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RIEMANN-LIOUVILLE FRACTIONAL INTEGRALS AND DERIVATIVES ON MORREY SPACES AND APPLICATIONS TO A CAUCHY-TYPE PROBLEM*

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Abstract We investigate the boundedness and compactness of Riemann-Liouville integral operators on Morrey spaces, a class of nonseparable function spaces. Instead of adopting dual or maximal viewpoints in integrable function spaces, our approach is based on the compactness of the truncated Riemann-Liouville fractional integrals, leveraging a criterion for strongly pre-compact sets. By constructing a truncated Marchaud fractional derivative function, we characterize the solution to Abel's equation on Morrey spaces. Utilizing the fixed-point theorem, we establish the existence and uniqueness of solutions to a Cauchy-type problem for fractional differential equations. Additionally, we provide an illustrative example to demonstrate the sufficiency of the conditions presented in our main result.

Keywords Morrey space, Riemann-Liouville fractional integral, fractional differential equation, fixed-point theorem.

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1. Introduction

In [32], Stein comprehensively categorized three fundamental classes of operators in harmonic analysis: singular integral operators, which contain Hilbert transforms, Riesz transforms, fractional Riesz transforms, Cauchy type operators and so on (c.f. [6,7,11,13,14,33]); average operators, which include Hardy-Littlewood maximal operators, Hardy operators, Hausdorff operators, etc. (see for instance [5,16,34,35]);

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and oscillatory integrals, which encompass the Fourier transform, the Bochner-Riesz means, the Radon transform in CT technology and so on (c.f. [13, 15, 19, 37, 38]). Fractional versions of these operators are also one of the hot topics now. We focus on fractional integrals, in particular the Riemann-Liouville fractional integral, in this paper.

In the past three decades, fractional calculus has grown in significance due to its wide applications in various fields of science and engineering, including fluid flow, image processing, probability theory, electrical networks and complex dynamics. The origin of fractional derivatives and integrals can be traced back to the letter of Leibniz and L'Hôspital in 1695. The study of fractional derivatives and integrals has proven to be a powerful tool for describing anomalous kinetics in various fields. Moreover, it offers potential methods for solving differential and integral equations, among other problems [3,9,10,21,22,25,27–31,38] and references therein. In fractional calculus, fractional derivatives are formulated within the framework of fractional integrals. The pivotal role played by the Riemann-Liouville fractional derivative in the field of fractional calculus is well-established. Consequently, the Riemann-Liouville fractional integral assumes a central position. Furthermore, the Caputo fractional derivative is defined through a modification of the Riemann-Liouville fractional integral.

The Riemann-Liouville integral of order α*,* $0 < \alpha < 1$*,* is represented as

$$I^{\alpha}_{0+} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(\tau)}{(t-\tau)^{1-\alpha}} d\tau.$$

Correspondingly, *The Riemann-Liouville derivative of order* α is defined by

$$D^{\alpha}_{0+} f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t \frac{f(\tau)}{(t-\tau)^{\alpha}} d\tau.$$

These definitions are valid for $0 < t < T$, where $T > 0$. It can be shown that $D^{\alpha}_{0+} f(t) = \frac{d}{dt} I^{1-\alpha}_{0+} f(t)$.

Morrey [23] introduced the classical Morrey spaces to investigate the local behavior of solutions to second-order elliptic partial differential equations. Various researchers have established that many properties of solutions to differential equations relate to the boundedness of certain operators on Morrey-type spaces. See, for instance, [1, 12, 17, 24, 36].

A real-valued function f is said to belong to Morrey space $L^{p,\lambda}(\Omega)$ if the following norm is finite:

$$\|f\|_{L^{p,\lambda}(\Omega)} := \sup_{x\in\Omega, r>0} \left(\frac{1}{r^{n\lambda}} \int_{B(x,r)\cap\Omega} |f(y)|^p dy \right)^{1/p}.$$

Here $1 \le p < \infty$ and $0 \le \lambda \le 1$ are parameters defining the space, and $\Omega \subset \mathbb{R}^n$ is an open subset. It is clear that $L^{p,0}(\Omega) = L^p(\Omega)$ and $L^{p,1}(\Omega) = L^{\infty}(\Omega)$. For $0 < \lambda < 1$, the space $L^{p,\lambda}(\Omega)$ has additional properties: it is a Banach space and it is nonseparable. However, standard functional spaces, such as those of continuous or integrable functions, are typically separable.

There are many authors studying the representation of functions by fractional integrals in continuous or Lebesgue spaces (see [25]). We are particularly interested in the case of $0 < \lambda < 1$, as the case of $\lambda = 0$ reduces to the more traditional

Lebesgue space, i.e., $L^{p,0}(0,T) = L^p(0,T)$. For convenience, we will work on a subspace of Morrey space $L^{p,\lambda}(\mathbb{R}^+)$ as follows:

$$\widetilde{L}^{p,\lambda}(0,T) = \left\{ f \in L^{p,\lambda}(\mathbb{R}^+) \mid f = 0,\ a.e.\ \text{ on } \ [\mathrm{T}, \infty) \right\},$$

since it is isometric to $L^{p,\lambda}(0,T)$.

This paper is organized as follows. In Section 2, we give the relation between the subspaces $\widetilde{L}^{p,\lambda}(0,T)$ and the local Morrey spaces $L^{p,\lambda}(0,T)$ which are nonseparable spaces. We observe that these spaces do not admit approximation or contractive properties. Furthermore, we establish the boundedness of Riemann-Liouville integral operators on local Morrey spaces. Section 3 focuses on the representation of functions by fractional integrals in Morrey spaces via the absolutely continuous functions and the truncated Marchaud fractional derivative functions. In Section 4, we also establish the existence and uniqueness of solutions to Cauchy problems for fractional differential equations using the Schauder fixed-point theorem. Unlike the typical approach using dual or maximal points in integrable function spaces, our proof strategy is based on the compactness of the truncated Riemann-Liouville fractional integrals, utilizing a criterion for strongly pre-compact sets. Finally, we provide an example to illustrate that the conditions in our main result are sufficient but not necessary.

2. Bounded and compact operators in Morrey spaces

In what follows, $B(x,r) = (x-r, x+r)$. We obtain the following relation between the space $\widetilde{L}^{p,\lambda}(0,T)$ and the local Morrey space $L^{p,\lambda}(0,T)$.

Proposition 2.1. *$\widetilde{L}^{p,\lambda}(0,T)$ is isometric to $L^{p,\lambda}(0,T)$, $1 \le p \le \infty$.*

Proof. Define $A : L^{p,\lambda}(0,T) \to \widetilde{L}^{p,\lambda}(0,T)$ by

$$A(f) = \begin{cases} f, & \text{in } \ (0,\mathrm{T}), \\ 0 & a.e. \ \text{ on } \ [\mathrm{T},\infty). \end{cases} \tag{2.1}$$

Then A is injective due to the fact that $f \equiv g$ in $L^{p,\lambda}(\mathbb{R}^+)$ if $f = g$ a.e. in $\mathbb{R}^+$.

It is clear that

$$\begin{aligned} &\sup_{x \ge T, r>0} \left(\frac{1}{r^\lambda} \int_{B(x,r)\cap(0,T)} |\widetilde{f}(y)|^p dy \right)^{1/p} \\ &\le \sup_{0<x<T, r>0} \left(\frac{1}{r^\lambda} \int_{B(x,r)\cap(0,T)} |\widetilde{f}(y)|^p dy \right)^{1/p} \end{aligned}$$

for any $\widetilde{f} \in \widetilde{L}^{p,\lambda}(0,T)$. Therefore,

$$\|\widetilde{f}\,|_{(0,T)}\,\|_{L^{p,\lambda}(0,T)} = \|\widetilde{f}\|_{L^{p,\lambda}(\mathbb{R}^+)}.$$

Thus A is surjective and

$$\|Af\|_{L^{p,\lambda}(\mathbb{R}^+)} = \|f\|_{\widetilde{L}^{p,\lambda}(0,T)}.$$

Consequently, A is an isometric mapping. □

The following theorem gives the boundedness of Riemann-Liouville integral operators on local Morrey spaces.

Theorem 2.1. *Let $1 < p, q < \infty$, $\frac{1}{p} < \alpha < 1$ and $0 \leq \beta, \mu \leq 1$. Then I^{α}_{0+} is bounded from $L^{p,\beta}(0,T)$ to $L^{q,\mu}(0,T)$.*

Proof. Assume that $f \in L^{p,\beta}(0,T)$. For any $r > 0$ and $x > 0$, by Hölder's inequality, we have

$$\begin{aligned} & r^{-\mu}\int_{B(x,r)\cap(0,T)} |I^{\alpha}_{0+}f(t)|^q dt \\ =& r^{-\mu}\int_{B(x,r)\cap(0,T)} \left|\frac{1}{\Gamma(\alpha)}\int_0^t \frac{f(\tau)}{(t-\tau)^{1-\alpha}}d\tau\right|^q dt \\ \leq& \frac{C\|f\|^q_{L^{p,\beta}(0,T)}}{r^{\mu}}\int_{B(x,r)\cap(0,T)} t^{\frac{\beta q}{p}}\left(\int_0^t (t-\tau)^{-\frac{(1-\alpha)p}{p-1}}d\tau\right)^{(1-\frac{1}{p})q} dt. \end{aligned} \tag{2.2}$$

Since $\frac{1}{p} < \alpha < 1$, we have $1 - \frac{(1-\alpha)p}{p-1} > 0$. Therefore,

$$r^{-\mu}\int_{B(x,r)\cap(0,T)} |I^{\alpha}_{0+}f(t)|^q dt \leq C\frac{\|f\|^q_{L^{p,\beta}(0,T)}}{r^{\mu}}\int_{B(x,r)\cap(0,T)} t^{\frac{\beta q+(\alpha p-1)q}{p}}dt.$$

(i) If $r \geq T$, then

$$\begin{aligned} r^{-\mu}\int_{B(x,r)\cap(0,T)} |I^{\alpha}_{0+}f(t)|^q dt &\leq C\frac{\|f\|^q_{L^{p,\beta}(0,T)}}{T^{\mu}}\int_{B(x,r)\cap(0,T)} t^{\frac{\beta q+(\alpha p-1)q}{p}}dt \\ &\leq C\|f\|^q_{L^{p,\beta}(0,T)}T^{\frac{\beta q+(\alpha p-1)q}{p}+1-\mu}. \end{aligned}$$

(ii) If $r < T$, then

$$\begin{aligned} r^{-\mu}\int_{B(x,r)\cap(0,T)} |I^{\alpha}_{0+}f(t)|^q dt &\leq C\frac{\|f\|^q_{L^{p,\beta}(0,T)}}{r^{\mu}}\int_{B(x,r)\cap(0,T)} t^{\frac{\beta q+(\alpha p-1)q}{p}}dt \\ &\leq C\frac{\|f\|^q_{L^{p,\beta}(0,T)}}{r^{\mu}}T^{\frac{\beta q+(\alpha p-1)q}{p}}\cdot 2r \\ &\leq C\|f\|^q_{L^{p,\beta}(0,T)}T^{\frac{\beta q+(\alpha p-1)q}{p}+1-\mu}. \end{aligned}$$

We arrive at

$$\|I^{\alpha}_{0+}u\|_{L^{q,\mu}(0,T)} \leq C\|f\|_{L^{p,\beta}(0,T)}.$$

□

By the same proof as that of Theorem 1.12 in [8], we have the following lemma.

Lemma 2.1. *Let $1 \leq p < \infty$ and $0 < \lambda < 1$. Suppose the subset G in $L^{p,\lambda}(\mathbb{R}^+)$ satisfies the following conditions:*
(I) *Norm boundedness uniformly*

$$\sup_{f\in G}\|f\|_{L^{p,\lambda}(\mathbb{R}^+)} < \infty. \tag{2.3}$$

(II) *Translation continuity*

$$\lim_{y\to 0^+} \|f(\cdot + y) - f(\cdot)\|_{L^{p,\lambda}(\mathbb{R}^+)} = 0, \tag{2.4}$$

uniformly in $f \in G$.
(III) *Control uniformly away from the origin*

$$\lim_{\gamma\to\infty} \|f\chi_{E_\gamma}\|_{L^{p,\lambda}(\mathbb{R}^+)} = 0, \tag{2.5}$$

uniformly in $f \in G$*, where* $E_\gamma = \{x \in \mathbb{R} : |x| > \gamma\}$*. Then* G *is strongly pre-compact set in* $L^{p,\lambda}(\mathbb{R}^+)$.

Proposition 2.2. *If* $1 < p < \infty$ *and* $\frac{1}{2}(1+\frac{1}{p}) < \alpha < 1$, $0 < \beta, \mu < 1$, *then*

$$\mathcal{T}u(t) = \frac{\chi_{(0,T)}(t)}{\Gamma(\alpha)} \int_0^t \frac{u(\tau)}{(t-\tau)^{1-\alpha}} d\tau,$$

is a compact operator from $\widetilde{L}^{p,\beta}(0,T)$ *to* $\widetilde{L}^{q,\mu}(0,T)$.

Proof. Suppose that K is an arbitrary bounded set in $\widetilde{L}^{p,\beta}(0,T)$, we only need to prove that the set $\mathcal{T}(K)$ is a strongly pre-compact set in $\widetilde{L}^{q,\mu}(0,T)$. By Lemma 2.1, it is suffice to show that (I)-(III) hold uniformly in $\mathcal{T}(K)$.

(I). Since K is bounded, there exists $M > 0$ such that $\|u\|_{\widetilde{L}^{p,\beta}(0,T)} \le M$ for every $u \in K$. For any $x \in \mathbb{R}^+$ and $r \in \mathbb{R}^+$, since $\mathcal{T}u(t) = \chi_{(0,T)}(t) I^\alpha_{0+} f(t)$, as in (2.2), by Hölder's inequality, we have

$$\begin{aligned} & r^{-\mu} \int_{x-r}^{x+r} |\mathcal{T}u(t)|^q dt \\ =& r^{-\mu} \int_{B(x,r)\cap(0,T)} |I^\alpha_{0+} f(t)|^q dt \\ \le& \frac{M^q}{r^\mu \Gamma(\alpha)^q} \int_{x-r}^{x+r} \chi_{(0,T)}(t) t^{\frac{\beta q}{p}} \left(\int_0^t (t-\tau)^{-\frac{(1-\alpha)p}{p-1}} d\tau \right)^{(1-\frac{1}{p})q} dt. \end{aligned} \tag{2.6}$$

Since $\frac{1}{p} < \frac{1}{2}(1+\frac{1}{p}) < \alpha < 1$, by the same discussion as in Theorem 2.1, we get

$$\|\mathcal{T}u\|_{L^{q,\mu}(\mathbb{R}^+)} \le CM.$$

(II) For all $x \ge T,\ y > 0$, by definition, we can see

$$|\mathcal{T}u(x+y) - \mathcal{T}u(x)| = 0. \tag{2.7}$$

For $x \in (0,T)$, $y > 0$ sufficiently small such that $x + y \in (0,T)$. It follows that

$$\begin{aligned} & |\mathcal{T}u(x+y) - \mathcal{T}u(x)| \\ =& \frac{1}{\Gamma(\alpha)} \left| \int_0^{x+y} \frac{u(\tau)}{(x+y-\tau)^{1-\alpha}} d\tau - \int_0^x \frac{u(\tau)}{(x-\tau)^{1-\alpha}} d\tau \right| \\ \le& \frac{1}{\Gamma(\alpha)} \left| \int_0^x \frac{u(\tau)}{(x+y-\tau)^{1-\alpha}} - \frac{u(\tau)}{(x-\tau)^{1-\alpha}} d\tau \right| \\ & + \frac{1}{\Gamma(\alpha)} \left| \int_x^{x+y} \frac{u(\tau)}{(x+y-\tau)^{1-\alpha}} d\tau \right| \\ :=& I_1 + I_2. \end{aligned} \tag{2.8}$$

We now consider the first integral. Recall that $|x^\lambda - y^\lambda| \le C|x-y|^\lambda,\ x, y \ge 0, 0 < \lambda < 1$. Using Hölder's inequality, we get

$$\begin{aligned} I_1 &= \frac{1}{\Gamma(\alpha)}\left|\int_0^x u(\tau)\frac{(x-\tau)^{1-\alpha}-(x+y-\tau)^{1-\alpha}}{(x+y-\tau)^{1-\alpha}(x-\tau)^{1-\alpha}}d\tau\right| \\ &\le C\int_0^x \frac{|u(\tau)|y^{1-\alpha}}{(x+y-\tau)^{1-\alpha}(x-\tau)^{1-\alpha}}d\tau \\ &\le Cy^{1-\alpha}Mx^{\frac{\beta}{p}}\left(\int_0^x \frac{1}{(x-\tau)^{\frac{2(1-\alpha)p}{p-1}}}d\tau\right)^{1-\frac{1}{p}} \\ &\le CMy^{1-\alpha}x^{\frac{\beta+2\alpha p-1-p}{p}}. \end{aligned}$$

For I_2, by Hölder's inequality, we have

$$\begin{aligned} I_2 &= \frac{1}{\Gamma(\alpha)}\left(\int_x^{x+y}|u(\tau)|^p d\tau\right)^{\frac{1}{p}}\left(\int_x^{x+y}\frac{1}{(x+y-\tau)^{\frac{(1-\alpha)p}{p-1}}}d\tau\right)^{1-\frac{1}{p}} \\ &\le CMy^{\frac{\beta}{p}}\left(\int_0^y \frac{1}{\tau^{\frac{(1-\alpha)p}{p-1}}}d\tau\right)^{1-\frac{1}{p}} \\ &= CMy^{\frac{\beta+\alpha p-1}{p}}. \end{aligned}$$

Combining (2.7), we get

$$|\mathcal{T}u(x+y)-\mathcal{T}u(x)| \le CM\chi_{(0,T)}(x)\left(y^{1-\alpha}x^{\frac{\beta+2\alpha p-1-p}{p}} + y^{\frac{\beta+\alpha p-1}{p}}\right). \tag{2.9}$$

(i) If $r \ge T$, then

$$\begin{aligned} &r^{-\mu}\int_{t-r}^{t+r}|\mathcal{T}u(x+y)-\mathcal{T}u(x)|^q dx \\ \le& CM^q r^{-\mu}\Bigg(y^{(1-\alpha)q}\int_{t-r}^{t+r}\chi_{(0,T)}(x)x^{(\frac{\beta+2\alpha p-1-p}{p})q}dx \\ &+ y^{(\frac{\beta+\alpha p-1}{p})q}\int_{t-r}^{t+r}\chi_{(0,T)}(x)dx\Bigg) \\ \le& CM^q T^{-\mu}\left(y^{(1-\alpha)q}T^{(\frac{\beta+2\alpha p-1-p}{p})q+1} + y^{(\frac{\beta+\alpha p-1}{p})q}T\right) \\ =& CM^q(y^{(1-\alpha)q} + y^{(\frac{\beta+\alpha p-1}{p})q}). \end{aligned} \tag{2.10}$$

(ii) When $r < T$, we get

$$\begin{aligned} &r^{-\mu}\int_{t-r}^{t+r}|\mathcal{T}u(x+y)-\mathcal{T}u(x)|^q dx \\ \le& CM^q r^{-\mu}\Bigg(y^{(1-\alpha)q}\int_{t-r}^{t+r}\chi_{(0,T)}(x)x^{(\frac{\beta+2\alpha p-1-p}{p})q}dx \\ &+ y^{(\frac{\beta+\alpha p-1}{p})q}\int_{t-r}^{t+r}\chi_{(0,T)}(x)dx\Bigg) \end{aligned}$$

$$\leq CM^q r^{-\mu}\left(y^{(1-\alpha)q}\int_{(x-t,x+t)\cap(0,T)} x^{(\frac{\beta+2\alpha p-1-p}{p})q}dx + 2ry^{(\frac{\beta+\alpha p-1}{p})q}\right) \tag{2.11}$$
$$\leq CM^q r^{-\mu}\left(2ry^{(1-\alpha)q}T^{(\frac{\beta+2\alpha p-1-p}{p})q} + 2ry^{(\frac{\beta+\alpha p-1}{p})q}\right)$$
$$\leq CM^q\left(y^{(1-\alpha)q} + y^{(\frac{\beta+\alpha p-1}{p})q}\right).$$

Therefore,

$$\|\mathcal{T}u(\cdot+y)-\mathcal{T}u(\cdot)\|_{L^{q,\mu}(\mathbb{R}^+)} \leq CM^q\left(y^{(1-\alpha)q} + y^{(\frac{\beta+\alpha p-1}{p})q}\right).$$

Since $\frac{1}{2}(1+\frac{1}{p})<\alpha<1$, we have

$$\lim_{y\to 0^+}\|\mathcal{T}u(\cdot+y)-\mathcal{T}u(\cdot)\|_{L^{q,\mu}(\mathbb{R}^+)}=0$$

uniformly in $\mathcal{T}u\in\mathcal{T}(K)$.

(III) Let $E_\gamma=\{x\in\mathbb{R}: x>\gamma\}$. It is clear that

$$\|\mathcal{T}(u)\chi_{E_\gamma}\|_{L^{q,\mu}(\mathbb{R}^+)} = \sup_{x,r\in\mathbb{R}^+}\left(r^{-\mu}\int_{x-r}^{x+r}|\mathcal{T}u(y)\chi_{E_\gamma}(y)|^q dy\right)^{\frac{1}{q}} = 0,$$

for any $u\in K$ and $\gamma>T$. Therefore,

$$\lim_{\gamma\to\infty}\|(\mathcal{T}u)\chi_{E_\gamma}\|_{L^{q,\mu}(\mathbb{R}^+)}=0$$

uniformly in $\mathcal{T}u\in\mathcal{T}(K)$.

From the above discussion and Lemma 2.1, it is easy to see that $\mathcal{T}(K)$ is strongly pre-compact in $\widetilde{L}^{q,\mu}(0,T)$. Thus $\mathcal{T}$ is a compact operator from $\widetilde{L}^{p,\beta}(0,T)$ to $\widetilde{L}^{q,\mu}(0,T)$. □

3. Representation of functions by fractional integrals in Morrey spaces

The integral equation

$$I^\alpha_{0^+}\varphi(x)=f(x),\quad x>0, \tag{3.1}$$

where $0<\alpha<1$, is called *Abel's equation* [25]. Denote by

$$f_{1-\alpha}(x)=I^{1-\alpha}_{0^+}f(x)=\frac{1}{\Gamma(1-\alpha)}\int_0^x\frac{f(t)dt}{(x-t)^\alpha}. \tag{3.2}$$

In continuous or Lebesgue spaces, the solvability condition of (3.1) is usually related to the absolutely continuous functions.

Definition 3.1. A function $f(x)$ is called absolutely continuous on an interval Ω, if for any $\varepsilon>0$ there exists $\delta>0$ such that for any finite set of pairwise nonintersecting intervals $[a_k,b_k]\subset\Omega$, $k=1,2,\cdots,n$, such that $\sum_{k=1}^n(b_k-a_k)<\delta$, the inequality $\sum_{k=1}^n|f(b_k)-f(a_k)|<\varepsilon$ holds. The space of these functions is denoted by $AC(\Omega)$.

It is known [25] that the space $AC(\Omega)$ coincides with the space of primitives of Lebesgue summable functions:

$$f(x) \in AC([a,b]) \Leftrightarrow f(x) = c + \int_a^x \varphi(t)dt, \quad \int_a^b |\varphi(t)|dt < \infty. \tag{3.3}$$

We first recall the representation of functions by fractional integrals in Lebesgue spaces, see Theorem 2.1 in [25].

Theorem 3.1. *Let $0 < \lambda < 1$. Abel equation (3.1) is solvable in $L^1(0,T)$ if and only if*

$$f_{1-\alpha}(x) \in AC([0,T]), \quad f_{1-\alpha}(0) = 0. \tag{3.4}$$

These conditions being satisfied the equation has a unique solution given by

$$\varphi(x) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_0^x \frac{f(t)dt}{(x-t)^\alpha} = D_{0+}^\alpha f(x). \tag{3.5}$$

We have the following characterization of the solution to (3.1) in Morrey space $L^{1,\lambda}(0,T)$.

Theorem 3.2. *Let $0 < \lambda < 1$. Abel equation (3.1) is solvable in $L^{1,\lambda}(0,T)$ if and only if (3.4) holds and*

$$f'_{1-\alpha}(x) \in L^{1,\lambda}(0,T),$$

where $f'_{1-\alpha}(x) = \frac{d}{dx} f_{1-\alpha}(x)$. These conditions being satisfied the equation has a unique solution given by (3.5).

Proof. The proof proceeds as that of Theorem 2.1 in [25]. Assume that (3.1) is solvable in $L^{1,\lambda}(0,T)$. We have

$$\int_0^x \frac{dt}{(x-t)^\alpha} \int_0^t \frac{\varphi(\tau)}{(t-\tau)^{1-\alpha}} d\tau = \Gamma(\alpha) \int_0^x \frac{f(t)dt}{(x-t)^\alpha}.$$

Using the Fubini theorem, we get

$$\int_0^x \varphi(\tau)d\tau \int_\tau^x \frac{dt}{(x-t)^\alpha (t-\tau)^{1-\alpha}} = \Gamma(\alpha) \int_0^x \frac{f(t)dt}{(x-t)^\alpha}.$$

Therefore,

$$B(\alpha, 1-\alpha) \int_0^x \varphi(\tau)d\tau = \Gamma(\alpha) \int_0^x \frac{f(t)dt}{(x-t)^\alpha}.$$

So we have

$$\int_0^x \varphi(\tau)d\tau = \frac{1}{\Gamma(1-\alpha)} \int_0^x \frac{f(t)dt}{(x-t)^\alpha}. \tag{3.6}$$

Since $\varphi \in L^{1,\lambda}(0,T) \subset L^1(0,T)$, by (3.3), $f_{1-\alpha}(x) \in AC([0,T])$ and $f_{1-\alpha}(0) = 0$.

After differentiation in (3.6), we have

$$\varphi(x) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^x \frac{f(t)dt}{(x-t)^\alpha} = f'_{1-\alpha}(x). \tag{3.7}$$

Thus $f'_{1-\alpha}(x) \in L^{1,\lambda}(0,T)$.

On the other hand, since $f'_{1-\alpha}(x) \in L^{1,\lambda}(0,T)$, so the function $\varphi(x)$ given by (3.7) exists and $\varphi(x) \in L^{1,\lambda}(0,T)$. Next we will show that it is indeed a solution of

(3.1). Now substituting $\varphi(x)$ into the left hand side of (3.1) and denote the result by $g(x)$, i.e.

$$\frac{1}{\Gamma(\alpha)}\int_0^x \frac{\varphi(t)}{(x-t)^{1-\alpha}}dt = \frac{1}{\Gamma(\alpha)}\int_0^x \frac{f_{1-\alpha}'(t)}{(x-t)^{1-\alpha}}dt = g(x). \tag{3.8}$$

We will show that $g(x) = f(x)$, which proves the theorem. It is clear to see that (3.8) is an equation of type (3.1) with respect to $f_{1-\alpha}'(x)$. So by (3.7) we have

$$f_{1-\alpha}'(x) = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_0^x \frac{g(t)dt}{(x-t)^{\alpha}} = g_{1-\alpha}'(x).$$

By assumption, $f_{1-\alpha}$ is absolutely continuous, and by virtue of (3.6) with $g(x)$ substituting for $f(x)$ on the right hand side, $g_{1-\alpha}$ is also absolutely continuous, therefore,

$$f_{1-\alpha}(x) - g_{1-\alpha}(x) = c.$$

Since $f_{1-\alpha}(0) = 0$ by conjecture, and $g_{1-\alpha}(0) = 0$ because (3.8) is a solvable equation. Hence $c = 0$, so

$$\int_0^x \frac{f(t)-g(t)}{(x-t)^{\alpha}}dt = 0,$$

which is also an equation of type (3.1). The uniqueness of its solution implies that $f(x) - g(x) = 0$. This completes the proof. □

The following relation between the fractional integration operator I_{0+}^{α} and the fractional differentiation operator D_{0+}^{α} is given by Lemma 2.5 in [22]. We only need the 1-dimensional case.

Lemma 3.1. *Let* $0 < \alpha < 1$. *If* $f(x) \in L^1(0,T)$ *and* $f_{1-\alpha}(x) \in AC[0,T]$, *then the equality*

$$\left(I_{0+}^{\alpha}D_{0+}^{\alpha}f\right)(x) = f(x) - \frac{f_{1-\alpha}(0)}{\Gamma(\alpha)}x^{\alpha-1}$$

holds almost everywhere on $[0,T]$, *where* $f_{1-\alpha}(x)$ *is defined in* (3.2).

An analogue of the Marchaud fractional derivative in the case of an interval $[a,b]$, $-\infty < a < b \le \infty$ is defined as

$$\mathbf{D}_{a^+}^{\alpha}f = \frac{f(x)}{\Gamma(1-\alpha)(x-a)^{\alpha}} + \frac{\alpha}{\Gamma(1-\alpha)}\int_a^x \frac{f(x)-f(t)}{(x-t)^{1+\alpha}}dt.$$

Its truncated fractional derivative is defined by

$$\mathbf{D}_{a^+,\varepsilon}^{\alpha}f = \frac{f(x)}{\Gamma(1-\alpha)(x-a)^{\alpha}} + \frac{\alpha}{\Gamma(1-\alpha)}\psi_{\varepsilon}(x), \tag{3.9}$$

where

$$\psi_{\varepsilon}(x) = \int_a^{x-\varepsilon} \frac{f(x)-f(t)}{(x-t)^{1+\alpha}}dt, \quad \varepsilon > 0. \tag{3.10}$$

We obtain that $\mathbf{D}_{a^+}^{\alpha}$ is indeed the left inverse operator of the operator of fractional integration within the frames of the Morrey spaces.

Theorem 3.3. *Suppose* $1 < p, q < \infty$, $0 < \alpha, \beta, \lambda < 1$, $\frac{1}{p} = \frac{1-\lambda}{q}$ *and* $\frac{1}{p} < \alpha < 1$.
(1) *If* f *can be represented as* $f = I^{\alpha}_{0+}\varphi$, $\varphi \in L^{p,\beta}(0,T)$, *then* $f \in L^{q,\lambda}(0,T)$ *and*

$$\mathbf{D}^{\alpha}_{0+} f = \lim_{\varepsilon \to 0} \mathbf{D}^{\alpha}_{0+,\varepsilon} f = \varphi, \tag{3.11}$$

holds in Morrey spaces $L^{q,\lambda}(0,T)$. *Conversely,*
(2) *If* $f \in L^{p,\beta}(0,T)$ *and there exists* $\varphi \in L^{q,\lambda}(0,T)$ *such that* (3.11) *holds in* $L^{q,\lambda}(0,T)$, *then* f *can be represented as* $f = I^{\alpha}_{0+}\varphi$.

Proof. (1) By assumption,

$$f(x) - f(x-t) = \frac{1}{\Gamma(\alpha)} \int_0^x y^{\alpha-1}\varphi(x-y)dy - \frac{1}{\Gamma(\alpha)} \int_t^x (y-t)^{\alpha-1}\varphi(x-y)dy.$$

Set

$$k(\xi) = \frac{1}{\Gamma(\alpha)} \begin{cases} \xi^{\alpha-1}, & 0 < \xi < 1, \\ \xi^{\alpha-1} - (\xi-1)^{\alpha-1}, & \xi > 1, \end{cases}$$

we have

$$f(x) - f(x-t) = t^{\alpha-1} \int_0^x k\left(\frac{y}{t}\right) \varphi(x-y)dy.$$

Thus, for $\varepsilon \le x \le T$, we get

$$\psi_\varepsilon(x) = \int_0^x \frac{\varphi(x-y)}{y} dy \int_{\frac{y}{x}}^{\frac{y}{\varepsilon}} k(s)ds.$$

By calculation (see P.125 in [25]),

$$\int_0^t k(s)ds = \alpha^{-1} t \Gamma(1-\alpha)\mathcal{K}(t),$$

where

$$\mathcal{K}(t) = \frac{\sin \alpha\pi}{\pi} \frac{t_+^{\alpha} - (t-1)_+^{\alpha}}{t}, \tag{3.12}$$

which has the properties:

$$\int_0^\infty \mathcal{K}(t)dt = 1, \tag{3.13}$$

here $t_+ = t$ if $t \ge 0$ and $t_+ = 0$ if $t < 0$. Therefore,

$$\frac{\alpha}{\Gamma(1-\alpha)} \psi_\varepsilon(x) = \int_0^x \varphi(x-y) \left[\frac{1}{\varepsilon}\mathcal{K}\left(\frac{y}{\varepsilon}\right) - \frac{1}{x}\mathcal{K}\left(\frac{y}{x}\right) \right] dy.$$

Since

$$\mathcal{K}\left(\frac{y}{x}\right) = \frac{\sin \alpha\pi}{\pi} \left(\frac{y}{x}\right)^{\alpha-1},$$

we have

$$\frac{\alpha}{\Gamma(1-\alpha)} \psi_\varepsilon(x) = \int_0^{\frac{x}{\varepsilon}} \mathcal{K}(y)\varphi(x-\varepsilon y)dy - \frac{f(x)}{\Gamma(1-\alpha)(x)^{\alpha}}.$$

For convenience we consider the function $\varphi(x)$ to be continued by zero beyond the interval $[0,T]$. Then by (3.13),

$$\mathbf{D}^{\alpha}_{0+,\varepsilon} f(x) - \varphi(x) = \int_0^\infty \mathcal{K}(y)[\varphi(x-\varepsilon y) - \varphi(x)]dy, \quad \varepsilon \le x < T. \tag{3.14}$$

For $0 < x < \varepsilon$, by (3.10) and $\varphi(x) = 0$ beyond the interval $[0, T]$, we have

$$\mathbf{D}^{\alpha}_{0^+,\varepsilon} f(x) - \varphi(x) = \frac{f(x)}{\varepsilon^{\alpha}\Gamma(1-\alpha)} = \frac{\sin\alpha\pi}{\pi\varepsilon^{\alpha}} \int_0^x \frac{\varphi(x-t)}{t^{1-\alpha}} dt. \tag{3.15}$$

For any $z \in (0, T),\ r > 0$, using the Minkowski inequality, we obtain

$$\begin{aligned} &\frac{1}{r^{\frac{\lambda}{q}}} \|\mathbf{D}^{\alpha}_{0^+,\varepsilon} f - \varphi\|_{L^q(B(z,r)\cap(0,T))} \\ \le& \frac{1}{r^{\frac{\lambda}{q}}} \int_0^{\infty} \mathcal{K}(y) \|\varphi(\cdot - \varepsilon y) - \varphi(\cdot)\|_{L^q(B(z,r)\cap[\varepsilon,T))} dy \\ &+ \frac{1}{r^{\frac{\lambda}{q}}} \frac{1}{\varepsilon^{\alpha}\Gamma(1-\alpha)} \|f\|_{L^q(B(z,r)\cap(0,\varepsilon))} \\ :=& I_1 + I_2. \end{aligned} \tag{3.16}$$

By Hölder's inequality, we have

$$\begin{aligned} I_1 &= \frac{1}{r^{\frac{\lambda}{q}}} \int_0^{\infty} \mathcal{K}(y) \left(\int_{B(z,r)\cap(\varepsilon,T)} |\varphi(x-\varepsilon y) - \varphi(x)|^q dx \right)^{\frac{1}{q}} dy \\ &\le \frac{C}{r^{\frac{\lambda}{q}}} \int_0^{\infty} \mathcal{K}(y) \left(\int_{B(z,r)\cap(0,T)} |\varphi(x-\varepsilon y) - \varphi(x)|^p dx \right)^{\frac{1}{p}} dy \cdot r^{\frac{1}{q}-\frac{1}{p}} \\ &= C \int_0^{\infty} \mathcal{K}(y) \left(\int_{B(z,r)\cap(0,T)} |\varphi(x-\varepsilon y) - \varphi(x)|^p dx \right)^{\frac{1}{p}} dy. \end{aligned}$$

For I_2, by Minkowski's inequality and Hölder's inequality, we get

$$\begin{aligned} I_2 &= \frac{1}{r^{\frac{\lambda}{q}}} \frac{1}{\varepsilon^{\alpha}\Gamma(1-\alpha)} \|f\|_{L^q(B(z,r)\cap(0,\varepsilon))} \\ &\le \frac{\sin\alpha\pi}{\pi\varepsilon^{\alpha}} \frac{1}{r^{\frac{\lambda}{q}}} \left(\int_{B(z,r)\cap(0,\varepsilon)} \left| \int_0^x \frac{\varphi(x-t)dt}{t^{1-\alpha}} \right|^q dx \right)^{\frac{1}{q}} \\ &= \frac{\sin\alpha\pi}{\pi\varepsilon^{\alpha}} \frac{1}{r^{\frac{\lambda}{q}}} \left(\int_0^{\varepsilon} \chi_{B(z,r)}(x) \left| \int_0^x \frac{\varphi(x-t)dt}{t^{1-\alpha}} \right|^q dx \right)^{\frac{1}{q}} \\ &\le \frac{\sin\alpha\pi}{\pi\varepsilon^{\alpha}} \frac{1}{r^{\frac{\lambda}{q}}} \int_0^{\varepsilon} \frac{1}{t^{1-\alpha}} \left(\int_t^{\varepsilon} \chi_{B(z,r)}(x) |\varphi(x-t)|^q dx \right)^{\frac{1}{q}} dt \\ &\le \frac{\sin\alpha\pi}{\pi\varepsilon^{\alpha}} \frac{1}{r^{\frac{\lambda}{q}}} \int_0^{\varepsilon} \frac{1}{t^{1-\alpha}} \left[\left(\int_t^{\varepsilon} |\varphi(x-t)|^p dx \right)^{\frac{q}{p}} \left(\int_{B(z,r)} dx \right)^{1-\frac{q}{p}} \right]^{\frac{1}{q}} dt \\ &\le C \frac{\sin\alpha\pi}{\pi\varepsilon^{\alpha}} \int_0^{\varepsilon} \frac{1}{t^{1-\alpha}} \left(\int_t^{\varepsilon} |\varphi(x-t)|^p dx \right)^{\frac{1}{p}} dt \\ &\le C \|\varphi\|_{L^p(0,\varepsilon)}. \end{aligned}$$

Therefore,

$$\begin{aligned} &\|\mathbf{D}^{\alpha}_{0^+,\varepsilon} f - \varphi\|_{L^{q,\lambda}(0,T)} \\ \le& C \int_0^{\infty} \mathcal{K}(y) \left(\int_{B(z,r)\cap(0,T)} |\varphi(x-\varepsilon y) - \varphi(x)|^p dx \right)^{\frac{1}{p}} dy + C\|\varphi\|_{L^p(0,\varepsilon)}. \end{aligned} \tag{3.17}$$

The first term on the right hand side goes to zero by the Lebesgue dominated convergence theorem. The second term tends to zero by the absolutely continuous property of the Lebesgue integral.

(2) Set $\varphi_\varepsilon = \mathbf{D}^\alpha_{0+,\varepsilon} f$. Due to the continuity of the operator I^α_{0+} in $L^{q,\lambda}(0,T)$, it is sufficient to prove that

$$f = \lim_{\varepsilon\to 0} I^\alpha_{0+}\varphi_\varepsilon.$$

By (3.9) and (3.10), when $\varepsilon \le x < T$, we have

$$I^\alpha_{0+}\varphi_\varepsilon(x) = \frac{\sin\alpha\pi}{\pi}\bigg(\int_\varepsilon^x \frac{f(y)dy}{(x-y)^{1-\alpha}y^\alpha} + \frac{1}{\varepsilon^\alpha}\int_0^\varepsilon \frac{f(y)dy}{(x-y)^{1-\alpha}} + \alpha\int_\varepsilon^x \frac{dy}{(x-y)^{1-\alpha}}\int_0^{y-\varepsilon}\frac{f(y)-f(t)}{(y-t)^{1+\alpha}}dt\bigg).$$

Then by simple calculation, we obtain

$$\begin{aligned} I^\alpha_{0+}\varphi_\varepsilon(x) &= \frac{\sin\alpha\pi}{\pi}\bigg(\frac{1}{\varepsilon^\alpha}\int_0^x \frac{f(y)dy}{(x-y)^{1-\alpha}} - \alpha\int_\varepsilon^x f(t)dt\int_{t+\varepsilon}^x \frac{dy}{(x-y)^{1-\alpha}(y-t)^{1+\alpha}}\bigg) \\ &= \frac{\sin\alpha\pi}{\pi\varepsilon^\alpha}\bigg(\int_0^x \frac{f(y)dy}{(x-y)^{1-\alpha}} - \int_0^{x-\varepsilon}\frac{f(t)(x-\varepsilon-t)^\alpha}{x-t}dt\bigg). \end{aligned}$$

Thus it is easy to show that

$$I^\alpha_{0+}\varphi_\varepsilon(x) = \int_0^{\frac{x}{\varepsilon}} \mathcal{K}(t)f(x-\varepsilon t)dt, \quad \varepsilon \le x < T, \tag{3.18}$$

where $\mathcal{K}(t)$ is defined in (3.12).

When $0 < x < \varepsilon$, by (3.10) considering $f(x)$ to be continued by zero beyond the interval $[0,T]$, we have

$$I^\alpha_{0+}\varphi_\varepsilon(x) = \frac{\sin\alpha\pi}{\pi\varepsilon^\alpha}\int_0^x \frac{f(y)dy}{(x-y)^{1-\alpha}}, \quad 0 < x < \varepsilon. \tag{3.19}$$

Having obtained (3.18) and (3.19), it is now shown similarly to (3.14)-(3.17) that

$$\lim_{\varepsilon\to 0} I^\alpha_{0+}\varphi_\varepsilon = f,$$

in $L^{q,\lambda}(0,T)$. The theorem is proved. □

4. Nonlinear fractional differential equations in Morrey spaces

Consider the classical Cauchy problem for the nonlinear fractional differential equation:

$$\begin{cases} D^\alpha_{0+}u(t) = f(t,u(t)), \\ I^{1-\alpha}_{0+}u(0) = 0. \end{cases} \tag{4.1}$$

The initial condition $I^{1-\alpha}_{0+}u(0) = 0$ in (4.1) is (more or less) equivalent to the following initial (weighted) condition:

$$\lim_{t\to 0^+} t^{1-\alpha}u(t) = 0.$$

For more details, see Lemma 4.2.

Numerous works are devoted to the study of (4.1) in continuous or integrable spaces. For the case of continuous or Hölder continuous, one can refer to [2, 19]. For the case of integrable spaces, see [4, 20]. In the present work, we will consider the existence of solutions to Cauchy problems for nonlinear fractional differential equations in nonseparable Morrey spaces.

Let us first recall the famous Schauder fixed-point theorem proved in 1930 (see [26]).

Lemma 4.1. *Let H be a convex and closed subset of a Banach space. Then any continuous and compact map $T: H \to H$ has a fixed point.*

Using this fixed-point theorem we obtain the existence and uniqueness of solutions to the Cauchy problem (4.1) in Morrey spaces.

Theorem 4.1. *Let $1 < p, q < \infty$, $0 \le \beta, \mu \le 1$. Suppose the operator $F: F(u) = f(t, u(t))$ is bounded, continuous from $\widetilde{L}^{p,\beta}(0,\delta)$ to $\widetilde{L}^{q,\mu}(0,\delta)$. If $\frac{1}{2}(1+\frac{1}{q}) < \alpha < 1$, then the Cauchy problem (4.1) has at least a solution $u \in \widetilde{L}^{p,\beta}(0,\delta)$ for a sufficiently small δ.*

Furthermore, if there exists a constant $C_F \in \mathbf{R}^+$, such that

$$\|Fu - Fv\|_{L^{q,\mu}(\mathbb{R}^+)} \le C_F \|u - v\|_{L^{p,\beta}(\mathbb{R}^+)}, \quad u, v \in L^{p,\beta}(\mathbb{R}^+), \tag{4.2}$$

then the solution of (4.1) is unique in $\widetilde{L}^{p,\beta}(0,\delta)$ for a sufficiently small δ.

Proof. Since $f(t,u(t)) \in L(0,\delta)$ and $D^{\alpha}_{0+}u(t) = \frac{d}{dt} I^{1-\alpha}_{0+} u(t)$, hence by (3.3), $u_{1-\alpha} = I^{1-\alpha}_{0+} u(t) \in AC([0,\delta])$. Then by Lemma 3.1, in $\widetilde{L}^{p,\beta}(0,\delta)$, the derivative equation (4.1) is equivalent to the following integral equation

$$\begin{aligned} u(t) &= \begin{cases} \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^t \frac{f(\tau, u(\tau))}{(t-\tau)^{(1-\alpha)}} d\tau, & t \le \delta \\ 0, & a.e. \quad t \ge \delta, \end{cases} \\ &= \frac{\chi_{(0,\delta)}(t)}{\Gamma(\alpha)} \int_0^t \frac{f(\tau, u(\tau))}{(t-\tau)^{1-\alpha}} d\tau \\ &= T(f(t,u)). \end{aligned} \tag{4.3}$$

Set

$$Au(t) = T(f(t,u)) = \frac{\chi_{(0,\delta)}(t)}{\Gamma(\alpha)} \int_0^t \frac{f(\tau, u(\tau))}{(t-\tau)^{1-\alpha}} d\tau.$$

Then the equation (4.3) has a solution in $\widetilde{L}^{p,\beta}(0,\delta)$ if and only if the operator A has a fixed point in $\widetilde{L}^{p,\beta}(0,\delta)$.

Firstly, we will show that A is completely continuous. By Proposition 2.2, T is a compact operator from $\widetilde{L}^{q,\mu}(0,\delta)$ to $\widetilde{L}^{p,\beta}(0,\delta)$. Since $F: u \to f(t,u)$ is bounded and continuous from $\widetilde{L}^{p,\beta}(0,\delta)$ to $\widetilde{L}^{q,\mu}(0,\delta)$ and $Au(t) = TFu(t)$. Therefore, A is a compact operator from $\widetilde{L}^{p,\beta}(0,\delta)$ to $\widetilde{L}^{p,\beta}(0,\delta)$ and is also continuous. Thus $A: \widetilde{L}^{p,\beta}(0,\delta) \to \widetilde{L}^{p,\beta}(0,\delta)$ is completely continuous.

Set $D = \{u : \|u\|_{\widetilde{L}^{p,\beta}(0,\delta)} \le R\}$. Then D is a bounded closed convex set. For any $x, r > 0$,

$$r^{-\beta} \int_{x-r}^{x+r} |Au|^p = r^{-\beta} \int_{x-r}^{x+r} \left| \frac{\chi_{(0,\delta)}(t)}{\Gamma(\alpha)} \int_0^t \frac{f(\tau, u(\tau))}{(t-\tau)^{1-\alpha}} d\tau \right|^p dt.$$

(i) When $r \geq \delta$, by Hölder's inequality, we have

$$\begin{aligned}
& r^{-\beta} \int_{x-r}^{x+r} |Au|^p \\
\leq & \delta^{-\beta} \int_{x-r}^{x+r} \left| \frac{\chi_{(0,\delta)}(t)}{\Gamma(\alpha)} \int_0^t \frac{f(\tau, u(\tau))}{(t-\tau)^{1-\alpha}} d\tau \right|^p dt \\
\leq & \frac{1}{\delta^\beta \Gamma(\alpha)^p} \int_0^\delta \left[\left(\int_0^t |f(\tau, u(\tau))|^q d\tau \right)^{\frac{1}{q}} \left(\int_0^t (t-\tau)^{\frac{-(1-\alpha)q}{q-1}} d\tau \right)^{1-\frac{1}{q}} \right]^p \\
\leq & \frac{\|Fu\|^p_{L^{q,\mu}(\mathbb{R}^+)}}{\delta^\beta \Gamma(\alpha)^p (\frac{\alpha q-1}{q-1})^{(1-\frac{1}{q})p}} \int_0^\delta t^{\frac{\mu p+(\alpha q-1)p}{q}} dt \\
\leq & \frac{\|F\|^p R^p \delta^{1-\beta+\frac{\mu p+(\alpha q-1)p}{q}}}{\Gamma(\alpha)^p (\frac{\alpha q-1}{q-1})^{(1-\frac{1}{q})p} (1+\frac{\mu p+(\alpha q-1)p}{q})}.
\end{aligned}$$

(ii) When $r < \delta$, by Hölder's inequality, we get

$$\begin{aligned}
& r^{-\beta} \int_{x-r}^{x+r} |Au|^p \\
\leq & \frac{1}{r^\beta \Gamma(\alpha)^p} \int_{x-r}^{x+r} \chi_{(0,\delta)}(t) \left[\left(\int_0^t |f(\tau, u(\tau))|^q d\tau \right)^{\frac{1}{q}} \left(\int_0^t (t-\tau)^{\frac{-(1-\alpha)q}{q-1}} d\tau \right)^{1-\frac{1}{q}} \right]^p dt \\
\leq & \frac{\|Fu\|^p_{L^{q,\mu}(\mathbb{R}^+)}}{r^\beta \Gamma(\alpha)^p (\frac{\alpha q-1}{q-1})^{(1-\frac{1}{q})p}} \int_{x-r}^{x+r} \chi_{(0,\delta)}(t) t^{\frac{\mu p+(\alpha q-1)p}{q}} dt \\
\leq & \frac{2\|F\|^p R^p}{\Gamma(\alpha)^p (\frac{\alpha q-1}{q-1})^{(1-\frac{1}{q})p}} \delta^{\frac{\mu p+(\alpha q-1)p}{q}} r^{1-\beta} \\
\leq & \frac{2\|F\|^p R^p}{\Gamma(\alpha)^p (\frac{\alpha q-1}{q-1})^{(1-\frac{1}{q})p}} \delta^{1-\beta+\frac{\mu p+(\alpha q-1)p}{q}}.
\end{aligned}$$

Therefore,

$$\|Au\|_{L^{p,\beta}(\mathbb{R}^+)} \leq \frac{2^{\frac{1}{p}} \|F\| R}{\Gamma(\alpha) (\frac{\alpha q-1}{q-1})^{(1-\frac{1}{q})} \min\{(1+\frac{\mu p+(\alpha q-1)p}{q})^{\frac{1}{p}}, 1\}} \delta^{\frac{1-\beta}{p}+\frac{\mu+(\alpha q-1)}{q}}.$$

Set

$$\delta = \left[\frac{\Gamma(\alpha) \left(\frac{\alpha q-1}{q-1}\right)^{(1-\frac{1}{q})} \min\{(1+\frac{\mu p+(\alpha q-1)p}{q})^{\frac{1}{p}}, 1\}}{2^{\frac{1}{p}} \|F\|} \right]^{\left[\frac{1-\beta}{p}+\frac{\mu+(\alpha q-1)}{q})\right]^{-1}}.$$

Then

$$\|Au\|_{L^{p,\beta}(\mathbb{R}^+)} \leq R.$$

That is to say $A : D \to D$. Therefore, A has a fixed point in D. Consequently, Equation (4.1) has at least a solution in $\widetilde{L}^{p,\beta}(0, \delta)$.

Furthermore, if (4.2) holds, then for $u_1,\ u_2 \in \widetilde{L}^{p,\beta}(0,\delta)$, by the similar calculations as those in (i) and (ii), we can obtain

$$\begin{aligned}
&\|Au_1 - Au_2\|_{L^{p,\beta}(\mathbb{R}^+)}\\
\leq&\frac{2^{\frac{1}{p}}\|Fu_1 - Fu_2\|_{\widetilde{L}^{q,\mu}(0,\delta)}}{\Gamma(\alpha)(\frac{\alpha q-1}{q-1})^{(1-\frac{1}{q})}\min\{(1+\frac{\mu p+(\alpha q-1)p}{q})^{\frac{1}{p}},1\}}\delta^{\frac{1-\beta}{p}+\frac{\mu+(\alpha q-1)}{q}}\\
\leq&\frac{2^{\frac{1}{p}}C_F}{\Gamma(\alpha)(\frac{\alpha q-1}{q-1})^{(1-\frac{1}{q})}\min\{(1+\frac{\mu p+(\alpha q-1)p}{q})^{\frac{1}{p}},1\}}\delta^{\frac{1-\beta}{p}+\frac{\mu+(\alpha q-1)}{q}}\|u_1-u_2\|_{L^{p,\beta}(\mathbb{R}^+)}.
\end{aligned} \tag{4.4}$$

Set

$$\delta < \left[\frac{\Gamma(\alpha)\left(\frac{\alpha q-1}{q-1}\right)^{(1-\frac{1}{q})}\min\{(1+\frac{\mu p+(\alpha q-1)p}{q})^{\frac{1}{p}},1\}}{2^{\frac{1}{p}}C_F}\right]^{\left[\frac{1-\beta}{p}+\frac{\mu+(\alpha q-1)}{q})\right]^{-1}}.$$

Then A is a contraction mapping in $\widetilde{L}^{p,\beta}(0,\delta)$, therefore, A has a unique fixed point in $\widetilde{L}^{p,\beta}(0,\delta)$. Consequently, the Cauchy problem (4.1) has a unique solution in $\widetilde{L}^{p,\beta}(0,\delta)$. □

The result of Theorem 4.1 remains true for the weighted Cauchy type problem:

$$\begin{cases} D^{\alpha}_{0^+}u(t) = f(t,u(t)),\\ \lim\limits_{t\to 0^+} t^{1-\alpha}u(t) = 0.\end{cases} \tag{4.5}$$

Its proof relies on the following assertion appeared in [22] with slight modifications.

Lemma 4.2. *Let $0<\alpha<1$; $b,c\in\mathbb{R}$ and let $u(t)$ be a Lebesgue measurable function on $[0,T]$.*

(I) *If there exits a.e. a limit*

$$\lim_{t\to 0^+} t^{1-\alpha}u(t) = c,$$

then there also exits a.e. a limit

$$I^{1-\alpha}_{0^+}u(0) := \lim_{t\to 0^+} I^{1-\alpha}_{0^+}u(t) = c\Gamma(\alpha).$$

(II) *If there exits a.e. a limit*

$$\lim_{t\to 0^+} I^{1-\alpha}_{0^+}u(t) = b,$$

and if there exits the limit $\lim_{t\to 0^+} t^{1-\alpha}u(t)$, then

$$\lim_{t\to 0^+} t^{1-\alpha}u(t) = \frac{b}{\Gamma(\alpha)}.$$

The following example shows that the conditions that the operator $F : F(u) = f(t,u(t))$ is bounded and continuous from $\widetilde{L}^{p,\beta}(0,\delta)$ to $\widetilde{L}^{q,\mu}(0,\delta)$, $\frac{1}{2}(1+\frac{1}{q})<\alpha<1$ are sufficient but not necessary. As is stated on Page 162 of [22], the spaces of continuous functions are preferable when considering the necessary condition for the solvability of fractional differential equations.

Example 4.1. Consider the following differential equation of fractional order $0 < \alpha < 1$,

$$\begin{cases} D^{\alpha}_{0+}u(t) = \lambda t^{\gamma}(u(t))^2, \\ I^{1-\alpha}_{0+}u(0) = 0, \end{cases} \tag{4.6}$$

where $t > 0,\ \lambda, \gamma \in \mathbb{R},\ \lambda \neq 0$.

By Property 2.1 in [22], it can be directly shown that equation (4.6) has the exact solution

$$u(t) = \begin{cases} \dfrac{\Gamma(1-\alpha-\gamma)}{\lambda\Gamma(1-2\alpha-\gamma)} t^{-(\alpha+\gamma)}, & 0 < t < \delta, \\ 0, \qquad a.e. \quad t \geq \delta \end{cases}$$

if $0 < \alpha + \gamma < 1$. In this case the right-hand side of the equation (4.6) takes the form

$$f(t, u(t)) = \begin{cases} \dfrac{1}{\lambda}\left[\dfrac{\Gamma(1-\alpha-\gamma)}{\Gamma(1-2\alpha-\gamma)}\right]^2 t^{-(2\alpha+\gamma)}, & 0 < t < \delta, \\ 0, \qquad a.e. \quad t \geq \delta. \end{cases}$$

We claim that $u \in \widetilde{L}^{p,\beta}(0,\delta)$ if $1-(\alpha+\gamma)p-\beta > 0$. While $f(t,u(t)) \notin \widetilde{L}^{q,\mu}(0,\delta)$ when $(2\alpha+\gamma)q > 1$.

In fact, for any $r > 0, x > 0$, we have

$$\begin{aligned} r^{-\beta}\int_{B(x,r)} |u(t)|^p dt &= r^{-\beta}\int_{B(x,r)\cap(0,\delta)} \left(\frac{\Gamma(1-\alpha-\gamma)}{\lambda\Gamma(1-2\alpha-\gamma)}\right)^p t^{-(\alpha+\gamma)p} dt \\ &= Cr^{-\beta}\int_{B(x,r)\cap(0,\delta)} t^{-(\alpha+\gamma)p} dt. \end{aligned}$$

(1) If $r \geq \delta$, then

$$r^{-\beta}\int_{B(x,r)} |u(t)|^p dt \leq C\delta^{-\beta}\int_0^{\delta} t^{-(\alpha+\gamma)p} dt = C\delta^{1-(\alpha+\gamma)p-\beta} < \infty.$$

(2) If $r < \delta$, when $x \leq r$, we obtain

$$\begin{aligned} r^{-\beta}\int_{B(x,r)} |u(t)|^p dt &\leq Cr^{-\beta}\int_0^{2r} t^{-(\alpha+\gamma)p} dt \\ &= Cr^{1-(\alpha+\gamma)p-\beta} \\ &< C\delta^{1-(\alpha+\gamma)p-\beta} \\ &< \infty. \end{aligned}$$

When $x > r$, then

$$\begin{aligned} r^{-\beta}\int_{B(x,r)} |u(t)|^p dt &\leq Cr^{-\beta}\int_{x-r}^{x+r} t^{-(\alpha+\gamma)p} dt \\ &= Cr^{-\beta}\left[(x+r)^{1-(\alpha+\gamma)p} - (x-r)^{1-(\alpha+\gamma)p}\right] \\ &\leq Cr^{-\beta}(2r)^{1-(\alpha+\gamma)p} \\ &\leq C\delta^{1-(\alpha+\gamma)p-\beta} \\ &< \infty. \end{aligned}$$

Therefore, $u \in \widetilde{L}^{p,\beta}(0,\delta)$.

On the other hand, if $(2\alpha+\gamma)q>1$, take $x=\delta$ and $r=2\delta$. We conclude that

$$r^{-\mu}\int_{B(x,r)}|f(t,u(t))|^q dt=\infty.$$

Thus $f(t,u(t))\notin \widetilde{L}^{q,\mu}(0,\delta)$.

Competing interests

The authors declare that they have no competing interests.

Author's contributions

All authors contributed equally to the manuscript and typed, read and approved the final manuscript.

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