}(\psi, \alpha A) = \mathbb{S}(\alpha\psi, A) = \|\alpha\|\mathbb{S}(\psi, A),$$

where α is $n \times n-$matrix.

The Hausdorff metric is related to the support function for $A, B \in conv(\mathbb{R}^n)$, since we have

$$H[A, B] = \sup\{\|\mathbb{S}(\psi, A) - \mathbb{S}(\psi, B)\|,\ \psi \in S^{n-1}\},$$

where $S^{n-1} = \{\psi \in \mathbb{R}^n : \|\psi\| = 1\}$ is the unit sphere in $\mathbb{R}^n$.

From the boundary conditions (3.7), (3.8) and the condition $(A_{3.13})$, the properties of the support function, we have

$$\begin{aligned}
&H\Big[\sum_{k=0}^{K}\alpha_k(\varepsilon)(Z_k), \sum_{k=0}^{K}\alpha_k(\varepsilon)(\bar{Z}_k)\Big]\\
=&\Big\|\sum_{k=0}^{K}\alpha_k(\varepsilon)\Big\|\Big(H[X_0, Y_0] + \varepsilon H[\varrho_k, \bar{\varrho}_k]\Big)\\
\leq&\sum_{k=0}^{K}\mu_k(\varepsilon)\Big(H[X_0, Y_0] + \varepsilon H[\varrho_k, \bar{\varrho}_k] + H[D_H X(t_k), D_H Y(t_k)]\Big).
\end{aligned}$$

Furthermore, we get

$$\begin{aligned}
&H[X_0, Y_0]\\
\leq&\frac{\|\sum\limits_{k=0}^{K}\alpha_k(\varepsilon)\|^{-1}}{1 - \|\big(\sum\limits_{k=0}^{K}\alpha_k(\varepsilon)\big)^{-1}\| \times \big(\sum\limits_{k=0}^{K}\mu_k(\varepsilon)\big)}\\
&\times\Big(\varepsilon\sum_{k=0}^{K}\big(\mu_k(\varepsilon) - \|\alpha_k(\varepsilon)\|\big)H[\varrho_k, \bar{\varrho}_k] + \sum_{k=0}^{K}\mu_k(\varepsilon)H[D_H X(t_k), D_H Y(t_k)]\Big)\\
\leq&A(\varepsilon)H[\varrho_k, \bar{\varrho}_k] + B(\varepsilon)H[D_H X(t_k), D_H Y(t_k)],
\end{aligned}\tag{3.15}$$

where

$$A(\varepsilon) = \frac{\|\sum\limits_{k=0}^{K}\alpha_k(\varepsilon)\|^{-1}\varepsilon\sum\limits_{k=0}^{K}\big(\mu_k(\varepsilon) - \|\alpha_k(\varepsilon)\|\big)}{1 - \|\big(\sum\limits_{k=0}^{K}\alpha_k(\varepsilon)\big)^{-1}\| \times \big(\sum\limits_{k=0}^{K}\mu_k(\varepsilon)\big)},$$

$$B(\varepsilon) = \frac{\|\sum\limits_{k=0}^{K}\alpha_k(\varepsilon)\|^{-1}\sum\limits_{k=0}^{K}\big(\mu_k(\varepsilon)\big)}{1 - \|\big(\sum\limits_{k=0}^{K}\alpha_k(\varepsilon)\big)^{-1}\| \times \big(\sum\limits_{k=0}^{K}\mu_k(\varepsilon)\big)}.$$

According to (3.1) and (3.4), we have

$$\begin{aligned}
&H[D_H X(t_k), D_H Y(t_k)]\\
\leq&\varepsilon H\Big[F\big(t_k, X(t_k), X(t_k - \delta(t_k))\big)\bigotimes G\big(t_k, X(t_k), X(t_k - \delta(t_k))\big),
\end{aligned}$$

$$
\begin{aligned}
&\overline{F}\big(Y(t_k), Y(t_k-\delta(t_k))\big) \bigotimes \overline{G}\big(Y(t_k)\big)\Big] \\
&\leq \varepsilon H\Big[F\big(t_k, X(t_k), X(t_k-\delta(t_k))\big) \bigotimes G\big(t_k, X(t_k), X(t_k-\delta(t_k))\big), \{0\}\Big] \\
&\quad + \varepsilon H\Big[\{0\}, \overline{F}\big(Y(t_k), Y(t_k-\delta(t_k))\big) \bigotimes \overline{G}\big(Y(t_k)\big)\Big] \\
&\leq 2\varepsilon M_1^2. \qquad (3.16)
\end{aligned}
$$

From the inequality (3.11) and (3.12), it can be obtained

$$
\begin{aligned}
H[\varrho_k, \bar{\varrho}_k] \leq\; & H\Big[\int_{t_0}^{t_k} \Big(F\big(s, X(s), X(s-\delta(s))\big) \bigotimes G\big(s, X(s), X(s-\delta(s))\big)\Big) ds, \\
& \int_{t_0}^{t_k} \Big(F\big(s, X(s), X(s-\delta(s))\big) \bigotimes G\big(s, Y(s), Y(s-\delta(s))\big)\Big) ds\Big] \\
& + \varepsilon H\Big[\int_{t_0}^{t_k} \Big(F\big(s, X(s), X(s-\delta(s))\big) \bigotimes G\big(s, Y(s), Y(s-\delta(s))\big)\Big) ds, \\
& \int_{t_0}^{t_k} \Big(F\big(s, Y(s), Y(s-\delta(s))\big) \bigotimes G\big(s, Y(s), Y(s-\delta(s))\big)\Big) ds\Big] \\
& + H\Big[\int_{t_0}^{t_k} \Big(F\big(s, Y(s), Y(s-\delta(s))\big) \bigotimes G\big(s, Y(s), Y(s-\delta(s))\big)\Big) ds, \\
& \int_{t_0}^{t_k} \Big(F\big(s, Y(s), Y(s-\delta(s))\big) \bigotimes \overline{G}\big(Y(s)\big)\Big) ds\Big] \\
& + H\Big[\int_{t_0}^{t_k} \Big(F\big(s, Y(s), Y(s-\delta(s))\big) \bigotimes \overline{G}\big(Y(s)\big)\Big) ds, \\
& \int_{t_0}^{t_k} \Big(\overline{F}\big(Y(s), Y(s-\delta(s))\big) \bigotimes \overline{G}\big(Y(s)\big)\Big) ds\Big] \\
\leq\; & M_1 H\Big[\int_{t_0}^{t_k} G\big(s, X(s), X(s-\delta(s))\big) ds, \int_{t_0}^{t_k} G\big(s, Y(s), Y(s-\delta(s))\big) ds\Big] \\
& + M_1 H\Big[\int_{t_0}^{t_k} F\big(s, X(s), X(s-\delta(s))\big) ds, \int_{t_0}^{t} F\big(s, Y(s), Y(s-\delta(s))\big) ds\Big] \\
& + M_1 H\Big[\int_{t_0}^{t_k} G\big(s, Y(s), Y(s-\delta(s))\big) ds, \int_{t_0}^{t_k} \overline{G}\big(Y(s)\big) ds\Big] \\
& + M_1 H\Big[\int_{t_0}^{t_k} F\big(s, Y(s), Y(s-\delta(s))\big) ds, \int_{t_0}^{t_k} \overline{F}\big(Y(s), Y(s-\delta(s))\big) ds\Big] \\
\leq\; & 2M_1(\lambda_{11}+\lambda_{12}) \int_{t_0}^{t_k} H[X(s), Y(s)] ds + 2M_1(t_k - t_0)\xi(t). \qquad (3.17)
\end{aligned}
$$

On the base of (3.15), (3.16) and (3.17), we get

$$
\begin{aligned}
H[X_0, Y_0] \leq\; & 2A(\varepsilon) M_1(\lambda_{11}+\lambda_{12}) \int_{t_0}^{t_k} H[X(s), Y(s)] ds \\
& + 2A(\varepsilon) M_1(t_k - t_0)\xi(t) + 2\varepsilon B(\varepsilon) M_1^2. \qquad (3.18)
\end{aligned}
$$

Using (3.9), (3.10), (3.13), (3.14) and (3.18), we have

$$
H[X(t), Y(t)] \leq 2\varepsilon M_1(\lambda_{11}+\lambda_{12}) \int_{t_0}^{t} H[X(s), Y(s)] ds + 2\varepsilon M_1(t - t_0)\xi(t)
$$

$$
\begin{aligned}
&+ 2A(\varepsilon)M_1(\lambda_{11}+\lambda_{12})\int_{t_0}^{t_k} H[X(s),Y(s)]ds \\
&+ 2A(\varepsilon)M_1(t_k-t_0)\xi(t) + 2\varepsilon B(\varepsilon)M_1^2 \\
\le\ & P(\varepsilon)\int_{t_0}^{t} H[X(s),Y(s)]ds + Q(\varepsilon,t),
\end{aligned} \tag{3.19}
$$

where

$$
\begin{aligned}
P(\varepsilon) &= 2M_1(\lambda_{11}+\lambda_{12})\big(\varepsilon + A(\varepsilon)\big), \\
Q(\varepsilon,t) &= 2M_1(A(\varepsilon)+\varepsilon)(t-t_0)\xi(t) + 2\varepsilon B(\varepsilon)M_1^2.
\end{aligned}
$$

In view of the Lemma 2.1, we can get following estimate

$$
H[X(t),Y(t)] \le Q(\varepsilon,t)\exp\{P(\varepsilon)L\}. \tag{3.20}
$$

Therefore, by selecting the appropriate k and ε_1, for all $\eta > 0$, $\varepsilon \in (0,\varepsilon_1]$, such that

$$
H[X(t),Y(t)] \le \eta.
$$

Theorem 3.1 is proved. □

3.2. When the average limit of right-hand side one function is absent

We notice that the study of the averaging method of set-valued differential equations is under the condition that the average limit of the right-hand side functions exists. When the average limit of the functions does not exist, in [13,15,18], the possibility of application of averaging method for differential inclusions is proved. Therefore we discuss the averaging method for the cases when the average limit of the functions on the right-hand side of the equations not exist.

Supposing the following limit not exist

$$
\lim_{T\to\infty}\frac{1}{T}\int_t^{t+T} G(s,X(s),U(s))ds.
$$

We introduce the concept of semi-deviation metric, which is defined as

$$
\beta_1[A,B] = \sup_{a\in A}\inf_{b\in B}\|a-b\|, \quad \beta_2[A,B] = \sup_{b\in B}\inf_{a\in A}\|a-b\|.
$$

Suppose that there be functions $\overline{G}_1$, $\overline{G}_2 : conv(\mathbb{R}^n) \to conv(\mathbb{R}^n)$, such that

$$
\lim_{T\to\infty}\beta_1\Big[\overline{G}_1(X), \frac{1}{T}\int_0^T G(s,X,U)ds\Big] = 0, \tag{3.21}
$$

$$
\lim_{T\to\infty}\beta_2\Big[\frac{1}{T}\int_0^T G(s,X,U)ds, \overline{G}_2(X)\Big] = 0, \tag{3.22}
$$

where for $\overline{G}_1(X)$ and $\overline{G}_2(X)$, the following relationships are satisfied respectively

$$
\overline{G}_1(X) \subset G_1(X), \quad \overline{\lim_{T\to\infty}}\beta_1\Big[G_1(X), \frac{1}{T}\int_t^{t+T} G(s,X,U)ds\Big] = 0,
$$

$$G_2(X) \subset \overline{G}_2(X), \quad \lim_{T\to\infty} \beta_2\Big[\frac{1}{T}\int_t^{t+T} G(s,X,U)ds, G_2(X)\Big] = 0.$$

Therefore, for any $\varepsilon' > 0$, there exists a $T > 0$, for any $T > T'$, we have

$$\begin{aligned}
&\overline{G}_1(X) \subset \frac{1}{T}\int_t^{t+T} G(s,X,U)ds + S_{\varepsilon'}(0),\\
&\frac{1}{T}\int_t^{t+T} G(s,X,U)ds \subset \overline{G}_2(X) + S_{\varepsilon'}(0),
\end{aligned}$$

where for $\varepsilon' \geq 0$, $S_{\varepsilon'}(0) = \{x \in \mathbb{R}^n, \|x\| \leq \varepsilon'\}$.

When the average limit of (3.1) does not exist, the average equations are respectively equivalent to

$$\begin{cases} D_H Y_1(t) = \varepsilon\Big(\overline{F}\big(Y_1(t), Y_1(t-\delta(t))\big) \bigotimes \overline{G}_1\big(Y_1(t)\big)\Big),\\ \sum\limits_{k=0}^{K} \alpha_k(\varepsilon) Y_1(t_k) = \Phi\big(Y_1(t_0), \cdots, Y_1(t_K), D_H Y_1(t_0), \cdots, D_H Y_1(t_K), \varepsilon\big), \end{cases} \tag{3.23}$$

$$\begin{cases} D_H Y_2(t) = \varepsilon\Big(\overline{F}\big(Y_2(t), Y_2(t-\delta(t))\big) \bigotimes \overline{G}_2\big(Y_2(t)\big)\Big),\\ \sum\limits_{k=0}^{K} \alpha_k(\varepsilon) Y_2(t_k) = \Phi\big(Y_2(t_0), \cdots, Y_2(t_K), D_H Y_2(t_0), \cdots, D_H Y_2(t_K), \varepsilon\big). \end{cases} \tag{3.24}$$

Now we present the following conditions that will be used in the following proof.

(H_1) The set-valued function $F(t,X,U), \overline{G}_1(X) \in conv(\mathbb{R}^n)$ are continuous and bounded, i.e. there exist $M_1, \lambda_{21}, \lambda_{22} > 0$, for $X', X'', U', U'' \in conv(\mathbb{R}^n)$, such that

$$\begin{aligned}
&\beta_1[F(t,X,U),\{0\}] \bigvee \beta_1[G(t,X,U),\{0\}] \leq M_1,\\
&\beta_1[\overline{F}(Y_1,U),\{0\}] \bigvee \beta_1[\overline{G}_1(Y_1),\{0\}] \leq M_1,\\
&\beta_1[F(t,X',U'),F(t,X'',U'')] \bigvee \beta_1[G(t,X',U'),G(t,X'',U'')]\\
\leq& \lambda_{21}\beta_1[X',X''] + \lambda_{22}\beta_1[U',U''],\\
&\beta_1[\overline{F}(Y_1',U'),\overline{F}(Y_1'',U'')] \leq \lambda_{21}\beta_1[Y_1',Y_1''] + \lambda_{22}\beta_1[U',U''],\\
&\beta_1[\overline{G}_1(Y_1'),\overline{G}_1(Y_1'')] \leq \lambda_{21}\beta_1[Y_1',Y_1'']
\end{aligned}$$

hold, where $a \bigvee b = \max\{a,b\}$;

(H_2) The following inequality

$$\begin{aligned}
&\beta_1\Big[\Phi\big(X(t_0), \cdots, X(t_K), D_H X(t_0), \cdots, D_H X(t_K), \varepsilon\big),\\
&\Phi\big(Y_1(t_0), \cdots, Y_1(t_K), D_H Y_1(t_0), \cdots, D_H Y_1(t_K), \varepsilon\big)\Big]\\
\leq& \sum_{k=0}^{K} \mu_k(\varepsilon)\Big(\beta_1\big[X(t_k), Y_1(t_k)] + \beta_1[D_H X(t_k), D_H Y_1(t_k)]\Big)
\end{aligned}$$

holds, and for $\varepsilon \in (0, \varepsilon_2]$ we have

$$0 < \big\|\big(\sum_{k=0}^{K} \alpha_k(\varepsilon)\big)^{-1}\big\| \times \big(\sum_{k=0}^{K} \mu_k(\varepsilon)\big) < 1,$$

where $\mu_k > 0$ are continuous functions, $k = 0, 1, \cdots, K$. $\sum\limits_{k=0}^{K} \alpha_k(\varepsilon)$ is $n \times n-$dimensional nonsingular matrix, A^{-1} is called the inverse of the matrix A.

we assume that F and G appearing in (3.1) all the conditions $(H_1) - (H_2)$ are satisfied. By applying the properties of the semi-deviation metric, we can get the following conclusions.

Theorem 3.2. *Suppose that the following conditions are satisfied in the domain D*

$(A_{3.21})$ *The limits (3.2) and (3.21) exist uniformly in $t \geq 0$;*

$(A_{3.22})$ *The solutions $Y_1(t)$ of the systems (3.23) at $t \in [0, T]$ together with its δ_2-neighbourhood belong to domain D, i.e. $O(Y(t), \delta_2) \subset D$, where $\delta_2 > 0$ is a constant.*

Then for any $\eta > 0$ and $L > 0$, there exists $\varepsilon_2(\eta, L) > 0$ such that, for $\varepsilon \in (0, \varepsilon_2]$, the following inequality is true

$$\beta_1[X(t), Y_1(t)] \leq \eta.$$

Proof. The solutions of the set-valued differential systems (3.1) satisfy the following equations

$$\begin{cases} X(t) = X_0 + \varepsilon \displaystyle\int_{t_0}^{t} \Big(F\big(s, X(s), X(s-\delta(s))\big) \bigotimes G\big(s, X(s), X(s-\delta(s))\big)\Big) ds, \\ \sum\limits_{k=0}^{K} \alpha_k(\varepsilon)(Z_k) = \Phi\big(Z_0, \cdots, Z_K, D_H X(t_0), \cdots, D_H X(t_K), \varepsilon\big), \end{cases} \tag{3.25}$$

where $X_0 = X(t_0)$, $\varrho_k = \int_{t_0}^{t_k} \Big(F\big(s, X(s), X(s-\delta(s))\big) \otimes G\big(s, X(s), X(s-\delta(s))\big)\Big) ds$, $Z_k = X_0 + \varepsilon \varrho_k$.

Similarly, the solutions of the average systems (3.23) satisfy the following integral equations

$$\begin{cases} Y_1(t) = Y_0 + \varepsilon \displaystyle\int_{t_0}^{t} \Big(\overline{F}\big(Y_1(s), Y_1(s-\delta(s))\big) \bigotimes \overline{G}_1(Y_1(s))\Big) ds, \\ \sum\limits_{k=0}^{K} \alpha_k(\varepsilon)(\bar{Z}_k) = \Phi\big(\bar{Z}_0, \cdots, \bar{Z}_K, D_H Y_1(t_0), \cdots, D_H Y_1(t_K), \varepsilon\big), \end{cases} \tag{3.26}$$

where $Y_0 = Y_1(t_0)$, $\bar{\varrho}_k = \int_{t_0}^{t_k} \Big(\overline{F}\big(Y_1(s), Y_1(s-\delta(s))\big) \otimes \overline{G}_1\big(Y_1(s)\big)\Big) ds$, $\bar{Z}_k = Y_0 + \varepsilon \bar{\varrho}_k$.

From the integral equations (3.25) and (3.26), we can write

$$\begin{aligned} &\beta_1[X(t), Y_1(t)] \qquad (3.27) \\ \leq &\varepsilon \beta_1\Big[\int_{t_0}^{t} \Big(F\big(s, X(s), X(s-\delta(s))\big) \bigotimes G\big(s, X(s), X(s-\delta(s))\big)\Big) ds, \\ &\int_{t_0}^{t} \Big(F\big(s, X(s), X(s-\delta(s))\big) \bigotimes G\big(s, Y_1(s), Y_1(s-\delta(s))\big)\Big) ds\Big] \\ &+ \varepsilon \beta_1\Big[\int_{t_0}^{t} \Big(F\big(s, X(s), X(s-\delta(s))\big) \bigotimes G\big(s, Y_1(s), Y_1(s-\delta(s))\big)\Big) ds, \end{aligned}$$

$$
\begin{aligned}
&\int_{t_0}^{t} \Big(F\big(s,Y_1(s),Y_1(s-\delta(s))\big) \bigotimes G\big(s,Y_1(s),Y_1(s-\delta(s))\big)\Big)ds\Big] \\
&+ \varepsilon\beta_1\Big[\int_{t_0}^{t} \Big(F\big(s,Y_1(s),Y_1(s-\delta(s))\big) \bigotimes G\big(s,Y_1(s),Y_1(s-\delta(s))\big)\Big)ds, \\
&\int_{t_0}^{t} \Big(F\big(s,Y_1(s),Y_1(s-\delta(s))\big) \bigotimes \overline{G}_1\big(Y_1(s)\big)\Big)ds\Big] \\
&+ \varepsilon\beta_1\Big[\int_{t_0}^{t} \Big(F\big(s,Y_1(s),Y_1(s-\delta(s))\big) \bigotimes \overline{G}_1\big(Y_1(s)\big)\Big)ds, \\
&\int_{t_0}^{t} \Big(\overline{F}\big(Y_1(s),Y_1(s-\delta(s))\big) \bigotimes \overline{G}_1\big(Y_1(s)\big)\Big)ds\Big] + \beta_1[X_0,Y_0] \\
\leq& \varepsilon M_1\beta_1\Big[\int_{t_0}^{t} G\big(s,X(s),X(s-\delta(s))\big)ds, \int_{t_0}^{t} G\big(s,Y_1(s),Y_1(s-\delta(s))\big)ds\Big] \\
&+ \varepsilon M_1\beta_1\Big[\int_{t_0}^{t} F\big(s,X(s),X(s-\delta(s))\big)ds, \int_{t_0}^{t} F\big(s,Y_1(s),Y_1(s-\delta(s))\big)ds\Big] \\
&+ \varepsilon M_1\beta_1\Big[\int_{t_0}^{t} G\big(s,Y_1(s),Y_1(s-\delta(s))\big)ds, \int_{t_0}^{t} \overline{G}_1\big(Y_1(s)\big)ds\Big] \\
&+ \varepsilon M_1\beta_1\Big[\int_{t_0}^{t} F\big(s,Y_1(s),Y_1(s-\delta(s))\big)ds, \int_{t_0}^{t} \overline{F}\big(Y_1(s),Y_1(s-\delta(s))\big)ds\Big] \\
&+ \beta_1[X_0,Y_0] \\
\leq& 2\varepsilon M_1\lambda_{21}\int_{t_0}^{t} \beta_1[X(s),Y_1(s)]ds \\
&+ 2\varepsilon M_1\lambda_{22}\int_{t_0}^{t} \beta_1[X(s-\delta(s)),Y_1(s-\delta(s))]ds \\
&+ \varepsilon M_1\beta_1\Big[\int_{t_0}^{t} G\big(s,Y_1(s),Y_1(s-\delta(s))\big)ds, \int_{t_0}^{t} \overline{G}_1\big(Y_1(s)\big)ds\Big] \\
&+ \varepsilon M_1\beta_1\Big[\int_{t_0}^{t} F\big(s,Y_1(s),Y_1(s-\delta(s))\big)ds, \int_{t_0}^{t} \overline{F}\big(Y_1(s),Y_1(s-\delta(s))\big)ds\Big] \\
&+ \beta_1[X_0,Y_0],
\end{aligned} \tag{3.28}
$$

where

$$
\begin{aligned}
&I_1 = 2\varepsilon M_1\lambda_{21}\int_{t_0}^{t} \beta_1[X(s),Y_1(s)]ds, \\
&I_2 = 2\varepsilon M_1\lambda_{22}\int_{t_0}^{t} \beta_1[X(s-\delta(s)),Y_1(s-\delta(s))]ds, \\
&I_3 = \varepsilon M_1\beta_1\Big[\int_{t_0}^{t} G\big(s,Y_1(s),Y_1(s-\delta(s))\big)ds, \int_{t_0}^{t} \overline{G}_1\big(Y_1(s)\big)ds\Big], \\
&I_4 = \varepsilon M_1\beta_1\Big[\int_{t_0}^{t} F\big(s,Y_1(s),Y_1(s-\delta(s))\big)ds, \int_{t_0}^{t} \overline{F}\big(Y_1(s),Y_1(s-\delta(s))\big)ds\Big], \\
&I_5 = \beta_1[X_0,Y_0].
\end{aligned}
$$

For $\delta : [0,T_1] \to \mathbb{R}_+$, such that $t-\delta(t) \in [t^*-T_1,T]$. When $t \in [t^*-T_1,0]$, obviously $X(t-\delta(t)) = Y(t-\delta(t))$. We can get

$$\begin{aligned}I_2 &\le 2\varepsilon M_1\lambda_{22}\int_{t^*-T_1}^{0}\beta_1[X(s-\delta(s)),Y_1(s-\delta(s))]ds\\&\quad+2\varepsilon M_1\lambda_{22}\int_0^t\beta_1[X(s-\delta(s)),Y_1(s-\delta(s))]ds\\&=2\varepsilon M_1\lambda_{22}\int_0^t\beta_1[X(\tau),Y_1(\tau)]d\tau,\end{aligned}\tag{3.29}$$

where $\tau = s-\delta(s)$. Since $\delta(t)\in\mathbb{R}_+$, obviously $ds = d\tau$.

By the limit (3.21) and conditions $(A_{3.21})$, for any $\eta_2 > 0$, there exist $\varepsilon_2(L,\eta_2) > 0$ and $T_2 > 0$ such that for $\varepsilon \le \varepsilon_2(L,\eta_2)$ and $t > T_2$, we have

$$\overline{G}_1(X) \subset \frac{1}{\Delta t}\int_t^{t+\Delta t} G(s,X,U)ds + S_{\eta_2}(0).$$

So there is a measurable set of choices $g_i(s,x,u)\in G(s,X,U)$ such that

$$\Big\|\frac{1}{\Delta t}\int_t^{t+\Delta t} g_i(s,x,u)ds - \overline{g}_i(x)\Big\| < \eta_2 \tag{3.30}$$

hold, where $\overline{g}_i(x)\in\overline{G}_1(X)$, $i=1,2,\cdots$.

Then

$$\Big\|\int_t^{t+\Delta t} g_i(s,x,u)ds - \int_t^{t+\Delta t}\overline{g}_i(x)ds\Big\| < \Delta t\eta_2. \tag{3.31}$$

From the (3.31), we can obtain

$$I_3 \le \varepsilon M_1(t-t_0)\eta_2. \tag{3.32}$$

By analogy, we get

$$I_4 \le \varepsilon M_1(t-t_0)\xi(t). \tag{3.33}$$

From the boundary conditions (3.25), (3.26) and the condition (H_2), we can know

$$\begin{aligned}&\beta_1\Big[\sum_{k=0}^K\alpha_k(\varepsilon)(Z_k),\sum_{k=0}^K\alpha_k(\varepsilon)(\bar{Z}_k)\Big]\\&\le\sum_{k=0}^K\mu_k(\varepsilon)\Big(\beta_1[X_0,Y_0]+\varepsilon H_1[\varrho_k,\bar{\varrho}_k]+\beta_1[D_HX(t_k),D_HY_1(t_k)]\Big).\end{aligned}$$

Furthermore, similar to (3.15) of Theorem 3.1, we have

$$\beta_1[X_0,Y_0]\le A(\varepsilon)\beta_1[\varrho_k,\bar{\varrho}_k]+B(\varepsilon)\beta_1[D_HX(t_k),D_HY_1(t_k)]. \tag{3.34}$$

According to (3.1) and (3.23), we have

$$
\begin{aligned}
&\beta_1[D_H X(t_k), D_H Y_1(t_k)]\\
\leq&\varepsilon\beta_1\Big[F\big(t_k, X(t_k), X(t_k-\delta(t_k))\big)\bigotimes G\big(t_k, X(t_k), X(t_k-\delta(t_k))\big),\\
&\overline{F}\big(Y_1(t_k), Y_1(t_k-\delta(t_k))\big)\bigotimes \overline{G}_1\big(Y_1(t_k)\big)\Big]\\
\leq&\varepsilon\beta_1\Big[F\big(t_k, X(t_k), X(t_k-\delta(t_k))\big)\bigotimes G\big(t_k, X(t_k), X(t_k-\delta(t_k))\big), \{0\}\Big]\\
&+\varepsilon\beta_1\Big[\{0\}, \overline{F}\big(Y_1(t_k), Y_1(t_k-\delta(t_k))\big)\bigotimes \overline{G}_1\big(Y_1(t_k)\big)\Big]\\
\leq&2\varepsilon M_1^2.
\end{aligned}
\tag{3.35}
$$

From the inequality (3.30) and (3.31), we have

$$
\begin{aligned}
&\beta_1[\varrho_k, \bar{\varrho}_k]\\
\leq&\beta_1\Big[\int_{t_0}^{t_k}\Big(F\big(s, X(s), X(s-\delta(s))\big)\bigotimes G\big(s, X(s), X(s-\delta(s))\big)\Big)ds,\\
&\int_{t_0}^{t_k}\Big(F\big(s, X(s), X(s-\delta(s))\big)\bigotimes G\big(s, Y_1(s), Y_1(s-\delta(s))\big)\Big)ds\Big]\\
&+\varepsilon\beta_1\Big[\int_{t_0}^{t_k}\Big(F\big(s, X(s), X(s-\delta(s))\big)\bigotimes G\big(s, Y_1(s), Y_1(s-\delta(s))\big)\Big)ds,\\
&\int_{t_0}^{t_k}\Big(F\big(s, Y_1(s), Y_1(s-\delta(s))\big)\bigotimes G\big(s, Y_1(s), Y_1(s-\delta(s))\big)\Big)ds\Big]\\
&+\beta_1\Big[\int_{t_0}^{t_k}\Big(F\big(s, Y_1(s), Y_1(s-\delta(s))\big)\bigotimes G\big(s, Y_1(s), Y_1(s-\delta(s))\big)\Big)ds,\\
&\int_{t_0}^{t_k}\Big(F\big(s, Y_1(s), Y_1(s-\delta(s))\big)\bigotimes \overline{G}_1\big(Y_1(s)\big)\Big)ds\Big]\\
&+\beta_1\Big[\int_{t_0}^{t_k}\Big(F\big(s, Y_1(s), Y_1(s-\delta(s))\big)\bigotimes \overline{G}_1\big(Y_1(s)\big)\Big)ds,\\
&\int_{t_0}^{t_k}\Big(\overline{F}\big(Y_1(s), Y_1(s-\delta(s))\big)\bigotimes \overline{G}_1\big(Y_1(s)\big)\Big)ds\Big]\\
\leq&M_1\beta_1\Big[\int_{t_0}^{t_k}G\big(s, X(s), X(s-\delta(s))\big)ds, \int_{t_0}^{t_k}G\big(s, Y_1(s), Y_1(s-\delta(s))\big)ds\Big]\\
&+M_1\beta_1\Big[\int_{t_0}^{t_k}F\big(s, X(s), X(s-\delta(s))\big)ds, \int_{t_0}^{t}F\big(s, Y_1(s), Y_1(s-\delta(s))\big)ds\Big]\\
&+M_1\beta_1\Big[\int_{t_0}^{t_k}G\big(s, Y_1(s), Y_1(s-\delta(s))\big)ds, \int_{t_0}^{t_k}\overline{G}_1\big(Y_1(s)\big)ds\Big]\\
&+M_1\beta_1\Big[\int_{t_0}^{t_k}F\big(s, Y_1(s), Y_1(s-\delta(s))\big)ds, \int_{t_0}^{t_k}\overline{F}\big(Y_1(s), Y_1(s-\delta(s))\big)ds\Big]\\
\leq&2M_1(\lambda_{21}+\lambda_{22})\int_{t_0}^{t_k}\beta_1[X(s), Y_1(s)]ds + M_1(t_k-t_0)(\xi(t)+\eta_2).
\end{aligned}
\tag{3.36}
$$

On the base of (3.34), (3.35) and (3.36), we get

$$\begin{aligned}\beta_1[X_0, Y_0] \leq 2A(\varepsilon)M_1(\lambda_{21}+\lambda_{22})\int_{t_0}^{t_k}\beta_1[X(s), Y_1(s)]ds \\ + A(\varepsilon)M_1(t_k - t_0)(\xi(t)+\eta_2) + 2\varepsilon B(\varepsilon)M_1^2.\end{aligned} \tag{3.37}$$

According to (3.27), (3.29), (3.32), (3.33) and (3.37), we have

$$\begin{aligned}\beta_1[X(t), Y_1(t)] &\leq 2\varepsilon M_1(\lambda_{21}+\lambda_{22})\int_{t_0}^{t}\beta_1[X(s), Y_1(s)]ds + 2\varepsilon M_1(t-t_0)\xi(t) \\ &\quad + 2A(\varepsilon)M_1(\lambda_{21}+\lambda_{22})\int_{t_0}^{t_k}\beta_1[X(s), Y_1(s)]ds \\ &\quad + 2A(\varepsilon)M_1(t_k-t_0)\xi(t) + 2\varepsilon B(\varepsilon)M_1^2 \\ &\leq P(\varepsilon)\int_{t_0}^{t}\beta_1[X(s), Y_1(s)]ds + Q(\varepsilon, t),\end{aligned}$$

where

$$\begin{aligned}&P(\varepsilon) = 2M_1(\lambda_{11}+\lambda_{12})\big(\varepsilon + A(\varepsilon)\big), \\ &Q(\varepsilon, t) = M_1(A(\varepsilon)+\varepsilon)(t-t_0)(\xi(t)+\eta_2) + 2\varepsilon B(\varepsilon)M_1^2.\end{aligned}$$

Using the Lemma 2.1, we get

$$\beta_1[X(t), Y_1(t)] \leq Q(\varepsilon, t)\exp\{P(\varepsilon)L\}. \tag{3.38}$$

So, by selecting the appropriate k and ε_2, For any $\eta > 0$ and $\varepsilon \in (0, \varepsilon_2]$, then we obtain

$$\beta_1[X(t), Y_1(t)] \leq \eta.$$

Theorem 3.2 is proved. □

Similar to the analysis of theorem 3.2, we can come to the following conclusions.

Theorem 3.3. *Let in the domain D, the following conditions hold*

($A_{3.31}$) *The limits (3.2) and (3.22) exist uniformly in $t \geq 0$;*

($A_{3.32}$) *The solutions $Y_2(t)$ of the systems (3.24) at $t \in [0, T]$ together with its δ_3-neighbourhood belong to domain D, i.e. $O(Y(t), \delta_3) \subset D$, where $\delta_3 > 0$ is a constant.*

Then for any $\eta > 0$ and $L > 0$, there exist $\varepsilon_3(\eta, L) > 0$ such that for $\varepsilon \in (0, \varepsilon_3]$ and $t \in [0, L\varepsilon^{-1}]$ the inequality holds

$$\beta_2[X(t), Y_2(t)] \leq \eta.$$

Proof. The proof of this theorem is similar to the proof in Theorem 3.1 and 3.2, so it is omitted. □

3.3. When the average limit of right-hand side two function are absent

When the following limit

$$\lim_{T\to\infty}\frac{1}{T}\int_t^{t+T} F(s, X(s), U(s))ds,$$
$$\lim_{T\to\infty}\frac{1}{T}\int_t^{t+T} G(s, X(s), U(s))ds$$

are absent, base on the concept of semi-deviation metric, suppose that $\overline{G}_1(X), \overline{G}_2(X)$ satisfy the limit (3.21) and (3.22). There exist functions $\overline{F}_1,\ \overline{F}_2 : conv(\mathbb{R}^n) \times conv(\mathbb{R}^n) \to conv(\mathbb{R}^n)$, such that

$$\lim_{T\to\infty} \beta_1\big[\overline{F}_1(X,U), \frac{1}{T}\int_0^T F(s,X,U)ds\big] = 0, \tag{3.39}$$

$$\lim_{T\to\infty} \beta_2\big[\frac{1}{T}\int_0^T F(s,X,U)ds, \overline{F}_2(X,U)\big] = 0 \tag{3.40}$$

hold.

The averging equations are equivalent to

$$\begin{cases} D_H Y_1(t) = \varepsilon\Big(\overline{F}_1\big(Y_1(t), Y_1(t-\delta(t))\big) \bigotimes \overline{G}_1\big(Y_1(t)\big)\Big), \\ \sum\limits_{k=0}^{K} \alpha_k(\varepsilon) Y_1(t_k) = \Phi\big(Y_1(t_0), \cdots, Y_1(t_K), D_H Y_1(t_0), \cdots, D_H Y_1(t_K), \varepsilon\big), \end{cases} \tag{3.41}$$

$$\begin{cases} D_H Y_2(t) = \varepsilon\Big(\overline{F}_2\big(Y_2(t), Y_2(t-\delta(t))\big) \bigotimes \overline{G}_2\big(Y_2(t)\big)\Big), \\ \sum\limits_{k=0}^{K} \alpha_k(\varepsilon) Y_2(t_k) = \Phi\big(Y_2(t_0), \cdots, Y_2(t_K), D_H Y_2(t_0), \cdots, D_H Y_2(t_K), \varepsilon\big). \end{cases} \tag{3.42}$$

Theorem 3.4. *Suppose that the following conditions are satisfied in the domain D, as following*

$(A_{3.41})$ *For the set-value mappings $\overline{F}_1(X,U), \overline{G}_1(X)$, all the conditions $(H_1)-(H_2)$ are satisfied;*

$(A_{3.42})$ *The limits (3.21) and (3.39) exist uniformly in $t \geq 0$;*

$(A_{3.43})$ *The solutions $Y_1(t)$ of the systems (3.41) at $t \in [0,T]$ together with its δ_4-neighbourhood belong to domain D, i.e. $O(Y(t), \delta_4) \subset D$, where $\delta_4 > 0$ is a constant.*

Then for any $\eta > 0$ and $L > 0$, there exist $\varepsilon_4(\eta, L) > 0$ such that for $\varepsilon \in (0, \varepsilon_4]$ and $t \in [0, L\varepsilon^{-1}]$ the inequality holds

$$\beta_1[X(t), Y_1(t)] \leq \eta.$$

Similar to the analysis of theorem 3.4, we can get the following conclusion.

Theorem 3.5. *Suppose that the following conditions are satisfied in the domain D, as following*

$(A_{3.51})$ *For the set-value mappings $\overline{F}_2(X,U), \overline{G}_2(X)$, all the conditions $(H_1)-(H_2)$ are satisfied;*

$(A_{3.52})$ *The limits (3.22) and (3.40) exist uniformly in $t \geq 0$;*

$(A_{3.53})$ *The solutions $Y_2(t)$ of the systems (3.42) at $t \in [0,T]$ together with its δ_5-neighbourhood belong to domain D, i.e. $O(Y(t), \delta_5) \subset D$, where $\delta_5 > 0$ is a constant.*

Then for any $\eta > 0$ and $L > 0$, there exist $\varepsilon_5(\eta, L) > 0$ such that for $\varepsilon \in (0, \varepsilon_5]$ and $t \in [0, L\varepsilon^{-1}]$ the following estimate is ture

$$\beta_2[X(t), Y_2(t)] \leq \eta.$$

Proof. The proof of the Theorem 3.4 and Theorem 3.5 are carried on similarly to the proof of Theorem 3.1 and Theorem 3.2, so they are omitted. $\square$

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