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EXACT SOLUTIONS AND DYNAMICS OF KUNDU-MUKHERJEE-NASKAR MODEL*

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Abstract For the Kundu-Mukherjee-Naskar model, to find its exact explicit solutions, it is necessary to analyze the dynamical behaviors of the corresponding differential system of the amplitude component, which is a planar dynamical system with a singular straight line. In this paper, by using the techniques from dynamical systems to analyze the parameter conditions of system and find the corresponding phase portraits, the dynamical behaviors of the amplitude component can be derived. Under different parameter conditions, exact explicit homoclinic solutions, periodic solution families as well as kink and anti-kink wave solutions can be found.

Keywords Singular nonlinear traveling wave equation, bifurcation, homoclinic solution, periodic solution, kink and anti-kink wave solutions, pseudo-peakon, Kundu-Mukherjee-Naskar model.

MSC(2010) 34C23, 35Q51-53, 58j55.

1. Introduction

In a recently published work [15], the authors studied the envelope solitons propagating in an optical waveguiding media governed by the Kundu-Mukherjee-Naskar (KMN) equation. They made the special ansatz $F(\xi) = \lambda - \rho \mathrm{sech}^2[\mu(\xi - \xi_0)]$, to derive the formation of a gray soliton on a continuous wave background in the system.

The KMN equation reads as:

$$iq_t + \alpha q_{xy} + i\gamma q(qq_x^* - q^* q_x) = 0, \tag{1.1}$$

where $q(x, y, t)$ indicates the soliton profile, x and y refer to the spatial variables, t is the temporal variable, while α and γ account for the dispersion and nonlinear parameters, respectively.

The seven authors of [15] considered the solutions of equation (1.1) with the form:

$$q(x, y, t) = \phi(\xi) \exp\left(i[-\kappa_1 x - \kappa_2 y + \omega t + \theta(\xi)]\right), \quad \xi = px + qy - vt, \tag{1.2}$$

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where $\kappa_j, j = 1, 2$ and ω are real-valued constants. ξ is the wave variable. $\phi(\xi)$ is the amplitude component. Substituting (1.2) into (1.1), they obtained the ordinary differential equation:

$$(v+\alpha(\kappa_1 q+\kappa_2 p)\phi+2p\gamma\phi^3)\theta'-pq\alpha\phi(\theta')^2+pq\alpha\phi''-2\kappa_1\gamma\phi^3-(\omega+\alpha\kappa_1\kappa_2)\phi = 0 \quad (1.3)$$

and

$$pq\alpha(\phi\theta'' + 2\phi'\theta') - [v + \alpha(\kappa_1 q + \kappa_2 p)]\phi' = 0. \quad (1.4)$$

Multiplying (1.4) by ϕ and integrating, yields

$$\theta' = \frac{A}{2pq\alpha\phi^2} + \frac{v + \alpha(\kappa_1 q + \kappa_2 p)}{2pq\alpha}, \quad (1.5)$$

where A is an integration constant. Hence, the phase modification $\theta(\xi)$ in (1.2) reads

$$\theta(\xi) = \frac{A}{2pq\alpha}\int \frac{d\xi}{\phi^2(\xi)} + \frac{v + \alpha(q\kappa_1 + p\kappa_2)}{2pq\alpha}\xi. \quad (1.6)$$

Directly substituting the expression (1.5) into (1.3), we obtain

$$\phi'' + a_1\phi + a_3\phi^3 - \frac{g}{\phi^3} = 0, \quad (1.7)$$

where $a_1 = \frac{[v+\alpha(q\kappa_1+p\kappa_2)]^2+4pA\gamma-4pq\alpha(\alpha\kappa_1\kappa_2+\omega)}{4\alpha^2p^2q^2}$, $a_3 = \frac{\gamma[v+\alpha(p\kappa_2-q\kappa_1)]}{\alpha^2pq^2}$, $g = \frac{A^2}{4\alpha^2p^2q^2} \geq 0$.

Equation (1.7) is equivalent to the planar dynamical system for $g > 0$:

$$\frac{d\phi}{d\xi} = y, \quad \frac{dy}{d\xi} = -\left(a_1\phi + a_3\phi^3 - \frac{g}{\phi^3}\right) = -F(\phi) \quad (1.8)$$

with the first integral

$$H(\phi, y) = y^2 + \frac{1}{2}a_3\phi^4 + a_1\phi^2 + \frac{g}{\phi^2} = h. \quad (1.9)$$

For $g = 0$, i.e., $A = 0$, equation (1.7) is equivalent to the planar dynamical system:

$$\frac{d\phi}{d\xi} = y, \quad \frac{dy}{d\xi} = -\left(a_1\phi + a_3\phi^3\right) \quad (1.10)$$

with the first integral

$$H_0(\phi, y) = y^2 + \frac{1}{2}a_3\phi^4 + a_1\phi^2 = h. \quad (1.11)$$

Unfortunately, the authors of [15] did not consider the dynamics of solutions of the corresponding systems (1.8) and (1.10) of equation (1.1). Therefore, their study results about exact solutions are not complete.

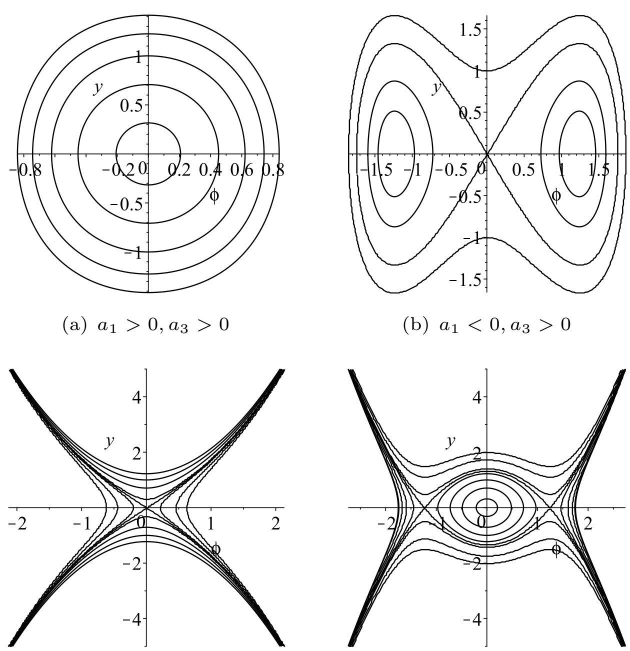
In this paper, we apply the method of dynamical systems to discuss the dynamical behaviors of solutions $\phi(\xi)$ and under different parameter conditions to find exact solutions of equation (1.8).

System (1.8) is a three-parameter system depending on parameter group (a_1, a_3, g). It has interesting dynamical behaviors. In addition, system (1.8) is a singular traveling wave system of the first kind defined by [8] and [7] with the singular straight lines $\phi = 0$. In the past three decades, nonlinear wave equations with non-smooth solitary wave solution (called peakon) and solution family having compact support (called compactons) (see [12]) attracted a lot of attention. Peakon was first coined by Cammasa and Holm [2,3], and thereafter other peakon equations were developed (see [4–11, 13, 14] and references therein).

The article is organized as follows. In section 2, we discuss the bifurcations of phase portraits for the systems (1.8) and (1.10). In section 3, we derive all possible exact explicit parametric representations for all bounded solutions (homoclinic orbits, periodic orbits and heteroclinic orbits) of system (1.10). In section 4, we study the exact solutions of system (1.8). All possible exact explicit parametric representations of the periodic orbits, homoclinic orbits as well as heteroclinic orbits of systems (1.8) and (1.10) can be given. Theorems 3.1 and 4.1 give rise to the main results of this paper.

2. The bifurcations of phase portraits of systems (1.8) and (1.10)

We first discuss the bifurcations of phase portraits of system (1.10). Clearly, when $a_1 a_3 > 0$, the cubic Hamiltonian system (1.10) only has a singular point $O(0,0)$. When $a_1 a_3 < 0$, it has three singular points $E_1(-\phi_1, 0)$, $O(0,0)$ and $E_2(\phi_1, 0)$, where $\phi_1 = \sqrt{-\frac{a_1}{a_3}}$. The bifurcations of phase portraits of system (1.10) are shown in Fig. 1.



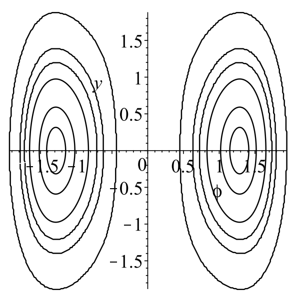
(a) $a_1 > 0, a_3 > 0$ (b) $a_1 < 0, a_3 > 0$

(c) $a_1 < 0, a_3 < 0$ (d) $a_1 > 0, a_3 < 0$

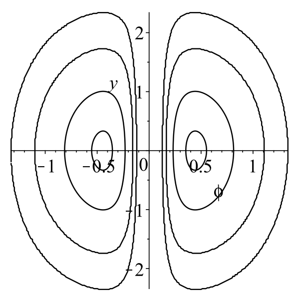
Figure 1. The bifurcations of phase portraits of system (1.10).

We next discuss the bifurcations of phase portraits of system (1.8). Write $F(\phi)$ in (1.8) as $f(\Phi) = a_3\Phi^3 + a_1\Phi^2 - g$, where $\Phi = \phi^2$. Clearly, $f'(\Phi) = 3a_3\Phi^2 + 2a_1\Phi$ has two zeros at $\Phi = 0$ and $\Phi = -\frac{2a_1}{3a_3}$. And we have $f\left(-\frac{2a_1}{3a_3}\right) = \frac{4a_1^3}{27a_3^2} - g$.

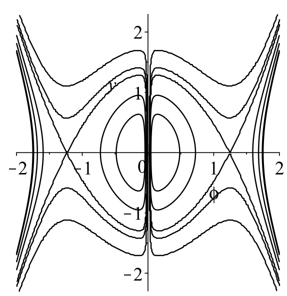
Obviously, when $a_1a_3 < 0, f'(\Phi)$ has a positive zero. For $g > 0, a_3 < 0$, when $f\left(-\frac{2a_1}{3a_3}\right) > 0$, i.e., $0 < g < \frac{4a_1^3}{27a_3^2}$, $f(\Phi)$ has two positive zeros. It follows that system (1.8) has four equilibrium points. When $a_1a_3 > 0, f(\Phi)$ has at most one positive zero. The bifurcations of phase portraits of system (1.8) are shown in Fig. 2.



(a) $a_1 < 0, a_3 > 0, g > 0$



(b) $a_1 > 0, a_3 > 0, g > 0$



(c) $a_1 > 0, a_3 < 0, 0 < g < \frac{4a_1^3}{27a_3^2}$

Figure 2. The bifurcations of phase portraits of system (1.8).

3. The exact explicit solutions of system (1.10)

We see from (1.11) that $h_1 = H_0(\mp\phi_1, 0) = -\frac{a_1^2}{2a_3}$ and $y^2 = h - a_1\phi^2 - \frac{1}{2}a_3\phi^4$. By using the first equation of (1.10), we have

$$\xi = \int_{\phi_0}^{\phi} \frac{d\phi}{\sqrt{h - a_1\phi^2 - \frac{1}{2}a_3\phi^4}}. \tag{3.1}$$

3.1. The case $a_1 > 0, a_3 > 0$ (see Fig. 1(a)).

Corresponding to the periodic orbit family defined by $H_0(\phi, y) = h, h \in (0, \infty)$, enclosing the origin $O(0,0)$, (3.1) can be written as $\sqrt{\frac{1}{2}a_3}\xi = \int_{\phi}^{\phi_b} \frac{d\phi}{\sqrt{(\phi_b^2-\phi^2)(\phi_a^2+\phi^2)}}$, where $\phi_a^2 = \frac{a_1}{a_3} + \sqrt{\Delta_1}, \phi_b^2 = -\frac{a_1}{a_3} + \sqrt{\Delta_1}, \Delta_1 = \frac{a_1^2+2ha_3}{a_3^2}$. Thus, we obtain the parametric representation of this periodic orbit family as follows:

$$\phi(\xi) = \phi_b \mathrm{cn}(\Omega_1 \xi, k), \tag{3.2}$$

where $k^2 = \frac{\phi_b^2}{\phi_a^2+\phi_b^2}, \Omega_1 = \sqrt{\frac{1}{2}a_3(\phi_a^2+\phi_b^2)}, \mathrm{cn}(\cdot, k)$ is the Jacobin elliptic function.

3.2. The case $a_1 < 0, a_3 > 0$ (see Fig. 1(b)).

(i) Corresponding to the two periodic orbit families defined by $H_0(\phi, y) = h, h \in (h_1, 0)$, enclosing the singular points E_1 and E_2, respectively, (3.1) can be written as $\sqrt{\frac{1}{2}a_3}\xi = \int_\phi^{\phi_a} \frac{d\phi}{\sqrt{(\phi_a^2-\phi^2)(\phi^2-\phi_b^2)}}$ where $\phi_b^2 = -\frac{|a_1|}{a_3} + \sqrt{\Delta_1}, \phi_a^2 = \frac{|a_1|}{a_3} + \sqrt{\Delta_1}, \Delta_1 = \frac{a_1^2+2ha_3}{a_3^2}$. It gives rise to the parametric representations of two periodic orbit families as follows:

$$\phi(\xi) = \mp\phi_a \mathrm{dn}(\Omega_2 \xi, k), \tag{3.3}$$

where $k^2 = \frac{\phi_a^2-\phi_b^2}{\phi_a^2}, \Omega_2 = \sqrt{\frac{1}{2}a_3\phi_a^2}, \mathrm{dn}(\cdot, k)$ is the Jacobin elliptic function.

(ii) Corresponding to the two homoclinic orbits to the origin $O(0,0)$ defined by $H_0(\phi, y) = 0$, enclosing the singular points E_1 and E_2, respectively, (3.1) can be written as $\sqrt{\frac{1}{2}a_3}\xi = \int_\phi^{\phi_M} \frac{d\phi}{\phi\sqrt{(\phi_M^2-\phi^2)}}$, where $\phi_M^2 = \frac{2|a_1|}{a_3}$. Hence, we obtain the parametric representations of the two homoclinic orbits as follows:

$$\phi(\xi) = \mp\phi_M \mathrm{sech}(\omega_0 \xi), \tag{3.4}$$

where $\omega_0 = \sqrt{\frac{1}{2}a_3\phi_M^2}$.

(iii) Corresponding to the global periodic orbit family defined by $H_0(\phi, y) = h, h \in (0, \infty)$, enclosing the three singular points $O(0,0), E_1$ and E_2, it has the same parametric representation as (3.2).

3.3. The case $a_1 > 0, a_3 < 0$ (see Fig. 1(d)).

(i) Corresponding to the periodic orbit family defined by $H_0(\phi, y) = h, h \in (0, h_1)$, enclosing the origin $O(0,0)$, (3.1) can be written as $\sqrt{\frac{1}{2}|a_3|}\xi = \int_0^\phi \frac{d\phi}{\sqrt{(\phi_a^2-\phi^2)(\phi_b^2-\phi^2)}}$ where $\phi_a^2 = \frac{a_1}{|a_3|} + \sqrt{\Delta_1}, \phi_b^2 = \frac{a_1}{|a_3|} - \sqrt{\Delta_1}, \Delta_1 = \frac{a_1^2-2h|a_3|}{a_3^2}$. Thus, we obtain the parametric representation of this periodic orbit family as follows:

$$\phi(\xi) = \phi_b \mathrm{sn}(\Omega_3 \xi, k), \tag{3.5}$$

where $k^2 = \frac{\phi_b^2}{\phi_a^2}, \Omega_3 = \sqrt{\frac{1}{2}|a_3|\phi_a^2}$.

(ii) Corresponding to the two heteroclinic orbits defined by $H_0(\phi, y) = h_1$, enclosing the origin $O(0,0)$, (3.1) can be written as $\sqrt{\frac{1}{2}|a_3|}\xi = \int_0^\phi \frac{d\phi}{\phi_1^2-\phi^2}$. Thus, we have the following parametric representations:

$$\phi(\xi) = \mp\phi_1 \tanh(\omega_1 \xi), \tag{3.6}$$

where $\omega_1 = \sqrt{\frac{1}{2}|a_3|\phi_1^2}$.

In summary, the following theorem is established.

Theorem 3.1. *Assume that $g = 0$ in system* (1.8)*. Then, equation* (1.1) *has exact explicit solutions:*

$$q(x, y, t) = \phi(\xi) \exp\left(i[-\kappa_1 x - \kappa_2 y + \omega t + \theta(\xi)]\right) \tag{3.7}$$

where $\theta(\xi) = \frac{v+\alpha(q\kappa_1+p\kappa_2)}{2pq\alpha}\xi, \phi(\xi)$ *is given by* (3.2), (3.3), (3.4), (3.5) *and* (3.6)*, respectively.*

4. The exact explicit solutions of system (1.8)

We know from (1.9) that $y^2 = \frac{h\phi^2 - g - a_1\phi^4 - \frac{1}{2}a_3\phi^6}{\phi^2}$. By using the first equation of (1.8), we have

$$\xi = \int_{\phi_0}^{\phi} \frac{\phi d\phi}{\sqrt{h\phi^2 - g - a_1\phi^4 - \frac{1}{2}a_3\phi^6}} = \int_{\psi_0}^{\psi} \frac{d\psi}{2\sqrt{(h\psi - g - a_1\psi^2 - \frac{1}{2}a_3\psi^3)}}, \tag{4.1}$$

where $\psi = \phi^2$.

4.1. The case $a_1 \neq 0, a_3 > 0, g > 0$ (see Fig. 2(a), (b))

Corresponding to the two periodic orbit families defined by $H(\phi, y) = h, h \in (h_1, \infty)$, enclosing two singular points, now, (4.1) can be written as $\sqrt{2a_3}\xi = \int_{\psi}^{\psi_a} \frac{d\psi}{\sqrt{(\psi_a-\psi)(\psi-\psi_b)(\psi+\psi_d)}}$. Thus, we have the following two periodic solution families:

$$\phi(\xi) = \mp\left(\psi_a - (\psi_a - \psi_b)\mathrm{sn}^2(\Omega_4\xi, k)\right)^{\frac{1}{2}}, \tag{4.2}$$

where $k^2 = \frac{\psi_a-\psi_b}{\psi_a+\psi_d}, \Omega_4 = \sqrt{\frac{1}{2}a_3(\psi_a + \psi_d)}$.

Notice that

$$\int \frac{d\xi}{\phi^2(\xi)} = \frac{1}{\psi_a}\int \frac{d\xi}{1 - \frac{\psi_a-\psi_b}{\psi_a}\mathrm{sn}^2(\Omega_4\xi, k)} = \frac{1}{\psi_a\Omega_4}\Pi\left(\arcsin\left(\mathrm{sn}(\Omega_4\xi, k)\right), \hat{\alpha}_1^2, k\right),$$

where $\Pi(\cdot, \cdot, k)$ is the elliptic integral of the third kind (see [1]) with $\hat{\alpha}_1^2 = \frac{\psi_a-\psi_b}{\psi_a}$.

Hence, we see from (1.6) that for the $\phi(\xi)$ given by (4.2), we have

$$\theta(\xi) = \theta_1(\xi) = \frac{v + \alpha(q\kappa_1 + p\kappa_2)}{2pq\alpha}\xi + \frac{A}{2pq\alpha\psi_a\Omega_4}\Pi\left(\arcsin\left(\mathrm{sn}(\Omega_4\xi, k)\right), \hat{\alpha}_1^2, k\right). \tag{4.3}$$

4.2. The case $a_1 > 0, a_3 < 0, 0 < g < \frac{4a_1^3}{27a_3^2}$ (see Fig. 2(c))

(i) Corresponding to the two periodic orbit families defined by $H(\phi, y) = h, h \in (h_1, h_2)$, enclosing two singular points, now, (4.1) can be written as $\sqrt{2|a_3|}\xi = \int_{\psi}^{\psi_b} \frac{d\psi}{\sqrt{(\psi_a-\psi)(\psi_b-\psi)(\psi-\psi_c)}}$. Thus, we have the following two periodic solution families:

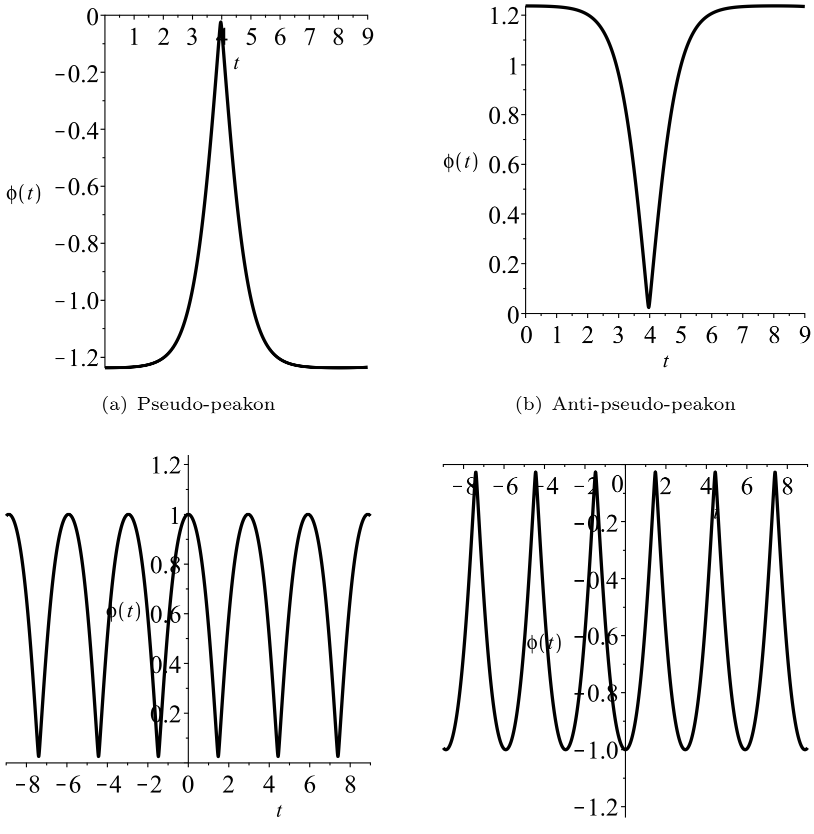
$$\phi(\xi) = \mp\left(\psi_a + \frac{\psi_b - \psi_a}{1 - k^2\mathrm{sn}^2(\Omega_5\xi, k)}\right)^{\frac{1}{2}}, \tag{4.4}$$

where $k^2 = \frac{\psi_b - \psi_c}{\psi_a - \psi_c}, \Omega_5 = \sqrt{\frac{1}{2}|a_3|(\psi_a - \psi_c)}$.

Then we obtain for the $\phi(\xi)$ given by (4.4)

$$\int \frac{d\xi}{\phi^2(\xi)} = \int \frac{d\xi}{\psi_b - \psi_a k^2 \mathrm{sn}^2(\Omega_5\xi, k)} - \int \frac{k^2 \mathrm{sn}^2(\Omega_5\xi, k)}{\psi_b - \psi_a k^2 \mathrm{sn}^2(\Omega_5\xi, k)} d\xi$$
$$= \frac{\xi}{\psi_a} + \frac{k^2}{\Omega_5}\Pi(\arcsin(\mathrm{sn}(\Omega_5\xi, k)), \hat{\alpha}_2^2, k),$$

where $\hat{\alpha}_2^2 = \frac{(\psi_b - \psi_c)\psi_a}{(\psi_a - \psi_c)\psi_b}$.



(a) Pseudo-peakon (b) Anti-pseudo-peakon

(c) Anti-pseudo-periodic-peakon (d) Pseudo-periodic-peakon

Figure 3. The pseudo-peakon and pseudo-periodic peakon of system (1.8).

Hence, we have from (1.6)

$$\theta(\xi) = \theta_2(\xi) = \left(\frac{A}{2pq\alpha\psi_a} + \frac{v + \alpha(q\kappa_1 + p\kappa_2)}{2pq\alpha}\right)\xi$$
$$+ \frac{A(\psi_a - \psi_b)}{2pq\alpha\psi_a\psi_b\Omega_5}\Pi\left(\arcsin\left(\mathrm{sn}(\Omega_5\xi, k)\right), \hat{\alpha}_2^2, k\right). \quad (4.5)$$

(ii) Corresponding to the two homoclinic orbits defined by $H(\phi, y) = h_2$, enclosing two singular points, now, (4.1) can be written as $\sqrt{2|a_3|}\xi = \int_{\psi_m}^{\psi} \frac{d\psi}{(\psi_2 - \psi)\sqrt{\psi - \psi_m}}$.

Thus, we have the following two solitary wave solutions:

$$\phi(\xi) = \mp \left(\psi_m + (\psi_2 - \psi_m) \tanh^2 \left(\frac{1}{2} \omega_2 \xi \right) \right)^{\frac{1}{2}}, \tag{4.6}$$

where $\omega_2 = \sqrt{2|a_3|(\psi_2 - \psi_m)}$.

We see that for the $\phi(\xi)$ given by (4.6), we have that

$$\begin{aligned} \int \frac{d\xi}{\phi^2(\xi)} &= \int \frac{d\xi}{\psi_m + (\psi_2 - \psi_m)\tanh^2\left(\frac{1}{2}\omega_2\xi\right)} \\ &= \frac{2}{\omega_2 \psi_2} \hat{\alpha}_3 \arctan\left(\hat{\alpha}_3 \tanh\left(\frac{1}{2}\omega_2 \xi \right) \right) + \frac{\xi}{\psi_2}, \end{aligned}$$

where $\hat{\alpha}_3^2 = \frac{\psi_2 - \psi_m}{\psi_m}$.

Therefore, we obtain

$$\begin{aligned} \theta(\xi) = \theta_3(\xi) = &\left(\frac{A}{2pq\alpha\psi_2} + \frac{v + \alpha(q\kappa_1 + p\kappa_2)}{2pq\alpha} \right) \xi \\ &+ \frac{A}{pq\alpha\omega_2\psi_2} \hat{\alpha}_3 \arctan\left(\hat{\alpha}_3 \tanh\left(\frac{1}{2}\omega_2\xi \right) \right). \end{aligned} \tag{4.7}$$

We see from Fig. 2(c) that if $0 < \phi_m \ll 1$ with $\phi_m = \sqrt{\psi_m}$, then there exist two segments of two homoclinic orbits which are very close to the singular straight line $\phi = 0$. This means that the two homoclinic orbits give rise to two pseudo-peakon solutions (see Fig. 3(a),(b)). When $|h - h_2| \ll 1$, the two periodic orbit families defined by $H(\phi, y) = h$ give rise to two families of pseudo-periodic peakon solutions (see Fig. 3(c),(d)).

To sum up, the following theorem is established.

Theorem 4.1. *Assume that $g > 0$ in system* (1.8). *Then, the following conclusions hold.*

- *(i) When $a_1 \neq 0, a_3 > 0, g > 0$, equation* (1.1) *has exact explicit solutions* (3.7), *where $\phi(\xi)$ is given by* (4.2), *$\theta(\xi) = \theta_1(\xi)$ is given by* (4.3);
- *(ii) When $a_1 > 0, a_3 < 0, 0 < g < \frac{4a_1^3}{27a_3^2}$, equation* (1.1) *has exact explicit solutions* (3.7), *where $\phi(\xi)$ is given by* (4.4) *and* (4.6), *$\theta(\xi) = \theta_j(\xi), j = 2, 3$ are given by* (4.5) *and* (4.7).

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