Journal of Applied Analysis and Computation
Volume 14, Number 1, February 2024, 84–105
Website:http://www.jaac-online.com
DOI:10.11948/20230034

MULTIPLE SOLUTIONS FOR NONHOMOGENEOUS KLEIN-GORDON EQUATION WITH SIGN-CHANGING POTENTIAL COUPLED WITH BORN-INFELD THEORY

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Abstract In this paper, we study the following nonhomogeneous Klein-Gordon equation with Born-Infeld theory

$$\begin{cases} -\Delta u + \lambda V(x)u - K(x)(2\omega + \phi)\phi u = f(x,u) + h(x), & x \in \mathbb{R}^3, \\ \Delta\phi + \beta\Delta_4\phi = 4\pi K(x)(\omega + \phi)u^2, & x \in \mathbb{R}^3, \end{cases}$$

where $\omega > 0$ is a constant, $\lambda > 0$ is a parameter and $\triangle_4\phi = div(|\nabla\phi|^2\nabla\phi)$. Under some suitable assumptions on V, K, f and h, the existence of multiple solutions is proved by using the Linking theorem and the Ekeland's variational principle in critical point theory. Especially, the potential V is allowed to be sign-changing.

Keywords Klein-Gordon equation, nonhomogeneous, Born-Infeld theory, Ekeland's variational principle, sign-changing potential.

MSC(2010) 35B33, 35J65, 35Q55.

1. Introduction

In this paper, we study the following nonhomogeneous Klein-Gordon equation with Born-Infeld theory

$$(KGBI)_\lambda \begin{cases} -\Delta u + \lambda V(x)u - K(x)(2\omega + \phi)\phi u = f(x,u) + h(x), & x \in \mathbb{R}^3, \\ \Delta\phi + \beta\Delta_4\phi = 4\pi K(x)(\omega + \phi)u^2, & x \in \mathbb{R}^3, \end{cases}$$

where $\omega > 0$ is a constant, $\lambda \geq 1$ is a parameter, $V \in C(\mathbb{R}^3, \mathbb{R})$ and $f \in C(\mathbb{R}^3 \times \mathbb{R}, \mathbb{R})$ and $\triangle_4\phi = div(|\nabla\phi|^2\nabla\phi)$.

It is well known that Klein-Gordon equation can be used to develop the theory of electrically charged fields (see [14]), and Born-Infeld theory is proposed by Born

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[5–7] to overcome the infinite energy problem associated with a point-charge source in the original Maxwell theory. The presence of the nonlinear term f simulates the interaction between many particles or external nonlinear perturbations. For more details in the physical aspects, we refer the readers to [4, 9, 15, 17, 25].

In recent years, the Born-Infeld nonlinear electromagnetism has become more important since its relevance in the theory of superstring and membranes. By using variational methods, several existence results for problem $(KGBI)_\lambda$ have been found with constant potential $V(x) = m^2 - \omega^2$. We recall some of them.

The case of $h \equiv 0$, that is the homogeneous case, has been widely studied in recent years. In 2002, the authors [12] considered for the following Klein-Gordon equation with Born-Infeld theory on $\mathbb{R}^3$

$$\begin{cases} -\Delta u + [m^2 - (\omega + \phi)^2]\phi u = f(x, u), & x \in \mathbb{R}^3, \\ \Delta\phi + \beta\Delta_4\phi = 4\pi(\omega + \phi)u^2, & x \in \mathbb{R}^3, \end{cases} \tag{1.1}$$

for the pure power of nonlinearity, i.e., $f(x, u) = |u|^{p-2}u$, where ω and m are constants. By using the mountain pass theorem, they proved that (1.1) has infinitely many radially symmetric solutions under $|m| > \omega$ and $4 < p < 6$. Mugnai [17] covered the case $2 < p \leq 4$ assuming $\sqrt{\frac{p-2}{2}}|m| > \omega > 0$. Later, the authors [20] and [16] considered the existence of solutions and ground state solutions for (1.1) with critical Sobolev exponent respectively. Zhang and Liu [29] consider the existence and multiplicity of sign-changing solutions by the method of invariant sets of descending flow.

Recently, for general potential $V(x)$, Chen and Song [11] obtained the existence of multiple nontrivial solutions for (1.1) with the nonlinearity with concave and convex nonlinearities. Other related results about homogeneous Klein-Gordon equation with Born-Infeld theory can be found in [1, 19, 21, 23, 27].

Next, we consider the nonhomogeneous case, that is $h \not\equiv 0$. In [10], Chen and Li proved that $(KGBI)_\lambda$ had two nontrivial radially symmetric solutions with $\lambda = 1$ if $f(x, u) = |u|^{p-2}u$ and $h(x)$ is radially symmetric. In [22], the authors obtain the existence of two solutions by the Mountain Pass Theorem and the Ekeland's variational principle in critical point theory for general $f(x, u)$.

Motivated by the above works, in the present paper we consider $(KGBI)_\lambda$ with more general potential $V(x)$ and $f(x, u)$. Precisely, we make the following assumptions.

$(V0)$ There is $b > 0$ such that $meas\{x \in \mathbb{R}^3 : V(x) \leq b\} < +\infty$, where $meas$ denotes the Lebesgue measures;

$(V1)$ $V \in C(\mathbb{R}^3, \mathbb{R})$ and V is bounded below;

$(V2)$ $\Omega = intV^{-1}(0)$ is nonempty and has smooth boundary and $\overline{\Omega} = V^{-1}(0)$;

$(f1)$ $F(x, u) = \int_0^u f(x, s)ds \geq 0$ for all (x, u) and $f(x, u) = o(u)$ uniformly in x as $u \to 0$;

$(f2)$ $F(x, u)/u^4 \to +\infty$ as $|u| \to +\infty$ uniformly in x;

$(f3)$ $\mathcal{F}(x, u) := \frac{1}{4}f(x, u)u - F(x, u) \geq 0$ for all $(x, u) \in \mathbb{R}^3 \times \mathbb{R}$;

$(f4)$ There exist $a_1, L_1 > 0$ and $\tau \in (3/2, 2)$ such that

$$|f(x, u)|^\tau \leq a_1\mathcal{F}(x, u)|u|^\tau, \quad \text{for all } x \in \mathbb{R}^3 \text{ and } |u| \geq L_1;$$

(K) $K(x) \in L^3(\mathbb{R}^3) \cup L^\infty(\mathbb{R}^3)$ and $K(x) \geq 0$ is not identically zero for a.e. $x \in \mathbb{R}^3$;

(h) $h(x) \in L^2(\mathbb{R}^3)$ and $h(x) \geq 0$ for a.e. $x \in \mathbb{R}^3$.

Remark 1.1. It follows from $(f3)$ and $(f4)$ that $|f(x,u)|^{\tau} \leq \frac{a_1}{4}|f(x,u)||u|^{\tau+1}$ for large u. Thus, by $(f1)$, for any $\varepsilon > 0$, there exists $C_{\varepsilon} > 0$ such that

$$|f(x,u)| \leq \varepsilon|u| + C_{\varepsilon}|u|^{q-1}, \ \forall(x,u) \in \mathbb{R}^3 \times \mathbb{R} \tag{1.2}$$

and

$$|F(x,u)| \leq \varepsilon|u|^2 + C_{\varepsilon}|u|^q, \ \forall(x,u) \in \mathbb{R}^3 \times \mathbb{R}, \tag{1.3}$$

where $q = 2\tau/(\tau-1) \in (4, 2^*)$ and $2^* = 6$ is the critical exponent for the Sobolev embedding in dimension 3.

Remark 1.2. It is not difficult to find out functions f satisfying $(f1)$–$(f4)$, for example,

$$f(x,t) = g(x)t^3\left(2In(1+t^2) + \frac{t^2}{1+t^2}\right), \forall(x,t) \in \mathbb{R}^3 \times \mathbb{R},$$

where g is a is a continuous bounded function with $\inf_{x\in\mathbb{R}^3} g(x) > 0$.

Before stating our main results, we give some notations.

$H^1(\mathbb{R}^3)$ is the usual Sobolev space endowed with the standard product and norm

$$(u,v)_{H^1} = \int_{\mathbb{R}^3}(\nabla u\nabla v + uv)dx; \qquad \|u\|_{H^1} = \left(\int_{\mathbb{R}^3}(|\nabla u|^2 + |u|^2)dx\right)^{1/2}.$$

For any $1 \leq s \leq +\infty$ and $\Omega \subset \mathbb{R}^3, L^s(\Omega)$ denotes a Lebesgue space; the norm in $L^s(\Omega)$ is denoted by $|u|_{s,\Omega}$, where Ω is a proper subset of $\mathbb{R}^3$, by $|\cdot|_s$ when $\Omega = \mathbb{R}^3$. Let $D^{1,2}(\mathbb{R}^3)$ be the completion of $C_0^{\infty}(\mathbb{R}^3)$ with respect to the norm

$$\|u\|_{D^{1,2}(\mathbb{R}^3)} = \left(\int_{\mathbb{R}^3}|\nabla u|^2dx\right)^{1/2}.$$

$D(\mathbb{R}^3)$ is the completion of $C_0^{\infty}(\mathbb{R}^3)$ with respect to the norm

$$\|u\|_D := |\nabla u|_2 + |\nabla u|_4.$$

It is clear that $D(\mathbb{R}^3)$ is continuously embedded in $D^{1,2}(\mathbb{R}^3)$. By the Sobolev inequality, we know that $D^{1,2}(\mathbb{R}^3)$ is continuously embedded in $L^6 = L^6(\mathbb{R}^3)$ and $D(\mathbb{R}^3)$ is continuously embedded in $L^{\infty} = L^{\infty}(\mathbb{R}^3)$.

$\bar{S}$ is the best Sobolev constant for the Sobolev embedding $D^{1,2}(\mathbb{R}^3) \hookrightarrow L^6(\mathbb{R}^3)$, that is,

$$\bar{S} = \inf_{u\in D^{1,2}(\mathbb{R}^3)\setminus\{0\}} \frac{\|u\|_{D^{1,2}}}{|u|_6}.$$

For any $r > 0$ and $z \in \mathbb{R}^3, B_r(z)$ denotes the ball of radius r centered at z.

We denote " $\rightharpoonup$ " the weak convergence and by " $\rightarrow$ " strong convergence. Also if we take a subsequence of a sequence $\{u_n\}$, we shall denote it again $\{u_n\}$. We use $o(1)$ to denote any quantity which tends to zero when $n \to \infty$. The letters d_i, C_i will be used to denote various positive constants which may vary from line to line and are not essential to the problem.

Now we can state our main results.

Theorem 1.1. *Assume that* $(V0)$–$(V1)$, $(f1)$–$(f4)$, (K) *and* (h) *are satisfied. If* $V(x) < 0$ *for some* $x \in \mathbb{R}^3$, *then for each* $k \in \mathbb{N}$, *there exist* $\lambda_k > k, b_k > 0$ *and* $\eta_k > 0$ *such that problem* $(KGBI)_\lambda$ *has at least two nontrivial solutions for every* $\lambda = \lambda_k, |K|_\infty < b_k$ *and* $|h|_2 \le \eta_k$.

Theorem 1.2. *Assume that* $(V0)$–$(V2)$, $(f1)$–$(f4)$, (K) *and* (h) *are satisfied. If* $V^{-1}(0)$ *has nonempty interior, then there exist* $\Lambda > 0, b_\lambda > 0$ *and* $\eta_\lambda > 0$ *such that problem* $(KGBI)_\lambda$ *has at least two nontrivial solutions for every* $\lambda > \Lambda, |h|_2 \le \eta_\lambda$ *and* $|K|_\infty < b_\lambda$.

If $V \ge 0$, we remove the restriction of the norm of K and we have the following theorem.

Theorem 1.3. *Assume that* $V \ge 0$, $(V0)$–$(V2)$, $(f1)$–$(f4)$, (K) *and* (h) *are satisfied. If* $V^{-1}(0)$ *has nonempty interior* Ω *and* $h \ne 0$, *then there exist* $\Lambda_* > 0$ *and* $\eta > 0$ *such that problem* $(KGBI)_\lambda$ *has at least two nontrivial solutions for every* $\lambda > \Lambda_*$ *and* $|h|_2 \le \eta$.

To obtain main results, we have to overcome some difficulties in using variational method. The main difficulty consists in the lack of compactness of the Sobolev embedding $H^1(\mathbb{R}^3)$ into $L^p(\mathbb{R}^3)$, $p \in (2,6)$. Since we assume that the potential is not radially symmetric, we cannot use the usual way to recover compactness, for example, restricting in the subspace $H^1_r(\mathbb{R}^3)$ of radially symmetric functions. To recover the compactness, we borrow some ideas used in [3, 13] and establish the parameter dependent compactness conditions.

To the best of our knowledge, it seems that our theorems are the first results about the existence of multiple solutions for the nonhomogeneous Klein-Gordon equation with Born-Infeld theory on $\mathbb{R}^3$ with general nonlinear term and sign-changing potential. In the following, we can see that many technical difficulties arise due to the presence of a non-local term ϕ, which is not homogeneous as it is in the Schrödinger-Poisson systems. In other words, the adaptation of the ideas to the procedure of our problem is not trivial at all, because of the presence of the nonlocal term ϕ_u. Hence, a more careful analysis of the interaction between the couple (u, ϕ) is required.

The paper is organized as follows. We introduce the variational setting and the compactness conditions in Section 2. In Section 3, we give the proofs of main results.

2. Variational setting and compactness condition

In this section, we firstly give the variational setting of $(KGBI)_\lambda$ and then establish the compactness conditions.

Let $V(x) = V^+(x) - V^-(x)$, where $V^\pm = \max\{\pm V(x), 0\}$. Let

$$E = \left\{ u \in H^1(\mathbb{R}^3) : \int_{\mathbb{R}^3} |\nabla u|^2 + V^+(x)u^2 dx < \infty \right\}$$

be equipped with the inner product $(u, v) = \int_{\mathbb{R}^3} (\nabla u \nabla v + V^+(x)uv)dx$ and the norm

$\|u\| = (u,u)^{1/2}$. For $\lambda > 0$, we also need the following inner product and norm

$$(u,v)_\lambda = \int_{\mathbb{R}^3} (\nabla u \nabla v + \lambda V^+(x)uv)dx, \quad \|u\|_\lambda = (u,u)_\lambda^{1/2}.$$

It is clear $\|u\| \le \|u\|_\lambda$ for $\lambda \ge 1$. Set $E_\lambda = (E, \|\cdot\|_\lambda)$. It follows from $(V0)$–$(V1)$ and the Poincaré inequality, we know that the embedding $E_\lambda \hookrightarrow H^1(\mathbb{R}^3)$ is continuous. Therefore, for $s \in [2,6]$, there exists $d_s > 0$ (independent of $\lambda \ge 1$) such that

$$|u|_s \le d_s \|u\|_\lambda, \quad \forall u \in E_\lambda. \tag{2.1}$$

Let

$$F_\lambda = \{u \in E_\lambda : suppu \subset V^{-1}([0,\infty))\},$$

and $F_\lambda^\perp$ denote the orthogonal complement of F_λ in E_λ. Clearly, $F_\lambda = E_\lambda$ if $V \ge 0$, otherwise $F_\lambda^\perp \ne \{0\}$. Define

$$A_\lambda := -\Delta + \lambda V,$$

then A_λ is formally self-adjoint in $L^2(\mathbb{R}^3)$ and the associated bilinear form

$$a_\lambda(u,v) = \int_{\mathbb{R}^3} (\nabla u \nabla v + \lambda V(x)uv)dx$$

is continuous in E_λ. As in [13], for fixed $\lambda > 0$, we consider the eigenvalue problem

$$-\Delta u + \lambda V^+(x)u = \mu\lambda V^-(x)u, \quad u \in F_\lambda^\perp. \tag{2.2}$$

By $(V0)$–$(V1)$, we know that the quadratic form $u \mapsto \int_{\mathbb{R}^3} \lambda V^-(x)u^2dx$ is weakly continuous. Hence following Theorem 4.45 and Theorem 4.46 in [24], we can deduce the following proposition, which is the spectral theorem for compact self-adjoint operators jointly with the Courant-Fischer minimax characterization of eigenvalues.

Proposition 2.1. *Suppose that $(V0)$–$(V1)$ hold, then for any fixed $\lambda > 0$, the eigenvalue problem (2.2) has a sequence of positive eigenvalues $\{\mu_j(\lambda)\}$, which may be characterized by*

$$\mu_j(\lambda) = \inf_{dimM \ge j, M \subset F_\lambda^\perp} \sup \left\{ \|u\|_\lambda^2 : u \in M, \int_{\mathbb{R}^3} \lambda V^-(x)u^2dx = 1 \right\}, \quad j = 1,2,3,....$$

Furthermore, $\mu_1(\lambda) \le \mu_2(\lambda) \le \cdots \le \mu_j(\lambda) \to +\infty$ as $j \to +\infty$, and the corresponding eigenfunctions $\{e_j(\lambda)\}$, which may be be chosen so that $(e_i(\lambda), e_j(\lambda))_\lambda = \delta_{ij}$, are a basis of $F_\lambda^\perp$.

Next, we give some properties for the eigenvalues $\{\mu_j(\lambda)\}$ defined above.

Proposition 2.2. ([13]) *Assume that $(V0)$–$(V1)$ hold and $V^- \not\equiv \{0\}$. Then, for each fixed $j \in \mathbb{N}$,*

(i) $\mu_j(\lambda) \to 0$ as $\lambda \to +\infty$;

(ii) $\mu_j(\lambda)$ is a non-increasing continuous function of λ.

Remark 2.1. By Proposition 2.2, there exists $\Lambda_0 > 0$ such that $\mu_1(\lambda) \le 1$ for all $\lambda > \Lambda_0$.

Denote

$$E_\lambda^- := span\{e_j(\lambda) : \mu_j(\lambda) \le 1\} \quad \text{and} \quad E_\lambda^+ := span\{e_j(\lambda) : \mu_j(\lambda) > 1\}.$$

Then $E_\lambda = E_\lambda^- \bigoplus E_\lambda^+ \bigoplus F_\lambda$ is an orthogonal decomposition. The quadratic form a_λ is negative semidefinite on E_λ^-, positive definite on $E_\lambda^+ \bigoplus F_\lambda$ and it is easy to see that $a_\lambda(u, v) = 0$ if u, v are in different subspaces of the above decomposition of E_λ.

From Remark 2.1, we have that $\dim E_\lambda^- \ge 1$ when $\lambda > \Lambda_0$. Moreover, since $\mu_j(\lambda) \to +\infty$ as $j \to +\infty$, $\dim E_\lambda^- < +\infty$ for every fixed $\lambda > 0$.

System $(KGBI)_\lambda$ has a variational structure. In fact, we consider the functional $\mathcal{J}_\lambda : E_\lambda \times D(\mathbb{R}^3) \to \mathbb{R}$ defined by

$$\begin{aligned}\mathcal{J}_\lambda(u, \phi) =& \frac{1}{2}\int_{\mathbb{R}^3} (|\nabla u|^2 + \lambda V(x)u^2)dx - \frac{1}{2}\int_{\mathbb{R}^3} K(x)(2\omega + \phi)\phi u^2 dx \\ &- \frac{1}{8\pi}\int_{\mathbb{R}^3} |\nabla \phi|^2 dx - \frac{\beta}{16\pi}\int_{\mathbb{R}^3} |\nabla \phi|^4 dx - \int_{\mathbb{R}^3} F(x, u)dx - \int_{\mathbb{R}^3} h(x)u dx.\end{aligned}$$

The solutions $(u, \phi) \in E_\lambda \times D(\mathbb{R}^3)$ of system $(KGBI)_\lambda$ are the critical points of $\mathcal{J}_\lambda$. By using the reduction method described in [4], we are led to the study of a new functional $I_\lambda(u)$ ((2.5)). We need the following technical result.

Proposition 2.3. *Let $K(x)$ satisfy the condition (K). Then for any $u \in E_\lambda$, there exists a unique $\phi = \phi_u \in D(\mathbb{R}^3)$ which satisfies*

$$\Delta\phi + \beta\Delta_4\phi = 4\pi K(x)(\phi + \omega)u^2.$$

Moreover, the map $\Phi : u \in E_\lambda \mapsto \phi_u \in D(\mathbb{R}^3)$ is continuously differentiable, and

(i) $\phi_u \le 0$, *moreover,* $-\omega \le \phi_u$ *on the set* $\{x \in \mathbb{R}^3 | u(x) \neq 0\}$;

(ii) $\int_{\mathbb{R}^3} (|\nabla\phi_u|^2 + \beta|\nabla\phi_u|^4)dx \le 4\pi\omega^2 d_2^2 |K|_\infty \|u\|_\lambda^2$, *if* $K \in L^\infty(\mathbb{R}^3)$.

(iii) $\int_{\mathbb{R}^3} (|\nabla\phi_u|^2 + \beta|\nabla\phi_u|^4)dx \le 4\pi\omega^2 d_3^2 |K|_3 \|u\|_\lambda^2$, *if* $K \in L^3(\mathbb{R}^3)$.

Proof. For every fixed $u \in E_\lambda$, the solutions of

$$\Delta\phi + \beta\Delta_4\phi = 4\pi K(x)(\phi + \omega)u^2 \tag{2.3}$$

are critical points of the functional

$$J(\phi) = \int_{\mathbb{R}^3} \left\{ \frac{1}{8\pi}|\nabla\phi|^2 + \frac{\beta}{16\pi}|\nabla\phi|^4 + K(x)\omega\phi u^2 + \frac{1}{2}K(x)\phi^2 u^2 \right\} dx \tag{2.4}$$

defined on D. The functional J is coercive. Indeed, by the continuous embedding of $D \in L^\infty(\mathbb{R}^3)$

$$J(\phi) \ge \frac{1}{8\pi}|\nabla\phi|_2^2 + \frac{\beta}{16\pi}|\nabla\phi|_4^4 - C|u^2|_1|K|_\infty(|\nabla\phi|_2 + |\nabla\phi|_4).$$

Furthermore J is weakly lower semicontinuous since each term in (2.4) is continuous and convex. Hence J admits a global minimum. The solution of is unique because the operator

$$A = -\Delta - \beta\Delta_4 + 4\pi K(x)u^2$$

is strictly monotone.

The result (i) can be proved similarly as Lemma 2.3 of [17].

(ii) After multiplying (2.3) by ϕ_u and integrating by parts, by (i) and (2.1), we can get that

$$\begin{aligned}\int_{\mathbb{R}^3}(|\nabla\phi_u|^2+\beta|\nabla\phi_u|^4)dx &= -4\pi\int_{\mathbb{R}^3}K(x)(\phi_u+\omega)\phi_u u^2dx\\ &\le -4\pi\omega\int_{\mathbb{R}^3}K(x)\phi_u u^2dx\\ &\le 4\pi\omega^2 d_2^2|K|_\infty\|u\|_\lambda^2.\end{aligned}$$

(iii) Similarly to (ii), we can get (iii) hold. The proof is complete. □

By above equality and the definition of $\mathcal{J}_\lambda$, we obtain a C^1 functional $I_\lambda = \mathcal{J}_\lambda(u,\phi_u): E_\lambda\to\mathbb{R}$ given by

$$\begin{aligned}I_\lambda(u) =&\frac{1}{2}\int_{\mathbb{R}^3}[|\nabla u|^2+\lambda V(x)u^2-K(x)(2\omega+\phi_u)\phi_u u^2]dx-\frac{1}{8\pi}\int_{\mathbb{R}^3}|\nabla\phi_u|^2dx\\ &-\frac{\beta}{16\pi}\int_{\mathbb{R}^3}|\nabla\phi_u|^4dx-\int_{\mathbb{R}^3}F(x,u)dx-\int_{\mathbb{R}^3}h(x)udx\\ =&\frac{1}{2}\int_{\mathbb{R}^3}(|\nabla u|^2+\lambda V(x)u^2-\omega K(x)\phi_u u^2)dx+\frac{\beta}{16\pi}\int_{\mathbb{R}^3}|\nabla\phi_u|^4dx\\ &-\int_{\mathbb{R}^3}F(x,u)dx-\int_{\mathbb{R}^3}h(x)udx\end{aligned}\tag{2.5}$$

and its Gateaux derivative is

$$\begin{aligned}\langle I_\lambda'(u),v\rangle =& \int_{\mathbb{R}^3}(\nabla u\cdot\nabla v+\lambda V(x)uv-K(x)(2\omega+\phi_u)\phi_u uvdx\\ &-\int_{\mathbb{R}^3}f(x,u)vdx-\int_{\mathbb{R}^3}h(x)vdx\end{aligned}$$

for all $v\in E_\lambda$. Set

$$M(u)=\int_{\mathbb{R}^3}K(x)\phi_u u^2dx.$$

Now we give some properties about the functional M and its derivative M' possess BL-splitting property, which is similar to Brezis-Lieb Lemma [8].

Proposition 2.4. *Let $K\in L^\infty(\mathbb{R}^3)\cup L^3(\mathbb{R}^3)$. If $u_n\rightharpoonup u$ in $H^1(\mathbb{R}^3)$ and $u_n(x)\to u(x)$ a.e. $x\in\mathbb{R}^3$, then*

(i) $\phi_{u_n}\rightharpoonup\phi_u$ in $D(\mathbb{R}^3)$;

(ii) $M(u_n-u)=M(u_n)-M(u)+o(1)$;

(iii) $M'(u_n-u)=M'(u_n)-M'(u)+o(1)$ in $H^{-1}(\mathbb{R}^3)$.

Proof. (i) A similar proof of Lemma 3.2 in [20]. The proof of (ii) and (iii) can be similar as Lemma 2.1 in [26]. We omit it here. □

Next, we investigate the compactness conditions for the functional I_λ. Recall that a C^1 functional J satisfies (PS) condition at level c if any sequence $\{u_n\}\subset E$ such that $J(u_n)\to c$ and $J'(u_n)\to 0$ has a convergent subsequence; and such sequence is called a $(PS)_c$ sequence.

We only consider the case $K \in L^\infty(\mathbb{R}^3)$, the other case $K \in L^3(\mathbb{R}^3)$ is similar.

Lemma 2.1. *Suppose that* $(V0)$–$(V1)$, $(f1)$–$(f4)$, (K) *and* (h) *are satisfied. Then every* $(PS)_c$ *sequence of* I_λ *is bounded in* E_λ *for each* $c \in \mathbb{R}$.

Proof. Let $\{u_n\} \subset E_\lambda$ be a $(PS)_c$ sequence of I_λ. Suppose by contradiction that

$$I_\lambda(u_n) \to c, \quad I'_\lambda(u_n) \to 0, \quad \|u_n\|_\lambda \to \infty \tag{2.6}$$

as $n \to \infty$ after passing to a subsequence. Take $w_n := u_n/\|u_n\|_\lambda$. Then $\|w_n\|_\lambda = 1, w_n \rightharpoonup w$ in E_λ and $w_n(x) \to w(x)$ a.e. $x \in \mathbb{R}^3$.

We first consider the case $w = 0$. By (2.6), $(f3)$, Proposition 2.3 and the fact $w_n \to 0$ in $L^2(\{x \in \mathbb{R}^3 : V(x) < 0\})$, we obtain

$$\begin{aligned}
&I_\lambda(u_n) - \frac{1}{4}\langle I'_\lambda(u_n), u_n\rangle \\
&= \frac{1}{4}\int_{\mathbb{R}^3}(|\nabla u_n|^2 + \lambda V(x)u_n^2)dx + \frac{1}{4}\int_{\mathbb{R}^3} K(x)\phi_{u_n}^2 u_n^2 dx + \frac{\beta}{16\pi}\int_{\mathbb{R}^3}|\nabla\phi_{u_n}|^4 dx \\
&\quad + \int_{\mathbb{R}^3}\mathcal{F}(x, u_n)dx - \frac{3}{4}\int_{\mathbb{R}^3} h(x)u_n dx.
\end{aligned}$$

Divided by $\|u_n\|_\lambda^2$ above inequality, by $\omega = 0$, we can obtain that

$$\begin{aligned}
o(1) &= \frac{1}{\|u_n\|_\lambda^2}\left(I_\lambda(u_n) - \frac{1}{4}\langle I'_\lambda(u_n), u_n\rangle\right) \\
&= \frac{1}{4}\|w_n\|_\lambda^2 - \frac{\lambda}{4}\int_{\mathbb{R}^3} V^-(x)w_n^2 dx + \frac{1}{4\|u_n\|_\lambda^2}\int_{\mathbb{R}^3} K(x)\phi_{u_n}^2 u_n^2 dx \\
&\quad + \frac{1}{\|u_n\|_\lambda^2}\frac{\beta}{16\pi}\int_{\mathbb{R}^3}|\nabla\phi_{u_n}|^4 dx + \frac{1}{\|u_n\|_\lambda^2}\int_{\mathbb{R}^3}\mathcal{F}(x, u_n)dx \\
&\quad - \frac{3}{4\|u_n\|_\lambda^2}\int_{\mathbb{R}^3} h(x)u_n dx \\
&\geq \frac{1}{4} - \frac{\lambda}{4}|V^-|_\infty \int_{suppV^-} w_n^2 dx - \frac{3}{4}|h|_2 d_2 \frac{1}{\|u_n\|_\lambda} \\
&= \frac{1}{4} + o(1),
\end{aligned}$$

which is a contradiction.

If $w \neq 0$, then $\Omega_1 := \{x \in \mathbb{R}^3 : w(x) \neq 0\}$ has positive Lebesgue measure. For $x \in \Omega_1$, one has $|u_n(x)| \to \infty$ as $n \to \infty$, and then, by $(f2)$,

$$\frac{F(x, u_n(x))}{u_n^4(x)} w_n^4(x) \to +\infty \quad \text{as } n \to \infty,$$

which, jointly with Fatou's lemma, shows that

$$\int_{\Omega_1}\frac{F(x, u_n)}{u_n^4} w_n^4 dx \to +\infty \quad \text{as } n \to \infty. \tag{2.7}$$

By (2.5) and Proposition 2.3, we have that

$$\begin{aligned}
&\frac{I_\lambda(u_n)}{\|u_n\|_\lambda^4}\\
&\le \frac{1+\omega^2 d_2^2|K|_\infty}{2\|u_n\|_\lambda^2} + \frac{\beta}{16\pi\|u_n\|_\lambda^4}\int_{\mathbb{R}^3}|\nabla\phi_{u_n}|^4dx - \int_{\mathbb{R}^3}\frac{F(x,u_n)}{\|u_n\|_\lambda^4}dx\\
&\quad - \int_{\mathbb{R}^3} h(x)\frac{u_n}{\|u_n\|_\lambda^4}dx\\
&\le \frac{2+3\omega^2 d_2^2|K|_\infty}{4\|u_n\|_\lambda^2} - \left(\int_{v=0}+\int_{v\neq 0}\right)\frac{F(x,u_n)}{u_n^4}v_n^4dx + \frac{|h|_2 d_2}{\|u_n\|_\lambda^3}\\
&\le \frac{2+3\omega^2 d_2^2|K|_\infty}{4\|u_n\|_\lambda^2} - \int_{v\neq 0}\frac{F(x,u_n)}{u_n^4}v_n^4dx + \frac{|h|_2 d_2}{\|u_n\|_\lambda^3}\\
&\to -\infty.
\end{aligned}$$

Combining this with $(f1)$, the first limit of (2.6), (K) and (h), we obtain that

$$0 \ge \limsup_{n\to\infty}\int_{\mathbb{R}^3}\frac{F(x,u_n)}{\|u_n\|_\lambda^4}dx \ge \limsup_{n\to\infty}\int_{\Omega_1}\frac{F(x,u_n)}{u_n^4}w_n^4dx = +\infty.$$

This is impossible.

Hence $\{u_n\}$ is bounded in E_λ.

For the case $K \in L^3(\mathbb{R}^3)$, we can use the Cauchy-Schwarz inequality and the boundedness of ϕ_{u_n} to get the result. □

Lemma 2.2. *Suppose that* $(V0)$–$(V1)$, (K), (h) *and* (1.2) *hold. If* $u_n \rightharpoonup u$ *in* E_λ, $u_n(x) \to u(x)$ *a.e. in* $\mathbb{R}^3$, *and we denote* $w_n := u_n - u$, *then*

$$I_\lambda(u_n) = I_\lambda(w_n) + I_\lambda(u) + o(1) \tag{2.8}$$

and

$$\langle I'_\lambda(u_n),\varphi\rangle = \langle I'_\lambda(w_n),\varphi\rangle + \langle I'_\lambda(u),\varphi\rangle + o(1),\ \text{uniformly for all } \varphi\in E_\lambda \tag{2.9}$$

as $n\to\infty$. *In particular, if* $I_\lambda(u_n)\to c(\in\mathbb{R})$ *and* $I'_\lambda(u_n)\to 0$ *in* E^*_λ *(the dual space of* E_λ*), then* $I'_\lambda(u) = 0$ *and*

$$\begin{aligned}
&I_\lambda(w_n) \to c - I_\lambda(u),\\
&\langle I'_\lambda(w_n),\varphi\rangle \to 0,\ uniformly\ for\ all\ \varphi\in E_\lambda
\end{aligned} \tag{2.10}$$

after passing to a subsequence.

Proof. Since $u_n \rightharpoonup u$ in E_λ, we have $(u_n - u, u)_\lambda \to 0$ as $n\to\infty$, which implies that

$$\|u_n\|_\lambda^2 = (w_n+u, w_n+u)_\lambda = \|w_n\|_\lambda^2 + \|u\|_\lambda^2 + o(1). \tag{2.11}$$

By $(V0)$, the Hölder inequality and $w_n \rightharpoonup 0$, we have

$$\left|\int_{\mathbb{R}^3} V^-(x)w_n u dx\right| = \left|\int_{suppV^-} V^- w_n u dx\right| \le |V^-|_\infty\left(\int_{suppV^-} w_n^2 dx\right)^{1/2}|u|_2 \to 0$$

as $n \to \infty$. Thus

$$\int_{\mathbb{R}^3} V^-(x)u_n^2 dx = \int_{\mathbb{R}^3} V^-(x)w_n^2 dx + \int_{\mathbb{R}^3} V^-(x)u^2 dx + o(1).$$

By Proposition 2.4 (ii), we have

$$M(u_n) = M(w_n) + M(u) + o(1).$$

Since $h \in L^2(\mathbb{R}^3)$,

$$\int_{\mathbb{R}^3} h(x)u_n dx = \int_{\mathbb{R}^3} h(x)w_n dx + \int_{\mathbb{R}^3} h(x)u dx,$$

therefore, to prove (2.8) and (2.9), it suffices to check that

$$\int_{\mathbb{R}^3} (F(x,u_n) - F(x,w_n) - F(x,u))dx = o(1) \tag{2.12}$$

and

$$\sup_{\|\phi\|_\lambda = 1} \int_{\mathbb{R}^3} (f(x,u_n) - f(x,w_n) - f(x,u))\phi dx = o(1). \tag{2.13}$$

We prove (2.12) firstly. Inspired by [2], we observe that

$$F(x,u_n) - F(x,u_n - u) = -\int_0^1 \left(\frac{d}{dt}F(x,u_n - tu)\right) dt = \int_0^1 f(x,u_n - tu)u dt,$$

and hence, by (1.2), we obtain that

$$|F(x,u_n) - F(x,u_n - u)| \le \varepsilon_1|u_n||u| + \varepsilon_1|u|^2 + C_{\varepsilon_1}|u_n|^{p-1}|u| + C_{\varepsilon_1}|u|^p,$$

where $\varepsilon_1, C_{\varepsilon_1} > 0$ and $p \in (4,6)$. Therefore, for each $\varepsilon > 0$, and the Young inequality, we get

$$|F(x,u_n) - F(x,w_n) - F(x,u)| \le C[\varepsilon|u_n|^2 + C_\varepsilon|u|^2 + \varepsilon|u_n|^p + C_\varepsilon|u|^p].$$

Next, we consider the function f_n given by

$$f_n(x) := \max\left\{|F(x,u_n) - F(x,w_n) - F(x,u)| - C\varepsilon(|u_n|^2 + |u_n|^p), 0\right\}.$$

Then $0 \le f_n(x) \le CC_\varepsilon(|u|^2 + |u|^p) \in L^1(\mathbb{R}^3)$. Moreover, by the Lebesgue dominated convergence theorem,

$$\int_{\mathbb{R}^3} f_n(x)dx \to 0 \quad \text{as } n \to \infty, \tag{2.14}$$

since $u_n \to u$ a.e. in $\mathbb{R}^3$. By the definition of f_n, it follows that

$$|F(x,u_n) - F(x,w_n) - F(x,u)| \le f_n(x) + C\varepsilon(|u_n|^2 + |u_n|^p).$$

Combining this with (2.14) and (1.3), shows that

$$\int_{\mathbb{R}^3} |F(x,u_n) - F(x,w_n) - F(x,u)|\, dx \le C\varepsilon$$

for n sufficiently large. It implies that

$$\int_{\mathbb{R}^3}[F(x,u_n)-F(x,w_n)-F(x,u)]dx=o(1).$$

The prove of (2.13) is similar to Lemma 4.7 in [28], we omit here.

Now, we check that $I'_\lambda(u)=0$. In fact, for each $\psi\in C_0^\infty(\mathbb{R}^3)$, we have

$$(u_n-u,\psi)_\lambda\to 0 \text{ as } n\to\infty. \tag{2.15}$$

and

$$\left|\int_{\mathbb{R}^3}V^-(x)(u_n-u)\psi dx\right|\le |V^-|_\infty\left(\int_{supp\psi}(u_n-u)^2dx\right)^{1/2}|\psi|_2\to 0 \text{ as } n\to\infty, \tag{2.16}$$

since $u_n\to u$ in $L^2_{loc}(\mathbb{R}^3)$. By Proposition 2.4 (i), $u_n\rightharpoonup u$ in E_λ yields $\phi_{u_n}\rightharpoonup\phi_u$ in $D(\mathbb{R}^3)$. So

$$\phi_{u_n}\rightharpoonup\phi_u \text{ in } L^6(\mathbb{R}^3).$$

For every $\psi\in C_0^\infty(\mathbb{R}^3)$, by the Hölder inequality we obtain

$$\int_{\mathbb{R}^3}|K(x)u\psi|^{6/5}dx\le |K|_\infty^{6/5}|\psi|_{12/5}^{6/5}|u|_{12/5}^{6/5},$$

that is $K(x)u\psi\in L^{6/5}(\mathbb{R}^3)$, and hence

$$\int_{\mathbb{R}^3}K(x)(\phi_{u_n}-\phi_u)u\psi dx\to 0.$$

By $u_n\to u$ in $L^3_{loc}(\mathbb{R}^3)$ and the Hölder inequality, we have

$$\begin{aligned}&|\int_{\mathbb{R}^3}K(x)\phi_{u_n}(u_n-u)\psi dx|\\ &\le |\psi|_2|K|_\infty|\phi_{u_n}|_6|u_n-u|_{3,\Omega_\psi}\\ &\le C|u_n-u|_{3,\Omega_\psi}\to 0 \quad \text{as } n\to\infty,\end{aligned}$$

here Ω_ψ is the support set of ψ. Consequently,

$$\begin{aligned}&|\int_{\mathbb{R}^3}[K(x)\phi_{u_n}u_n\psi-K(x)\phi_u u\psi]dx|\\ &\le\int_{\mathbb{R}^3}|K(x)\phi_{u_n}(u_n-u)\psi|dx+\int_{\mathbb{R}^3}|K(x)(\phi_{u_n}-\phi_u)u\psi|dx\\ &=o(1).\end{aligned} \tag{2.17}$$

For every $\psi\in C_0^\infty(\mathbb{R}^3)$ and Proposition 2.4 (ii), we obtain

$$\int_{\mathbb{R}^3}2\omega K(x)\phi_{u_n}u_n\psi dx=\int_{\mathbb{R}^3}2\omega K(x)\phi_{w_n}w_n\psi dx+\int_{\mathbb{R}^3}2\omega K(x)\phi_u u\psi dx+o(1).$$

Now we need to prove

$$\int_{\mathbb{R}^3}K(x)\phi_{u_n}^2u_n\psi dx=\int_{\mathbb{R}^3}K(x)\phi_{w_n}^2w_n\psi dx+\int_{\mathbb{R}^3}K(x)\phi_u^2u\psi dx+o(1).$$

By $u_n \to u$ in $L^s_{loc}(\mathbb{R}^3), 1 \le s < 6$; $\phi_{u_n} \to \phi_u$ in $L^s_{loc}(\mathbb{R}^3), 1 \le s < 6$, the boundedness of (ϕ_{u_n}) and the Hölder inequality, we have

$$\begin{aligned}
&\int_{\mathbb{R}^3} K(x)(\phi^2_{u_n} u_n - \phi^2_u u)\psi dx \\
&= \int_{\mathbb{R}^3} K(x)\phi^2_{u_n}(u_n - u)\psi dx + \int_{\mathbb{R}^3} K(x)(\phi^2_{u_n} - \phi^2_u)u\psi dx \\
&\le C|K|_\infty \|\nabla \phi_{u_n}\|^2 \left(\int_{\Omega_\psi} |u_n - u|^{3/2} dx\right)^{2/3} + |K|_\infty \int_{\Omega_\psi} (\phi^2_{u_n} - \phi^2_u)u\psi dx \\
&\to 0,
\end{aligned} \tag{2.18}$$

as $n \to \infty$, here Ω_ψ is the support set of ψ.

Furthermore, by the dominated convergence theorem and (1.2), we have

$$\int_{\mathbb{R}^3} [f(x, u_n) - f(x, u)]\psi dx = \int_{\Omega_\psi} [f(x, u_n) - f(x, u)]\psi dx = o(1).$$

Since $u_n \rightharpoonup u$ in $L^2(\mathbb{R}^3)$ and $h \in L^2(\mathbb{R}^3)$, we obtain $\int_{\mathbb{R}^3} h(u_n - u)dx = o(1)$. This jointly with (2.15), (2.16), (2.18) and the dominated convergence theorem, shows that

$$\langle I'_\lambda(u), \psi\rangle = \lim_{n\to\infty} \langle I'_\lambda(u_n), \psi\rangle = 0, \quad \forall \psi \in C^\infty_0(\mathbb{R}^3).$$

Hence $I'_\lambda(u) = 0$. Combining with (2.8)-(2.9) and Proposition 2.4 (iii), we obtain (2.10). The proof is complete. □

Lemma 2.3. *Assume that $V \ge 0, (V0)$–$(V1)$, $(f1)$–$(f4)$, (K) and (h) hold. Then, for any $M > 0$, there is $\Lambda = \Lambda(M) > 0$ such that I_λ satisfies $(PS)_c$ condition for all $c < M$ and $\lambda > \Lambda$.*

Proof. Let $\{u_n\} \subset E_\lambda$ be a $(PS)_c$ sequence with $c < M$. By Lemma 2.1, we know that $\{u_n\}$ is bounded in E_λ, and there exists $C > 0$ such that $\|u_n\|_\lambda \le C$. Therefore, up to a subsequence, we can assume that

$$\begin{aligned}
&u_n \rightharpoonup u \text{ in } E_\lambda; \\
&u_n \to u \text{ in } L^s_{loc}(\mathbb{R}^3)(1 \le s < 2^*); \\
&u_n(x) \to u(x) \text{ a.e. } x \in \mathbb{R}^3.
\end{aligned} \tag{2.19}$$

Now we can show that $u_n \to u$ in E_λ for $\lambda > 0$ large. Denote $w_n := u_n - u$, then $w_n \rightharpoonup 0$ in E_λ. According to Lemma 2.2 and the fact (2.10) holds uniformly for all $\varphi \in E_\lambda$, we have $I'_\lambda(u) = 0$, and

$$I_\lambda(w_n) \to c - I_\lambda(u), \quad I'_\lambda(w_n) \to 0 \quad \text{as } n \to \infty. \tag{2.20}$$

According to $V \ge 0$ and $(f3)$, we obtain

$$\begin{aligned}
I_\lambda(u) &= I_\lambda(u) - \frac{1}{4}\langle I'_\lambda(u), u\rangle \\
&= \frac{1}{4}\|u\|^2_\lambda + \frac{1}{4}\int_{\mathbb{R}^3} K(x)\phi^2_u u^2 dx + \frac{\beta}{16\pi}\int_{\mathbb{R}^3} |\nabla \phi_u|^4 dx + \int_{\mathbb{R}^3} \mathcal{F}(x, u) dx
\end{aligned}$$

$$
\begin{aligned}
&-\frac{3}{4}\int_{\mathbb{R}^3} hudx\\
&=\Phi_\lambda(u)-\frac{3}{4}\int_{\mathbb{R}^3} hudx,
\end{aligned}
$$

here $\Phi_\lambda(u)=\frac{1}{4}\|u\|_\lambda^2+\frac{1}{4}\int_{\mathbb{R}^3}K(x)\phi_u^2u^2dx+\frac{\beta}{16\pi}\int_{\mathbb{R}^3}|\nabla\phi_u|^4dx+\int_{\mathbb{R}^3}\mathcal{F}(x,u)dx\geq 0$.

Again by (2.19), (2.20) and Proposition 2.3 (i), we have

$$
\begin{aligned}
&\frac{1}{4}\|w_n\|_\lambda^2+\int_{\mathbb{R}^3}\mathcal{F}(x,w_n)dx\\
&\leq \frac{1}{4}\int_{\mathbb{R}^3}K(x)\phi_{w_n}^2w_n^2dx+\frac{\beta}{16\pi}\int_{\mathbb{R}^3}|\nabla\phi_{w_n}|^4dx+\int_{\mathbb{R}^3}\mathcal{F}(x,w_n)dx\\
&=I_\lambda(w_n)-\frac{1}{4}\langle I_\lambda'(w_n),w_n\rangle+\frac{3}{4}\int_{\mathbb{R}^3}hw_ndx+o(1)\\
&\leq c-I_\lambda(u)+o(1)\\
&=c-\left[\Phi_\lambda(u)-\frac{3}{4}\int_{\mathbb{R}^3}hudx\right]+\frac{3}{4}\int_{\mathbb{R}^3}hw_ndx+o(1)\\
&\leq M+\tilde{M}+o(1). \qquad (2.21)
\end{aligned}
$$

Here we use the fact $c<M$ and

$$
\frac{3}{4}|h|_2|u|_2\leq\frac{3}{4}|h|_2d_2\|u\|_\lambda\leq\frac{3}{4}|h|_2d_2\liminf_{n\to\infty}\|u_n\|_\lambda\leq|h|_2d_2C\leq\tilde{M},
$$

where $\tilde{M}$ is a positive constant independent of λ. Hence

$$
\int_{\mathbb{R}^3}\mathcal{F}(x,w_n)dx\leq M+\tilde{M}+o(1). \qquad (2.22)
$$

Because $V(x)<b$ on a set of finite measure and $w_n\rightharpoonup 0$, we obtain

$$
|w_n|_2^2\leq\frac{1}{\lambda b}\int_{V\geq b}\lambda V^+(x)w_n^2dx+\int_{V<b}w_n^2dx\leq\frac{1}{\lambda b}\|w_n\|_\lambda^2+o(1). \qquad (2.23)
$$

For $2<s<2^*$, by the Hölder and Sobolev inequality and (2.23), we have

$$
\begin{aligned}
|w_n|_s^s&=\int_{\mathbb{R}^3}|w_n|^sdx\\
&\leq\left(\int_{\mathbb{R}^3}|w_n|^2dx\right)^{\frac{6-s}{s}}\left(\int_{\mathbb{R}^3}|w_n|^6dx\right)^{\frac{9s-18}{s}}\\
&\leq\left[\frac{1}{\lambda b}\int_{\mathbb{R}^3}\left(|\nabla w_n|^2+\lambda V^+w_n^2\right)dx\right]^{\frac{6-s}{s}}\left(\bar{S}^{-6}\left[\int_{\mathbb{R}^3}|\nabla w_n|^2dx\right]^3\right)^{\frac{9s-18}{s}}+o(1)\\
&\leq\left(\frac{1}{\lambda b}\right)^{\frac{6-s}{4}}\bar{S}^{-\frac{3(s-2)}{2}}\|w_n\|_\lambda^s+o(1). \qquad (2.24)
\end{aligned}
$$

According to $(f1)$, for any $\varepsilon>0$, there exists $\delta=\delta(\varepsilon)>0$ such that $|f(x,t)|\leq\varepsilon|t|$ for all $x\in\mathbb{R}^3$ and $|t|\leq\delta$, and $(f4)$ is satisfied for $|t|\geq\delta$ (with the same τ but possibly larger than a_1). Hence we have that

$$
\int_{|w_n|\leq\delta}f(x,w_n)w_ndx\leq\varepsilon\int_{|w_n|\leq\delta}w_n^2dx\leq\frac{\varepsilon}{\lambda b}\|w_n\|_\lambda^2+o(1), \qquad (2.25)
$$

and

$$\begin{aligned}\int_{|w_n|\geq\delta} f(x,w_n)w_n dx &\leq \left(\int_{|w_n|\geq\delta} |\frac{f(x,w_n)}{w_n}|^\tau dx\right)^{1/\tau} |w_n|_s^2 \\ &\leq \left(\int_{|w_n|\geq\delta} a_1\mathcal{F}(x,w_n)dx\right)^{1/\tau} |w_n|_s^2 \\ &\leq [a_1(M+\tilde{M})]^{1/\tau}\bar{S}^{-\frac{3(2s-4)}{2s}}\left(\frac{1}{\lambda b}\right)^\theta \|w_n\|_\lambda^2 + o(1) \qquad (2.26)\end{aligned}$$

by $(f4)$, (2.22), (2.24) with $s = 2\tau/(\tau-1)$ and the Hölder inequality, where $\theta = \frac{6-s}{2s} > 0$.

Since $u_n \rightharpoonup u$ in $L^2(\mathbb{R}^3)$ and $h \in L^2(\mathbb{R}^3)$, we obtain that

$$\int_{\mathbb{R}^3} h(u_n-u)dx \to 0 \quad \text{as } n\to\infty. \qquad (2.27)$$

By $-\omega \leq \phi_u \leq 0$, we have that $2\omega+\phi_u \geq 0$. Therefore, by (2.25), (2.26), (2.27) and Proposition 2.3 (i), we obtain that

$$\begin{aligned}o(1) &= \langle I'_\lambda(w_n), w_n\rangle \\ &\geq \|w_n\|_\lambda^2 - \int_{\mathbb{R}^3} K(x)(2\omega+\phi_{w_n})\phi_{w_n}w_n^2dx - \int_{\mathbb{R}^3} f(x,w_n)w_n dx - \int_{\mathbb{R}^3} hw_n dx \\ &\geq \left[1-\frac{\varepsilon}{\lambda b} - [a_1(M+\tilde{M})]^{1/\tau}\bar{S}^{-\frac{3(2s-4)}{2s}}\left(\frac{1}{\lambda b}\right)^\theta\right]\|w_n\|_\lambda^2 + o(1). \qquad (2.28)\end{aligned}$$

So, there exists $\Lambda = \Lambda(M) > 0$ such that $w_n \to 0$ in E_λ when $\lambda > \Lambda$. Since $w_n = u_n - u$, it follows that $u_n \to u$ in E_λ. This completes the proof. □

Lemma 2.4. *Assume* $(V0)$*–*$(V1)$*,* $(f1)$*–*$(f4)$*,* (K) *and* (h) *hold. Let* $\{u_n\}$ *be a* $(PS)_c$ *sequence of* I_λ *with level* $c > 0$*. Then for any* $M > 0$*, there is* $\Lambda = \Lambda(M) > 0$ *such that, up to a subsequence,* $u_n \rightharpoonup u$ *in* E_λ *with* u *being a nontrivial critical point of* I_λ *and satisfying* $I_\lambda(u) \leq c$ *for all* $c < M$ *and* $\lambda > \Lambda$*.*

Proof. We modify the proof of Lemma 2.3. By Lemma 2.2, we obtain

$$I'_\lambda(u) = 0, \quad I_\lambda(w_n) \to c - I_\lambda(u), \quad I'_\lambda(u_n) \to 0 \quad \text{as } n\to\infty. \qquad (2.29)$$

However, since V is allowed to be sign-changing and the appearance of nonlinear term h, from

$$\begin{aligned}I_\lambda(u) &= I_\lambda(u) - \frac{1}{4}\langle I'_\lambda(u), u\rangle \\ &= \frac{1}{4}\|u\|_\lambda^2 - \frac{\lambda}{4}\int_{\mathbb{R}^3} V^-(x)u^2dx + \frac{1}{4}\int_{\mathbb{R}^3} K(x)\phi_u^2u^2dx + \frac{\beta}{16\pi}\int_{\mathbb{R}^3}|\nabla\phi_u|^4dx \\ &\quad + \int_{\mathbb{R}^3}\mathcal{F}(x,u)dx - \frac{3}{4}\int_{\mathbb{R}^3} hudx\end{aligned}$$

we cannot deduce that $I_\lambda(u) \geq 0$. We consider two possibilities:

(i) $I_\lambda(u) < 0$;

(ii) $I_\lambda(u) \geq 0$.

If $I_\lambda(u) < 0$, then $u \neq 0$ is nontrivial and the proof is done. If $I_\lambda(u) \geq 0$, following the argument in the proof of Lemma 2.3 step by step, we can get $u_n \to u$ in E_λ. Indeed, by $(V0)$ and $w_n \to 0$ in $L^2(\{x \in \mathbb{R}^3 : V(x) < b\})$, we obtain

$$\left| \int_{\mathbb{R}^3} V^-(x) w_n^2(x) dx \right| \leq |V^-|_\infty \int_{suppV^-} w_n^2 dx = o(1),$$

which jointly this with (2.29) and Proposition 2.3(i), we have

$$\begin{aligned}
&\int_{\mathbb{R}^3} \mathcal{F}(x, w_n) dx \\
&= I_\lambda(w_n) - \frac{1}{4}\langle I_\lambda'(w_n), w_n\rangle - \frac{1}{4}\|w_n\|_\lambda^2 + \frac{1}{4}\int_{\mathbb{R}^3} \lambda V^-(x) w_n^2 dx \\
&\quad - \frac{1}{4}\int_{\mathbb{R}^3} K(x)\phi_{w_n}^2 w_n^2 dx - \frac{\beta}{16\pi}\int_{\mathbb{R}^3} |\nabla \phi_{w_n}|^4 dx + \frac{3}{4}\int_{\mathbb{R}^3} h w_n dx \\
&\leq c - I_\lambda(u) + o(1) \leq M + o(1).
\end{aligned}$$

It follows that (2.26), (2.27) and (2.28) remain valid. Therefore $u_n \to u$ in E_λ and $I_\lambda(u) = c(> 0)$. The proof is complete. □

3. Proofs of main results

If V is sign-changing, we first verify that the functional I_λ have the linking geometry to apply the following linking theorem [18].

Proposition 3.1. *Let $E = E_1 \bigoplus E_2$ be a Banach space with $\dim E_2 < \infty, \Phi \in C^1(E, \mathbb{R}^3)$. If there exist $R > \rho > 0, \alpha > 0$ and $e_0 \in E_1$ such that*

$$\alpha := \inf \Phi(E_1 \cap S_\rho) > \sup \Phi(\partial Q)$$

where $S_\rho = \{u \in E : \|u\| = \rho\}, Q = \{u = v + te_0 : v \in E_2, t \geq 0, \|u\| \leq R\}$. Then Φ has a $(PS)_c$ sequence with $c \in [\alpha, \sup \Phi(Q)]$.

In our paper, we use Proposition 3.1 with $E_1 = E_\lambda^+ \bigoplus F_\lambda$ and $E_2 = E_\lambda^-$. By Proposition 2.2, $\mu_j(\lambda) \to 0$ as $\lambda \to \infty$ for every fixed j. By Remark 2.1, there is $\Lambda_1 > 0$ such that $E_\lambda^- \neq \emptyset$ and E_λ^- is finite dimensional for $\lambda > \Lambda_1$. Now we can investigate the linking structure of the functional I_λ.

Lemma 3.1. *Assume that $(V0)$–$(V1)$, (K), (h) and (1.2) with $p \in (4, 2^*)$ are satisfied. Then, for each $\lambda > \Lambda_1$(is the constant given in Remark 2.1), there exist $\alpha_\lambda, \rho_\lambda$ and $\eta_\lambda > 0$ such that*

$$I_\lambda(u) \geq \alpha_\lambda \text{ for all } u \in E_\lambda^+ \bigoplus F_\lambda \text{ with } \|u\|_\lambda = \rho_\lambda \text{ and } |h|_2 < \eta_\lambda. \tag{3.1}$$

Furthermore, if $V \geq 0$, we can choose $\alpha, \rho, \eta > 0$ independent of λ.

Proof. For any $u \in E_\lambda^+ \bigoplus F_\lambda$, writing $u = u_1 + u_2$ with $u_1 \in E_\lambda^+$ and $u_2 \in F_\lambda$. Clearly, $(u_1, u_2)_\lambda = 0$, and

$$\int_{\mathbb{R}^3} (|\nabla u|^2 + \lambda V(x) u^2) dx = \int_{\mathbb{R}^3} (|\nabla u_1|^2 + \lambda V(x) u_1^2) dx + \|u_2\|_\lambda^2. \tag{3.2}$$

By Proposition 2.1, we obtain that $\mu_j(\lambda) \to +\infty$ as $j \to +\infty$ for each fixed $\lambda > \Lambda_1$. So there is a positive integer n_λ such that $\mu_j(\lambda) \le 1$ for $j \le n_\lambda$ and $\mu_j(\lambda) > 1$ for $j > n_\lambda + 1$. For $u_1 \in E_\lambda^+$, we set $u_1 = \Sigma_{j=n_\lambda+1}^{\infty}\mu_j(\lambda)e_j(\lambda)$. Thus

$$\int_{\mathbb{R}^3}(|\nabla u_1|^2 + \lambda V(x)u_1^2)dx = \|u_1\|_\lambda^2 - \int_{\mathbb{R}^3}\lambda V^-(x)u_1^2dx \ge \left(1 - \frac{1}{\mu_{n_\lambda+1}(\lambda)}\right)\|u_1\|_\lambda^2. \tag{3.3}$$

By using (1.3), (2.1), (3.2), (3.3) and $-\omega \le \phi_u \le 0$ on the set $\{x \in \mathbb{R}^3 | u(x) \ne 0\}$, we have

$$\begin{aligned} I_\lambda(u) &\ge \frac{1}{2}\left(1 - \frac{1}{\mu_{n_\lambda+1}(\lambda)}\right)\|u\|_\lambda^2 - \varepsilon|u|_2^2 - C_\varepsilon|u|_q^q - |h|_2|u|_2 \\ &\ge \frac{1}{2}\left(1 - \frac{1}{\mu_{n_\lambda+1}(\lambda)}\right)\|u\|_\lambda^2 - \varepsilon d_2^2\|u\|_\lambda^2 - C_\varepsilon d_q^q\|u\|_\lambda^q - d_2|h|_2\|u\|_\lambda \\ &\ge \|u\|_\lambda\left\{\left[\frac{1}{2}\left(1 - \frac{1}{\mu_{n_\lambda+1}(\lambda)}\right) - \varepsilon d_2^2\right]\|u\|_\lambda - C_\varepsilon d_q^q\|u\|_\lambda^{q-1} - d_2|h|_2\right\}. \end{aligned}$$

Let $g(t) = \left[\frac{1}{2}(1 - \frac{1}{\mu_{n_\lambda+1}(\lambda)} - \varepsilon d_2^2\right]t - C_\varepsilon d_q^q t^{q-1}$, for $t > 0$, $q \in (4,6)$ there exists $\rho(\lambda) = \left[\frac{\frac{1}{2}(1-\frac{1}{\mu_{n_\lambda+1}(\lambda)}) - \varepsilon d_2^2}{C_\varepsilon d_q^q(q-1)}\right]^{\frac{1}{q-2}}$ such that $\max_{t\ge0} g(t) = g(\rho(\lambda)) > 0$. It follows from above inequality, $I_\lambda(u)\mid_{\|u\|_\lambda=\rho(\lambda)} > 0$ for all $|h|_2 < \eta_\lambda := \frac{g(\rho(\lambda))}{2d_2}$. Of course, $\rho(\lambda)$ can be chosen small enough, we can obtain the same result: there exists $\alpha_\lambda > 0$, such that $I_\lambda(u) \ge \alpha_\lambda$, here $\|u\|_\lambda = \rho_\lambda$.

If $V \ge 0$, since $E_\lambda = F_\lambda$, and

$$\int_{\mathbb{R}^3}(|\nabla u|^2 + \lambda V(x)u^2)dx = \|u\|_\lambda^2,$$

we can choose $\alpha, \rho, \eta > 0$ (independent of λ) such that (3.1) holds. □

Lemma 3.2. *Suppose that $(V0), (V1)$, $(f1)$–$(f2)$, (K) and (h) are satisfied. Then, for any finite dimensional subspace $\tilde{E}_\lambda \subset E_\lambda$, there holds*

$$I_\lambda(u) \to -\infty \quad \text{as} \quad \|u\|_\lambda \to \infty,\ u \in \tilde{E}_\lambda.$$

Proof. Arguing indirectly, we can assume that there is a sequence $(u_n) \subset \tilde{E}_\lambda$ with $\|u_n\|_\lambda \to \infty$ such that

$$-\infty < \inf_n I_\lambda(u_n). \tag{3.4}$$

Take $v_n := u_n/\|u_n\|_\lambda$. Since $\dim \tilde{E}_\lambda < +\infty$, there exists $v \in \tilde{E}_\lambda \setminus \{0\}$ such that

$$v_n \to v \quad \text{in} \quad \tilde{E}_\lambda, \qquad v_n(x) \to v(x) \text{ a.e. } x \in \mathbb{R}^3$$

after passing to a subsequence. If $v(x) \ne 0$, then $|u_n(x)| \to +\infty$ as $n \to \infty$, and hence by $(f2)$, we obtain that

$$\frac{F(x, u_n(x))}{u_n^4(x)}v_n^4(x) \to +\infty \quad \text{as } n \to \infty,$$

which jointly this with $(f1)$, (2.5), Proposition 2.3 (ii) and Fatou's lemma, we obtain

$$\begin{aligned}\frac{I_\lambda(u_n)}{\|u_n\|_\lambda^4} &\le \frac{1}{2\|u_n\|_\lambda^2} + \frac{\omega^2 d_2^2}{2\|u_n\|_\lambda^2}|K|_\infty - \int_{\mathbb{R}^3} \frac{F(x,u_n)}{\|u_n\|_\lambda^4}dx - \int_{\mathbb{R}^3} h(x)\frac{u_n}{\|u_n\|_\lambda^4}dx \\ &\le \frac{1}{2\|u_n\|_\lambda^2} + \frac{\omega^2 d_2^2}{2\|u_n\|_\lambda^2}|K|_\infty - \left(\int_{v=0} + \int_{v\neq 0}\right)\frac{F(x,u_n)}{u_n^4}v_n^4 dx + \frac{|h|_2 d_2}{\|u_n\|_\lambda^3} \\ &\le \frac{1}{2\|u_n\|_\lambda^2} + \frac{\omega^2 d_2^2}{2\|u_n\|_\lambda^2}|K|_\infty - \int_{v\neq 0}\frac{F(x,u_n)}{u_n^4}v_n^4 dx + \frac{|h|_2 d_2}{\|u_n\|_\lambda^3} \\ &\to -\infty.\end{aligned}$$

This contradicts (3.4).

If $K \in L^3(\mathbb{R}^3)$, we can similarly get the result. □

Lemma 3.3. *Suppose that* $(V0), (V1), (h), (K)$ *and* $(f1)$*–*$(f2)$ *are satisfied. If* $V(x) < 0$ *for some* x*, then, for each* $k \in \mathbb{N}$*, there exist* $\lambda_k > k, b_k > 0, w_k \in E_{\lambda_k}^+ \bigoplus F_{\lambda_k}, R_{\lambda_k} > \rho_{\lambda_k}(\rho_{\lambda_k}$ *is the constant given in Lemma 3.1) and* $\eta_k > 0$ *such that, for* $|h|_2 < \eta_k$*,* $|K|_\infty < b_k(or|K|_3 < b_k)$*,*
$(a)\sup I_{\lambda_k}(\partial Q_k) \le 0$;
$(b)\sup I_{\lambda_k}(Q_k)$ *is bounded above by a constant independent of* λ_k*,*
where $Q_k := \{u = v + tw_k : v \in E_{\lambda_k}^-, t \ge 0, \|u\|_{\lambda_k} \le R_{\lambda_k}\}$.

Proof. We adapt an argument in Ding and Szulkin [13]. For each $k \in \mathbb{N}$, since $\mu_j(k) \to +\infty$ as $j \to \infty$, there exists $j_k \in \mathbb{N}$ such that $\mu_{j_k}(k) > 1$. By Proposition 2.2, there exists $\lambda_k > k$ such that

$$1 < \mu_{j_k}(\lambda_k) < 1 + \frac{1}{\lambda_k}.$$

Taking $w_k := e_{j_k}(\lambda_k)$ be an eigenfunction of $\mu_{j_k}(\lambda_k)$, then $w_k \in E_{\lambda_k}^+$ as $\mu_{j_k}(\lambda_k) > 1$. Because $\dim E_{\lambda_k}^- \bigoplus \mathbb{R}w_k < +\infty$, it follows directly from Lemma 3.2 that (a) holds with $R_{\lambda_k} > 0$ large enough.

According to $(f2)$, for each $\tilde\eta > |V^-|_\infty$, there is $r_{\tilde\eta} > 0$ such that $F(x,t) \ge \frac{1}{2}\tilde\eta t^2$ if $|t| \ge r_{\tilde\eta}$. For $u = v + w \in E_{\lambda_k}^- \bigoplus \mathbb{R}w_k$, we have

$$\int_{\mathbb{R}^3} V^-(x)u^2dx = \int_{\mathbb{R}^3} V^-(x)v^2dx + \int_{\mathbb{R}^3} V^-(x)w^2dx$$

by the orthogonality of $E_{\lambda_k}^-$ and $\mathbb{R}w_k$. Therefore, by Proposition 2.3 (ii), we obtain

$$\begin{aligned}I_{\lambda_k}(u) \le{}& \frac{1}{2}\int_{\mathbb{R}^3}\left(|\nabla w|^2 + \lambda_k V(x)w^2\right)dx - \frac{1}{2}\int_{\mathbb{R}^3} K(x)\omega\phi_u u^2dx + \frac{\beta}{16\pi}\int_{\mathbb{R}^3}|\nabla\phi_u|^4dx \\ & - \int_{suppV^-} F(x,u)dx - \int_{\mathbb{R}^3} hudx \\ \le{}& \frac{1}{2}[\mu_{j_k}(\lambda_k) - 1]\lambda_k\int_{\mathbb{R}^3} V^-(x)w^2dx - \frac{1}{2}\int_{suppV^-}\tilde\eta u^2dx + \frac{3\omega^2 d_2^2}{4}|K|_\infty\|u\|_{\lambda_k}^2 \\ & + d_2|h|_2\|u\|_{\lambda_k} - \int_{suppV^-,|u|\le r_{\tilde\eta}}\left(F(x,u) - \frac{1}{2}\tilde\eta u^2\right)dx\end{aligned}$$

$$\leq \frac{1}{2}\int_{\mathbb{R}^3} V^-(x)w^2dx - \frac{\tilde{\eta}}{2|V^-|_\infty}\int_{\mathbb{R}^3} V^-(x)w^2dx + C_{\tilde{\eta}} + \frac{3\omega^2 d_2^2}{4}|K|_\infty R_{\lambda_k}^2$$

$$+ d_2|h|_2 R_{\lambda_k}$$

$$\leq C_{\tilde{\eta}} + 1$$

for $u = v + w \in E_{\lambda_k}^- \bigoplus \mathbb{R}w_k$ with $\|u\|_{\lambda_k} \leq R_{\lambda_k}$, $|K|_\infty < b_k := \frac{2}{3}(\omega d_2 R_{\lambda_k})^{-2}$ and $|h|_2 < \eta_k := \frac{1}{2d_2 R_{\lambda_k}}$, where $C_{\tilde{\eta}}$ depends on $\tilde{\eta}$ but not λ_k.

If $K \in L^3(\mathbb{R}^3)$, by the Hölder inequality, we obtain that $|K|_3 < b_k := \frac{2}{3}(\omega d_3 R_{\lambda_k})^{-2}$. □

Lemma 3.4. *Suppose that $(V0), (V1), (h), (K)$ and $(f1)$–$(f2)$ are satisfied. If $\Omega := intV^{-1}(0)$ is nonempty, then, for each $\lambda > \Lambda_1($ is the constant given in Remark 2.1), there exist $w \in E_\lambda^+ \bigoplus F_\lambda, R_\lambda > 0, b_\lambda > 0$ and $\eta_\lambda > 0$ such that for $|h|_2 < \eta_\lambda, |K|_\infty < b_\lambda or(|K|_3 < b_\lambda)$,*
$(a)\sup I_\lambda(\partial Q) \leq 0$;
$(b)\sup I_\lambda(Q)$ is bounded above by a constant independent of λ,
where $Q := \{u = v + tw : v \in E_\lambda^-, t \geq 0, \|u\|_\lambda \leq R_\lambda\}$.

Proof. Choose $e_0 \in C_0^\infty(\Omega) \setminus \{0\}$, then $e_0 \in F_\lambda$. By Lemma 3.2, there is $R_\lambda > 0$ large such that $I_\lambda(u) \leq 0$ where $u \in E_\lambda^- \bigoplus \mathbb{R}e_0$ and $\|u\|_\lambda \geq R_\lambda$.

For $u = v + w \in E_\lambda^- \bigoplus \mathbb{R}e_0$, we have

$$\begin{aligned} I_\lambda(u) \leq & \frac{1}{2}\int_{\mathbb{R}^3} |\nabla u|^2 dx - \frac{1}{2}\int_{\mathbb{R}^3} K(x)\omega\phi_u u^2 dx - \int_\Omega F(x,u)dx \\ & + \frac{\beta}{16\pi}\int_{\mathbb{R}^3} |\nabla\phi_u|^4 dx - \int_{\mathbb{R}^3} hudx \\ \leq & \frac{1}{2}\int_{\mathbb{R}^3} |\nabla w|^2 dx - \frac{\tilde{\eta}}{2}\int_\Omega u^2 dx - \int_{\Omega,|u|\leq r_{\tilde{\eta}}} \left(F(x,u) - \frac{\tilde{\eta}}{2}u^2\right)dx \\ & + \frac{3\omega^2 d_2^2}{4}|K|_\infty \|u\|_\lambda^2 + |h|_2 d_2 \|u\|_\lambda \\ \leq & \frac{1}{2}\int_{\mathbb{R}^3} |\nabla w|^2 dx - \frac{\tilde{\eta}}{2}\int_\Omega u^2 dx + C_{\tilde{\eta}} + \frac{3\omega^2 d_2^2}{2}|K|_\infty \|u\|_\lambda^2 + |h|_2 d_2 \|u\|_\lambda. \end{aligned} \tag{3.5}$$

Observing $w \in C_0^\infty(\Omega)$, we have

$$\int_{\mathbb{R}^3} |\nabla w|^2 dx = \int_\Omega (-\Delta w)udx \leq |\Delta w|_2 |u|_{2,\Omega} \leq c_0|\nabla w|_2|u|_{2,\Omega} \leq \frac{c_0^2}{2\tilde{\eta}}|\nabla w|_2^2 + \frac{\tilde{\eta}}{2}|u|_{2,\Omega}^2, \tag{3.6}$$

where c_0 is a constant depending on e_0. Choosing $\tilde{\eta} > c_0^2$, we have $|\nabla w|_2^2 \leq \tilde{\eta}|u|_{2,\Omega}^2$, and it follows from (3.5) that

$$I_\lambda(u) \leq C_{\tilde{\eta}} + \frac{3\omega^2 d_2^2}{4}|K|_\infty R_\lambda^2 + |h|_2 d_2 R_\lambda \leq C_{\tilde{\eta}} + 1$$

for all $u \in E_\lambda^- \bigoplus \mathbb{R}e_0$ with $\|u\|_\lambda \leq R_\lambda, |h|_2 < \eta_\lambda := \frac{1}{2d_2 R_\lambda}$ and $|K|_\infty < b_\lambda := \frac{2}{3}(\omega d_2 R_\lambda)^{-2}$, where $C_{\tilde{\eta}}$ depends on $\tilde{\eta}$ but not λ.

If $K \in L^3(\mathbb{R}^3)$, by the Hölder inequality, we get that for $|K|_3 < b_\lambda := \frac{2}{3}(\omega d_3 R_\lambda)^{-2}$. □

Now we are in a position to prove our main results.

Proof of Theorem 1.1. The proof of Theorem 1.1 is divided in two steps.

Step 1. There exists a function $u_\lambda \in E_\lambda$ such that $I'_\lambda(u_\lambda) = 0$ and $I_\lambda(u_\lambda) < 0$.

Since $h \in L^2(\mathbb{R}^3)$ and $h \geq 0 (\not\equiv 0)$, we can choose a function $\psi \in E_\lambda$ such that

$$\int_{\mathbb{R}^3} h(x)\psi(x)dx > 0.$$

Hence, by $-\omega \leq \phi_u \leq 0$ we obtain that

$$\begin{aligned} I_\lambda(t\psi) &= \frac{t^2}{2}\|\psi\|_\lambda^2 - \frac{\lambda t^2}{2}\int_{\mathbb{R}^3} V^-(x)\psi^2 dx - \frac{t^2}{2}\int_{\mathbb{R}^3} K(x)(2\omega + \phi_{t\psi})\phi_{t\psi}\psi^2 dx \\ &\quad - \frac{1}{8\pi}\int_{\mathbb{R}^3} |\nabla\phi_{t\psi}|^2 dx - \frac{\beta}{16\pi}\int_{\mathbb{R}^3} |\nabla\phi_{t\psi}|^4 dx - \int_{\mathbb{R}^3} F(x, t\psi)dx \\ &\quad - t\int_{\mathbb{R}^3} h(x)\psi dx \\ &\leq \frac{t^2}{2}\|\psi\|_\lambda^2 + \frac{t^2}{2}\int_{\mathbb{R}^3} K(x)\omega^2\psi^2 dx + \frac{t^4}{4}C_1\|\psi\|_\lambda^4 + Ct^2\|\psi\|_\lambda^4 - t\int_{\mathbb{R}^3} h(x)\psi dx \\ &< 0 \quad \text{for } t > 0 \text{ small enough.} \end{aligned}$$

Thus, there exists u_λ small enough such that $I_\lambda(u_\lambda) < 0$. By Lemma 3.2, we have

$$c_{0,\lambda} = \inf\{I_\lambda(u) : u \in \overline{B}_{\rho_\lambda}\} < 0,$$

where $\rho_\lambda > 0$ is given by Lemma 3.1, $B_{\rho_\lambda} = \{u \in E_\lambda : \|u\|_\lambda < \rho_\lambda\}$. By the Ekeland's variational principle, there exists a minimizing sequence $\{u_{n,\lambda}\} \subset \overline{B}_{\rho_\lambda}$ such that

$$c_{0,\lambda} \leq I_\lambda(u_{n,\lambda}) < c_{0,\lambda} + \frac{1}{n_\lambda},$$

and

$$I_\lambda(w_\lambda) \geq I_\lambda(u_{n,\lambda}) - \frac{1}{n_\lambda}\|w_\lambda - u_{n,\lambda}\|_\lambda$$

for all $w_\lambda \in \overline{B}_{\rho_\lambda}$. Therefore, $\{u_{n,\lambda}\}$ is a bounded Palais-Smale sequence of I_λ. Then, by a standard procedure, Lemma 2.3 and Lemma 2.2 imply that there is a function $u_\lambda \in E_\lambda$ such that $I'_\lambda(u_\lambda) = 0$ and $I_\lambda(u_\lambda) = c_{0,\lambda} < 0$.

If $V \geq 0$, we can get $\rho_\lambda, c_{0,\lambda}, u_{0,\lambda}$ are independent of λ.

Step 2. There exists a function $\widetilde{u}_\lambda \in E_\lambda$ such that $I'_\lambda(\widetilde{u}_\lambda) = 0$ and $I_\lambda(\widetilde{u}_\lambda) > 0$.

It follows from Lemmas 3.1, 3.3 and Proposition 3.1 that, for each $k \in \mathbb{N}, \lambda = \lambda_k$ and $|h|_2 < \eta_k$, I_{λ_k} has a $(PS)_c$ sequence with $c \in [\alpha_{\lambda_k}, \sup I_{\lambda_k}(Q_k)]$. Setting $M := \sup I_{\lambda_k}(Q_k)$, then I_{λ_k} has a nontrivial critical point according to Lemmas 2.1, 2.4 and Proposition 3.1. Hence there exists a function $\widetilde{u}_\lambda \in E_\lambda$ such that $I'_\lambda(\widetilde{u}_\lambda) = 0$ and $I_\lambda(\widetilde{u}_\lambda) = c \geq \alpha_{\lambda_k} > 0$. The proof is complete. □

Proof of Theorem 1.2. The first solution is similar to the first solution of Theorem 1.1. The second solution follows from Lemmas 2.1, 2.4, 3.1, 3.4 and Proposition 3.1. The proof is complete. □

Proof of Theorem 1.3. The proof of Theorem 1.3 is divided in two steps.

Step 1. There exists a function $u_0 \in E_\lambda$ such that $I'_\lambda(u_0) = 0$ and $I_\lambda(u_0) < 0$.

In the proof of Theorem 1.1, we can choose $c_0 = c_{0,\lambda}, B_\rho = B_{\rho,\lambda}$, then by the Ekeland's variational principle, there exists a sequence $\{u_n\} \subset \overline{B}_\rho$ such that

$$c_0 \leq I_\lambda(u_n) < c_0 + \frac{1}{n},$$

and

$$I_\lambda(w) \geq I_\lambda(u_n) - \frac{1}{n}\|w - u_n\|_\lambda$$

for all $w \in \overline{B}_\rho$. Then by a standard procedure, we can show that $\{u_n\}$ is a bounded Palais-Smale sequence of I_λ. Therefore Lemma 2.2 and Lemma 2.3 imply that there exists a function $u_0 \in E_\lambda$ such that $I'_\lambda(u_0) = 0$ and $I_\lambda(u_0) = c_0 < 0$.

Step 2. There exists a function $\widetilde{u}_\lambda \in E_\lambda$ such that $I'_\lambda(\widetilde{u}_\lambda) = 0$ and $I_\lambda(\widetilde{u}_\lambda) > 0$.

Since we suppose $V \geq 0$, the functional I_λ has mountain pass geometry and the existence of nontrivial solutions can be obtained by mountain pass theorem [18, 24, 30]. Indeed, by Lemma 3.1, there exist constants $\alpha, \rho, \eta > 0$ (independent of λ) such that, for each $\lambda > \Lambda_0$,

$$I_\lambda(u) \geq \alpha \ \text{ for } \ u \in E_\lambda \text{ with } \ \|u\|_\lambda = \rho \ \text{ and } \ |h|_2 < \eta.$$

Take $e \in C_0^\infty(\Omega) \setminus \{0\}$, by $(f1), (f2)$ and Fatou's lemma, we get

$$\begin{aligned}\frac{I_\lambda(te)}{t^4} \leq &\frac{1}{2t^2}\int_\Omega |\nabla e|^2 dx - \frac{1}{2t^2}\int_\Omega K(x)\omega^2 e^2 dx - \int_{\{x\in\Omega: e(x)\neq 0\}} \frac{F(x,te)}{(te)^4} e^4 dx \\ &+ \frac{\beta}{16\pi}\int_{\mathbb{R}^3} |\nabla\phi_{te}|^4 dx - t^{-3}\int_\Omega he dx \to -\infty\end{aligned}$$

as $t \to +\infty$, which yields that $I_\lambda(te) < 0$ for $t > 0$ large. Clearly, there is $C > 0$ (independent of λ) such that

$$c_\lambda := \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} I_\lambda(\gamma(t)) \leq \sup_{t\geq 0} I_\lambda(te_0) \leq C$$

where $\Gamma = \{\gamma \in C([0,1], E_\lambda) : \gamma(0) = 0, \|\gamma(1)\|_\lambda \geq \rho, I_\lambda(\gamma(1)) < 0\}$. By Mountain pass theorem and Lemma 2.3, we obtain a nontrivial critical point $\tilde{u}_\lambda$ of I_λ with $I_\lambda(\tilde{u}_\lambda) \in [\alpha, C]$ for λ large. The proof is complete. □

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