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INTEGRABILITY AND BIFURCATION OF LIMIT CYCLES FOR A CLASS OF QUASI-HOMOGENEOUS SYSTEMS

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Abstract The integrability and bifurcation of limit cycles for a class of quasi-homogeneous systems are studied, with four integrability conditions being obtained, and the existence of seven limit cycles in the neighborhood of origin being proved.

Keywords Quasi-homogeneous, center, focal value, limit cycle.

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1. Introduction

Consider the following ordinary differential system

$$\begin{aligned}\frac{dx}{dt} &= \sum_{i+j=1}^{\infty} a_{ij}x^i y^j = \sum_{k=1}^{\infty} X_k(x,y) = X(x,y),\\ \frac{dy}{dt} &= \sum_{i+j=1}^{\infty} b_{ij}x^i y^j = \sum_{k=1}^{\infty} Y_k(x,y) = Y(x,y),\end{aligned} \tag{1.1}$$

where $X(x,y), Y(x,y)$ are series of x, y with non-zero convergence radius and real coefficients, and

$$X_k(x,y) = \sum_{i+j=k} a_{ij}x^i y^j, \quad Y_k(x,y) = \sum_{i+j=k} b_{ij}x^i y^j, \tag{1.2}$$

are $k-$th order homogeneous polynomials in x, y.

In qualitative theory of planar ordinary differential systems,the center problem and bifurcation of limit cycles at the origin of system (1.1) are important and extensively studied because they are closely related to the well-known Hilbert's 16th problem. Those problems are far from being solved by now. However, for elementary singular points, there have been many good results. For example, the

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quadratic system was studied in [7], the systems with homogeneous nonlinearities of degree 3, 4 and 5 in [9, 10], the Kukles system in [11], and quadratic-like cubic systems [8, 23], etc. All results show that it is a provocative task to solve the center problem for a concrete system. The center and cyclicity problems for quartic linear-like reversible systems were considered in [13].

For degenerate singular points, the center problem becomes more difficult than that of elementary singular points because the classical methods can not be used. Many mathematicians have been devoted to solving this problem for many years and some results about nilpotent singular points associated with a double-zero eigenvalue and nonzero linear part were obtained, see [3–6]. In 2009, the inverse integrating factor method was proposed to solve the center problem and analytical center problem of the nilpotent singular points with multiplicity three in [22]. As a class of special systems, Z_2-equivalent cubic planar differential systems with two nilpotent singular points or other kinds of singular points were studied in [16–19]. Bifurcation diagrams for the nilpotent centers of Hamiltonian systems with linear and cubic homogeneous polynomial vector fields were obtained in [12]. Some new bifurcation problems were considered in [15, 26, 29].

The most difficult case is that the singular point is associated with a double-zero eigenvalue, having zero linear part.There are only few results for some special systems. For instance, the results for the homogeneous polynomial differential systems

$$\frac{dx}{dt} = X_n(x, y), \quad \frac{dy}{dt} = Y_n(x, y) \tag{1.3}$$

and its perturbed system

$$\begin{aligned} \frac{dx}{dt} &= X_n(x, y) + \sum_{i+j=n+1}^{\infty} a_{ij}x^i y^j, \\ \frac{dy}{dt} &= Y_n(x, y) + \sum_{i+j=n+1}^{\infty} b_{ij}x^i y^j, \end{aligned} \tag{1.4}$$

can be found in [1, 2]. The center problem and first integral of quasi-homogeneous polynomial differential systems were considered in [20], and its bifurcation of limit cycles was also discussed in [24, 25, 27, 28, 30]. Especially, a special case of system (1.4) with $n = 1$ was investigated in [21] where the relation between the focus value and the weights was obtained.

In this paper, a class of quasi-homogenous systems which can be written as

$$\begin{aligned} \frac{dx}{dt} =& -2y^3 + a_{50}x^5 + a_{60}x^6 - \frac{2}{5}b_{42}x^5y + a_{32}x^3y^2 - \frac{3}{5}b_{43}x^5y^2 - \frac{4}{3}b_{24}x^3y^3 \\ & - \frac{2}{5}a_{60}y^4 - \frac{1}{90}b_{23}(3a_{60} - 5b_{23})(3a_{60} + 5b_{23})x^3y^4 - 6b_{06}xy^5, \\ \frac{dy}{dt} =& 3x^5 + b_{60}x^6 + \frac{12}{5}a_{60}x^5y + b_{42}x^4y^2 + b_{23}x^2y^3 + b_{43}x^4y^3 \\ & + b_{24}x^2y^4 + \frac{1}{150}b_{23}(3a_{60} - 5b_{23})(3a_{60} + 5b_{23})x^2y^5 + b_{06}y^6 \end{aligned} \tag{1.5}$$

will be studied. This paper is divided into three sections. In the next section, a class of quasi-homogeneous polynomial differential systems is investigated. The first eight focus values are computed to determine the center conditions and their sufficiency is proved. Finally, a conclusion is given.

2. Center problem of a class of quasi-homogenous systems with weights $(2, 3)$

In this section, we study the center problem and bifurcation of limit cycles of system (1.5) which can be transformed to

$$\frac{dr}{d\theta} = \frac{\cos\theta\sin\theta(3\cos^4\theta - 2\sin^2\theta)}{6(\cos^6\theta + \sin^4\theta)} r + o(r), \tag{2.1}$$

by $x = r^2\cos\theta,\ y = r^3\sin\theta$.

The solution of system (2.1) that satisfies $r|_{\theta=0} = h$ can be written as

$$r = \tilde{r}(\theta, h) = \sum_{k=1}^{\infty} \nu_k(\theta) h^k.$$

Then, we can compute $\nu_k(\theta)$ step by step. At first, it is easy to get

$$\nu_1(\theta) = \frac{1}{(\cos^6\theta + \sin^4\theta)^{\frac{1}{12}}}.$$

From the equation

$$\nu_2'(\theta) = \frac{a_{50}\cos^{10}\theta(2\cos^2\theta + 3\sin^2\theta)}{12(\cos^6\theta + \sin^4\theta)^{\frac{25}{12}}},$$

we can get that

$$\nu_2(2\pi) = \frac{1}{12} a_{50} I_2,$$

where I_2 is given by (2.2). Suppose $\nu_2(2\pi) = 0$, then $a_{50} = 0$.

Equation

$$\nu_3'(\theta) = \frac{b_{60}\cos^6\theta\sin^3\theta(2\cos^2\theta + 3\sin^2\theta)}{18(\cos^6\theta + \sin^4\theta)^{\frac{13}{6}}}$$

yields that

$$\nu_3(\theta) = \frac{-b_{60}\cos^7\theta}{42(\cos^6\theta + \sin^4\theta)^{\frac{7}{6}}},$$

and $\nu_4(\theta)$ satisfies

$$\begin{aligned}\nu_4'(\theta) =& (15a_{60}\cos^9\theta + 15a_{32}\cos^6\theta\sin^2\theta + 18a_{60}\cos^3\theta\sin^4\theta + 10b_{23}\sin^6\theta) \\ &\times \frac{\cos^2\theta(2\cos^2\theta + 3\sin^2\theta)}{180(\cos^6\theta + \sin^4\theta)^{\frac{9}{4}}}.\end{aligned}$$

We can solve $\nu_4(\theta)$ by using integration by parts and obtain

$$\nu_4(2\pi) = \frac{13}{80}(a_{32} + b_{23}) I_4,$$

where I_4 is given by (2.2). $\nu_4(2\pi) = 0$ yields $a_{32} = -b_{23}$.

Similarly, $\nu_5(\theta)$ satisfies

$$\nu_5'(\theta) = -\left(126b_{42}\cos^{12}\theta + 85b_{60}^2\cos^9\theta\sin^2\theta - 84b_{42}\cos^6\theta\sin^4\theta - 210b_{42}\sin^8\theta\right) \times \frac{\cos^4\theta\sin\theta(2\cos^2\theta + 3\sin^2\theta)}{3780(\cos^6\theta + \sin^4\theta)^{\frac{10}{3}}},$$

which shows that

$$\nu_5(\theta) = \frac{\cos^5\theta(85b_{60}^2\cos^9\theta - 588b_{42}\cos^6\theta\sin^2\theta - 588b_{42}\sin^6\theta)}{17640(\cos^6\theta + \sin^4\theta)^{\frac{7}{3}}}.$$

Furthermore, $\nu_6(\theta)$ can be solved from

$$\begin{aligned}\nu_6'(\theta) =& \frac{-b_{60}\cos^6\theta(2\cos^2\theta + 3\sin^2\theta)}{6300(\cos^6\theta + \sin^4\theta)^{\frac{41}{12}}} \\ & \times (225a_{60}\cos^{12}\theta - 225b_{23}\cos^9\theta\sin^2\theta + 200a_{60}\cos^6\theta\sin^4\theta \\ & + 500b_{23}\cos^3\theta\sin^6\theta - 112a_{60}\sin^8\theta)\end{aligned}$$

using integration by parts, and

$$\nu_6(2\pi) = -\frac{63017}{1413600}a_{60}b_{60}I_6,$$

where I_6 is given by (2.2).

By similar calculation, we can get the following results.

Theorem 2.1. *The first eight focal values at the origin of system* (1.5) *are*

$$\begin{aligned}\nu_2(2\pi) &= \frac{1}{12}a_{50}I_2, \\ \nu_4(2\pi) &= \frac{13}{80}(a_{32} + b_{23})I_4, \\ \nu_6(2\pi) &= -\frac{63017}{1413600}a_{60}b_{60}I_6, \\ \nu_8(2\pi) &= -\frac{642056779}{4463030400}a_{60}b_{42}I_8, \\ \nu_{10}(2\pi) &= -\frac{4991}{261120}a_{60}b_{24}I_{10}, \\ \nu_{12}(2\pi) &= -\frac{11681848243563}{20641272524800}a_{60}b_{06}I_{12}, \\ \nu_{14}(2\pi) &= -\frac{3310132185787063}{501793556232806400}a_{60}^2b_{43}I_{14}, \\ \nu_{16}(2\pi) &= -\frac{3124141657}{4258013184000}a_{60}^2b_{23}(3a_{60} - 5b_{23})(3a_{60} + 5b_{23})I_{16},\end{aligned}$$

where

$$\begin{aligned}I_{4k-2} &= \int_0^{2\pi} \frac{\cos^{26k-16}\theta\left(2\cos^2\theta + 3\sin^2\theta\right)}{\left(\cos^6\theta + \sin^4\theta\right)^{\frac{52k-27}{12}}}\, d\theta \ > 0, \\ I_{4k} &= \int_0^{2\pi} \frac{\cos^{26k-6}\theta\sin^2\theta\left(2\cos^2\theta + 3\sin^2\theta\right)}{\left(\cos^6\theta + \sin^4\theta\right)^{\frac{52k-1}{12}}}\, d\theta \ > 0.\end{aligned} \tag{2.2}$$

Theorem 2.2 yields that

Proposition 2.1. *The first eight focal values at the origin of system* (1.5) *are all zero if and only if*

$$\begin{aligned} &a_{50}=0,\ a_{32}=-b_{23},\ b_{60}=0,\\ &b_{42}=0,\ \ b_{24}=0,\ b_{06}=0,\ b_{43}=0,\\ &a_{60}b_{23}(3a_{60}-5b_{23})(3a_{60}+5b_{23})=0. \end{aligned} \tag{2.3}$$

Moreover, we have the following theorem.

Theorem 2.2. *The origin of system* (1.5) *is a center if and only if one of the following conditions holds*

(I) $a_{50}=b_{60}=b_{42}=b_{24}=b_{06}=b_{43}=a_{60}=0,\ a_{32}=-b_{23}$;

(II) $a_{50}=b_{60}=b_{42}=b_{24}=b_{06}=b_{43}=b_{23}=a_{32}=0$;

(III) $a_{50}=b_{60}=b_{42}=b_{24}=b_{06}=b_{43}=0,\ a_{32}=-b_{23}, a_{60}=\frac{5}{3}b_{23}$;

(IV) $a_{50}=b_{60}=b_{42}=b_{24}=b_{06}=b_{43}=0,\ a_{32}=-b_{23}, a_{60}=-\frac{5}{3}b_{23}$.

Proof. The necessity has been proved, we only need to prove the sufficiency one by one.

When condition (I) holds, system (1.5) can be simplified to

$$\frac{dx}{dt}=-y^2(b_{23}x^3+2y),\quad \frac{dy}{dt}=x^2(3x^3+b_{23}y^3),$$

which is a Hamiltonian.

When condition (II) holds, (1.5) can be rewritten as

$$\frac{dx}{dt}=\frac{1}{5}(5a_{60}x^6-10y^3-2a_{60}y^4),\quad \frac{dy}{dt}=\frac{3}{5}x^5(5+4a_{60}y),$$

which is symmetric with respect to the $y-$axis and has the integrating factor

$$M_1=f_1^{\frac{-7}{2}}$$

and the first integral

$$F_1=f_1^{-5}f_2,$$

where

$$\begin{aligned} f_1=&5+4a_{60}y,\\ f_2=&9a_{60}^4x^{12}+3(125+250a_{60}y+150a_{60}^2y^2+20a_{60}^3y^3+4a_{60}^4y^4)x^6\\ &+(375+720a_{60}y+400a_{60}^2y^2+40a_{60}^3y^3+4a_{60}^4y^4)y^4. \end{aligned}$$

When one of condition (III) and (IV) holds, (1.5) can be transformed to

$$\begin{aligned} \frac{dx}{dt}&=-2y^3+a_{60}x^6-b_{23}x^3y^2-\frac{2}{5}a_{60}y^4,\\ \frac{dy}{dt}&=3x^5+\frac{12}{5}a_{60}x^5y+b_{23}x^2y^3, \end{aligned}$$

which has the integrating factor

$$M_2 = (f_3 f_4)^{-1}$$

and the first integral

$$F = f_3^9 f_4^{-5},$$

where

$$\begin{aligned} f_3 &= 1 + \frac{1}{6} a_{60} b_{23} x^3 + a_{60} y + \frac{1}{10} a_{60}^2 y^2, \\ f_4 &= 1 + \frac{3}{10} a_{60} b_{23} x^3 + \frac{9}{5} a_{60} y + \frac{6}{25} a_{60}^2 b_{23} x^3 y + \frac{9}{10} a_{60}^2 y^2 + \frac{12}{125} a_{60}^3 y^3. \end{aligned}$$

So the origin of system (1.5) is a center when one of the four conditions holds. □

Now, we study the number of limit cycles in small neighborhood of the origin of system (1.5). According to Theorem 3.7 in [21] or Theorem 3.2 in [14], the following theorem can be obtained easily.

Theorem 2.3. *The origin of system* (1.5) *is a* 8*th-order weak focus when*

$$\begin{aligned} &a_{50} = 0,\ a_{32} = -b_{23},\ b_{60} = 0, \\ &b_{42} = 0,\ \ b_{24} = 0,\ b_{06} = 0,\ b_{43} = 0, \\ &a_{60} b_{23} (3a_{60} - 5b_{23})(3a_{60} + 5b_{23}) \neq 0, \end{aligned}$$

and there exist at most 7 limit cycles near the origin of system (1.5)*, multiplicity taken into account.*

3. Conclusion

In summary, based on symbolic computation of singular point quantities at the degenerate Hopf singularity for a class of quasi-homogenous systems, we have found its all center conditions and determined that the highest order as a fine focus is 8. By the classical method, we got its all integrable conditions which include analytical integrability and non-analytical integrability. Furthermore, we obtained that the system has at most 7 small limit cycles from the equilibrium via Hopf bifurcation, multiplicity taken into account. In our future work, we will consider the perturbation phenomenons with different perturbation terms, and study their dynamic behaviors.

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