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ASYMPTOTIC BEHAVIORS OF A HEROIN EPIDEMIC MODEL WITH NONLINEAR INCIDENCE RATE INFLUENCED BY STOCHASTIC PERTURBATIONS*

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Abstract In this paper, the dynamical behaviors of a stochastic heroin epidemic model with Lévy noises is investigated. First, we prove that this system has a unique global positive solution. Second, we derive the conditions of persistence in the mean and asymptotic stability in mean square using the Lyapunov and inequalities technique, and establish a criterion for positive recurrence. The results show that asymptotic behaviors are closely related to the Lévy measure. Finally, numerical simulations are used to illustrate the theoretical results.

Keywords Heroin epidemic model, persistence in the mean, positive recurrence, Lévy noises.

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1. Introduction

Heroin is an illegal, highly addictive drug derived from morphine, a naturally occurring substance extracted from the seed pods of certain varieties of poppy plants in [1]. Heroin is both the most abused and the most rapidly acting of the opiates, and the abuse of heroin has already become an increasingly global social problem. As of the end of 2019, approximately 270 million people worldide took drugs every year, and nearly 600,000 people died from drug abuse in [2]. It is easy to see that the prevalence of drug abuse is a problem cannot be neglected and needs to be solved urgently.

Over the past several decades, many mathematical models have been built to assist policy makers in preventing drug abuse availability and treatment resources. For example, in [19], Macktintosh and Stewart presented an exponential model illustrating how the use of heroin spread in an epidemic manner. Subsequently,

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White and Comiskey in [30] proposed an ODE model for heroin addiction as follows:

$$\begin{cases} \dfrac{dS(t)}{dt} = \Lambda - \dfrac{\beta_1 U_1(t)S(t)}{N(t)} - \mu S(t), \\ \dfrac{dU_1(t)}{dt} = \dfrac{\beta_1 U_1(t)S(t)}{N(t)} + \dfrac{\beta_3 U_1(t)U_2(t)}{N(t)} - (p+\mu+\delta_1)U_1(t), \\ \dfrac{dU_2(t)}{dt} = pU_1(t) - \dfrac{\beta_3 U_1(t)U_2(t)}{N(t)} - (\mu+\delta_2)U_2(t). \end{cases} \tag{1.1}$$

Muroya et al. in [24] extended their heroin model by introducing a linear term $\sigma U_2(t)$ to replace the nonlinear term $\frac{\beta_3 U_1(t)U_2(t)}{N(t)}$, indicating that the global dynamics of the extended heroin model depend on the basic reproduction number. However, in reality, people who become drug addicts are not intricately tied by one drug use, but by at least two or more repeated uses. Based on this fact, Ma et al. in [18] further considered the nonlinear incidence rate $\beta S(t)U_1^2(t)$ and obtained the following model

$$\begin{cases} \dfrac{dS(t)}{dt} = b - \beta S(t)U_1^2(t) - dS(t) + \delta U_2(t), \\ \dfrac{dU_1(t)}{dt} = \beta S(t)U_1^2(t) - \alpha U_1(t) + \sigma U_2(t) - dU_1(t), \\ \dfrac{dU_2(t)}{dt} = \alpha U_1(t) - \sigma U_2(t) - (d+\delta)U_2(t), \end{cases} \tag{1.2}$$

where $S(t)$, $U_1(t)$, $U_2(t)$ represent the number of susceptible individuals, drug users not undergoing treatment, and drug users undergoing treatment at time t, respectively. b denotes the recruitment rate of individuals in the general population entering the susceptible population, and d is the natural death rate of the population. $\beta U_1^2(t)$ denotes the infection force, α is the proportion of drug users who enter treatment, σ is the probability of outflow of individuals from drug users in treatment, and δ denotes the probability of drug users becoming the susceptible population again through treatment. For this model, the authors performed a detailed mathematical analysis and investigated a wide range of dynamical behaviors from saddle-node bifurcation to Hopf bifurcation and Bogdanov-Takens bifurcation of codimension 2. In addition, there is a non-exhaustive list of papers on the epidemic dynamics of deterministic heroin models (see e.g. [7,23,25,26] references therein).

To the best of our knowledge, drug abusers are inevitably affected by environmental noise, because physical health, education, vocational opportunities, and many other physical factors embedded in society are usually unpredictable. Hence, stochastic heroin models may be more realistic because the varying environmental effects cannot be neglected. Wei et al. in [29] proposed a stochastic heroin population model under non-degenerate conditions, while Liu et al. in [16] also studied the dynamics of a stochastic heroin population model. In [14], Liu and Wang established a stochastic non-autonomous heroin model and obtained sufficient conditions for extinction and permanence in mean. Subsequently, Jovanović and Vujović in [11] provided stability for the stochastic heroin model with two disdtributed delays. Wei et al. in [28] investigated a stochastic heroin population model with standard incidence rates between distinct patches. In addition, many authors have studied the effect of continuous environmental fluctuations on epidemic models (for example [5,6,9,10,12,15,17,21,27,32,35] references therein). However, stochastic heroin models with Brownian motion alone can not describe massive diseases such as HIV and SARS may break the continuity of solutions, such as [3,8,13,22,31,33,34,36] and the references therein. To explain these phenomena, introducing a discontinuous Lévy noise into the heroin models provides a feasible and realistic model.

Note that people who become drug addicts are intricately tied by at least two or more repeated uses, and the infection force may suffer sudden environmental perturbations, such as earthquakes, hurricanes, SARS, COVID-19, etc. Based on this fact, we introduce random perturbations into system (1.2) by replacing the coefficient β of the infection force with

$$\beta dt \to \beta dt + \gamma dB(t) + \int_Z H(z)\tilde{N}_q(dt, dz),$$

where $B(t)$ denotes Brownian motion, $\tilde{N}_q(dt, dz)$ denotes a compensated Poisson random measure corresponding to a Poisson measure $N_q(dt, dz)$ with characteristic measure $dt\,\nu(dz)$ on the product space $[0, \infty) \times Z$ and ν is a Lévy measure with $\nu(Z) < \infty$. Meanwhile, by taking the change in the variables,

$$S = \frac{b}{d}S^*, \quad U_1 = \frac{b}{d}U_1^*, \quad U_2 = \frac{b}{d}U_2^*, \quad \beta = \left(\frac{b}{d}\right)^2 \beta^*$$

and still denote $(S^*, U_1^*, U_2^*) = (S, U_1, U_2)$, $\beta^* = \beta$. Therefore, the heroin epidemic system (1.2) becomes the following stochastic differential equation driven by Lévy noises:

$$\begin{cases} dS(t) =(d - \beta S(t)U_1^2(t) - dS(t) + \delta U_2(t))dt \\ \qquad\qquad - \gamma S(t)U_1^2(t)dB(t) - \displaystyle\int_Z H(z)S(t-)U_1^2(t-)\tilde{N}_q(dt, dz), \\ dU_1(t) =(\beta S(t)U_1^2(t) - \alpha U_1(t) + \sigma U_2(t) - dU_1(t))dt \\ \qquad\qquad + \gamma S(t)U_1^2(t)dB(t) + \displaystyle\int_Z H(z)S(t-)U_1^2(t-)\tilde{N}_q(dt, dz), \\ dU_2(t) =(\alpha U_1(t) - \sigma U_2(t) - (d + \delta)U_2(t))dt, \end{cases} \tag{1.3}$$

where $x(t-) = \lim_{s\uparrow t} x(s)$, and further assume that $B(t)$ and $N_q(dt, dz)$ are independent.

The remainder of this paper is organized as follows. In Sec. 2, some necessary preliminaries are recalled, and the theorem concerning the existence-uniqueness positive solution of the stochastic heroin epidemic model with Lévy noises is established. In Sec. 3, the persistence in the mean and the asymptotic stability in mean square of system (1.3) are further verified. In Sec. 4, the sufficient condition for positive recurrence to system is established. In Sec. 5, the numerical simulations are performed to illustrate the presented results.

2. Some preliminaries and existence-uniqueness of the positive solution

This section recalls several basic concepts and notations that are required throughout the paper.

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space with the filtration $\{\mathcal{F}_t\}_{t\geq 0}$, which satisfies the usual conditions (it is right continuous, whereas $\mathcal{F}_0$ contains all P-null sets). Consider the following 3-dimensional stochastic differential equations with Lévy noises:

$$dX(t) = a(X(t))dt + b(X(t))dB(t) + \int_Z c(X(t-), z)\tilde{N}_q(dt, dz), \tag{2.1}$$

where $X(0) \in \mathbb{R}^3$. Assuming $h \in C^2(\mathbb{R}^3)$, the infinitesimal generator of processes $X(t)$ to (2.1) is

$$\begin{aligned}\mathcal{L}f(X) =&\sum_{i=1}^{3} a_i(X)\frac{\partial}{\partial X_i}f(X) + \frac{1}{2}\sum_{i,j=1}^{3}[b^T(X)b(X)]_{ij}\frac{\partial^2}{\partial X_i \partial X_j}h(X)\\ &+\int_Z \left[h(\widetilde{c}(X,z)) - h(X) - \sum_{i=1}^{3} c_i(X,z)\frac{\partial}{\partial X_i}h(X)\right]\nu(dz),\end{aligned}$$

where $\widetilde{c}(X, z) = X + c(X, z)$, T denotes transiposition. Meanwhile, define the subsets

$$\mathbb{R}^3_+ = \{X \in \mathbb{R}^3 : X_i > 0,\, i = 1, 2, 3\}, \quad \Lambda = \{X \in \mathbb{R}^3_+, X_1 + X_2 + X_3 = 1\},$$

and denote $X(t) = (S(t), U_1(t), U_2(t))$. In addition, $\langle h(t)\rangle$ denotes the mean value of the function $h(t)$ on $[0, \infty)$, that is, $\langle h(t)\rangle = \frac{1}{t}\int_0^t h(s)ds$.

Remark 2.1. Add all equations in system (1.3), we have

$$d(S(t) + U_1(t) + U_2(t)) = [d - d(S(t) + U_1(t) + U_2(t))]dt,$$

therefore, $\lim_{t\to\infty}(S(t) + U_1(t) + U_2(t)) = 1$. From [17], it is natural to assume that the population is constant and study the dynamical behavior of this system on Λ. Moreover, the solution of system (1.3) has the following properties

$$\limsup_{t\to\infty}\frac{\log S(t)}{t} \leq 0,\;\; \limsup_{t\to\infty}\frac{\log U_1(t)}{t} \leq 0 \text{ and } \limsup_{t\to\infty}\frac{\log U_2(t)}{t} \leq 0.$$

The following is about the existence-uniqueness of the global solution.

Theorem 2.1. *Assume that $(H_1) : |H(z)| \leq \omega < 1,$ for all $z \in Z$. Then there exists a unique positive solution $(S(t), U_1(t), U_2(t))$ of system (1.3) for any initial value $(S(0), U_1(0), U_2(0)) \in \Lambda$, $t \geq 0$ and the solution $(S(t), U_1(t), U_2(t)) \in \Lambda$ almost surely.*

Proof. Note that the coefficients of system (1.3) is locally Lipschitz continuous, and there exists a unique local solution $(S(t), U_1(t), U_2(t))$ for any initial value $(S(0), U_1(0), U_2(0)) \in \Lambda$ and $t \in [0, \tau_e]$, where τ_e is the explosion time (see [20]). To complete the proof, it is necessary to prove $\tau_\infty = \infty$, a.s. Namely, by [21], we only need to construct a nonnegative C^2-function V which satisfies $\mathcal{L}V \leq L_1$, where $L_1 > 0$ is a constant. Define a C^2-function $V : \Lambda \to \mathbb{R}_+ \cup \{\infty\}$ as follows:

$$V(S, U_1, U_2) = (S - 1 - \log S) + (U_1 - 1 - \log U_1) + (U_2 - 1 - \log U_2),$$

and it is easy to know that $V(S,U_1,U_2) \geq 0$ for any $(S,U_1,U_2) \in \Lambda$. Applying Itô's formula to (1.3), we obtain

$$\begin{aligned}
dV =& \mathcal{L}V dt - \left(1 - \frac{1}{S(t)}\right)\gamma S(t)U_1^2(t)dB(t) + \left(1 - \frac{1}{U_1(t)}\right)\gamma S(t)U_1^2(t)dB(t)\\
&+ \int_Z \Big[S(t-) - H(z)S(t-)U_1^2(t-) - 1 - \log(S(t-) - H(z)S(t-)U_1^2(t-))\\
&-(S(t-) - 1 - \log S(t-))\Big]\tilde{N}_q(dt,dz) + \int_Z \Big[U_1(t-) + H(z)S(t-)U_1^2(t-) - 1\\
&- \log(U_1(t-) + H(z)S(t-)U_1^2(t-)) - (U_1(t-) - 1 - \log U_1(t-))\Big]\tilde{N}_q(dt,dz)\\
=& \mathcal{L}V dt + U_1(t)(U_1(t) - \gamma S(t))dB(t)\\
&- \int_Z [\log(1 - H(z)U_1^2(t-)) + \log(1 + H(z)S(t-)U_1(t-))]\tilde{N}_q(dt,dz),
\end{aligned}$$

where $\mathcal{L}V:\ \Lambda \to \mathbb{R}$ is defined by

$$\begin{aligned}
&\mathcal{L}V\\
=&\left(1 - \frac{1}{S}\right)(d - \beta SU_1^2 - dS + \delta U_2) + \frac{1}{2}\gamma^2 U_1^4 + \int_Z \Big[S - H(z)SU_1^2 - 1\\
&- \log(S - H(z)SU_1^2) - (S - 1 - \log S) + H(z)SU_1^2\left(1 - \frac{1}{S}\right)\Big]\nu(dz)\\
&+ \left(1 - \frac{1}{U_1}\right)(\beta SU_1^2 + \sigma U_2 - (\alpha + d)U_1) + \frac{1}{2}\gamma^2 S^2 U_1^2 + \int_Z \Big[U_1 + H(z)SU_1^2\\
&-1 - \ln(U_1 + H(z)SU_1^2) - (U_1 - 1 - \log U_1) - H(z)SU_1^2\left(1 - \frac{1}{U_1}\right)\Big]\nu(dz)\\
:=& I_1 + I_2 + I_3,
\end{aligned}$$

in which

$$\begin{aligned}
I_1 =& \alpha + \sigma + \delta + 4d - d(S + U_1 + U_2) - \frac{d + \delta U_2}{S}\\
&- \frac{\sigma U_2}{U_1} - \frac{\alpha U_1}{U_2} + \frac{\gamma^2}{2}U_1^2(U_1^2 + S^2),\\
I_2 =& -\int_Z [H(z)U_1^2 + \log(1 - H(z)U_1^2)]\nu(dz),\\
I_3 =& -\int_Z [\log(1 + H(z)SU_1) - H(z)SU_1]\nu(dz).
\end{aligned}$$

Note that $(S,U_1,U_2) \in \Lambda$, thereby

$$\begin{aligned}
I_1 \leq& \alpha + \sigma + \delta + 4d - \frac{\sigma U_2}{U_1} - \frac{\alpha U_1}{U_2} + \beta + \gamma^2\\
\leq& \alpha + \sigma + \delta + 4d + \beta + \gamma^2 - 2\sqrt{\sigma\alpha}.
\end{aligned} \tag{2.2}$$

Then by (H_1), we have

$$1 - H(z)U_1^2 > 0, \text{ and } 1 + H(z)SU_1 > 0, \text{ for all } z \in Z.$$

In addition, Taylor formula and (H_1) imply that

$$\begin{aligned} I_2 \leq & -\int_Z \left[H(z)U_1^2 - H(z)U_1^2 - \frac{H^2(z)U_1^4}{2(1-\theta H(z)U_1^2)^2} \right] \nu(dz) \\ = & \int_Z \frac{H^2(z)U_1^4}{2(1-\theta H(z)U_1^2)^2} \nu(dz) \\ \leq & \frac{\omega^2}{2(1-\omega)^2} \nu(Z), \end{aligned} \tag{2.3}$$

where $\theta \in (0,1)$ is an arbitrary number.

Similarly, there exist an arbitrary number $\theta \in (0,1)$ such that

$$\begin{aligned} I_3 \leq & -\int_Z \left[H(z)SU_1 - H(z)SU_1 - \frac{H^2(z)S^2U_1^2}{2(1+\theta H(z)SU_1)^2} \right] \nu(dz) \\ = & \int_Z \frac{H^2(z)S^2U_1^2}{2(1+\theta H(z)SU_1)^2} \nu(dz) \\ \leq & \frac{\omega^2}{2(1-\omega)^2} \nu(Z). \end{aligned} \tag{2.4}$$

Therefore, according to (2.2)-(2.4), we get

$$\mathcal{L}V \leq \alpha + \sigma + \delta + 4d + \beta + \gamma^2 - 2\sqrt{\sigma\alpha} + \frac{\omega^2}{(1-\omega)^2} \nu(Z) := L_1.$$

Arguing similarly as in [21], we obtain the desired assertion. □

Remark 2.2. Theorem 2.1 ensures that the solution will remain in Λ, that is, the system has a unique positive solution and is biologically meaningful. This nice property provides us with a great opportunity to discuss how the solutions vary in Λ.

3. Persistence in the mean and asymptotic stability in mean square

In this section, we describe the dynamical behavior of system (1.3). Suppose that $L_2 = \int_Z H(z)\nu(dz) - \beta - \frac{\gamma^2}{2} > 0$, for simplicity, we define the following:

$$R_0^S = \frac{\alpha + d + \sigma + \delta}{(d + \sigma + \delta)L_2} \left[\delta - \log(1-\omega)\nu(Z) \right]^{\frac{1}{2}}.$$

Theorem 3.1. *Under condition (H_1), let $(S(t), U_1(t), U_2(t))$ be the solution of system (1.3) with initial value $(S(0), U_1(0), U_2(0)) \in \Lambda$. Suppose further that (H_2): $R_0^S > 1$, then system (1.3) is persistent in the mean, that is,*

$$\liminf_{t\to\infty} \frac{1}{t} \int_0^t S(s)ds > 0, \ \ a.s.$$

Proof. Under conditions (H_1) and $(S(0), U_1(0), U_2(0)) \in \Lambda$, Theorem 2.1 implies $(S(t), U_1(t), U_2(t)) \in \Lambda$. Combining $d(S(t))$ and $d(U_1(t))$ in system (1.3) and $U_2(t) = 1 - S(t) - U_1(t)$ yields

$$d(S(t) + U_1(t)) = [d + \sigma + \delta - (d + \sigma + \delta)S(t) - (\alpha + d + \sigma + \delta)U_1(t)]dt. \tag{3.1}$$

Integrating (3.1) from 0 to t, it is easy to verify that

$$\langle U_1(t)\rangle = \frac{d+\sigma+\delta}{\alpha+d+\sigma+\delta} - \frac{d+\sigma+\delta}{\alpha+d+\sigma+\delta}\langle S(t)\rangle + \varphi(t), \tag{3.2}$$

where

$$\varphi(t) = -\frac{1}{\alpha+d+\sigma+\delta}\left[\frac{S(t)-S(0)}{t} + \frac{U_1(t)-U_1(0)}{t}\right]$$

which satisfies

$$\lim_{t\to\infty}\varphi(t) = 0. \tag{3.3}$$

Applying Itô's formula to $\log S(t)$ and $U_2(t) = 1 - S(t) - U_1(t)$ gives

$$\begin{aligned} d(\log S(t)) =& \Big[\frac{d}{S(t)} + \frac{\delta}{S(t)}(1-U_1(t)) - \beta U_1^2(t) - (d+\delta) - \frac{\gamma^2}{2}U_1^4(t) \\ &+ \int_Z (\log(1-H(z)U_1^2(t)) + H(z)U_1^2(t))\nu(dz)\Big]dt - \gamma U_1^2(t)dB(t) \\ &+ \int_Z \log(1-H(z)U_1^2(t-))\tilde{N}_q(dt,dz). \end{aligned}$$

Then by (H_1) and the comparison theorem, we have

$$\begin{aligned} d(\log S(t)) \geq& \Big[-\delta U_1(t) - \beta U_1^2(t) - \frac{\gamma^2}{2}U_1^4(t) + \int_Z H(z)\nu(dz)U_1^2(t) \\ &+ \log(1-\omega)\nu(Z)\Big]dt - \gamma U_1^2(t)dB(t) \\ &+ \int_Z \log(1-H(z)U_1^2(t-))\tilde{N}_q(dt,dz). \end{aligned} \tag{3.4}$$

Integrating from 0 to t and dividing by t on both sides of (3.4) yields

$$\begin{aligned} &\frac{\log S(t) - \log S(0)}{t} \\ \geq& -\delta\langle U_1(t)\rangle - \beta\langle U_1^2(t)\rangle - \frac{\gamma^2}{2}\langle U_1(t)^4\rangle + \log(1-\omega)\nu(Z) \\ &+ \int_Z H(z)\nu(dz)\langle U_1^2(t)\rangle + \frac{M_1(t)}{t} + \frac{M_2(t)}{t} \\ \geq& -\delta + \log(1-\omega)\nu(Z) + L_2\langle U_1(t)\rangle^2 + \frac{M_1(t)}{t} + \frac{M_2(t)}{t}, \end{aligned} \tag{3.5}$$

where

$$M_1(t) = -\int_0^t \gamma U_1^2(s)dB(s), M_2(t) = \int_0^t\int_Z \log(1-H(z)U_1^2(s-))\tilde{N}_q(ds,dz).$$

Substituting (3.3) into (3.5), we obtain

$$\begin{aligned} \frac{\log S(t) - \log S(0)}{t} \geq& -\delta + \log(1-\omega)\nu(Z) + \frac{M_1(t)}{t} + \frac{M_2(t)}{t} \\ &+ L_2\left[\frac{d+\sigma+\delta}{\alpha+d+\sigma+\delta} - \frac{d+\sigma+\delta}{\alpha+d+\sigma+\delta}\langle S(t)\rangle + \varphi(t)\right]^2. \end{aligned} \tag{3.6}$$

According to Remark 2.1, (3.6) can be written as

$$\begin{aligned}\langle S(t)\rangle \geq & \frac{\alpha+d+\sigma+\delta}{(d+\sigma+\delta)L_2}\Big[\frac{\log S(t)-\log S(0)}{t}+\delta-\log(1-\omega)\nu(Z) \\ & -\frac{M_1(t)}{t}-\frac{M_2(t)}{t}\Big]^{\frac{1}{2}}-1-\frac{\alpha+d+\sigma+\delta}{d+\sigma+\delta}\varphi(t). \end{aligned} \tag{3.7}$$

Note that the quadratic variation of $M_i(t),\ i=1,2$ can be calculated by

$$\begin{aligned}\langle M_1, M_1\rangle_t =& \int_0^t \gamma^2 U_1^4(s)ds \leq \gamma^2 t, \\ \langle M_2, M_2\rangle_t =& \int_0^t \int_Z \left[\log(1-H(z)U_1^2(s-))\right]^2 \nu(dz)ds \\ \leq & \max\{[\log(1-\omega)]^2, \log(1+\omega)]^2\}t. \end{aligned}$$

According to the strong law of large numbers in [13], we get

$$\lim_{t\to\infty}\frac{M_i(t)}{t}=0,\ i=1,2,\ a.s. \tag{3.8}$$

Taking the inferior limit on both sides of (3.7) and combining (3.4) and (3.8), (H_2) yields

$$\liminf_{t\to\infty}\langle S(t)\rangle \geq \frac{\alpha+d+\sigma+\delta}{(d+\sigma+\delta)L_2}\left[\delta-\log(1-\omega)\nu(Z)\right]^{\frac{1}{2}}-1>0,\ a.s.,$$

implies system (1.3) is persistent in the mean. This completes this proof. □

Remark 3.1. According to Theorem 3.1, if the coefficient of the infection force β and the diffusion coefficient γ satisfy $L_2=\int_Z H(z)\nu(dz)-\beta-\frac{\gamma^2}{2}>0$, and further the proportion of drug users who enter treatment α and the probability of drug users becoming the susceptible population again through treatment δ make $R_0^S>1$ hold, we obtain the number of susceptible individuals $S(t)$ will eventually persist in the mean, which implies that the heroin drug is persistent in the mean affected by the Lévy measure.

Moreover, the following theorem obtains asymptotic stability in mean square of system (1.3).

Theorem 3.2. *If the assumption (H_1) is satisfied, and further assume that (H_3): $\beta>\max\{\frac{1}{2}\left(\gamma^2+\int_Z H^2(z)\nu(dz)\right),\delta\}$ and $d+\delta>\frac{1}{2}$. Then there exist positive constants L_3 and L_4 such that the solution $X(t)$ of system (1.3) is asymptotically stable in mean square, that is,*

$$\mathbb{E}|X(t)|^2 \leq L_3 + L_4|X(0)|^2 e^{-t}.$$

Proof. Define a C^2-function $V:\ \mathbb{R}_+^3 \to \mathbb{R}_+\cup\{+\infty\}$ by

$$V(X)=V(S,U_1,U_2)=l_0S^2+(U_1+U_2)^2,$$

where $l_0=\frac{\beta-\delta}{\delta}>0$. Obviously

$$(l_0+1)|X|^2 \leq V(X) \leq (l_0+2)|X|^2. \tag{3.9}$$

Applying Itô's formula, we obtain

$$\mathbb{E}(e^t V(X(t))) = V(X(0)) + \mathbb{E}\int_0^t e^s[V(X(s)) + \mathcal{L}V(X(s))]ds,$$

where

$$\begin{aligned}\mathcal{L}V(X) =& 2l_0 S\left[d - \beta S U_1^2 - dS + \delta U_2\right] + (l_0+1)(\gamma S U_1^2)^2 \\ &+2(U_1+U_2)\left[\beta S U_1^2 - dU_1 - (d+\delta)U_2\right] + (l_0+1)\int_Z H^2(z) S^2 U_1^4 \nu(dz).\end{aligned}$$

Hence

$$\begin{aligned}\mathcal{L}V(X) + V(X) =& 2l_0 S\left[d - \beta S U_1^2 - dS + \delta(1 - S - U_1)\right] + (l_0+1)(\gamma S U_1^2)^2 \\ &+2(1-S)\left[\beta S U_1^2 - dU_1 - (d+\delta)(1-S-U_1)\right] \\ &+(l_0+1)\int_Z H^2(z) S^2 U_1^4 \nu(dz) + l_0 S^2 + (U_1+U_2)^2 \\ =& \left(\gamma^2 + \int_Z H^2(z)\nu(dz)\right)(l_0+1)S^2U_1^4 - 2\beta(l_0+1)S^2U_1^2 + 2\beta S U_1^2 \\ &-[2(d+\delta)-1]S^2 - 2\delta(l_0+1)SU_1 + 2[(d+\delta)(l_0+2)-1]S \\ &+2dU_1 + 1 - 2(d+\delta).\end{aligned}$$

Moreover, we have by $(S(0), U_1(0), U_2(0)) \in \Lambda$

$$\begin{aligned}\mathcal{L}V(X) + V(X) \le& -\left(2\beta - \gamma^2 - \int_Z H^2(z)\nu(dz)\right)(l_0+1)S^2U_1^2 \\ &-2[\delta(l_0+1)-\beta]SU_1 + 2[(d+\delta)(l_0+2)-1]S \\ &-[2(d+\delta)-1]S^2 + 2dU_1 + 1 - 2(d+\delta) \\ \le& -\left(2\beta - \gamma^2 - \int_Z H^2(z)\nu(dz)\right)\frac{\beta}{\delta}S^2U_1^2 - [2(d+\delta)-1]S^2 \\ &+2\left[\frac{(d+\delta)(\beta+\delta)}{\delta} - 1\right]S + 2dU_1 + 1 - 2(d+\delta) \\ =:& R(X).\end{aligned}$$

Furthermore, (H_3) implies $R(X) \le \hat{L}_3$, where $\hat{L}_3$ is a positive constant independent of X. Hence

$$\mathbb{E}(e^t V(X(t))) \le V(X(0)) + \mathbb{E}\int_0^t \hat{L}_3 e^s ds = V(X(0)) + \hat{L}_3(e^t - 1). \tag{3.10}$$

Combining (3.9) and (3.10), we have

$$\mathbb{E}|X(t)|^2 \le \frac{\delta}{\beta}\mathbb{E}V(X(t)) \le \frac{\delta}{\beta}[e^{-t}V(X(0)) + \hat{L}_3(1 - e^{-t})] \le L_4|X(0)|^2 e^{-t} + L_3,$$

where $L_3 = \frac{\delta}{\beta}\hat{L}_3$ and $L_4 = \frac{\beta+\delta}{\beta}$. This completes the proof. □

Remark 3.2. When the natural death rate of the population d and the probability of drug users becoming the susceptible population again through treatment δ satisfy $d + \delta > 1$ and further the coefficient of the infection force β make

$\beta > \max\{\frac{1}{2}\left(\gamma^2 + \int_Z H^2(z)\nu(dz)\right), \delta\}$ hold, from Theorem 3.2 and Chebyshev's inequality, it easily follows that the number of drug users of system (1.3) is stable in probability in the large, i.e., for any $X(0)$, $\varepsilon > 0$ and $\zeta > 0$, there exists a $T = T(X(0), \varepsilon, \zeta)$ such that $P(|X(t, T, X(0))| > \varepsilon) < \zeta$ for all $t > T$.

4. Positive recurrence

In this section, we establish a domain $D \subset \Lambda$, which is positive recurrent for processes $(S(t), U_1(t), U_2(t))$.

Definition 4.1. Process $X(t-, x)$ with $X(0) = x$ is recurrent with respect to D, if for any $x \notin D$, $\mathbb{P}(\tau_D < \infty) = 1$, where τ_D is the hitting time of D for the process $X(t-, x)$, that is,

$$\tau_D = \inf\{t > 0, X(t-, x) \in D\}.$$

Process $X(t-, x)$ is said to be positive recurrent with respect to D if $E(\tau_D) < \infty$ for any $x \notin D$.

For the simplicity of discussion, define

$$R_1^S = \frac{\alpha}{(\alpha + d - \log(1-\omega)\nu(Z))(\sigma + d + \delta)}.$$

Theorem 4.1. *Under condition (H_1), let $(S(t), U_1(t), U_2(t))$ be the solution of system (1.3) with initial value $(S(0), U_1(0), U_2(0)) \in \Lambda$. Suppose further that (H_4): $R_1^S > 1$ and $\int_Z H(z)\nu(dz) > \beta$, then system (1.3) is positive recurrent with respect to the domain*

$$D = \{(S, U_1, U_2) \in \Lambda, S \geq \epsilon,\ U_1 \geq \epsilon,\ U_2 \geq \epsilon\},$$

where $\epsilon > 0$ is a sufficiently small constant.

Proof. Define a nonnegative C^2-function $V : \Lambda \to \mathbb{R}_+ \cup \{\infty\}$ as follows

$$V(S, U_1, U_2) = L(-l_1 \log U_1 - l_2 \log U_2) - \log S - \log U_2,$$

where $l_1 = \frac{1}{\alpha + d - \log(1-\omega)\nu(Z)}$, $l_2 = \frac{1}{\sigma + d + \delta}$ and $L > 0$ satisfies the following condition

$$L\lambda - \sigma - 2d - \delta - \beta - \frac{\gamma^2}{2} + \int_Z H(z)\nu(dz) + \log(1-\omega)\nu(Z) > 2 \tag{4.1}$$

in which

$$\lambda = 2\left[\left(\frac{\alpha}{(\alpha + d - \log(1-\omega)\nu(Z))(\sigma + d + \delta)}\right)^{\frac{1}{2}} - 1\right] = \left[(R_1^S)^{\frac{1}{2}} - 1\right] > 0. \tag{4.2}$$

Let $V(S, U_1, U_2) = LV_1(U_1, U_2) + V_2(S, U_2)$, where $V_1(U_1, U_2) = -l_1 \log U_1 - l_2 \log U_2$, $V_2(S, U_2) = -\log S - \log U_2$. Making use of Itô's formula to $-\log S(t)$ gives

$$\begin{aligned} &d(-\log S(t)) \\ =&-\frac{1}{S(t)}\Big[\left(d - \beta S(t)U_1^2(t) - dS(t) + \delta U_2(t)\right)dt - \gamma S(t)U_1^2(t)dB(t)\Big] \end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{2S^2(t)}(\gamma S(t)U_1^2(t))^2dt+\int_Z\Big[-\log(S(t-)-H(z)S(t-)U_1^2(t-))\\
&+\log S(t-)+\frac{1}{S(t)}H(z)S(t-)U_1^2(t-)\Big]\nu(dz)dt\\
&+\int_Z\left[-\log(S(t-)-H(z)S(t-)U_1^2(t-))+\log S(t-)\right]\tilde{N}_q(dt,dz)\\
=&\mathcal{L}(-\log S(t))dt+\gamma U_1^2(t)dB(t)-\int_Z[\log(1-H(z)U_1^2(t-))]\tilde{N}_q(dt,dz),
\end{aligned}
$$

where

$$
\begin{aligned}
\mathcal{L}(-\log S)=&-\frac{d}{S}-\frac{\delta U_2}{S}+\beta U_1^2+d+\frac{\gamma^2}{2}U_1^4\\
&-\int_Z\Big[\log(1-H(z)U_1^2)+H(z)U_1^2\Big]\nu(dz).
\end{aligned}
$$

According to the proof of Theorem 3.1, we obtain by $(S(0),U_1(0),U_2(0))\in\Lambda$ and (H_1)

$$
\begin{aligned}
\mathcal{L}(-\log S)\le&-\frac{d}{S}-\frac{\delta U_2}{S}+\beta U_1^2+d+\frac{\gamma^2}{2}U_1^4\\
&-\log(1-\omega)\nu(Z)-\int_Z H(z)\nu(dz)U_1^2.
\end{aligned}
\tag{4.3}
$$

Similar to $\mathcal{L}(-\log S)$, applying Itô's formula to $\mathcal{L}(-\log U_1)$ gets

$$
\begin{aligned}
\mathcal{L}(-\log U_1)\le&-\beta SU_1+\alpha+d-\frac{U_2}{U_1}+\frac{\gamma^2}{2}S^2U_1^2\\
&-\log(1-\omega)\nu(Z)+\int_Z H(z)\nu(dz)SU_1.
\end{aligned}
\tag{4.4}
$$

Therefore, combining (4.3)-(4.4) obtains

$$
\begin{aligned}
\mathcal{L}(V_1(U_1,U_2))\le&l_1\Big(-\beta SU_1+\alpha+d-\frac{U_2}{U_1}+\frac{\gamma^2}{2}S^2U_1^2-\log(1-\omega)\nu(Z)\\
&+\int_Z H(z)\nu(dz)SU_1\Big)+l_2\left(-\frac{\alpha U_1}{U_2}+\sigma+d+\delta\right)\\
=&-\left(l_1\frac{U_2}{U_1}+l_2\frac{\alpha U_1}{U_2}\right)+l_1(\alpha+d-\log(1-\omega)\nu(Z))\\
&+l_2(\sigma+d+\delta)+l_1\left(-\beta SU_1+\frac{\gamma^2}{2}S^2U_1^2+\int_Z H(z)\nu(dz)SU_1\right)\\
\le&-2\sqrt{\alpha l_1l_2}+l_1(\alpha+d-\log(1-\omega)\nu(Z))\\
&+l_2(\sigma+d+\delta)+l_1\left[\left(\int_Z H(z)\nu(dz)-\beta\right)SU_1+\frac{\gamma^2}{2}S^2U_1^2\right].
\end{aligned}
$$

Moreover, (H_3) implies

$$
\mathcal{L}(V_1(U_1,U_2))\le-\lambda+l_1\left[\left(\int_Z H(z)\nu(dz)-\beta\right)U_1+\frac{\gamma^2}{2}U_1^2\right].
\tag{4.5}
$$

In addition, similar to the proof of Theorem 3.1, we have by $(S(0), U_1(0), U_2(0)) \in \Lambda$ and (H_1)

$$\begin{aligned}\mathcal{L}(V_2(S,U_2)) \leq& -\frac{d}{S} - \frac{\alpha U_1}{U_2} + \sigma + 2d + \delta \\ &+\beta + \frac{\gamma^2}{2} - \int_Z H(z)\nu(dz) - \log(1-\omega)\nu(Z). \end{aligned} \tag{4.6}$$

Combining (4.5) and (4.6), we obtain

$$\begin{aligned}&\mathcal{L}(V(S,U_1,U_2)) \\ \leq& Ll_1\left[\left(\int_Z H(z)\nu(dz) - \beta\right)U_1 + \frac{\gamma^2}{2}U_1^2\right] - L\lambda - \frac{d}{S} - \frac{\alpha U_1}{U_2} \\ &+\sigma + 2d + \delta + \beta + \frac{\gamma^2}{2} - \int_Z H(z)\nu(dz) - \log(1-\omega)\nu(Z). \end{aligned} \tag{4.7}$$

Next, define a compact subset

$$\hat{D} = \{(S,U_1,U_2) \in \Lambda : S \geq \epsilon_1,\ U_1 \geq \epsilon_2,\ U_2 \geq \epsilon_3\},$$

where $\epsilon_i > 0,\ i = 1,\ 2,\ 3$ is a sufficiently small constant such the following conditions hold

$$-\frac{d}{\epsilon_1} + F \leq -1, \tag{4.8}$$

$$Ll_1\left[\left(\int_Z H(z)\nu(dz) - \beta\right)\epsilon_1 + \frac{\gamma^2}{2}\epsilon_1^2\right] \leq 1, \tag{4.9}$$

$$-\frac{\sigma}{\epsilon_1} + F \leq -1, \tag{4.10}$$

where

$$\begin{aligned}F = \sup_{(S,U_1,U_2)\in\Lambda} \Big\{ &Ll_1\left[\left(\int_Z H(z)\nu(dz) - \beta\right)U_1 + \frac{\gamma^2}{2}U_1^2\right] + \sigma + 2d \\ &+\delta + \beta + \frac{\gamma^2}{2} - \int_Z H(z)\nu(dz) - \log(1-\omega)\nu(Z)\Big\}. \end{aligned} \tag{4.11}$$

For convenience, one divides $\Lambda \backslash \hat{D}$ into three domains as following

$D_1 = \{(S,U_1,U_2) \in \Lambda : 0 < S < \epsilon_1\}, D_2 = \{(S,U_1,U_2) \in \Lambda : 0 < U_1 < \epsilon_2,\ S \geq \epsilon_1\}$,
$D_3 = \{(S,U_1,U_2) \in \Lambda : 0 < U_2 < \epsilon_3,\ S \geq \epsilon_1,\ U_1 \geq \epsilon_2\}$.

Case 1. If $(S,U_1,U_2) \in D_1$, we have by (26) and (4.11)

$$\begin{aligned}\mathcal{L}(V(S,U_1,U_2)) \leq& Ll_1\left[\left(\int_Z H(z)\nu(dz) - \beta\right)U_1 + \frac{\gamma^2}{2}U_1^2\right] - L\lambda - \frac{d}{S} \\ &+\sigma + 2d + \delta + \beta + \frac{\gamma^2}{2} - \int_Z H(z)\nu(dz) - \log(1-\omega)\nu(Z) \\ \leq& -\frac{d}{\epsilon_1} + F \leq -1, \text{ for any } (S,U_1,U_2) \in D_1. \end{aligned} \tag{4.12}$$

Case 2. If $(S, U_1, U_2) \in D_2$, it is easy to see that

$$\begin{aligned}\mathcal{L}(V(S,U_1,U_2)) \leq & Ll_1\left[\left(\int_Z H(z)\nu(dz)-\beta\right)U_1+\frac{\gamma^2}{2}U_1^2\right]-L\lambda \\ & +\sigma+2d+\delta+\beta+\frac{\gamma^2}{2}-\int_Z H(z)\nu(dz)-\log(1-\omega)\nu(Z).\end{aligned}$$

Choosing $\epsilon_2 = \epsilon_1$, (4.1) and (26) imply

$$\mathcal{L}(V(S,U_1,U_2)) \leq -1, \text{ for any } (S,U_1,U_2) \in D_2. \tag{4.13}$$

Case 3. If $(S, U_1, U_2) \in D_3$, we get

$$\begin{aligned}\mathcal{L}(V(S,U_1,U_2)) \leq & Ll_1\left[\left(\int_Z H(z)\nu(dz)-\beta\right)U_1+\frac{\gamma^2}{2}U_1^2\right]-L\lambda-\frac{\alpha U_1}{U_2} \\ & +\sigma+2d+\delta+\beta+\frac{\gamma^2}{2}-\int_Z H(z)\nu(dz)-\log(1-\omega)\nu(Z).\end{aligned}$$

Choosing $\epsilon_3 = \epsilon_2^2 = \epsilon_1^2$, (27) and (4.11) imply

$$\mathcal{L}(V(S,U_1,U_2)) \leq -\frac{\sigma}{\epsilon_1}+F \leq -1, \text{ for any } (S,U_1,U_2) \in D_3. \tag{4.14}$$

Moreover, from (4.12)-(4.14), for a sufficiently small $\epsilon = \epsilon_1$, it has $D = \hat{D}$ and we further obtain

$$\mathcal{L}(V(S,U_1,U_2)) \leq -1, \text{ for any } (S,U_1,U_2) \in \Lambda \backslash D. \tag{4.15}$$

Now let $(S(0), U_1(0), U_2(0)) \in \Lambda \backslash D$, then Itô's formula and (4.15) imply

$$\begin{aligned}& \mathbb{E}V(S(\tau_D),U_1(\tau_D),U_2(\tau_D)) - V(S(0),U_1(0),U_2(0)) \\ =& \mathbb{E}\int_0^{\tau_D} \mathcal{L}V(S(t),U_1(t),U_2(t))dt \\ \leq & -\mathbb{E}(\tau_D).\end{aligned} \tag{4.16}$$

Therefore, it is easy to get

$$\mathbb{E}(\tau_D) \leq V(S(0),U_1(0),U_2(0)).$$

This completes the proof. □

Remark 4.1. As positive recurrence can provide a better description and display of the persistence of system (1.3), it allows us to have a deeper understanding of how environmental noise affects the steady-state for persistence. According to Theorem 4.1, if the proportion of drug users who enter treatment α, the natural death rate of the population d, the probability of drug users becoming the susceptible population again through treatment δ and the probability of outflow of individuals from drug users in treatment σ make $R_1^S > 1$ hold, it obtains that the heroin drug users of system (1.3).

Table 1. Detailed list of parameter values

Parameters	Units	Values	References
α: the proportion of drug users who enter treatment	day^{-1}	0.15	[4]
σ: the probability of outflow of individuals from drug users in treatmen	day^{-1}	0.45	[4]
b: the recruitment rate of individuals in the general population entering the susceptible population	day^{-1}	0.25	[4]
d: the natural death rate of the population	day^{-1}	0.25	[4]
δ: the probability of drug users becoming the susceptible population again through treatment	day^{-1}	0.46	[4]

5. Examples and computer simulations

In this section, we introduce mainly some examples and numerical simulations to support the main results.

Example 5.1. Let us illustrate the asymptotic stability in mean square of system (1.3) in Theorem 3.2. Choosing the initial value $(S(0), U_1(0), U_2(0)) = (0.45, 0.15, 0.4)$ and

$$\alpha = 0.15,\ \beta = 0.6,\ \sigma = 0.45,\ d = 0.25,\ \delta = 0.46,$$
$$\gamma = 0.7,\ H(z) = 0.55,\ \nu(Z) = 0.65,\ \text{(Table 1 for details).}$$

It is easy to see that $|H(z)| = 0.55 < 1$, $d + \delta = 0.25 + 0.46 = 0.71 > \frac{1}{2}$ and $\beta = 0.6 > \max\{\frac{1}{2}\left(\gamma^2 + \int_Z H^2(z)\nu(dz)\right), \delta\} = \max\{0.3433, 0.46\} = 0.46$, which implies (H_1) and (H_3) hold. It follows from Theorem 3.2 that system (1.3) becomes asymptotic stability in mean square. The simulated of system (1.3) is shown Fig. 1 (a). Meanwhile, the numerical simulations of the different samples $S(t)$, $U_1(t)$ and $U_2(t)$ are shown as in Figs. 1 (b). From Fig. 1 (a), it's interesting to find the number of drug users not in treatment $U_1(t)$ and drug users undergoing treatment $U_2(t)$ tends to 0.

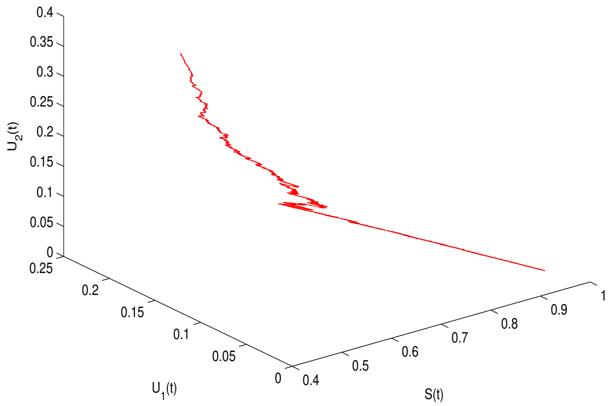
Example 5.2. Let us illustrate positive recurrent of system (1.3) in Theorem 4.1. Choosing the initial value $(S(0), U_1(0), U_2(0)) = (0.25, 0.45, 0.3)$ and

$$\alpha = 0.065,\ \beta = 0.3,\ \sigma = 0.05,\ d = 0.01,\ \delta = 0.02,$$
$$\gamma = 0.15,\ H(z) = 0.8,\ \nu(Z) = 0.4.$$

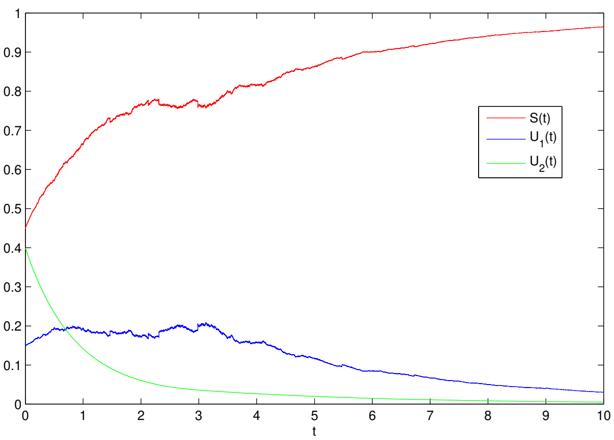
It is easy to verify that $\int_Z H(z)\nu(dz) = 0.8 \times 0.4 = 0.32 > \beta = 0.3$, and $R_1^S = \frac{0.65}{(0.65+0.01-\log(1-0.8)\times 0.4)(0.05+0.01+0.02)} = 1.1305 > 1$, implies that (H_1) and (H_4) hold. It follows from Theorem 4.1 that system (1.3) is positive recurrent with respect to the domain $D = \{(S, U_1, U_2) \in \Lambda, S \geq \epsilon,\ U_1 \geq \epsilon,\ U_2 \geq \epsilon\}$. The simulated of system (1.3) is shown Fig. 2. Meanwhile, the numerical simulations of the different samples $S(t)$, $U_1(t)$ and $U_2(t)$ are shown as in Figs. 3 (a)-(c), respectively.

6. Conclusions

Note that drug abuses could be affected by the exogenous perturbations, for example, sudden changes of the temperature and humidity. Based on this fact, we



(a)



(b)

Figure 1. $\gamma = 0.7,\ H(z) = 0.55,\ \nu(Z) = 0.65, \Delta = 0.01$. (a) The simulation of system (1.3) with $T = 100$. (b) The different samples $S(t)$, $U_1(t)$ and $U_2(t)$ with $T = 10$.

establish a heroin epidemic model with nonlinear incidence rate influenced by discontinuous stochastic perturbations in this paper. Precisely, We have not only proved that the stochastic heroin epidemic model with Lévy noises has a unique global positive solution, but also obtained the persistence in the mean to system (1.3) as $R_0^S > 1$. Meanwhile, we analyze and establish sufficient condition for the asymptotic stability in mean square of system (1.3). Moreover, it is interesting to perform the positive recurrent for system (1.3) as $R_1^S > 1$.

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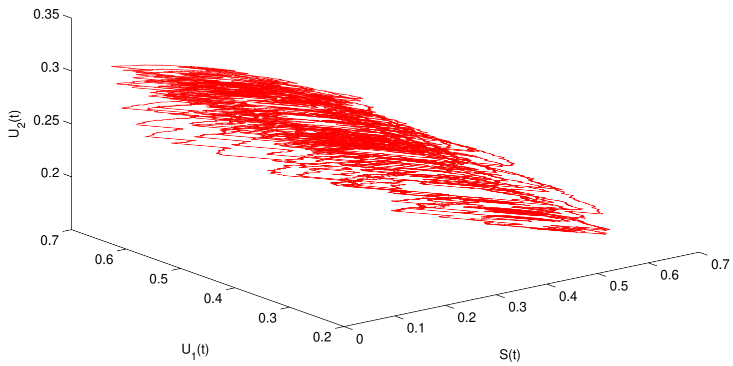

Figure 2. The simulation of system (1.3) with $\alpha = 0.065,\ \beta = 0.3,\ \sigma = 0.05,\ d = 0.01,\ \delta = 0.02,\ \gamma = 0.15,\ H(z) = 0.8,\ \nu(Z) = 0.4,\ T = 1000$ and $\Delta = 0.001$.

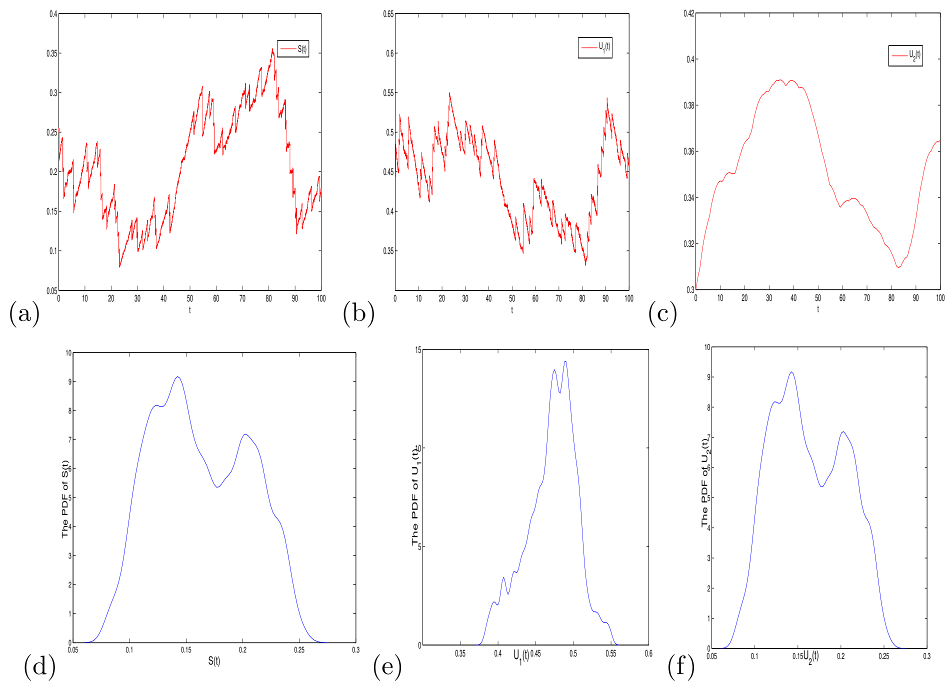

Figure 3. The simulation of system (1.3) with $\alpha = 0.065,\ \beta = 0.3,\ \sigma = 0.05,\ d = 0.01,\ \delta = 0.02,\ \gamma = 0.15,\ H(z) = 0.8,\ \nu(Z) = 0.4, T = 100$ and $\Delta = 0.001$. (a), (b) and (c) denote the simulation of $S(t)$, $U_1(t)$, $U_2(t)$, respectively. (d), (e) and (f) denote the the density function of $S(t)$, $U_1(t)$, $U_2(t)$, respectively.

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