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INVERSE VARIATIONAL PRINCIPLES FOR TOPOLOGICAL PRESSURES ON MEASURES

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Abstract In this paper, we generalize the various types of topological pressures and measure-theoretical pressures for non-additive continuous potential with tempered distortion. We show inverse variational principles of measures for this non-additive topological pressures. Furthermore, we apply the inverse variational principles for topological pressures on measures to give the estimate of Hausdorff dimension of measures supported on average conformal repellers.

Keywords Topological pressure, measure-theoretic pressure, variational principle.

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1. Introduction

Throughout this paper, a *topological dynamical system* (TDS for short) is a pair (X,T), where X is a compact metric space with a metric d and $T:X\to X$ is a continuous map. Let $\mathcal{M}(X)$, $\mathcal{M}(X,T)$ and $\mathcal{E}(X,T)$ denote respectively the set of Borel probability measures, T-invariant Borel probability measures and T-invariant ergodic Borel probability measures on X. By a *Borel measure theoretical dynamical system*$(X,\mathcal{B}(X),\mu,T)$ we mean $(X,\mathcal{B}(X),\mu)$ is a Borel measure space and T is a Borel measure preserving transformation.

Kolmogorov [10] introduced measure-theoretical entropy $h_\mu(T)$ for any $(X,\mathcal{B}(X),\mu,T)$. Later, Adler etc [1] introduced topological entropy $h_{top}(T)$ for any TDS (X,T). Dinaburg [7] and Bowen [3] independently gave equivalent definitions for topological entropy by using separating and spanning sets. The variational principle (see [15, Thm 8.6]) reveals the basic relation between topological entropy and measure-theoretical entropy: if (X,T) is a TDS then

$$h_{top}(T)=\sup\{h_\mu(T):\mu\in\mathcal{M}(X,T)\}.$$

Bowen [4] presented a type of topological entropy $h^B_{top}(Z,T)$ for any set Z in a TDS (X,T) in a way resembling Hausdorff dimension, which is the so-called Bowen topological entropy. Particularly, he showed that $h^B_{top}(X,T)=h_{top}(T)$ for any TDS (X,T).

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Topological pressure, as a non-trivial extension of topological entropy, was first introduced by Ruelle [13] and extended to compact spaces with continuous transformations by Walters [15]. There is also a variational principle of topological pressure [15, Thm 9.10]: if (X,T) is a TDS then $P(T,f) = \sup\{h_\mu(T) + \int f \,\mathrm{d}\,\mu | \mu \in \mathcal{M}(X,T)\}$, where f is a continuous functions of X and $P(T,f)$ is the pressure of T with respect to f. Pesin etc [11] further extended Bowen's results [4] to topological pressure and introduced a type of topological pressure $P^B(Z,T,f)$ for any set Z in a TDS (X,T), which we call Pesin-Pitskel topological pressure. They in [11] showed that, among other things, if (X,T) is a TDS then $P(T,f) = P^B(X,T,f)$.

Inspired by the variational relation between topological entropy and measure-theoretical entropy, Feng etc [8] introduced measure-theoretical lower and upper entropies and packing topological entropy, and they obtained two variational principles for Bowen entropy and packing entropy: if $Z \subset X$ is nonempty and compact then

$$h^B_{top}(Z,T) = \sup\{\underline{h}_\mu(T) : \mu \in \mathcal{M}(X),\ \mu(Z) = 1\},$$
$$h^P_{top}(Z,T) = \sup\{\overline{h}_\mu(T) : \mu \in \mathcal{M}(X),\ \mu(Z) = 1\},$$

where $h^P_{top}(Z,T)$, $\underline{h}_\mu(T)$ and $\overline{h}_\mu(T)$ denote respectively the packing topological entropy of Z, measure-theoretical lower and upper entropies of μ. Tang etc [14] generalized Feng-Huang's variational principle of Bowen topological entropy to Pesin-Pitskel topological pressure: if $Z \subset X$ is nonempty and compact then

$$P^B(Z,T,f) = \sup\{\underline{P}_\mu(T,f) : \mu \in \mathcal{M}(X),\ \mu(Z) = 1\},$$

where $P^B(Z,T,f)$ denotes the Pesin-Pitskel pressure of Z (see Definition 2.1) and $\underline{P}_\mu(T,f)$ denotes the measure-theoretical lower pressure of μ (see Definition 3.1). Recently, Wang in [17] obtained two new variational principles for Bowen and packing topological entropies by introducing Bowen entropy and packing entropy of measures in the sense of Katok. He showed that if $Z \subset X$ is nonempty and compact then

$$h^B_{top}(Z,T) = \sup\{P^{KB}_\mu(T,0) : \mu \in \mathcal{M}(X),\ \mu(Z) = 1\},$$
$$h^P_{top}(Z,T) = \sup\{P^{KP}_\mu(T,0) : \mu \in \mathcal{M}(X),\ \mu(Z) = 1\},$$

where $P^{KB}_\mu(T,0)$ and $P^{KP}_\mu(T,0)$ denote respectively the Bowen and packing topological entropies of μ in the sense of Katok (see Definition 3.3).

Let $C(X,\mathbb{R})$ denote the set of all continuous functions of X, and let $P^B_\mu(T,f)$ and $P^{KB}_\mu(T,f)$ denote the Pesin-Pitskel pressure of μ (see Definition 3.2) and the Pesin-Pitskel pressure of μ in the sense of Katok (see Definition 3.3) respectively. Zhong etc in [18] proved the following theorem.

Theorem 1.1. *Let (X,T) be a TDS, $f \in C(X,\mathbb{R})$ and $Z \subset X$ be a nonempty compact set. Then*

$$\begin{aligned} P^B(Z,T,f) &= \sup\{P^B_\mu(T,f) : \mu \in \mathcal{M}(X),\ \mu(Z) = 1\} \\ &= \sup\{P^{KB}_\mu(T,f) : \mu \in \mathcal{M}(X),\ \mu(Z) = 1\}. \end{aligned}$$

Let $P^P(Z,T,f)$, $\overline{P}_\mu(T,f)$, $P^P_\mu(T,f)$ and $P^{KP}_\mu(T,f)$ denote respectively the packing topological pressure of Z (see Definition 2.1), measure-theoretical upper pressure of μ (see Definition 3.1), packing pressure of μ (see Definition 3.2) and packing pressure of μ in the sense of Katok (see Definition 3.3).

Zhong etc in [18] also proved the following theorems.

Theorem 1.2. *Let (X,T) be a TDS, $f \in C(X,\mathbb{R})$ and $Z \subset X$ be a nonempty compact set. If $P^P(Z,T,f) > \|f\|_\infty$, where $\|f\|_\infty := \sup_{x\in X} f(x)$, then*

$$\begin{aligned} P^P(Z,T,f) &= \sup\{\overline{P}_\mu(T,f) : \mu \in \mathcal{M}(X),\ \mu(Z)=1\} \\ &= \sup\{P^P_\mu(T,f) : \mu \in \mathcal{M}(X),\ \mu(Z)=1\} \\ &= \sup\{P^{KP}_\mu(T,f) : \mu \in \mathcal{M}(X),\ \mu(Z)=1\}. \end{aligned}$$

From Theorem 1.1 in [18], we have the following result.

Theorem 1.3. *Let (X,T) be a TDS and $f \in C(X,\mathbb{R})$, $\mu \in \mathcal{M}(X)$, then*

$$\begin{aligned} P^B_\mu(T,f) = P^{KB}_\mu(T,f) \le \inf\{P^B(Z,T,f) : \mu(Z)=1\}, \\ P^P_\mu(T,f) = P^{KP}_\mu(T,f) \le \inf\{P^P(Z,T,f) : \mu(Z)=1\}. \end{aligned}$$

Wang in [17] proved that for $f=0$, $P^{KB}_\mu(T,0) \le \inf\{P^B(Z,T,0) : \mu(Z)=1\}$ and $P^{KP}_\mu(T,0) \le \inf\{P^P(Z,T,0) : \mu(Z)=1\}$. The results as above generalized the results in [17] to topological pressure.

The questions are whether inequality as above can be equality? In this paper, we prove the following inverse variational principles.

Theorem 1.4. *Let (X,T) be a TDS and $f \in C(X,\mathbb{R})$ and $\mu \in \mathcal{M}(X)$, then*

$$\begin{aligned} P^B_\mu(T,f) = P^{KB}_\mu(T,f) = \inf\{P^B(Z,T,f) : \mu(Z)=1\}, \\ P^P_\mu(T,f) = P^{KP}_\mu(T,f) = \inf\{P^P(Z,T,f) : \mu(Z)=1\}. \end{aligned}$$

Furthermore, in [2], the notion of average conformal repeller is introduced. In [6], Hausdorff dimension for every subset of average conformal repeller Λ is given as root of Bowen equation. We apply the results of Theorem1.4 as above to give the estimate of Hausdorff dimension for every $\mu \in \mathcal{M}(\Lambda)$.

Theorem 1.5. *Let M be a C^∞ Riemannian manifold and $T : M \to M$ be a C^1 map. Suppose $\Lambda \subset M$ is an average conformal repeller. Then for every $\mu \in \mathcal{M}(\Lambda)$, $dim_H\mu = t^*$, where t^* is the unique solution of equation $P^B_\mu(T,-tf(x)) = 0$ in $[0,d]$ and $f(x) = \log(|det(DT(x))|)^{\frac{1}{d}}$.*

Next, we introduce a class of non-additive continuous potential. Let (X,T) be a TDS, $\mathcal{F} = \{f_n : X \to \mathbb{R}\}$ be a sequence of continuous potentials. We say that $\{f_n\}_{n\in\mathbb{N}}$ satisfy tempered distortion, if

$$\lim_{\varepsilon\to 0}\limsup_{n\to\infty}\frac{f_n(\varepsilon)}{n} = 0, \tag{1.1}$$

where $f_n(\varepsilon) = \sup\{|f_n(x) - f_n(y)| : x \in X, y \in B_n(x,\varepsilon)\}$.

Remark 1.1. If $f \in C(X,\mathbb{R})$, then $\mathcal{F} = \{f_n = \sum_{i=0}^{n-1} f(T^i(x))\}$ is a sequence of continuous potentials with tempered distortion.

In this paper, we will define various types of topological pressures and measure-theoretical pressures for non-additive continuous potential with tempered distortion, and investigate the relations between various types of topological pressures and different versions of measure-theoretical pressures.

We arrange the rest of this paper as follows. In Section 2, we recall the notions of topological pressures and discuss their relations. Section 3 presents various kinds of measure-theoretical pressures, and section 4 presents the proof of main theorem. In section 5, we apply the main theorem to give the estimate of Hausdorff dimension for every $\mu \in \mathcal{M}(\Lambda)$.

2. Topological pressures

In this section, we follow the forms in [18] to give four types of topological pressures: Pesin-Pitskel topological pressure, lower capacity topological pressure, upper capacity topological pressure and packing topological pressure, and present some basic properties for these pressures.

For any $n \in \mathbb{N}$ and $x, y \in X$, let

$$d_n(x,y) = \max\{d(T^i(x), T^i(y)) : 0 \le i < n\}.$$

For any $n \in \mathbb{N}$, $\varepsilon > 0$, $x \in X$, let $B_n(x,\varepsilon) = \{y \in X : d_n(x,y) < \varepsilon\}$ and $\overline{B}_n(x,\varepsilon) = \{y \in X : d_n(x,y) \le \varepsilon\}$.

Given a non-additive continuous potential with tempered distortion $\mathcal{F} = \{f_n\}$, let

$$f_n(x,\varepsilon) = \sup_{y \in B_n(x,\varepsilon)} f_n(y), \quad \text{and} \quad \overline{f}_n(x,\varepsilon) = \sup_{y \in \overline{B}_n(x,\varepsilon)} f_n(y).$$

Let $Z \subset X$ be a nonempty set. Given $n \in \mathbb{N}$, $\alpha \in \mathbb{R}$, $\varepsilon > 0$ and $\mathcal{F}$, define

$$M(n,\alpha,\varepsilon,Z,T,\mathcal{F}) = \inf\{\sum_i e^{-\alpha n_i + f_{n_i}(x_i)} : Z \subset \cup_i B_{n_i}(x_i,\varepsilon)\}, \tag{2.1}$$

where the infimum is taken over all finite or countable collections of $\{B_{n_i}(x_i,\varepsilon)\}_i$ such that $x_i \in X$, $n_i \ge n$ and $\bigcup_i B_{n_i}(x_i,\varepsilon) \supset Z$. Likewise, we define

$$R(n,\alpha,\varepsilon,Z,T,\mathcal{F}) = \inf\{\sum_i e^{-\alpha n + f_n(x_i)} : Z \subset \cup_i B_n(x_i,\varepsilon)\}, \tag{2.2}$$

where the infimum is taken over all finite or countable collections of $\{B_n(x_i,\varepsilon)\}_i$ such that $x_i \in X$ and $\bigcup_i B_n(x_i,\varepsilon) \supset Z$.

Define

$$M^P(n,\alpha,\varepsilon,Z,T,\mathcal{F}) = \sup\{\sum_i e^{-\alpha n_i + f_{n_i}(x_i)}\}, \tag{2.3}$$

where the supremum is taken over all finite or countable pairwise disjoint families $\{\overline{B}_{n_i}(x_i,\varepsilon)\}$ such that $x_i \in Z$, $n_i \ge n$ for all i, where $\overline{B}_{n_i}(x_i,\varepsilon) = \{y \in X : d_{n_i}(x,y) \le \varepsilon\}$.

Let

$$\begin{aligned} M(\alpha,\varepsilon,Z,T,\mathcal{F}) &= \lim_{n\to\infty} M(n,\alpha,\varepsilon,Z,T,\mathcal{F}), \\ \underline{R}(\alpha,\varepsilon,Z,T,\mathcal{F}) &= \liminf_{n\to\infty} R(n,\alpha,\varepsilon,Z,T,\mathcal{F}), \\ \overline{R}(\alpha,\varepsilon,Z,T,\mathcal{F}) &= \limsup_{n\to\infty} R(n,\alpha,\varepsilon,Z,T,f), \end{aligned}$$

$$M^P(\alpha,\varepsilon,Z,T,\mathcal{F})=\lim_{n\to\infty}M^P(n,\alpha,\varepsilon,Z,T,\mathcal{F}).$$

Define

$$M^{\mathcal{P}}(\alpha,\varepsilon,Z,T,\mathcal{F})=\inf\{\sum_{i=1}^{\infty}M^P(\alpha,\varepsilon,Z_i,T,\mathcal{F}):Z\subset\cup_{i=1}^{\infty}Z_i\}.$$

It is routine to check that when α goes from $-\infty$ to $+\infty$, the quantities

$$M(\alpha,\varepsilon,Z,T,\mathcal{F}),\ \underline{R}(\alpha,\varepsilon,Z,T,\mathcal{F}),\overline{R}(\alpha,\varepsilon,Z,T,\mathcal{F}),\ M^{\mathcal{P}}(\alpha,\varepsilon,Z,T,\mathcal{F})$$

jump from $+\infty$ to 0 at unique critical values respectively. Hence we can define the numbers

$$\begin{aligned}P^B(\varepsilon,Z,T,\mathcal{F})&=\sup\{\alpha:M(\alpha,\varepsilon,Z,T,\mathcal{F})=+\infty\}\\&=\inf\{\alpha:M(\alpha,\varepsilon,Z,T,\mathcal{F})=0\},\\ \underline{CP}(\varepsilon,Z,T,\mathcal{F})&=\sup\{\alpha:\underline{R}(\alpha,\varepsilon,Z,T,\mathcal{F})=+\infty\}\\&=\inf\{\alpha:\underline{R}(\alpha,\varepsilon,Z,T,\mathcal{F})=0\},\\ \overline{CP}(\varepsilon,Z,T,\mathcal{F})&=\sup\{\alpha:\overline{R}(\alpha,\varepsilon,Z,T,\mathcal{F})=+\infty\}\\&=\inf\{\alpha:\overline{R}(\alpha,\varepsilon,Z,T,\mathcal{F})=0\},\\ P^P(\varepsilon,Z,T,\mathcal{F})&=\sup\{\alpha:M^{\mathcal{P}}(\alpha,\varepsilon,Z,T,\mathcal{F})=+\infty\}\\&=\inf\{\alpha:M^{\mathcal{P}}(\alpha,\varepsilon,Z,T,\mathcal{F})=0\}.\end{aligned}$$

Definition 2.1. We call the following quantities

$$\begin{aligned}P^B(Z,T,\mathcal{F})&=\lim_{\varepsilon\to0}P^B(\varepsilon,Z,T,\mathcal{F}),\\ \underline{CP}(Z,T,\mathcal{F})&=\lim_{\varepsilon\to0}\underline{CP}(\varepsilon,Z,T,\mathcal{F}),\\ \overline{CP}(Z,T,\mathcal{F})&=\lim_{\varepsilon\to0}\overline{CP}(\varepsilon,Z,T,\mathcal{F}),\\ P^P(Z,T,\mathcal{F})&=\lim_{\varepsilon\to0}P^P(\varepsilon,Z,T,\mathcal{F}).\end{aligned}$$

Pesin-Pitskel, *lower capacity*, *upper capacity* and *packing topological pressures of T on the set Z with respect to $\mathcal{F}$.*

Remark 2.1. The definitions of Pesin-Pitskel, lower capacity and upper capacity topological pressures follow the generalized Carathéodory construction described in [12]. For more details, see [12, p74]. Wang etc in [16] introduced packing topological pressure.

Replacing $f_{n_i}(x_i)$ in Eqs. (2.1), (2.3) by $f_{n_i}(x_i,\varepsilon)$ and $\overline{f}_{n_i}(x_i,\varepsilon)$ respectively and $f_n(x_i)$ in Eq. (2.2) by $f_n(x_i,\varepsilon)$, we can define new functions $\mathcal{M}$, $\mathcal{R}$, $\mathcal{M}^P$. For any set $Z\subset X$ and $\varepsilon>0$, we denote the respective critical values by

$$P^{B'}(\varepsilon,Z,T,\mathcal{F}),\ \underline{CP}'(\varepsilon,Z,T,\mathcal{F}),\ \overline{CP}'(\varepsilon,Z,T,\mathcal{F}),\ P^{P'}(\varepsilon,Z,T,\mathcal{F}).$$

Proposition 2.1. *Let (X,T) be a TDS, $\mathcal{F}$ satisfy tempered distortion and $Z\subset X$. Then*

$$P^B(Z,T,\mathcal{F})=\lim_{\varepsilon\to0}P^{B'}(\varepsilon,Z,T,\mathcal{F}),$$

$$\underline{CP}(Z,T,\mathcal{F}) = \lim_{\varepsilon\to 0} \underline{CP}'(\varepsilon,Z,T,\mathcal{F}),$$
$$\overline{CP}(Z,T,\mathcal{F}) = \lim_{\varepsilon\to 0} \overline{CP}'(\varepsilon,Z,T,\mathcal{F}),$$
$$P^P(Z,T,\mathcal{F}) = \lim_{\varepsilon\to 0} P^{P'}(\varepsilon,Z,T,\mathcal{F}).$$

Proof. Fix $\varepsilon > 0$. It is clear that $P^B(\varepsilon,Z,T,\mathcal{F}) \le P^{B'}(\varepsilon,Z,T,\mathcal{F})$. Since $\mathcal{F}$ satisfies tempered distortion, then for every $\lambda > 0$, there exists $\varepsilon_0 > 0$, for any $0 < \varepsilon < \varepsilon_0$, there exists $N(\varepsilon)$, if $n \ge N(\varepsilon)$, then $f_n(\varepsilon) < \lambda n$. Hence we have

$$f_n(x,\varepsilon) \le f_n(x) + f_n(\varepsilon) \le f_n(x) + \lambda n, \forall n \ge N(\varepsilon).$$

It then follows that for $n \ge N(\varepsilon)$,

$$\begin{aligned}
M(n,\alpha,\varepsilon,Z,T,\mathcal{F}) &= \inf\{\sum_i e^{-\alpha n_i + f_{n_i}(x_i)} : Z \subset \cup_i B_{n_i}(x_i,\varepsilon)\}\\
&\ge \inf\{\sum_i e^{-\alpha n_i + f_{n_i}(x_i,\varepsilon) - n_i\lambda} : Z \subset \cup_i B_{n_i}(x_i,\varepsilon)\}\\
&= \inf\{\sum_i e^{-(\alpha+\lambda) n_i + f_{n_i}(x_i,\varepsilon)} : Z \subset \cup_i B_{n_i}(x_i,\varepsilon)\}\\
&= \mathcal{M}(n,\alpha+\lambda,\varepsilon,Z,T,\mathcal{F}).
\end{aligned}$$

Letting $n \to \infty$ yields

$$M(\alpha,\varepsilon,Z,T,\mathcal{F}) \ge \mathcal{M}(\alpha+\lambda,\varepsilon,Z,T,\mathcal{F}).$$

This implies that

$$P^B(\varepsilon,Z,T,\mathcal{F}) \ge P^{B'}(\varepsilon,Z,T,\mathcal{F}) - \lambda.$$

It then follows that

$$P^B(\varepsilon,Z,T,\mathcal{F}) \le P^{B'}(\varepsilon,Z,T,\mathcal{F}) \le P^B(\varepsilon,Z,T,\mathcal{F}) + \lambda.$$

Let $\varepsilon \to 0$, the arbitrariness of λ implies the desired equality. The other equalities can be proven similarly. □

The following are some basic properties of these pressures.

Proposition 2.2. *Let (X,T) be a TDS and $\mathcal{F}$ satisfy tempered distortion. Then the following assertions hold:*

1. *For any $Z \subset X$, $P^B(Z,T,\mathcal{F}) \le \underline{CP}(Z,T,\mathcal{F}) \le \overline{CP}(Z,T,\mathcal{F})$.*
2. *If $Z_1 \subset Z_2$, then $\mathscr{P}(Z_1,T,\mathcal{F}) \le \mathscr{P}(Z_2,T,\mathcal{F})$, where $\mathscr{P} \in \{P^B, \underline{CP}, \overline{CP}, P^P\}$.*
3. *If $Z = \cup_{i\in I} Z_i$ is a union of sets $Z_i \subset X$, with I at most countable, then*
   - *3-a. $M(\alpha,\varepsilon,Z,T,\mathcal{F}) \le \sum_i M(\alpha,\varepsilon,Z_i,T,\mathcal{F})$;*
   - *3-b. $M^{\mathcal{P}}(\alpha,\varepsilon,Z,T,\mathcal{F}) \le \sum_i M^{\mathcal{P}}(\alpha,\varepsilon,Z_i,T,\mathcal{F})$;*
   - *3-c. $P^B(Z,T,\mathcal{F}) = \sup_{i\in I} P^B(Z_i,T,\mathcal{F})$;*
   - *3-d. $P^P(Z,T,\mathcal{F}) = \sup_{i\in I} P^P(Z_i,T,\mathcal{F})$;*
   - *3-e. $\underline{CP}(Z,T,\mathcal{F}) \ge \sup_{i\in I} \underline{CP}(Z_i,T,\mathcal{F})$;*
   - *3-f. $\overline{CP}(Z,T,\mathcal{F}) \ge \sup_{i\in I} \overline{CP}(Z_i,T,\mathcal{F})$.*
4. *For any $Z \subset X$, $P^B(Z,T,\mathcal{F}) \le P^P(Z,T,\mathcal{F}) \le \overline{CP}(Z,T,\mathcal{F})$.*
5. *If Z is T-invariant and compact, then*

$$P^B(Z,T,\mathcal{F}) = P^P(Z,T,\mathcal{F}) = \underline{CP}(Z,T,\mathcal{F}) = \overline{CP}(Z,T,\mathcal{F}).$$

The proof of Proposition 2.2 can follow the proof in [18] for additive potentials.

3. Measure-theoretic pressures

In this section, we discuss the relations among various types of measure-theoretic topological pressure.

Let (X,T) be a TDS, $f \in C(X,\mathbb{R})$ and $\mu \in \mathcal{M}(X)$. The *measure-theoretic lower and upper local pressures of $x \in X$ with respect to μ and f* are defined by

$$\underline{P}_\mu(x,T,f) := \lim_{\varepsilon\to 0} \liminf_{n\to\infty} \frac{-\log \mu(B_n(x,\varepsilon)) + f_n(x)}{n},$$
$$\overline{P}_\mu(x,T,f) := \lim_{\varepsilon\to 0} \limsup_{n\to\infty} \frac{-\log \mu(B_n(x,\varepsilon)) + f_n(x)}{n}.$$

Definition 3.1. The *measure-theoretic lower and upper local pressures of μ with respect to f* are defined as

$$\underline{P}_\mu(T,f) := \int \underline{P}_\mu(x,T,f)\,\mathrm{d}\,\mu(x),$$
$$\overline{P}_\mu(T,f) := \int \overline{P}_\mu(x,T,f)\,\mathrm{d}\,\mu(x).$$

Definition 3.2. We call the following quantities

$$\begin{aligned} P_\mu^B(T,\mathcal{F}) &:= \lim_{\varepsilon\to 0}\lim_{\delta\to 0} \inf\{P^B(\varepsilon,Z,T,\mathcal{F}) : \mu(Z)\ge 1-\delta\} \\ &= \lim_{\varepsilon\to 0}\lim_{\delta\to 0} \inf\{P^{B'}(\varepsilon,Z,T,\mathcal{F}) : \mu(Z)\ge 1-\delta\}, \\ \underline{CP}_\mu(T,\mathcal{F}) &:= \lim_{\varepsilon\to 0}\lim_{\delta\to 0} \inf\{\underline{CP}(\varepsilon,Z,T,\mathcal{F}) : \mu(Z)\ge 1-\delta\} \\ &= \lim_{\varepsilon\to 0}\lim_{\delta\to 0} \inf\{\underline{CP}'(\varepsilon,Z,T,\mathcal{F}) : \mu(Z)\ge 1-\delta\}, \\ \overline{CP}_\mu(T,\mathcal{F}) &:= \lim_{\varepsilon\to 0}\lim_{\delta\to 0} \inf\{\overline{CP}(\varepsilon,Z,T,\mathcal{F}) : \mu(Z)\ge 1-\delta\} \\ &= \lim_{\varepsilon\to 0}\lim_{\delta\to 0} \inf\{\overline{CP}'(\varepsilon,Z,T,\mathcal{F}) : \mu(Z)\ge 1-\delta\}, \\ P_\mu^P(T,\mathcal{F}) &:= \lim_{\varepsilon\to 0}\lim_{\delta\to 0} \inf\{P^P(\varepsilon,Z,T,\mathcal{F}) : \mu(Z)\ge 1-\delta\} \\ &= \lim_{\varepsilon\to 0}\lim_{\delta\to 0} \inf\{P^{P'}(\varepsilon,Z,T,\mathcal{F}) : \mu(Z)\ge 1-\delta\}. \end{aligned}$$

Pesin-Pitskel, lower capacity, upper capacity and packing pressures of μ with respect to non-additive potential $\mathcal{F}$ with tempered distortion.

Katok in [9] introduced a type of measure-theoretic entropy. Recently, Wang in [17] studied the dimension types of this entropy. Following along the line of topological pressures in Section 2, we shall introduce four dimension types of measure-theoretic pressure in the sense of Katok.

Let $Z \subset X$ be a nonempty set. Given $\mu \in \mathcal{M}(X)$, $n \in \mathbb{N}$, $\alpha \in \mathbb{R}$, $\varepsilon > 0$, $0 < \delta < 1$ and $\mathcal{F}$ satisfy tempered distortion, define

$$M_\mu(n,\alpha,\varepsilon,\delta,T,\mathcal{F}) = \inf\{\sum_i e^{-\alpha n_i + f_{n_i}(x_i)} : \mu\big(\cup_i B_{n_i}(x_i,\varepsilon)\big) \ge 1-\delta\}, \tag{3.1}$$

where the infimum is taken over all finite or countable collections of $\{B_{n_i}(x_i,\varepsilon)\}_i$ such that $x_i \in X$, $n_i \ge n$ and $\mu\big(\cup_i B_{n_i}(x_i,\varepsilon)\big) \ge 1-\delta$. Likewise, we define

$$R_\mu(n,\alpha,\varepsilon,\delta,T,\mathcal{F}) = \inf\{\sum_i e^{-\alpha n + f_n(x_i)} : \mu\big(\cup_i B_n(x_i,\varepsilon)\big) \ge 1-\delta\}, \tag{3.2}$$

where the infimum is taken over all finite or countable collections of $\{B_n(x_i,\varepsilon)\}_i$ such that $x_i \in X$ and $\mu\big(\cup_i B_n(x_i,\varepsilon)\big) \geq 1-\delta$.

Let

$$\begin{aligned} M_\mu(\alpha,\varepsilon,\delta,T,\mathcal{F}) &= \lim_{n\to\infty} M_\mu(n,\alpha,\varepsilon,\delta,T,\mathcal{F}),\\ \underline{M}_\mu(\alpha,\varepsilon,\delta,T,\mathcal{F}) &= \liminf_{n\to\infty} R_\mu(n,\alpha,\varepsilon,\delta,T,\mathcal{F}),\\ \overline{M}_\mu(\alpha,\varepsilon,\delta,T,\mathcal{F}) &= \limsup_{n\to\infty} R_\mu(n,\alpha,\varepsilon,\delta,T,\mathcal{F}). \end{aligned}$$

Define

$$M_\mu^{\mathcal{P}}(\alpha,\varepsilon,\delta,T,\mathcal{F}) = \inf\{\sum_{i=1}^{\infty} M^P(\alpha,\varepsilon,Z_i,T,\mathcal{F}) : \mu\big(\cup_{i=1}^{\infty} Z_i\big) \geq 1-\delta\}.$$

Thus, when α goes from $-\infty$ to $+\infty$, the quantities

$$M_\mu(\alpha,\varepsilon,\delta,T,\mathcal{F}),\ \underline{M}_\mu(\alpha,\varepsilon,\delta,T,\mathcal{F}), \overline{M}_\mu(\alpha,\varepsilon,\delta,T,\mathcal{F}), \text{and } M_\mu^{\mathcal{P}}(\alpha,\varepsilon,\delta,T,\mathcal{F})$$

jump from $+\infty$ to 0 at unique critical values respectively. Hence we can define the numbers

$$\begin{aligned} P_\mu^{KB}(\varepsilon,\delta,T,\mathcal{F}) &= \sup\{\alpha : M_\mu(\alpha,\varepsilon,\delta,T,\mathcal{F}) = +\infty\}\\ &= \inf\{\alpha : M_\mu(\alpha,\varepsilon,\delta,T,\mathcal{F}) = 0\},\\ \underline{CP}_\mu^K(\varepsilon,\delta,T,f) &= \sup\{\alpha : \underline{M}_\mu(\alpha,\varepsilon,\delta,T,\mathcal{F}) = +\infty\}\\ &= \inf\{\alpha : \underline{M}_\mu(\alpha,\varepsilon,\delta,T,\mathcal{F}) = 0\},\\ \overline{CP}_\mu^K(\varepsilon,\delta,T,\mathcal{F}) &= \sup\{\alpha : \overline{M}_\mu(\alpha,\varepsilon,\delta,T,\mathcal{F}) = +\infty\}\\ &= \inf\{\alpha : \overline{M}_\mu(\alpha,\varepsilon,\delta,T,\mathcal{F}) = 0\},\\ P_\mu^{KP}(\varepsilon,\delta,T,\mathcal{F}) &= \sup\{\alpha : M_\mu^{\mathcal{P}}(\alpha,\varepsilon,\delta,T,\mathcal{F}) = +\infty\}\\ &= \inf\{\alpha : M_\mu^{\mathcal{P}}(\alpha,\varepsilon,\delta,T,\mathcal{F}) = 0\}. \end{aligned}$$

Definition 3.3. We call the following quantities

$$\begin{aligned} P_\mu^{KB}(T,\mathcal{F}) &= \lim_{\varepsilon\to 0}\lim_{\delta\to 0} P_\mu^{KB}(\varepsilon,\delta,T,f),\\ \underline{CP}_\mu^K(T,\mathcal{F}) &= \lim_{\varepsilon\to 0}\lim_{\delta\to 0} \underline{CP}_\mu^K(\varepsilon,\delta,T,\mathcal{F}),\\ \overline{CP}_\mu^K(T,\mathcal{F}) &= \lim_{\varepsilon\to 0}\lim_{\delta\to 0} \overline{CP}_\mu^K(\varepsilon,\delta,T,\mathcal{F}),\\ P_\mu^{KP}(T,\mathcal{F}) &= \lim_{\varepsilon\to 0}\lim_{\delta\to 0} P_\mu^{KP}(\varepsilon,\delta,T,\mathcal{F}). \end{aligned}$$

Pesin-Pitskel, lower capacity, upper capacity and packing pressures of μ in the sense of Katok with respect to non-additive potential $\mathcal{F}$ with tempered distortion, respectively.

Remark 3.1. If we replace $f_{n_i}(x_i)$ in Eqs. (3.1), (2.3) by $f_{n_i}(x_i,\varepsilon)$ and $\overline{f}_{n_i}(x_i,\varepsilon)$ respectively and $f_n(x_i,\varepsilon)$ by $f_n(x_i)$ in Eq. (3.2), we can define new functions $\mathcal{M}_\mu$, $\mathcal{R}_\mu$, $\mathcal{M}_\mu^{\mathcal{P}}$. For any $\varepsilon > 0$ and $0 < \delta < 1$, we denote the respective critical values by

$$P_\mu^{KB'}(\varepsilon,\delta,T,\mathcal{F}),\ \underline{CP}_\mu^{K'}(\varepsilon,\delta,T,\mathcal{F}),\ \overline{CP}_\mu^{K'}(\varepsilon,\delta,T,\mathcal{F}),\ P_\mu^{KP'}(\varepsilon,\delta,T,\mathcal{F}).$$

Proposition 3.1. *Let (X,T) be a TDS, $\mu \in \mathcal{M}(X)$ and $\mathcal{F}$ satisfy tempered distortion. Then*

$$\begin{aligned}
P_\mu^{KB}(T,\mathcal{F}) &= \lim_{\varepsilon\to 0}\lim_{\delta\to 0} P_\mu^{KB'}(\varepsilon,\delta,T,\mathcal{F}),\\
\underline{CP}_\mu^{K}(T,\mathcal{F}) &= \lim_{\varepsilon\to 0}\lim_{\delta\to 0} \underline{CP}_\mu^{K'}(\varepsilon,\delta,T,\mathcal{F}),\\
\overline{CP}_\mu^{K}(T,\mathcal{F}) &= \lim_{\varepsilon\to 0}\lim_{\delta\to 0} \overline{CP}_\mu^{K'}(\varepsilon,\delta,T,\mathcal{F}),\\
P_\mu^{KP}(T,\mathcal{F}) &= \lim_{\varepsilon\to 0}\lim_{\delta\to 0} P_\mu^{KP'}(\varepsilon,\delta,T,\mathcal{F}).
\end{aligned}$$

Proof. The proof is analogous to that of Proposition 2.1, so we omit it. □

Proposition 3.2. *Let (X,T) be a TDS, $\mu \in \mathcal{M}(X)$ and $\mathcal{F}$ satisfy tempered distortion. Then*

$$\begin{aligned}
&P_\mu^{KB}(T,\mathcal{F}) = P_\mu^{B}(T,\mathcal{F}),\ \underline{CP}_\mu^{K}(T,\mathcal{F}) = \underline{CP}_\mu(T,\mathcal{F}),\\
&\overline{CP}_\mu^{K}(T,\mathcal{F}) \le \overline{CP}_\mu(T,\mathcal{F}),\ P_\mu^{KP}(T,\mathcal{F}) = P_\mu^{P}(T,\mathcal{F}).
\end{aligned}$$

The proof is analogous to the proof in [18] for additive potentials, so we omit it.

4. Proofs of Theorem 1.4

First, we prove the following proposition. Then Theorem 1.4 can be the corollary of this proposition.

Proposition 4.1. *Let (X,T) be a TDS, $\mathcal{F}$ satisfy tempered distortion and $\mu \in \mathcal{M}(X)$, then*

$$\begin{aligned}
P_\mu^{B}(T,\mathcal{F}) &= P_\mu^{KB}(T,\mathcal{F})\\
&= \lim_{\epsilon\to 0}\inf\{P^B(\epsilon,Z,T,\mathcal{F}) : \mu(Z)=1\}\\
&= \inf\{P^B(Z,T,\mathcal{F}) : \mu(Z)=1\},
\end{aligned}$$

$$\begin{aligned}
P_\mu^{P}(T,\mathcal{F}) &= P_\mu^{KP}(T,\mathcal{F})\\
&= \lim_{\epsilon\to 0}\inf\{P^P(\epsilon,Z,T,\mathcal{F}) : \mu(Z)=1\}\\
&= \inf\{P^P(Z,T,\mathcal{F}) : \mu(Z)=1\}.
\end{aligned}$$

Proof. First, we prove that

$$P_\mu^{B}(T,\mathcal{F}) = \lim_{\epsilon\to 0}\inf\{P^B(\epsilon,Z,T,\mathcal{F}) : \mu(Z)=1\}.$$

Denote $\lim\limits_{\epsilon\to 0}\lim\limits_{\delta\to 0}\inf\{P^B(\epsilon,Z,T,\mathcal{F}) : \mu(Z)>1-\delta\}$ by C. From the definition

$$P_\mu^{B}(T,\mathcal{F}) = C,$$

we know that for any $\alpha>0$, there exists $\epsilon_0>0$ such that for any $0<\epsilon<\epsilon_0$, it has

$$\lim_{\delta\to 0}\inf\{P^B(\epsilon,Z,T,\mathcal{F}) : \mu(Z)>1-\delta\} < C+\frac{\alpha}{3}.$$

Hence, for any $0 < \epsilon < \epsilon_0$, there exists $\delta(\epsilon) > 0$, such that for any $0 < \delta < \delta(\epsilon)$, we have

$$\inf\{P^B(\epsilon, Z, T, \mathcal{F}) : \mu(Z) > 1 - \delta\} < C + \frac{2\alpha}{3}.$$

Therefore for every k large enough, there exists Z_k with $\mu(Z_k) > 1 - \frac{1}{k}$, such that

$$P^B(\epsilon, Z_k, T, \mathcal{F}) < C + \alpha.$$

Take $Z = \cup Z_k$. Then we have $\mu(Z) = 1$ and by Proposition 2.2,

$$P^B(\epsilon, Z, T, \mathcal{F}) = \sup_k P^B(\epsilon, Z_k, T, \mathcal{F}) \le C + \alpha.$$

Hence we have

$$\inf\{P^B(\epsilon, Z, T, \mathcal{F}) : \mu(Z) = 1\} \le C + \alpha.$$

This implies that

$$\lim_{\epsilon \to 0} \inf\{P^B(\epsilon, Z, T, \mathcal{F}) : \mu(Z) = 1\} \le C = P^B_\mu(T, \mathcal{F}).$$

Furthermore, for any $\delta > 0$,

$$\inf\{P^B(\epsilon, Z, T, \mathcal{F}) : \mu(Z) = 1\} \ge \inf\{P^B(\epsilon, Z, T, \mathcal{F}) : \mu(Z) > 1 - \delta\}.$$

Then

$$\inf\{P^B(\epsilon, Z, T, \mathcal{F}) : \mu(Z) = 1\} \ge \lim_{\delta \to 0} \inf\{P^B(\epsilon, Z, T, \mathcal{F}) : \mu(Z) > 1 - \delta\}.$$

Hence

$$\begin{aligned} &\lim_{\epsilon \to 0} \inf\{P^B(\epsilon, Z, T, \mathcal{F}) : \mu(Z) = 1\} \\ \ge &\lim_{\epsilon \to 0} \lim_{\delta \to 0} \inf\{P^B(\epsilon, Z, T, \mathcal{F}) : \mu(Z) > 1 - \delta\} \\ = &C. \end{aligned}$$

Therefore

$$P^B_\mu(T, \mathcal{F}) = \lim_{\epsilon \to 0} \inf\{P^B(\epsilon, Z, T, \mathcal{F}) : \mu(Z) = 1\} = C.$$

Next, we prove that

$$P^B_\mu(T, \mathcal{F}) = \inf\{P^B(Z, T, \mathcal{F}) : \mu(Z) = 1\}.$$

For any $\alpha > 0$, there exists $\epsilon_0 > 0$ such that for any $0 < \epsilon < \epsilon_0$,

$$\inf\{P^B(\epsilon, Z, T, \mathcal{F}) : \mu(Z) = 1\} < C + \frac{\alpha}{2}.$$

Hence, $\exists Z_\epsilon \subset X$ with $\mu(Z_\epsilon) = 1$ satisfies

$$P^B(\epsilon, Z_\epsilon, T, \mathcal{F}) < C + \alpha.$$

Take $\epsilon = \frac{1}{k}$ for k sufficiently large. Then $\exists Z_k \subset X$ with $\mu(Z_k) = 1$ such that

$$P^B(\frac{1}{k}, Z_k, T, \mathcal{F}) < C + \alpha.$$

Let $Z = \cap_k Z_k$. Then we have $\mu(Z) = 1$ and

$$P^B(\frac{1}{k}, Z, T, \mathcal{F}) \leq P^B(\frac{1}{k}, Z_k, T, \mathcal{F}) < C + \alpha.$$

Therefore

$$\inf\{P^B(Z, T, \mathcal{F}) : \mu(Z) = 1\} \leq P^B(\frac{1}{k}, Z, T, \mathcal{F}) \leq C + \alpha.$$

The arbitrariness of α implies that

$$\inf\{P^B(Z, T, \mathcal{F}) : \mu(Z) = 1\} \leq P^B(\frac{1}{k}, Z, T, \mathcal{F}) \leq C.$$

On the other hand, for every $Z \subset X$ with $\mu(Z) = 1$, we have

$$P^B_\mu(T, \mathcal{F}) = P^{KB}_\mu(T, \mathcal{F}) \leq P^B(Z, T, \mathcal{F}).$$

It implies

$$P^B_\mu(T, \mathcal{F}) = P^{KB}_\mu(T, \mathcal{F}) = \inf\{P^B(Z, T, \mathcal{F}) : \mu(Z) = 1\}.$$

Analogously, we can prove the following result.

$$\begin{aligned} P^P_\mu(T, \mathcal{F}) &= P^{KP}_\mu(T, \mathcal{F}) \\ &= \lim_{\epsilon \to 0} \inf\{P^P(\epsilon, Z, T, \mathcal{F}) : \mu(Z) = 1\} \\ &= \inf\{P^P(Z, T, \mathcal{F}) : \mu(Z) = 1\}. \end{aligned}$$

These complete the proof of Proposition 4.1. □

5. Application to dimension estimate of measures supported on average conformal repellers.

In this section, we first introduce the definition of average conformal repellers. Let M be an d-dimensional C^∞ Riemannian manifold with Riemannian measure ν, T be a C^1 map from M to itself. Let $\Lambda \subset M$ be an T-invariant compact subset, where T invariance means $T\Lambda = \Lambda$. Assume that T is expanding on Λ, i.e., there exist $C > 0, \kappa > 1$, such that for any $x \in \Lambda, n \geq 1$,

$$\|DT^n(x)u\| \geq C\kappa^n \|u\|, \ \forall u \in T_x M.$$

Without loss generality we can take $C = 1$.

Let $\mathcal{E}(\Lambda, T)$ denote the all ergodic invariant measures supported on Λ. By the Oseledec multiplicative ergodic theorem, for any $\mu \in \mathcal{E}(\Lambda, T)$, we can define Lyapunov exponents $\lambda_1(\mu) \leq \lambda_2(\mu) \leq \cdots \leq \lambda_d(\mu)$. Let now μ be an T-invariant measure on Λ. By work of Brin etc in [5], for μ-almost every $x \in \Lambda$ there exists the limit

$$h_\mu(x) = \lim_{\epsilon \to 0} \lim_{n \to \infty} -\frac{1}{n} \log \mu(B_n(x, \epsilon)).$$

The number $h_\mu(x)$ is called the local entropy of μ at the point x.

Definition 5.1. An invariant repeller is called average conformal if for any $\mu \in \mathcal{E}(\Lambda, T)$, $\lambda_1(\mu) = \lambda_2(\mu) = \cdots = \lambda_d(\mu) > 0$.

It is obvious that a conformal repeller is an average conformal repeller, but reverse is not true.

Proposition 5.1. *If Λ is an average conformal repeller, then*

$$\lim_{n\to\infty} \frac{1}{n}(\log |DT^n(x)| - \log m(DT^n(x))) = 0$$

uniformly on Λ.

Proof. The proof can be found in [2]. □

In fact, we always have

$$m(DT^n(x)) \leq (|det(DT^n(x))|)^{\frac{1}{d}} \leq |DT^n(x)|.$$

Let $f(x) = \log(|det(DT(x))|)^{\frac{1}{d}}$ and $\mathcal{F} = \{f_n(x) = \log |DT^n(x)|\}$. If Λ is an average conformal repeller, then by proposition 5.1, we know that $\mathcal{F}$ is a sequence of subadditive continuous potentials with tempered distortion and

$$P^B(Z, T, -tf(x)) = P^B(Z, T, -t\mathcal{F}).$$

In [6], the author gave the dimension estimate for every set of an average conformal repeller.

Proposition 5.2. *Let M be a C^∞ Riemannian manifold and $T : M \to M$ be a C^1 map. Suppose $\Lambda \subset M$ is an average conformal repeller. Then for every set $Z \subset \Lambda$, $dim_H Z = t^*$, where t^* is the unique solution of equation $P^B(Z, T, -tf(x)) = 0$ in $[0, d]$.*

Next, we give the proof of Theorem 1.5.

The proof of Theorem 1.5. From the definition, we have $dim_H \mu = \inf\{dim_H Z : \mu(Z) = 1\}$. Theorem 1.4 tells us that

$$P^B_\mu(T, -tf(x)) = \inf\{P^B(Z, T, -tf(x)) : \mu(Z) = 1\}.$$

Suppose that t^* is the unique solution of equation $P^B_\mu(T, -tf(x)) = 0$ in $[0, d]$. Then for every set Z with $\mu(Z) = 1$, it has

$$P^B(Z, T, -t^* f(x)) \geq 0.$$

Thus $dim_H Z \geq t^*$, and hence

$$t^* \leq \inf\{dim_H Z : \mu(Z) = 1\}.$$

This implies that $dim_H \mu \geq t^*$.

On the other hand, for every $t > t^*$, we have $P^B_\mu(T, -tf(x)) < 0$. Then there exists subset Z with $\mu(Z) = 1$ such that

$$P^B(Z, T, -tf(x)) < 0.$$

It means $dim_H Z \leq t$, and hence

$$\inf\{dim_H Z : \mu(Z) = 1\} \leq t.$$

Therefore $\inf\{dim_H Z : \mu(Z) = 1\} \le t^*$. Thus

$$t^* \ge dim_H \mu.$$

So we obtain

$$dim_H \mu = t^*.$$

This completes the proof of Theorem 1.5.

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