Journal of Applied Analysis and Computation
Volume 14, Number 2, April 2024, 1111–1132
Website:http://www.jaac-online.com
DOI:10.11948/20230387

THE NUMBER OF LIMIT CYCLES NEAR A DOUBLE HOMOCLINIC LOOP FOR A NEAR-HAMILTONIAN SYSTEM*

Xiaoyu Xu[1], Junmin Yang[1,†] and Tong Han[2]

Abstract In this paper, for a general near-Hamiltonian system we study the number and distributions of limit cycles near a double homoclinic loop. For a cubic Hamiltonian system with general polynomial perturbations, we obtain a lower bound of the maximum number of limit cycles near a double homoclinic loop.

Keywords Limit cycle, Melnikov function, homoclinic loop, bifurcation.

MSC(2010) 34C07, 34C23.

1. Introduction and main results

It is well known that the second part of the Hilbert's 16th problem proposed by Hilbert [14] is to study the maximum number and related locations of limit cycles for a planar polynomial system with degree n. There have been many works on studying the number of limit cycles near a center, a homoclinic loop, a heteroclinic loop or periodic orbits for a planar differential system with perturbations, see [5,13,17–19].

In this paper, we consider a near-Hamiltonian system of the form

$$\dot{x} = H_y + \varepsilon f(x, y, \delta), \quad \dot{y} = -H_x + \varepsilon g(x, y, \delta), \tag{1.1}$$

where H, f and g are analytic functions in (x, y), $\varepsilon > 0$ is a small parameter and $\delta \in D \subset \mathbf{R}^n$ with D compact. When $\varepsilon = 0$, (1.1) becomes

$$\dot{x} = H_y, \quad \dot{y} = -H_x, \tag{1.2}$$

which is a Hamiltonian system.

Arnold [1] proposed the weak Hilbert's 16th problem, which is to ask for the maximum number of isolated zeros of the Melnikov function,

$$M(h, \delta) = \oint_{H(x,y)=h} g\mathrm{d}x - f\mathrm{d}y,$$

†the corresponding author.
[1]School of Mathematical Sciences, Hebei Normal University, Shijiazhuang, Hebei 050024, China
[2]Golden Education, Hongkou District, Shanghai, 200083, China
*This project was supported by the National Natural Science Foundation of China (11971145).
Email: 13832174692@qq.com(X. Xu), jmyang@hebtu.edu.cn(J. Yang), hantong@gaodun.com(T. Han)

where the equation $H(x, y) = h$ defines a family of periodic orbits of system (1.2). To find the number of zeros of the Melnikov function, an important tool is to study the expansions of the Melnikov function near a center, a homoclinic loop or a heteroclinic loop with hyperbolic saddles or nilpotent singular points, see [9, 12, 15, 21] for example.

For the case that the equation $H(x, y) = 0$ defines a homoclinic loop L_0 with a hyperbolic saddle, Roussarie [21] proved that for $0 < |h| \ll 1$ the Melnikov function has an expansion of the form:

$$M(h, \delta) = \sum_{i \geq 0} (c_{2i}(\delta) + c_{2i+1}(\delta) h \ln |h|) h^i, \tag{1.3}$$

where $c_0 = \oint_{L_0} g\mathrm{d}x - f\mathrm{d}y$, and the formulas of c_1, c_2 and c_3 were respectively given by [10] and [8]. For $c_{2i+1} (i \geq 1)$, Han and Yu [11] gave a method of computing them. Later, Tian and Han [22] developed a method of computing c_{2i+1} and c_{2i} for $i \geq 2$ under some assumptions. Geng and Tian [3] generalized this method to calculate the coefficients appearing in the expansion of the Melnikov function near a heteroclinic loop with hyperbolic saddles. Some authors used the expansion given in (1.3) and the first few coefficients to study the number of limit cycles near a heteroclinic loop or a compound loop with hyperbolic saddles, see [20, 26]. In recent decades, the expansion of the Melnikov function was used to study the number of limit cycles near a generalized homoclinic loop or a generalized heteroclinic loop for a piecewise near-Hamiltonian system, see [23, 24].

In this paper, suppose that the equation $H(x, y) = 0$ defines a double homoclinic loop $L_0 (= L_{10} \cup L_{20})$ with a hyperbolic saddle at the origin, and the equation $H(x, y) = h$ defines a family of periodic orbits $L(h)$ for $0 < h \ll 1$ and two families of periodic orbits $L_1(h)$ and $L_2(h)$ for $0 < -h \ll 1$. See Figure 1.

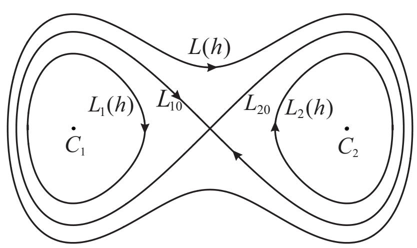

Figure 1. The double homoclinic loop L_0.

Correspondingly, there are three Melnikov functions below

$$\begin{aligned} M_j(h, \delta) &= \oint_{L_j(h)} g\mathrm{d}x - f\mathrm{d}y, \quad -h_0 < h < 0, \quad j = 1, 2, \\ M(h, \delta) &= \oint_{L(h)} g\mathrm{d}x - f\mathrm{d}y, \quad 0 < h < h_0, \end{aligned} \tag{1.4}$$

where h_0 is a small positive constant. For the expansions of $M_j(h,\delta)(j=1,2)$ and $M(h,\delta)$ near L_0 we have ([25])

$$\begin{aligned} M_j(h,\delta) &= \sum_{i\geq 0}\left(c_{2i,j}(\delta)+c_{2i+1}(\delta)h\ln|h|\right)h^i, \quad j=1,2, \quad 0<-h\ll 1, \\ M(h,\delta) &= \sum_{i\geq 0}\left(c_{2i}(\delta)+2c_{2i+1}(\delta)h\ln h\right)h^i, \quad 0<h\ll 1, \end{aligned} \tag{1.5}$$

where the first four coefficients were obtained in [25].

Recently, Han et al. [7] found the relation between $c_{2i,1}$, $c_{2i,2}$ and c_{2i} for $i\geq 0$ as given below

$$c_{2i} = c_{2i,1} + c_{2i,2}. \tag{1.6}$$

If system (1.1) is centrally symmetric, Han et al. [7] gave a way of obtaining limit cycles near L_0 as shown in the following lemma.

Lemma 1.1. *Suppose that system* (1.1) *is centrally symmetric, i.e.,* H,f,g *satisfy* $H(x,y)=H(-x,-y)$, $f(x,y,\delta)=-f(-x,-y,\delta)$, $g(x,y,\delta)=-g(-x,-y,\delta)$, *and the equation* $H(x,y)=0$ *defines a double homoclinic loop* L_0. *If there exist* $\delta_0\in D$ *and* $k\geq 1$ *such that*

$$c_k(\delta_0)\neq 0,\ c_j(\delta_0)=0,\ j=0,\cdots,k-1$$

and

$$\operatorname{rank}\left.\frac{\partial(c_0,c_1,\cdots,c_{k-1})}{\partial(\delta_1,\delta_2,\cdots,\delta_n)}\right|_{\delta=\delta_0}=k,$$

then for any given neighborhood V *of* L_0 *there exists* δ *near* δ_0 *such that for* $0<\varepsilon\ll 1$ *system* (1.1) *has at least* $\left[\frac{5}{2}k\right]$ *limit cycles in* V.

In this paper, suppose that system (1.1) is not centrally symmetric. We study the number of limit cycles near L_0 and obtain the following theorem.

Theorem 1.1. *Consider system* (1.1) *and let* (1.4)-(1.6) *hold. If there exists* δ_0 *such that*

$$c_{k,1}(\delta_0)c_{k,2}(\delta_0)>0$$

and

$$\begin{aligned} &c_i(\delta_0)=0,\ i=1,3,\cdots,k-1, \\ &c_{l,j}(\delta_0)=0,\ l=0,2,\cdots,k-2, \quad j=1,2, \\ &\operatorname{rank}\left.\frac{\partial(c_{0,1},c_{0,2},c_1,c_{2,1},c_{2,2},\cdots,c_{k-1})}{\partial\delta}\right|_{\delta=\delta_0}=\frac{3k}{2} \end{aligned} \tag{1.7}$$

for an even k, *or*

$$\begin{aligned} &c_{l,j}(\delta_0)=0,\ l=0,2,\cdots,k-1, \quad j=1,2, \\ &c_i(\delta_0)=0,\ i=1,3,\cdots,k-2, \quad c_k(\delta_0)\neq 0, \\ &\operatorname{rank}\left.\frac{\partial(c_{0,1},c_{0,2},c_1,c_{2,1},c_{2,2},\cdots,c_{k-1,1},c_{k-1,2})}{\partial\delta}\right|_{\delta=\delta_0}=\frac{3k+1}{2} \end{aligned}$$

for an odd k, *then for any given neighborhood* V *of* L_0 *there exists* δ *near* δ_0 *such that for* $0<\varepsilon\ll 1$ *system* (1.1) *has at least* $\left[\frac{5k}{2}\right]$ *limit cycles in* V *with three* *distributions* $(k,k)+\left[\frac{k}{2}\right]$, $(k,k-1)+(\left[\frac{k}{2}\right]+1)$ *and* $(k-1,k)+(\left[\frac{k}{2}\right]+1)$, *where* $(l_1,l_2)+l_3$ *means that* l_1 *limit cycles are near and inside* L_{10}, l_2 *limit cycles are near and inside* L_{20}, *and* l_3 *limit cycles are near and outside* L_0.

It should be noted that Han and Chen [6] proved the above theorem for $k=2$. They further proved that 5 is the maximal number of limit cycles near L_0. Iliev et al. [16] studied system (1.1) with

$$H(x,y)=\frac{1}{2}y^2-\frac{1}{2}x^2+\frac{1}{4}x^4, \tag{1.8}$$

and f,g being arbitrary cubic polynomials in (x,y). By the higher-order Melnikov functions they obtained the number of limit cycles bifurcated from $L_j(h)$ for $0<-h<\frac{1}{4}$ under some conditions.

Now, suppose that in (1.1) H is given by (1.8) and f,g are given by

$$f(x,y,\delta)=\sum_{i+j=1}^{n}a_{i,j}x^iy^j,\quad g(x,y,\delta)=\sum_{i+j=1}^{n}b_{i,j}x^iy^j,\quad n\geq 2, \tag{1.9}$$

or

$$f(x,y,\delta)=\sum_{\substack{i+j=1\\ i+j\ odd}}^{n}a_{i,j}x^iy^j,\quad g(x,y,\delta)=\sum_{\substack{i+j=1\\ i+j\ odd}}^{n}b_{i,j}x^iy^j,\quad n\geq 3. \tag{1.10}$$

In this case, the phase portrait of system (1.2) is shown in Figure 1, where $C_1(-1,0)$, $C_2(1,0)$ are elementary centers, the double homoclinic loop L_0 is defined by the equation $H(x,y)=0$ and

$$L_i(h)=\{(x,y)|H(x,y)=h,-\tfrac{1}{4}<h<0,(-1)^ix>0\},$$
$$L(h)=\{(x,y)|H(x,y)=h,0<h<h_0\}.$$

To obtain the number of limit cycles near L_0 by Theorem 1.1, a key step is to obtain the coefficients appearing in the expansions of M_1,M_2 and M. The method given in Tian and Han [22] is effective under some conditions. Motivated by [2], in this paper we obtain all the desired coefficients under some conditions. Then, by using these coefficients we obtain the number and distributions of limit cycles near the double homoclinic loop L_0 as shown in the following two theorems.

Theorem 1.2. *For system* (1.1), *let* H *satisfy* (1.8) *and* f,g *satisfy* (1.9). *There exists* δ_0 *such that for some* (ε,δ) *near* $(0,\delta_0)$ *system* (1.1) *has at least* $\left[\frac{5(n-1)}{2}\right]-\frac{1}{2}(1+(-1)^n)$, *denoted by* κ_n, *limit cycles near* L_0.

(1) *If* n *is even and* $n\geq 4$, *the three distributions of the* κ_n *limit cycles are* $(n-2,n-2)+\frac{n}{2}$, $(n-1,n-3)+\frac{n}{2}$ *and* $(n-1,n-2)+\frac{n}{2}-1$ (*or* $(n-2,n-2)+\frac{n}{2}$, $(n-3,n-1)+\frac{n}{2}$ *and* $(n-2,n-1)+\frac{n}{2}-1$). *If* $n=2$, *the distribution of the* $\kappa_n(=1)$ *limit cycle is* $(1,0)+0$ (*or* $(0,1)+0$).

(2) *If* n *is odd, the three distributions of the* κ_n *limit cycles are* $(n-1,n-1)+\frac{n-1}{2}$, $(n-1,n-2)+\frac{n+1}{2}$ *and* $(n-2,n-1)+\frac{n+1}{2}$.

Theorem 1.3. *For system* (1.1) *with* H *being given by* (1.8) *and* f,g *given by* (1.10), *there exists* δ_0 *such that for some* (ε,δ) *near* $(0,\delta_0)$ *system* (1.1) *has at least* $\frac{5(n-1)}{2}$ *limit cycles near* L_0 *with the distribution* $(n-1,n-1)+\frac{n-1}{2}$.

2. The number of limit cycles near L_0

Proof of Theorem 1.1. We first suppose that k is even with $k = 2m, m \in \mathbf{Z}_+$. By (1.7) and the inverse function theorem, the equations

$$c_{2i+1}(\delta) = c_{2i+1}, \quad c_{2i,j}(\delta) = c_{2i,j}, \quad i = 0, 1, \cdots, m-1, \quad j = 1, 2,$$

have a unique solution $\delta = \delta_0 + O(|c_{0,1}, c_{0,2}, c_1, c_{2,1}, c_{2,2}, \cdots, c_{2m-1}|)$, which means that $c_{0,1}, c_{0,2}, c_1, c_{2,1}, c_{2,2} \cdots, c_{2m-1}$ can be taken as free parameters. Then, the expansions of $M_j(h, \delta)$ $(j = 1, 2)$ and $M(h, \delta)$ in (1.5) can be written as the following form:

$$M_j(h, \delta) = \sum_{i=0}^{m-1} (c_{2i,j} + c_{2i+1} h \ln |h|) h^i + c_{2m,j} h^m + O(h^{m+1} \ln |h|) \tag{2.1}$$

for $0 < -h \ll 1$, and

$$M(h, \delta) = \sum_{i=0}^{m-1} (c_{2i} + 2c_{2i+1} h \ln h) h^i + c_{2m} h^m + O(h^{m+1} \ln h)$$

for $0 < h \ll 1$, where

$$c_{2m,j} = c_{2m,j}(\delta_0) + O(|c_{0,1}, c_{0,2}, c_1, c_{2,1}, c_{2,2}, \cdots, c_{2m-1}|)$$

and $c_{2m} = c_{2m,1} + c_{2m,2}$.

For $\delta = \delta_0$, i.e., $c_{0,1} = c_{0,2} = c_1 = c_{2,1} = c_{2,2} = \cdots = c_{2m-1} = 0$ and $c_{2m,j} = c_{2m,j}(\delta_0)$, we have

$$M_j(h, \delta) = c_{2m,j} h^m + O(h^{m+1} \ln |h|), \quad 0 < -h \ll 1, \quad j = 1, 2,$$
$$M(h, \delta) = c_{2m} h^m + O(h^{m+1} \ln h), \quad 0 < h \ll 1.$$

Note that $c_{2m,j} \neq 0$. There exist $h^*_{0,j}$ and $\bar{h}^*_0$ such that $M_j(h^*_{0,j}, \delta_0) \neq 0$ and $M(\bar{h}^*_0, \delta_0) \neq 0$.

Next, we will take suitable values of parameter δ to find simple zeros of $M_j(h, \delta)$ and $M(h, \delta)$.

Firstly, let $c_{0,1} = c_{0,2} = c_1 = c_{2,1} = c_{2,2} = \cdots = c_{2m-2,1} = c_{2m-2,2} = 0$ and

$$0 < |c_{2m-1}| \ll 1, \quad c_{2m-1} c_{2m,j} > 0, \quad j = 1, 2.$$

In this case we have

$$M_j(h, \delta) = h^m (c_{2m-1} \ln |h| + c_{2m,j}) + O(h^{m+1} \ln |h|), \quad 0 < -h \ll 1, \quad j = 1, 2,$$
$$M(h, \delta) = h^m (2c_{2m-1} \ln h + c_{2m}) + O(h^{m+1} \ln h), \quad 0 < h \ll 1.$$

It is easy to see that $M(h, \delta)$ has a simple zero $\bar{h}^*_1 \in (0, \bar{h}^*_0)$, and $M_j(h, \delta)$ has a simple zero $h^*_{1,j} \in (h^*_{0,j}, 0)$ for each j.

Secondly, let $c_{0,1} = c_{0,2} = c_1 = c_{2,1} = c_{2,2} = \cdots = c_{2m-3} = 0$ and

$$|c_{2m-2,1}| \ll |c_{2m-1}|, \quad |c_{2m-2,2}| \ll |c_{2m-1}|.$$

Then, for $0 < |h| \ll 1$, $M_j(h,\delta)$ and $M(h,\delta)$ can be written as given below:

$$M_j(h,\delta) = h^{m-1}(c_{2m-2,j} + c_{2m-1} h \ln|h|) + O(h^m),$$
$$M(h,\delta) = h^{m-1}(c_{2m-2} + 2c_{2m-1} h \ln h) + O(h^m).$$

To find the maximal number of simple zeros of $M_j(h,\delta)$ and $M(h,\delta)$ obtained by this step, we consider the following cases according to the sign of $c_{2m-2,1}, c_{2m-2,2}$ and c_{2m-1}:

(1.i) $c_{2m-2,1}c_{2m-2,2} < 0$, $c_{2m-2,1}c_{2m-1} > 0$, $(c_{2m-2,1} + c_{2m-2,2})c_{2m-1} > 0$ (resp., < 0);

(1.ii) $c_{2m-2,1}c_{2m-2,2} < 0$, $c_{2m-2,1}c_{2m-1} < 0$, $(c_{2m-2,1} + c_{2m-2,2})c_{2m-1} > 0$ (resp., < 0);

(1.iii) $c_{2m-2,1}c_{2m-2,2} > 0$ and $c_{2m-2,1}c_{2m-1} > 0$;

(1.iv) $c_{2m-2,1}c_{2m-2,2} > 0$ and $c_{2m-2,1}c_{2m-1} < 0$.

We denote the number of simple zeros of $M_1(h,\delta)$, $M_2(h,\delta)$ and $M(h,\delta)$ obtained in the second step by $\mu_1^{[2]}, \mu_2^{[2]}$ and $\mu_3^{[2]}$, respectively. And let $\mu^{[2]} = \mu_1^{[2]} + \mu_2^{[2]} + \mu_3^{[2]}$. Similar to the first step, we obtain the following table, which shows the values of $\mu_1^{[2]}, \mu_2^{[2]}, \mu_3^{[2]}$ and $\mu^{[2]}$ in each one of the above cases.

The sign of $c_{2m-2,1}c_{2m-2,2}$	The sign of $c_{2m-2,1}c_{2m-1}$	The sign of $c_{2m-2}c_{2m-1}$	$\mu_1^{[2]}$	$\mu_2^{[2]}$	$\mu_3^{[2]}$	$\mu^{[2]}$
$-$	$+$	$+$	0	1	1	2
		$-$	0	1	0	1
	$-$	$+$	1	0	1	2
		$-$	1	0	0	1
$+$	$+$	$+$	0	0	1	1
	$-$	$-$	1	1	0	2

Thirdly, take $c_{0,1} = c_{0,2} = c_1 = c_{2,1} = c_{2,2} = \cdots = c_{2m-4,1} = c_{2m-4,2} = 0$ and

$$|c_{2m-3}| \ll |c_{2m-2,1}|, \quad |c_{2m-3}| \ll |c_{2m-2,2}|.$$

For the sign of $c_{2m-3}, c_{2m-2,1}$ and $c_{2m-2,2}$, the following cases need to be considered:

(2.i) $c_{2m-2,1}c_{2m-2,2} < 0$, $c_{2m-2,1}c_{2m-3} > 0$, $(c_{2m-2,1}+c_{2m-2,2})c_{2m-3} > 0$ (resp., < 0);

(2.ii) $c_{2m-2,1}c_{2m-2,2} < 0$, $c_{2m-2,1}c_{2m-3} < 0$, $(c_{2m-2,1} + c_{2m-2,2})c_{2m-3} > 0$ (resp., < 0);

(2.iii) $c_{2m-2,1}c_{2m-2,2} > 0$, $c_{2m-2,1}c_{2m-3} > 0$;

(2.iv) $c_{2m-2,1}c_{2m-2,2} > 0$, $c_{2m-2,1}c_{2m-3} < 0$.

We denote the number of simple zeros of $M_1(h,\delta)$, $M_2(h,\delta)$ and $M(h,\delta)$ obtained in the third step by $\mu_1^{[3]}, \mu_2^{[3]}$ and $\mu_3^{[3]}$, respectively. And let $\mu^{[3]} = \mu_1^{[3]} + \mu_2^{[3]} + \mu_3^{[3]}$. In the following table we give the values of $\mu_1^{[3]}, \mu_2^{[3]}, \mu_3^{[3]}$ and $\mu^{[3]}$ in each one of the above cases.

From the two steps described above, it can be seen that the maximum number of simple zeros of M_1, M_2 and M obtained by the above two steps is 5, of which M_1, M_2 and M have 2, 2 and 1 simple zeros, respectively, denoted by $(2,2)+1$.

The sign of $c_{2m-2,1}c_{2m-2,2}$	The sign of $c_{2m-2,1}c_{2m-3}$	The sign of $c_{2m-2}c_{2m-3}$	$\mu_1^{[3]}$	$\mu_2^{[3]}$	$\mu_3^{[3]}$	$\mu^{[3]}$
$-$	$+$	$+$	1	0	1	2
		$-$	1	0	0	1
	$-$	$+$	0	1	1	2
		$-$	0	1	0	1
$+$	$+$	$+$	1	1	1	3
	$-$	$-$	0	0	0	0

Further, let $c_{0,1} = c_{0,2} = c_1 = c_{2,1} = c_{2,2} = \cdots = c_{2i-2,1} = c_{2i-2,2} = 0$ and

$$|c_{2i-1}| \ll |c_{2i,1}| \ll |c_{2i+1}|, \quad |c_{2i-1}| \ll |c_{2i,2}| \ll |c_{2i+1}|,$$
$$c_{2i,1}c_{2i,2} > 0, \quad c_{2i,1}c_{2i+1} < 0, \quad c_{2i,1}c_{2i-1} > 0$$

for $i = m-2, m-3, \cdots, 1$ one by one. For each i, we can obtain two more simple zeros of $M_j(h,\delta)(j = 1,2)$ and one more simple zero of $M(h,\delta)$. Now, we have found $5(m-1)+3$ simple zeros of $M_j(j = 1,2)$ and M with distribution $(2m-1, 2m-1)+m$.

Finally, for fixed $c_1, c_{2,1}, c_{2,2}, \cdots, c_{2m-1}$, let $c_{0,1}$ and $c_{0,2}$ satisfy

$$|c_{0,1}| \ll |c_1|, \quad |c_{0,2}| \ll |c_1|.$$

By changing the sign of $c_{0,1}$ and $c_{0,2}$, we can obtain the number of simple zeros of M_1, M_2 and M, denoted by $\mu_1^{[2m]}, \mu_2^{[2m]}$ and $\mu_3^{[2m]}$ respectively. By the following table, it is easy to see that if $c_{0,1}c_{0,2} < 0, c_{0,1}c_1 > 0, c_0c_1 > 0$ or $c_{0,1}c_{0,2} < 0, c_{0,1}c_1 < 0, c_0c_1 > 0$ or $c_{0,1}c_{0,2} > 0, c_{0,1}c_1 < 0, c_0c_1 < 0$, the total number of simple zeros of M_1, M_2 and M obtained by this step is 2 with distributions $(0,1)+1$, $(1,0)+1$ and $(1,1)+0$.

The sign of $c_{0,1}c_{0,2}$	The sign of $c_{0,1}c_1$	The sign of c_0c_1	$\mu_1^{[2m]}$	$\mu_2^{[2m]}$	$\mu_3^{[2m]}$	$\mu^{[2m]}$
$-$	$+$	$+$	0	1	1	2
		$-$	0	1	0	1
	$-$	$+$	1	0	1	2
		$-$	1	0	0	1
$+$	$+$	$+$	0	0	1	1
	$-$	$-$	1	1	0	2

Summarizing the above, there exists δ near δ_0 such that the total number of simple zeros of M_1, M_2 and M is $5m$ with distributions $(2m,2m)+m$, $(2m-1, 2m)+(m+1)$ and $(2m, 2m-1)+(m+1)$. Then, for some (ε, δ) near $(0, \delta_0)$, system (1.1) has at least $5m$ limit cycles near the double homoclinic loop L_0 with distributions $(2m,2m)+m$, $(2m-1,2m)+(m+1)$ and $(2m,2m-1)+(m+1)$.

For $k = 2m - 1$, the proof is similar to the above. This ends the proof.

By the proof of Theorem 1.1, we can obtain the following corollary.

Corollary 2.1. *Suppose* (1.4)-(1.6) *hold. If there exists* δ_0 *such that for an even* k (1.7) *holds and*

$$\begin{aligned} &c_{k,1}(\delta_0)c_{k,2}(\delta_0) < 0, \\ &[(c_{k,1} + c_{k,2})c_{k,1}]_{\delta=\delta_0} < 0 \quad (resp., [(c_{k,1} + c_{k,2})c_{k,1}]_{\delta=\delta_0} > 0), \end{aligned}$$

then, for any given neighborhood V *of* L_0 *there exists* δ *near* δ_0 *such that for* $0 < \varepsilon \ll 1$ *system* (1.1) *has at least* $\frac{5k}{2} - 1$ *limit cycles in* V *with three distributions* $(k-1, k) + \frac{k}{2}$, $(k-1, k-1) + \frac{k}{2} + 1$ *and* $(k-2, k) + \frac{k}{2} + 1$ (*resp.*, $(k, k-1) + \frac{k}{2}$, $(k-1, k-1) + \frac{k}{2} + 1$ *and* $(k, k-2) + \frac{k}{2} + 1$).

Similar to Theorem 1.1, we can prove the following theorem.

Theorem 2.1. *Consider system* (1.1) *and let* (1.4)-(1.6) *hold. If there exists* δ_0 *such that for an even* k ($k \geq 2$),

$$\begin{aligned} &c_i(\delta_0) = 0, \ i = 1, 3, \cdots, k-1, \\ &c_{l,j}(\delta_0) = 0, \ l = 0, 2, \cdots, k-2, \quad j = 1, 2, \\ &c_{k,1}(\delta_0) = 0 \ (resp., \ c_{k,2}(\delta_0) = 0), \end{aligned} \tag{2.2}$$

and

$$\begin{aligned} &\text{rank } \frac{\partial(c_{0,1}, c_{0,2}, c_1, c_{2,1}, c_{2,2}, \cdots, c_{k-1}, c_{k,j})}{\partial \delta}\bigg|_{\delta=\delta_0} \\ =&\frac{3k}{2} + 1, \quad for \ j = 1 \ (resp., \ for \ j = 2), \end{aligned} \tag{2.3}$$

$$c_{k+1}(\delta_0)c_{k,2}(\delta_0) < 0 \ (resp., \ c_{k+1}(\delta_0)c_{k,1}(\delta_0) < 0), \tag{2.4}$$

then for any given neighborhood V *of* L_0 *there exists* δ *near* δ_0 *such that for* $0 < \varepsilon \ll 1$ *system* (1.1) *has at least* $\frac{5k}{2} + 1$ *limit cycles in* V. *Further, the* $\frac{5k}{2} + 1$ *limit cycles have three distributions* $(k+1, k) + \frac{k}{2}$, $(k+1, k-1) + \frac{k}{2} + 1$ *and* $(k, k) + \frac{k}{2} + 1$ (*resp.*, $(k, k+1) + \frac{k}{2}$, $(k-1, k+1) + \frac{k}{2} + 1$ *and* $(k, k) + \frac{k}{2} + 1$).

Proof. We first suppose

$$c_{k,1}(\delta_0) = 0, \quad c_{k+1}(\delta_0)c_{k,2}(\delta_0) < 0$$

and

$$\text{rank } \frac{\partial(c_{0,1}, c_{0,2}, c_1, c_{2,1}, c_{2,2}, \cdots, c_{k-1}, c_{k,1})}{\partial \delta}\bigg|_{\delta=\delta_0} = \frac{3k}{2} + 1.$$

By the inverse function theorem, the equations

$$c_{0,1}(\delta) = c_{0,1}, \ c_{0,2}(\delta) = c_{0,2}, \ c_1(\delta) = c_1, \ \cdots, c_{k-1}(\delta) = c_{k-1}, \ c_{k,1}(\delta) = c_{k,1}$$

have a unique solution $\delta = \delta_0 + O(|c_{0,1}, c_{0,2}, c_1, c_{2,1}, c_{2,2}, \cdots, c_{k-1}, c_{k,1}|)$, which means that $c_{0,1}, c_{0,2}, c_1, c_{2,1}, c_{2,2}, \cdots, c_{k-1}, c_{k,1}$ can be taken as free parameters.

Firstly, let $c_{0,1} = c_{0,2} = c_1 = \cdots = c_{k-2,1} = c_{k-2,2} = c_{k-1} = 0$ and

$$|c_{k,1}| \ll |c_{k+1}(\delta_0)|, \quad c_{k,1}c_{k+1}(\delta_0) < 0.$$

Note that

$$M_1(h,\delta) = c_{k,1}h^{\frac{k}{2}} + [c_{k+1}(\delta_0) + O(c_{k,1})]h^{\frac{k}{2}+1}\ln|h| + O(h^{\frac{k}{2}+1}).$$

By this step, we obtain a simple zero of $M_1(h,\delta)$, denoted by $\hat{h}_1^{[1]}$ with $0 < -\hat{h}_1^{[1]} \ll 1$.

Secondly, let $c_{0,1} = c_{0,2} = c_1 = \cdots = c_{k-2,1} = c_{k-2,2} = 0$ and

$$|c_{k-1}| \ll |c_{k,1}|, \quad |c_{k-1}| \ll |c_{k,2}(\delta_0)|, \quad c_{k-1}c_{k,1} > 0.$$

Under this condition, we have

$$\begin{aligned}
M_1(h,\delta) &= c_{k-1}h^{\frac{k}{2}}\ln|h| + c_{k,1}h^{\frac{k}{2}} + O(h^{\frac{k}{2}+1}\ln|h|),\\
M_2(h,\delta) &= c_{k-1}h^{\frac{k}{2}}\ln|h| + [c_{k,2}(\delta_0) + O(|c_{k-1}, c_{k,1}|)]h^{\frac{k}{2}} + O(h^{\frac{k}{2}+1}\ln|h|),\\
M(h,\delta) &= 2c_{k-1}h^{\frac{k}{2}}\ln h + [c_{k,1} + (c_{k,2}(\delta_0) + O(|c_{k-1}, c_{k,1}|))]h^{\frac{k}{2}} + O(h^{\frac{k}{2}+1}\ln h).
\end{aligned}$$

Note that $c_{k+1}(\delta_0)c_{k,2}(\delta_0) < 0$ and $c_{k+1}(\delta_0)c_{k,1} < 0$, which leads to $c_{k,1}c_{k,2}(\delta_0) > 0$. Then, for each of M_1, M_2 and M we can obtain a simple zero by this step, denoted by $\hat{h}_1^{[2]}, \hat{h}_2^{[1]}, \hat{h}^{[1]}$ respectively with $\hat{h}_1^{[1]} < \hat{h}_1^{[2]} < 0$.

Note that $k-1$ is odd. Directly by Theorem 1.1 we can obtain $\left[\frac{5(k-1)}{2}\right]$ more simple zeros of M_1, M_2 and M by taking suitable $c_{0,1}$, $c_{0,2}$, c_1, $c_{2,1}$, $c_{2,2}$, $\cdots$, $c_{k-2,1}$, $c_{k-2,2}$, with distributions $(k-1,k-1) + \left[\frac{k-1}{2}\right]$, $(k-1,k-2) + (\left[\frac{k-1}{2}\right]+1)$ and $(k-2,k-1) + (\left[\frac{k-1}{2}\right]+1)$. Meanwhile, $\hat{h}_1^{[1]}, \hat{h}_1^{[2]}, \hat{h}_2^{[1]}, \hat{h}^{[1]}$ still exist.

Thus, for $0 < \varepsilon \ll 1$ and δ near δ_0, system (1.1) has at least $\frac{5k}{2}+1$ limit cycles near L_0. And the $\frac{5k}{2}+1$ limit cycles have distributions $(k+1,k) + \frac{k}{2}$, $(k+1,k-1) + \frac{k}{2}+1$ and $(k,k) + \frac{k}{2}+1$.

The other case can be proved similarly. The proof is completed. □

Similarly, if $c_{k+1}(\delta_0)c_{k,2}(\delta_0) > 0$ or $c_{k+1}(\delta_0)c_{k,1}(\delta_0) > 0$ in (2.4), we obtain the following theorem.

Theorem 2.2. *Consider system* (1.1) *and let* (1.4)-(1.6) *hold. If there exists* δ_0 *such that* (2.2) *and* (2.3) *hold, and*

$$c_{k+1}(\delta_0)c_{k,2}(\delta_0) > 0 \ (resp., c_{k+1}(\delta_0)c_{k,1}(\delta_0) > 0),$$

then for any given neighborhood V of L_0 there exists δ near δ_0 such that for $0 < \varepsilon \ll 1$ system (1.1) *has at least $\frac{5k}{2}$ limit cycles in V with three distributions $(k,k) + \frac{k}{2}$, $(k,k-1) + \frac{k}{2}+1$ and $(k-1,k) + \frac{k}{2}+1$.*

3. Proof of Theorem 1.2

Note that f and g are given by (1.9), which means that

$$f_x + g_y = \sum_{\substack{i+j=1,\\ i\geq 1}}^{n} i a_{i,j}x^{i-1}y^j + \sum_{\substack{i+j=1,\\ j\geq 1}}^{n} j b_{i,j}x^i y^{j-1}.$$

For convenience, we introduce $I_{i,j}^{[s]}(h), I_{i,j}(h)$ and $\sigma_{i,j}$ for $i,j \geq 0$ with

$$I_{i,j}^{[s]}(h) = \oint_{L_s(h)} x^i y^j \mathrm{d}x, \quad I_{i,j}(h) = \oint_{L(h)} x^i y^j \mathrm{d}x, \quad s=1,2,\ i+j \geq 1,$$
$$\sigma_{i,j} = (i+1)a_{i+1,j} + (j+1)b_{i,j+1}.$$

Then, it is obvious that $f_x + g_y \equiv \sum\limits_{i+j=0}^{n-1} \sigma_{i,j} x^i y^j$. By using Green's formula twice, the function M_1 in (1.4) can be written as

$$M_1(h,\delta) = \iint_U (f_x + g_y)\mathrm{d}x\mathrm{d}y = \sum_{i+j=0}^{n-1} \frac{1}{j+1} \sigma_{i,j} I_{i,j+1}^{[1]}, \tag{3.1}$$

where $U = \{(x,y) | H(x,y) \leq h, -\frac{1}{4} < h < 0, x < 0\}$. Similarly, M_2 and M in (1.4) can be written as

$$M_2(h,\delta) = \sum_{i+j=0}^{n-1} \frac{1}{j+1} \sigma_{i,j} I_{i,j+1}^{[2]}, \quad M(h,\delta) = \sum_{i+j=0}^{n-1} \frac{1}{j+1} \sigma_{i,j} I_{i,j+1}. \tag{3.2}$$

Noticing that the coefficients in the expressions of $M_s (s=1,2)$ and M have been changed from $a_{i,j}, b_{i,j}$ to $\sigma_{i,j}$, so we replace $M_s(h,\delta)$ and $M(h,\delta)$ by $M_s(h,\sigma)$ and $M(h,\sigma)$, respectively, where $\sigma = (\sigma_{0,0}, \sigma_{1,0}, \sigma_{0,1}, \cdots, \sigma_{0,n-1}) \in \mathbf{R}^{\frac{n(n+1)}{2}}$.

On the properties of $I_{i,j}^{[s]}(h) (s=1,2)$ and $I_{i,j}(h)$, we give the following lemma.

Lemma 3.1. (i) $I_{i,j}(h) = 0,\ I_{i,j}^{[s]}(h) = 0\ (s=1,2)$ *if* $i \geq 0$ *and* j *is even.*

(ii) $I_{i,j}(h) = I_{i-2,j}(h) + \frac{i-3}{j+2} I_{i-4,j+2}(h)$ *for* $i \geq 3$*;*

$I_{i,j+2}(h) = \frac{4(j+2)}{i+2j+5}(hI_{i,j}(h) + \frac{1}{4} I_{i+2,j}(h))$ *for* $i,j \geq 0$.

(iii) $I_{i,j}^{[s]}(h) = I_{i-2,j}^{[s]}(h) + \frac{i-3}{j+2} I_{i-4,j+2}^{[s]}(h)$ *for* $i \geq 3$*;*

$I_{i,j+2}^{[s]}(h) = \frac{4(j+2)}{i+2j+5}(hI_{i,j}^{[s]}(h) + \frac{1}{4} I_{i+2,j}^{[s]}(h))$ *for* $i,j \geq 0$.

Proof. (i) Note that $y^2 = 2h + x^2 - \frac{1}{2}x^4$ along $L(h)$ or $L_s(h), s=1,2$. For even j, we easily get $I_{i,j}(h) = 0$ and $I_{i,j}^{[s]}(h) = 0\ (s=1,2)$.

(ii) Noticing (1.8), for $i \geq 3$ we have

$$\begin{aligned}
x^i y^j \mathrm{d}x =& x^{i-3} y^j \mathrm{d}\left(\frac{1}{4}x^4\right) \\
=& x^{i-3} y^j \mathrm{d}\left(H + \frac{1}{2}x^2 - \frac{1}{2}y^2\right) \\
=& x^{i-3} y^j \mathrm{d}H + x^{i-2} y^j \mathrm{d}x - x^{i-3} y^{j+1} \mathrm{d}y \\
=& x^{i-3} y^j \mathrm{d}H + x^{i-2} y^j \mathrm{d}x - \mathrm{d}\left(\frac{1}{j+2} x^{i-3} y^{j+2}\right) + \frac{i-3}{j+2} x^{i-4} y^{j+2} \mathrm{d}x.
\end{aligned}$$

And thus,

$$\oint_{L(h)} x^i y^j \mathrm{d}x = \oint_{L(h)} x^{i-2} y^j \mathrm{d}x + \frac{i-3}{j+2} \oint_{L(h)} x^{i-4} y^{j+2} \mathrm{d}x,$$

which gives

$$I_{i,j}(h) = I_{i-2,j}(h) + \frac{i-3}{j+2} I_{i-4,j+2}(h), \quad i \geq 3.$$

For $i \geq 0$, we further have

$$\begin{aligned} h \oint_{L(h)} x^i y^j \,\mathrm{d}x = & \oint_{L(h)} \left(\frac{1}{2} y^2 - \frac{1}{2} x^2 + \frac{1}{4} x^4 \right) x^i y^j \,\mathrm{d}x \\ = & \frac{1}{2} I_{i,j+2}(h) - \frac{1}{2} I_{i+2,j}(h) + \frac{1}{4} I_{i+4,j}(h), \end{aligned} \tag{3.3}$$

where $I_{i+4,j}(h)$ satisfies

$$I_{i+4,j}(h) = I_{i+2,j}(h) + \frac{i+1}{j+2} I_{i,j+2}(h). \tag{3.4}$$

Substituting (3.4) into (3.3) gives that

$$I_{i,j+2}(h) = \frac{4(j+2)}{i+2j+5} \left(h I_{i,j} + \frac{1}{4} I_{i+2,j} \right).$$

This finishes the proof of (ii). And (iii) can be proved similarly.

□

For $I^{[s]}_{0,1}, I^{[s]}_{1,1}, I^{[s]}_{2,1}$ with $s = 1, 2$, we further have

$$\begin{aligned} I^{[1]}_{0,1}(h) = & \int_{-\sqrt{1+\sqrt{1+4h}}}^{-\sqrt{1-\sqrt{1+4h}}} \sqrt{-2x^4 + 4x^2 + 8h} \,\mathrm{d}x = \frac{16h}{3} \zeta_1(h) + \frac{4}{3} \zeta_2(h), \\ I^{[1]}_{1,1}(h) = & \int_{-\sqrt{1+\sqrt{1+4h}}}^{-\sqrt{1-\sqrt{1+4h}}} x \sqrt{-2x^4 + 4x^2 + 8h} \,\mathrm{d}x = -\frac{\sqrt{2}}{4} (1+4h)\pi, \\ I^{[1]}_{2,1}(h) = & \int_{-\sqrt{1+\sqrt{1+4h}}}^{-\sqrt{1-\sqrt{1+4h}}} x^2 \sqrt{-2x^4 + 4x^2 + 8h} \,\mathrm{d}x = \frac{16h}{15} \zeta_1(h) + \left(\frac{16h}{5} + \frac{16}{15} \right) \zeta_2(h) \end{aligned} \tag{3.5}$$

and

$$\begin{aligned} I^{[2]}_{0,1}(h) &= \int_{\sqrt{1-\sqrt{1+4h}}}^{\sqrt{1+\sqrt{1+4h}}} \sqrt{-2x^4 + 4x^2 + 8h} \,\mathrm{d}x = I^{[1]}_{0,1}(h), \\ I^{[2]}_{1,1}(h) &= \int_{\sqrt{1-\sqrt{1+4h}}}^{\sqrt{1+\sqrt{1+4h}}} x \sqrt{-2x^4 + 4x^2 + 8h} \,\mathrm{d}x = -I^{[1]}_{1,1}(h), \\ I^{[2]}_{2,1}(h) &= \int_{\sqrt{1-\sqrt{1+4h}}}^{\sqrt{1+\sqrt{1+4h}}} x^2 \sqrt{-2x^4 + 4x^2 + 8h} \,\mathrm{d}x = I^{[1]}_{2,1}(h), \end{aligned}$$

where

$$\zeta_1(h) = \frac{\mathrm{EllipticK}\left(\sqrt{\frac{2\sqrt{1+4h}}{1+\sqrt{1+4h}}} \right)}{\sqrt{2+2\sqrt{1+4h}}}, \quad \zeta_2(h) = \frac{(1-\sqrt{1+4h}) \cdot \mathrm{EllipticE}\left(\sqrt{\frac{2\sqrt{1+4h}}{1+\sqrt{1+4h}}} \right)}{\sqrt{2+2\sqrt{1+4h}} \left(1 - \frac{2\sqrt{1+4h}}{1+\sqrt{1+4h}} \right)}.$$

By (3.1), (3.2) and Lemma 3.1, for each $s=1,2$, $M_s(h,\sigma)$ can be written as a combination of $I_{0,1}^{[s]}(h), I_{1,1}^{[s]}(h)$ and $I_{2,1}^{[s]}(h)$ with

$$M_s(h,\sigma)=\sum_{k=0}^{\left[\frac{n-1}{2}\right]}\alpha_{0,k}h^kI_{0,1}^{[s]}(h)+\sum_{k=0}^{\left[\frac{n-2}{2}\right]}\alpha_{1,k}h^kI_{1,1}^{[s]}(h)+\sum_{k=0}^{\left[\frac{n-3}{2}\right]}\alpha_{2,k}h^kI_{2,1}^{[s]}(h), \tag{3.6}$$

where

$$\begin{aligned}
&\alpha_{0,0}(\sigma)=\sigma_{0,0},\\
&\alpha_{0,k}(\sigma)=\lambda_{0,2k}^{[k]}\sigma_{0,2k}+\sum_{2k+2\le\bar n\le n-1}\ \sum_{\substack{0\le i\le\tilde n\\ i+j=\bar n\\ i,j\ even}}\lambda_{i,j}^{[k]}\sigma_{i,j},\quad \tilde n=\min\{2\bar n-4k,\bar n\},\\
&\alpha_{1,k}(\sigma)=\eta_{1,2k}^{[k]}\sigma_{1,2k}+\sum_{2k+3\le\bar n\le n-1}\ \sum_{\substack{1\le i\le\tilde n\\ i+j=\bar n\\ i\ odd,j\ even}}\eta_{i,j}^{[k]}\sigma_{i,j},\quad \tilde n=\min\{\bar n,2\bar n-4k-1\},\\
&\alpha_{2,k}(\sigma)=\tau_{2,2k}^{[k]}\sigma_{2,2k}+\tau_{0,2k+2}^{[k]}\sigma_{0,2k+2}\\
&\qquad\qquad+\sum_{2k+4\le\bar n\le n-1}\ \sum_{\substack{0\le i\le\tilde n\\ i+j=\bar n\\ i,j\ even}}\tau_{i,j}^{[k]}\sigma_{i,j},\quad \tilde n=\min\{\bar n,2\bar n-4k-2\}
\end{aligned} \tag{3.7}$$

with some constants $\lambda_{i,j}^{[k]},\eta_{i,j}^{[k]},\tau_{i,j}^{[k]}$ and $\eta_{1,0}^{[0]}=1,\tau_{2,0}^{[0]}=1$. It is easy to see that

$$\frac{\partial\left(\alpha_{0,0},\alpha_{1,0},\alpha_{2,0}\right)}{\partial\left(\sigma_{0,0},\sigma_{1,0},\sigma_{2,0}\right)}=\boldsymbol{I}_{3\times3}\equiv\boldsymbol{B}_0, \tag{3.8}$$

and

$$\frac{\partial\left(\alpha_{0,k},\alpha_{1,k},\alpha_{2,k}\right)}{\partial\left(\sigma_{0,2k},\sigma_{1,2k},\sigma_{2,2k}\right)}=\begin{bmatrix}\lambda_{0,2k}^{[k]} & 0 & \lambda_{2,2k}^{[k]}\\ 0 & \eta_{1,2k}^{[k]} & 0\\ 0 & 0 & \tau_{2,2k}^{[k]}\end{bmatrix}\equiv\boldsymbol{B}_k,\quad 1\le k\le\left[\frac{n-3}{2}\right], \tag{3.9}$$

where

$$\lambda_{0,2k}^{[k]}=\frac{1}{2k+1}\prod_{i=1}^{k}\frac{4(2i+1)}{4i+3},\quad \eta_{1,2k}^{[k]}=\frac{1}{2k+1}\prod_{i=1}^{k}\frac{4(2i+1)}{4i+4},$$
$$\tau_{2,2k}^{[k]}=\frac{1}{2k+1}\prod_{i=1}^{k}\frac{4(2i+1)}{4i+5}.$$

Next, we will use the first few coefficients appearing in the expansions of $I_{0,1}^{[s]}(h)$, $I_{1,1}^{[s]}(h)$ and $I_{2,1}^{[s]}(h)$ to obtain the first few coefficients appearing in the expansions of $M_s(h,\sigma)$ and $M(h,\sigma)$. By [12] we know that $I_{0,1}^{[s]}(h), I_{1,1}^{[s]}(h)$ and $I_{2,1}^{[s]}(h)$ are analytic functions, and for $0 < -h \ll 1$ one may suppose

$$\begin{aligned} I_{0,1}^{[1]}(h) &= \sum_{i\geq 0} \left(r_{0,2i} + r_{0,2i+1} h \ln |h|\right) h^i, \\ I_{1,1}^{[1]}(h) &= \sum_{i\geq 0} \left(r_{1,2i} + r_{1,2i+1} h \ln |h|\right) h^i, \\ I_{2,1}^{[1]}(h) &= \sum_{i\geq 0} \left(r_{2,2i} + r_{2,2i+1} h \ln |h|\right) h^i. \end{aligned} \tag{3.10}$$

By (3.5), we easily get

$$r_{1,0} = -\frac{\sqrt{2}}{4}\,\pi, \quad r_{1,1} = 0, \quad r_{1,2} = -\sqrt{2}\,\pi, \quad r_{1,2i+1} = r_{1,2i+2} = 0, \quad i \geq 1. \tag{3.11}$$

Directly by Maple or Theorem 2.2 in [12], we have

$$r_{0,0} = \frac{4}{3}, \quad r_{0,1} = -1, \quad r_{2,0} = \frac{16}{15}, \quad r_{2,1} = 0. \tag{3.12}$$

To prove Theorem 1.2, in the following we first suppose $n = 2m$ with $m \geq 1$, and then suppose $n = 2m - 1$ with $m \geq 2$.

3.1. Case 1: $n = 2m$

In this subsection we first suppose $m \geq 2$. By (3.6), $M_s(h, \sigma)$ can be rewritten as the following form:

$$\begin{aligned} M_s(h,\sigma) = &\sum_{j=0}^{2} \left(\alpha_{j,0}(\sigma) + h\alpha_{j,1}(\sigma) + \cdots + h^{m-2}\alpha_{j,m-2}(\sigma)\right) I_{j,1}^{[s]}(h) \\ &+ h^{m-1}\left(\alpha_{0,m-1}(\sigma) I_{0,1}^{[s]}(h) + \alpha_{1,m-1}(\sigma) I_{1,1}^{[s]}(h)\right), \quad m \geq 2. \end{aligned} \tag{3.13}$$

Then, by (2.1), (3.10) and (3.13), we can obtain the coefficients appearing in the expansion of $M_s(h, \sigma)$ as follows:

$$\begin{aligned} c_{2i,1}(\sigma) &= \sum_{j=0}^{2} \sum_{l=0}^{i} \alpha_{j,l}(\sigma) r_{j,2i-2l}, \\ c_{2i,2}(\sigma) &= \sum_{j=0}^{2} \sum_{l=0}^{i} (-1)^j \alpha_{j,l}(\sigma) r_{j,2i-2l}, \\ c_{2i+1}(\sigma) &= \sum_{j=0}^{2} \sum_{l=0}^{i} \alpha_{j,l}(\sigma) r_{j,2i-2l+1} \end{aligned} \tag{3.14}$$

for $0 \le i \le m-2$, and

$$\begin{aligned}
c_{2(m-1),1}(\sigma) &= \sum_{j=0}^{2}\sum_{l=0}^{m-2}\alpha_{j,l}(\sigma)r_{j,2(m-1)-2l}+\alpha_{0,m-1}(\sigma)r_{0,0}+\alpha_{1,m-1}(\sigma)r_{1,0},\\
c_{2(m-1),2}(\sigma) &= \sum_{j=0}^{2}\sum_{l=0}^{m-2}(-1)^j\alpha_{j,l}(\sigma)r_{j,2(m-1)-2l}+\alpha_{0,m-1}(\sigma)r_{0,0}-\alpha_{1,m-1}(\sigma)r_{1,0},\\
c_{2m-1}(\sigma) &= \sum_{j=0}^{2}\sum_{l=0}^{m-2}\alpha_{j,l}(\sigma)r_{j,2m-2l-1}+\alpha_{0,m-1}(\sigma)r_{0,1}+\alpha_{1,m-1}(\sigma)r_{1,1}.
\end{aligned} \tag{3.15}$$

By (3.14), it can be seen that

$$\frac{\partial(c_{2i,1},c_{2i,2},c_{2i+1})}{\partial(\alpha_{0,j},\alpha_{1,j},\alpha_{2,j})}=\begin{bmatrix} r_{0,2(i-j)} & r_{1,2(i-j)} & r_{2,2(i-j)}\\ r_{0,2(i-j)} & -r_{1,2(i-j)} & r_{2,2(i-j)}\\ r_{0,2(i-j)+1} & r_{1,2(i-j)+1} & r_{2,2(i-j)+1}\end{bmatrix}\equiv \boldsymbol{A}_{i-j} \tag{3.16}$$

with $0\le i\le m-2,\ 0\le j\le i$. Especially, by (3.11), (3.12) and (3.16) we have

$$\det \boldsymbol{A}_0=\begin{vmatrix} \frac{4}{3} & -\frac{\sqrt{2}}{4}\pi & \frac{16}{15}\\ \frac{4}{3} & \frac{\sqrt{2}}{4}\pi & \frac{16}{15}\\ -1 & 0 & 0\end{vmatrix}=\frac{8\sqrt{2}}{15}\pi.$$

By (3.16) and the formula of $c_{2(m-1),1}(\sigma)$ in (3.15), we further obtain

$$\begin{aligned}
&\det\frac{\partial\left(c_{0,1},c_{0,2},c_1,\cdots,c_{2(m-2),1},c_{2(m-2),2},c_{2m-3},c_{2(m-1),1}\right)}{\partial\left(\alpha_{0,0},\alpha_{1,0},\alpha_{2,0},\cdots,\alpha_{0,m-2},\alpha_{1,m-2},\alpha_{2,m-2},\alpha_{0,m-1}\right)}\\
&=\det\begin{bmatrix} \boldsymbol{A}_0 & & & \\ \boldsymbol{A}_1 & \boldsymbol{A}_0 & & \\ \vdots & \vdots & \ddots & \\ \boldsymbol{A}_{m-2} & \boldsymbol{A}_{m-1} & \dots & \boldsymbol{A}_0 \\ \boldsymbol{P}_{m-1} & \boldsymbol{P}_{m-2} & \cdots & \boldsymbol{P}_1 & r_{0,0}\end{bmatrix}_{(3m-2)\times(3m-2)}\neq 0,
\end{aligned} \tag{3.17}$$

where $\boldsymbol{P}_l=(r_{0,2l},r_{1,2l},r_{2,2l}),\quad 1\le l\le m-1$.

On the other hand, by (3.7), (3.8) and (3.9) we have

$$\begin{aligned}
&\det\frac{\partial\left(\alpha_{0,0},\alpha_{1,0},\alpha_{2,0},\cdots,\alpha_{0,m-2},\alpha_{1,m-2},\alpha_{2,m-2},\alpha_{0,m-1}\right)}{\partial\left(\sigma_{0,0},\sigma_{1,0},\sigma_{2,0},\cdots,\sigma_{0,2(m-2)},\sigma_{1,2(m-2)},\sigma_{2,2(m-2)},\sigma_{0,2(m-1)}\right)}\\
&=\det\begin{bmatrix} \boldsymbol{B}_0 & \boldsymbol{C}_{1,1} & \cdots & \boldsymbol{C}_{1,m-2} & \boldsymbol{Q}_0\\ & \boldsymbol{B}_1 & \cdots & \boldsymbol{C}_{2,m-2} & \boldsymbol{Q}_1\\ & & \ddots & \vdots & \vdots\\ & & & \boldsymbol{B}_{m-2} & \boldsymbol{Q}_{m-2}\\ & & & & \lambda_{0,2(m-1)}^{[m-1]}\end{bmatrix}_{(3m-2)\times(3m-2)}\neq 0,
\end{aligned} \tag{3.18}$$

where $\boldsymbol{Q}_l (0 \le l \le m-2)$ is a 3×1 matrix, and $\boldsymbol{C}_{i,l} (i = 1, 2, \cdots, m-2, i \le l \le m-2)$ is a 3×3 matrix. Then, it follows from (3.17) and (3.18) that

$$\det \frac{\partial \left(c_{0,1}, c_{0,2}, c_1, \cdots, c_{2(m-2),1}, c_{2(m-2),2}, c_{2m-3}, c_{2(m-1),1}\right)}{\partial \left(\sigma_{0,0}, \sigma_{1,0}, \sigma_{2,0}, \cdots, \sigma_{0,2(m-2)}, \sigma_{1,2(m-2)}, \sigma_{2,2(m-2)}, \sigma_{0,2(m-1)}\right)} \ne 0, \quad (3.19)$$

which means that the equations $c_{2i,1} = c_{2i,2} = c_{2i+1} = 0\,(0 \le i \le m-2), c_{2(m-1),1} = 0$ of σ have a unique solution of the form

$$\begin{aligned} &\left(\sigma_{0,0}, \sigma_{1,0}, \sigma_{2,0}, \cdots, \sigma_{0,2(m-2)}, \sigma_{1,2(m-2)}, \sigma_{2,2(m-2)}, \sigma_{0,2(m-1)}\right) \\ &= \varphi(\sigma_{0,1}, \sigma_{0,3}, \cdots, \sigma_{0,2m-1}, \cdots, \sigma_{2m-1,0}). \end{aligned} \quad (3.20)$$

Let $\sigma|_{(3.20)\text{ holds}} \equiv \sigma_0$. From the above we know that

$$c_{2(m-1),1}(\sigma_0) = 0,\ c_{2i,1}(\sigma_0) = c_{2i,2}(\sigma_0) = c_{2i+1}(\sigma_0) = 0,\ 0 \le i \le m-2. \quad (3.21)$$

Next, we will give $c_{2(m-1),2}(\sigma_0)$ and $c_{2m-1}(\sigma_0)$.

By (3.14), it is easy to obtain that

$$c_{0,1}(\sigma) = \sum_{l=0}^{2} \alpha_{l,0} r_{l,0}, \quad c_{0,2}(\sigma) = \sum_{l=0}^{2} (-1)^l \alpha_{l,0} r_{l,0}, \quad c_1(\sigma) = \sum_{l=0}^{2} \alpha_{l,0} r_{l,1}.$$

Note that (3.16) holds for $i = 0, j = 0$ and $\det \boldsymbol{A}_0 \ne 0$, which means that

$$c_{0,1}(\sigma) = c_{0,2}(\sigma) = c_1(\sigma) = 0 \Longleftrightarrow \alpha_{0,0}(\sigma) = \alpha_{1,0}(\sigma) = \alpha_{2,0}(\sigma) = 0. \quad (3.22)$$

If $\alpha_{0,0}(\sigma) = \alpha_{1,0}(\sigma) = \alpha_{2,0}(\sigma) = 0$, by (3.14) we further obtain

$$c_{2,1}(\sigma) = \sum_{l=0}^{2} \alpha_{l,1} r_{l,0}, \quad c_{2,2}(\sigma) = \sum_{l=0}^{2} (-1)^l \alpha_{l,1} r_{l,0}, \quad c_3(\sigma) = \sum_{l=0}^{2} \alpha_{l,1} r_{l,1}.$$

Note that (3.16) holds for $i = 1, j = 1$ and $\det \boldsymbol{A}_0 \ne 0$, which yields that

$$c_{2,1}(\sigma) = c_{2,2}(\sigma) = c_3(\sigma) = 0 \Longleftrightarrow \alpha_{0,1}(\sigma) = \alpha_{1,1}(\sigma) = \alpha_{2,1}(\sigma) = 0. \quad (3.23)$$

Similarly, one can prove that if $\alpha_{0,l}(\sigma) = \alpha_{1,l}(\sigma) = \alpha_{2,l}(\sigma) = 0$ for $0 \le l \le i-1$ and $i = 2, \cdots, m-2$, then

$$c_{2i,1}(\sigma) = c_{2i,2}(\sigma) = c_{2i+1}(\sigma) = 0 \Longleftrightarrow \alpha_{0,i}(\sigma) = \alpha_{1,i}(\sigma) = \alpha_{2,i}(\sigma) = 0. \quad (3.24)$$

If $\alpha_{0,i}(\sigma) = \alpha_{1,i}(\sigma) = \alpha_{2,i}(\sigma) = 0$ for $0 \le i \le m-2$, by (3.15), (3.11) and (3.12) we obtain

$$\begin{aligned} c_{2(m-1),1}(\sigma) &= \frac{4}{3}\alpha_{0,m-1}(\sigma) - \frac{\sqrt{2}}{4}\pi\alpha_{1,m-1}(\sigma), \\ c_{2(m-1),2}(\sigma) &= \frac{4}{3}\alpha_{0,m-1}(\sigma) + \frac{\sqrt{2}}{4}\pi\alpha_{1,m-1}(\sigma), \\ c_{2m-1}(\sigma) &= -\alpha_{0,m-1}(\sigma). \end{aligned} \quad (3.25)$$

Solving the equation $c_{2(m-1),1}(\sigma) = 0$ for $\alpha_{0,m-1}(\sigma)$ we obtain

$$\alpha_{0,m-1}(\sigma) = \frac{3}{16}\sqrt{2}\,\pi\alpha_{1,m-1}(\sigma), \quad (3.26)$$

by (3.7) which further gives

$$\sigma_{0,2(m-1)} = \frac{3}{16}\sqrt{2}\,\pi \cdot \frac{\eta_{1,2m-2}^{[m-1]}}{\lambda_{0,2m-2}^{[m-1]}}\,\sigma_{1,2m-2}. \tag{3.27}$$

Now, by (3.25), (3.26) and (3.7), we easily obtain

$$\begin{aligned} c_{2(m-1),2}(\sigma_0) &= \frac{\sqrt{2}\,\pi}{2}\eta_{1,2m-2}^{[m-1]}\sigma_{1,2m-2}, \\ c_{2m-1}(\sigma_0) &= -\frac{3\sqrt{2}\pi}{16}\eta_{1,2m-2}^{[m-1]}\sigma_{1,2m-2}. \end{aligned} \tag{3.28}$$

By (3.13) and (3.26), we further have

$$\begin{aligned} M_s(h,\sigma_0) &= h^{m-1}\alpha_{1,m-1}\left(\frac{3}{16}\sqrt{2}\,\pi I_{0,1}^{[s]}(h) + I_{1,1}^{[s]}(h)\right) \\ &= \left[\sum_{i=0}^{m-1}\binom{m-1}{i}\left(h+\frac{1}{4}\right)^i\left(-\frac{1}{4}\right)^{m-1-i}\right]\alpha_{1,m-1} \\ &\quad\times\left(\frac{3}{16}\sqrt{2}\,\pi I_{0,1}^{[s]}(h) + I_{1,1}^{[s]}(h)\right), \quad s=1,2. \end{aligned} \tag{3.29}$$

By [4], for $0 < h + \frac{1}{4} \ll 1$, $M_s(h,\sigma), I_{0,1}^{[s]}(h)$ and $I_{1,1}^{[s]}(h)$ can be expanded as the forms below

$$M_s(h,\sigma) = b_0^{[s]}(\sigma)(h+\frac{1}{4}) + O((h+\frac{1}{4})^2), \tag{3.30}$$

$$I_{j,1}^{[s]}(h) = b_{j,0}^{[s]}(h+\frac{1}{4}) + O((h+\frac{1}{4})^2), \quad j=0,1, \tag{3.31}$$

where

$$b_{0,0}^{[1]} = \sqrt{2}\,\pi, \quad b_{1,0}^{[1]} = -\sqrt{2}\,\pi, \quad b_{0,0}^{[2]} = \sqrt{2}\,\pi, \quad b_{1,0}^{[2]} = \sqrt{2}\,\pi.$$

Then, by (3.29) and (3.31) we obtain

$$\begin{aligned} b_0^{[1]}(\sigma_0) &= \sqrt{2}\pi\left(-\frac{1}{4}\right)^{m-1}\left(\frac{3}{16}\sqrt{2}\,\pi - 1\right)\eta_{1,2m-2}^{[m-1]}\sigma_{1,2m-2}, \\ b_0^{[2]}(\sigma_0) &= \sqrt{2}\pi\left(-\frac{1}{4}\right)^{m-1}\left(\frac{3}{16}\sqrt{2}\,\pi + 1\right)\eta_{1,2m-2}^{[m-1]}\sigma_{1,2m-2}. \end{aligned} \tag{3.32}$$

It follows from (1.5) and (3.21) that

$$\begin{aligned} M_1(h,\sigma_0) &= c_{2m-1}(\sigma_0)h^m\ln|h| + O(h^m), \quad 0 < -h \ll 1, \\ M_2(h,\sigma_0) &= c_{2(m-1),2}(\sigma_0)h^{m-1} + O(h^m\ln|h|), \quad 0 < -h \ll 1. \end{aligned}$$

Hence, let $\sigma_{1,2m-2}$, the element of σ_0, satisfy $\sigma_{1,2m-2} \neq 0$. By (3.28), (3.30) and (3.32), we obtain

$$M_1(\varepsilon,\sigma_0)M_1(-\frac{1}{4}+\varepsilon,\sigma_0) > 0, \quad M_2(\varepsilon,\sigma_0)M_2(-\frac{1}{4}+\varepsilon,\sigma_0) > 0$$

for $0 < \varepsilon \ll 1$, which means that we can not find simple zeros of $M_s(h,\sigma_0)(s=1,2)$ for $h \in (-\frac{1}{4}, 0)$.

Note that $c_{2(m-1),2}(\sigma_0)c_{2m-1}(\sigma_0) < 0$. Then, by Theorem 2.1 system (1.1) has at least $5m-4$ limit cycles near L_0 for some (ε,σ) near $(0,\sigma_0)$ with three distributions: $(2m-2,2m-2)+m$, $(2m-1,2m-3)+m$ and $(2m-1,2m-2)+m-1$.

Next, we will prove that for $m\ge 2$ there exists another parameter $\tilde\sigma_0$ such that system (1.1) has at least $5m-4$ limit cycles near L_0 for some (ε,σ) near $(0,\tilde\sigma_0)$ with three distributions: $(2m-2,2m-2)+m$, $(2m-3,2m-1)+m$ and $(2m-2,2m-1)+m-1$.

Similar to (3.19), we have

$$\det\frac{\partial\left(c_{0,1},c_{0,2},c_1,\cdots,c_{2(m-2),1},c_{2(m-2),2},c_{2m-3},c_{2(m-1),2}\right)}{\partial\left(\sigma_{0,0},\sigma_{1,0},\sigma_{2,0},\cdots,\sigma_{0,2(m-2)},\sigma_{1,2(m-2)},\sigma_{2,2(m-2)},\sigma_{0,2(m-1)}\right)}\neq 0.$$

Therefore, the equations $c_{2i,1}=c_{2i,2}=c_{2i+1}=0\,(0\le i\le m-2), c_{2(m-1),2}=0$ of σ have a unique solution of the form

$$\begin{aligned}&\left(\sigma_{0,0},\sigma_{1,0},\sigma_{2,0},\cdots,\sigma_{0,2(m-2)},\sigma_{1,2(m-2)},\sigma_{2,2(m-2)},\sigma_{0,2(m-1)}\right)\\&=\psi(\sigma_{0,1},\sigma_{0,3},\cdots,\sigma_{0,2m-1},\cdots,\sigma_{2m-1,0}).\end{aligned}\tag{3.33}$$

Let $\sigma|_{(3.33)\text{ holds}}\equiv\tilde\sigma_0$. By (3.22)-(3.24) and (3.7) we obtain that $c_{2i,1}(\tilde\sigma_0)=c_{2i,2}(\tilde\sigma_0)=c_{2i+1}(\tilde\sigma_0)=0$ for $0\le i\le m-2$, $c_{2(m-1),2}(\tilde\sigma_0)=0$ and

$$\begin{aligned}c_{2(m-1),1}(\tilde\sigma_0)&=-\frac{\sqrt2}{2}\pi\eta^{[m-1]}_{1,2m-2}\sigma_{1,2m-2},\\c_{2m-1}(\tilde\sigma_0)&=\frac{3\sqrt2}{16}\pi\eta^{[m-1]}_{1,2m-2}\sigma_{1,2m-2},\end{aligned}\tag{3.34}$$

$$\begin{aligned}M_s(h,\tilde\sigma_0)=&\left[\sum_{i=0}^{m-1}\binom{m-1}{i}\left(h+\frac14\right)^i\left(-\frac14\right)^{m-1-i}\right]\alpha_{1,m-1}\\&\times\left(-\frac{3}{16}\sqrt2\,\pi I^{[s]}_{0,1}(h)+I^{[s]}_{1,1}(h)\right).\end{aligned}\tag{3.35}$$

Then, by (3.30), (3.31) and (3.35) we obtain

$$\begin{aligned}b_0^{[1]}(\tilde\sigma_0)&=-\sqrt2\,\pi\left(1+\frac{3\sqrt2}{16}\pi\right)\left(-\frac14\right)^{m-1}\eta^{[m-1]}_{1,2m-2}\sigma_{1,2m-2},\\b_0^{[2]}(\tilde\sigma_0)&=\sqrt2\,\pi\left(1-\frac{3\sqrt2}{16}\pi\right)\left(-\frac14\right)^{m-1}\eta^{[m-1]}_{1,2m-2}\sigma_{1,2m-2}.\end{aligned}\tag{3.36}$$

Suppose $\sigma_{1,2m-2}\neq0$. In this case, we can not find simple zeros of $M_s(h,\tilde\sigma_0)(s=1,2)$ for $h\in(-\frac14,0)$. On the other hand, note that $c_{2(m-1),1}(\tilde\sigma_0)c_{2m-1}(\tilde\sigma_0)<0$. Then, the conclusion follows from Theorem 2.1.

Next, suppose $m=1$. In this case, note that $\alpha_{1,0}=\sigma_{1,0}$ and $\alpha_{0,0}=\sigma_{0,0}$. Then, by (3.25) we can obtain the coefficients $c_{0,1},c_{0,2}$ and c_1 appearing in (1.5) with

$$c_{0,1}(\sigma)=\frac43\sigma_{0,0}-\frac{\sqrt2}{4}\pi\sigma_{1,0},\quad c_{0,2}(\sigma)=\frac43\sigma_{0,0}+\frac{\sqrt2}{4}\pi\sigma_{1,0},\quad c_1(\sigma)=-\sigma_{0,0}.\tag{3.37}$$

Solving the equation $c_{0,1}(\sigma)=0$ for $\sigma_{0,0}$ gives $\sigma_{0,0}=\frac{3\sqrt2}{16}\pi\sigma_{1,0}$. Let $\sigma_0=(\frac{3\sqrt2}{16}\pi\sigma_{1,0}, \sigma_{1,0},\ \sigma_{0,1})$. Then, (3.28) and (3.32) hold for $m=1$.

It is obvious that $c_{0,2}(\sigma_0)c_1(\sigma_0) < 0$ if $\sigma_{1,0} \neq 0$. Then, we can easily prove that for some (ε, σ) near $(0, \sigma_0)$ with $\sigma_{1,0} \neq 0$ system (1.1) has at least 1 limit cycle near L_0 with distribution $(1,0)+0$.

If we solve the equation $c_{0,2}(\sigma) = 0$ in (3.37) for $\sigma_{0,0}$, then $\sigma_{0,0} = -\frac{3\sqrt{2}}{16}\pi\sigma_{1,0}$, and (3.34) and (3.36) hold for $m = 1$, where $\widetilde{\sigma}_0 = (-\frac{3\sqrt{2}}{16}\pi\sigma_{1,0}, \sigma_{1,0}, \sigma_{0,1})$. Similarly, we can prove that for some (ε, σ) near $(0, \widetilde{\sigma}_0)$ with $\sigma_{1,0} \neq 0$ system (1.1) has at least 1 limit cycle near L_0 with distribution $(0,1)+0$.

3.2. Case 2: $n = 2m - 1$

In this case, by (3.6), $M_s(h, \sigma)$ can be written as the following form:

$$\begin{aligned} M_s(h,\sigma) = &\sum_{j=0}^{2} \left(\alpha_{j,0}(\sigma) + h\alpha_{j,1}(\sigma) + \cdots + h^{m-2}\alpha_{j,m-2}(\sigma)\right) I_{j,1}^{[s]}(h) \\ &+ h^{m-1}\alpha_{0,m-1}(\sigma) I_{0,1}^{[s]}(h). \end{aligned} \tag{3.38}$$

Similar to the case of $n = 2m$, (3.14) still holds for $0 \le i \le m-2$, and

$$\begin{aligned} c_{2(m-1),1}(\sigma) &= \sum_{j=0}^{2}\sum_{l=0}^{m-2} \alpha_{j,l}(\sigma) r_{j,2(m-1)-2l} + r_{0,0}\alpha_{0,m-1}(\sigma), \\ c_{2(m-1),2}(\sigma) &= \sum_{j=0}^{2}\sum_{l=0}^{m-2} (-1)^j \alpha_{j,l}(\sigma) r_{j,2(m-1)-2l} + r_{0,0}\alpha_{0,m-1}(\sigma), \\ c_{2m-1}(\sigma) &= \sum_{j=0}^{2}\sum_{l=0}^{m-2} \alpha_{j,l}(\sigma) r_{j,2m-2l-1} + r_{0,1}\alpha_{0,m-1}(\sigma). \end{aligned} \tag{3.39}$$

By (3.19), it is easy to obtain that

$$\det \frac{\partial\left(c_{0,1}, c_{0,2}, c_1, \cdots, c_{2(m-2),1}, c_{2(m-2),2}, c_{2m-3}\right)}{\partial\left(\sigma_{0,0}, \sigma_{1,0}, \sigma_{2,0}, \cdots, \sigma_{0,2(m-2)}, \sigma_{1,2(m-2)}, \sigma_{2,2(m-2)}\right)} \neq 0.$$

Then, the equations $c_{2i,1} = c_{2i,2} = c_{2i+1} = 0\,(0 \le i \le m-2)$ of σ have a unique solution of the form

$$\begin{aligned} &\left(\sigma_{0,0}, \sigma_{1,0}, \sigma_{2,0}, \cdots, \sigma_{0,2(m-2)}, \sigma_{1,2(m-2)}, \sigma_{2,2(m-2)}\right) \\ &= \phi(\sigma_{0,1}, \sigma_{0,3}, \cdots, \sigma_{0,2(m-1)}, \cdots, \sigma_{2(m-1),0}). \end{aligned} \tag{3.40}$$

Note that (3.22)-(3.24) still hold in this case. Let $\sigma|_{(3.40)\text{ holds}} \equiv \hat{\sigma}_0$. By (3.7), (3.38)-(3.40) and (3.22)-(3.24), we obtain

$$\begin{aligned} &c_{2i,1}(\hat{\sigma}_0) = c_{2i,2}(\hat{\sigma}_0) = c_{2i+1}(\hat{\sigma}_0) = 0, \quad 0 \le i \le m-2, \\ &c_{2(m-1),1}(\hat{\sigma}_0) = c_{2(m-1),2}(\hat{\sigma}_0) = \frac{4}{3}\,\lambda_{0,2m-2}^{[m-1]}\sigma_{0,2m-2}, \end{aligned}$$

and

$$M_s(h, \hat{\sigma}_0) = \left[\sum_{i=0}^{m-1} \binom{m-1}{i} \left(h + \frac{1}{4}\right)^i \left(-\frac{1}{4}\right)^{m-1-i}\right] \lambda_{0,2m-2}^{[m-1]}.\sigma_{0,2m-2} I_{0,1}^{[s]}(h), \ s = 1, 2.$$

Hence, by (3.30) and (3.31) we obtain

$$b_0^{[1]}(\hat{\sigma}_0) = b_0^{[2]}(\hat{\sigma}_0) = \sqrt{2}\,\pi\left(-\frac{1}{4}\right)^{m-1}\lambda_{0,2m-2}^{[m-1]}\sigma_{0,2m-2}.$$

Let $\sigma_{0,2m-2} \neq 0$. In this case, we can not find simple zeros of $M_s(h,\hat{\sigma}_0)(s=1,2)$ for $h \in (-\frac{1}{4},0)$. Note that $c_{2(m-1),1}(\hat{\sigma}_0)c_{2(m-1),2}(\hat{\sigma}_0) > 0$. By Theorem 1.1, system (1.1) has at least $5m-5$ limit cycles near L_0 for some (ε,σ) near $(0,\hat{\sigma}_0)$ with distributions $(2m-2,2m-2)+m-1$, $(2m-2,2m-3)+m$ and $(2m-3,2m-2)+m$.

4. Proof of Theorem 1.3

In this section, for system (1.1) we suppose that H satisfies (1.8) and f,g satisfy (1.10) and $n=2m-1$, $m\geq 2$. In this case, $M_1 = M_2$.

Similar to (3.1) and (3.6), the function M_1 in (3.1) can be written as

$$\begin{aligned} M_1(h,\delta) &= \sum_{\substack{i+j=0\\ i+j\ even}}^{2m-2}\frac{1}{j+1}\sigma_{i,j}I_{i,j+1}^{[1]} \\ &= \sum_{k=0}^{m-1}\alpha_{0,k}(\sigma)h^k I_{0,1}^{[1]}(h) + \sum_{k=0}^{m-2}\alpha_{2,k}(\sigma)h^k I_{2,1}^{[1]}(h) \\ &= \sum_{k=0}^{m-2}h^k\left(\alpha_{0,k}(\sigma)I_{0,1}^{[1]}(h)+\alpha_{2,k}(\sigma)I_{2,1}^{[1]}(h)\right) \\ &\quad + h^{m-1}\alpha_{0,m-1}(\sigma)I_{0,1}^{[1]}(h), \end{aligned} \tag{4.1}$$

where $\alpha_{0,k}(\sigma)$ and $\alpha_{2,k}(\sigma)$ satisfy (3.7). By (4.1) and (3.10), we can obtain the coefficients appearing in (1.5) with

$$\begin{aligned} c_{2i,1}(\sigma) &= \sum_{l=0}^{i}(\alpha_{0,l}(\sigma)r_{0,2i-2l}+\alpha_{2,l}(\sigma)r_{2,2i-2l}), \\ c_{2i+1}(\sigma) &= \sum_{l=0}^{i}(\alpha_{0,l}(\sigma)r_{0,2i-2l+1}+\alpha_{2,l}(\sigma)r_{2,2i-2l+1}) \end{aligned}$$

for $0\leq i\leq m-2$, and

$$c_{2(m-1),1}(\sigma) = \sum_{l=0}^{m-2}(\alpha_{0,l}(\sigma)r_{0,2(m-1)-2l}+\alpha_{2,l}(\sigma)r_{2,2(m-1)-2l}) + \alpha_{0,m-1}(\sigma)r_{0,0}. \tag{4.2}$$

Similar to (3.16), (3.17) and (3.18), we have

$$\frac{\partial(c_{2i,1},c_{2i+1})}{\partial(\alpha_{0,j},\alpha_{2,j})} = \begin{bmatrix} r_{0,2(i-j)} & r_{2,2(i-j)} \\ r_{0,2(i-j)+1} & r_{2,2(i-j)+1}\end{bmatrix} \equiv \widetilde{\boldsymbol{A}}_{i-j}, \quad 0\leq i\leq m-2,\ 0\leq j\leq i,$$

where $\det \widetilde{\boldsymbol{A}}_0 = \frac{16}{15}$, and further

$$\begin{aligned}
&\det \frac{\partial\left(c_{0,1}, c_1, \cdots, c_{2(m-2),1}, c_{2m-3}\right)}{\partial\left(\alpha_{0,0}, \alpha_{2,0}, \cdots, \alpha_{0,m-2}, \alpha_{2,m-2}\right)} \\
&= \det \begin{bmatrix} \widetilde{\boldsymbol{A}}_0 & & & \\ \widetilde{\boldsymbol{A}}_1 & \widetilde{\boldsymbol{A}}_0 & & \\ \vdots & \vdots & \ddots & \\ \widetilde{\boldsymbol{A}}_{m-2} & \widetilde{\boldsymbol{A}}_{m-1} & \ldots & \widetilde{\boldsymbol{A}}_0 \end{bmatrix}_{(2m-2)\times(2m-2)} \neq 0.
\end{aligned} \tag{4.3}$$

Noting that

$$\begin{aligned}
&\frac{\partial\left(\alpha_{0,0}, \alpha_{2,0}\right)}{\partial\left(\sigma_{0,0}, \sigma_{2,0}\right)} = \boldsymbol{I}_{2\times 2} \equiv \widetilde{\boldsymbol{B}}_0, \\
&\frac{\partial\left(\alpha_{0,i}, \alpha_{2,i}\right)}{\partial\left(\sigma_{0,2i}, \sigma_{2,2i}\right)} = \begin{bmatrix} \lambda_{0,2i}^{[i]} & * \\ 0 & \tau_{2,2i}^{[i]} \end{bmatrix} \equiv \widetilde{\boldsymbol{B}}_i, \quad 1 \leq i \leq m-2,
\end{aligned}$$

and $\det \widetilde{\boldsymbol{B}}_i \neq 0$ for $0 \leq i \leq m-2$, we have

$$\begin{aligned}
&\det \frac{\partial\left(\alpha_{0,0}, \alpha_{2,0}, \cdots, \alpha_{0,m-2}, \alpha_{2,m-2}\right)}{\partial\left(\sigma_{0,0}, \sigma_{2,0}, \cdots, \sigma_{0,2(m-2)}, \sigma_{2,2(m-2)}\right)} \\
&= \det \begin{bmatrix} \widetilde{\boldsymbol{B}}_0 & \widetilde{\boldsymbol{C}}_{1,1} & \cdots & \widetilde{\boldsymbol{C}}_{1,m-2} \\ & \widetilde{\boldsymbol{B}}_1 & \cdots & \widetilde{\boldsymbol{C}}_{2,m-2} \\ & & \ddots & \vdots \\ & & & \widetilde{\boldsymbol{B}}_{m-2} \end{bmatrix}_{(2m-2)\times(2m-2)} \neq 0,
\end{aligned} \tag{4.4}$$

where $\widetilde{\boldsymbol{C}}_{i,l} (i = 1, 2, \cdots, m-2, i \leq l \leq m-2)$ is a 2×2 matrix. Further, by (4.3) and (4.4), we obtain

$$\det \frac{\partial\left(c_{0,1}, c_1, \cdots, c_{2(m-2),1}, c_{2m-3}\right)}{\partial\left(\sigma_{0,0}, \sigma_{2,0}, \cdots, \sigma_{0,2(m-2)}, \sigma_{2,2(m-2)}\right)} \neq 0. \tag{4.5}$$

Similar to Section 3, we can prove that

$$c_{0,1}(\sigma) = c_1(\sigma) = 0 \Longleftrightarrow \alpha_{0,0}(\sigma) = \alpha_{2,0}(\sigma) = 0, \tag{4.6}$$

and

$$c_{2i,1}(\sigma) = c_{2i+1}(\sigma) = 0 \Longleftrightarrow \alpha_{0,i}(\sigma) = \alpha_{2,i}(\sigma) = 0,\ i = 1, \cdots, m-2 \tag{4.7}$$

if $\alpha_{0,l}(\sigma) = \alpha_{2,l}(\sigma) = 0$ for $0 \leq l \leq i-1$.

By (4.5), the equations $c_{2i,1} = c_{2i+1} = 0\,(0 \leq i \leq m-2)$ of σ have a unique solution of the form

$$\begin{aligned}
&\left(\sigma_{0,0}, \sigma_{2,0}, \cdots, \sigma_{0,2(m-2)}, \sigma_{2,2(m-2)}\right) \\
&= \vartheta(\sigma_{1,1}, \sigma_{1,3}, \cdots, \sigma_{1,2m-3}, \cdots, \sigma_{2m-2,0}).
\end{aligned} \tag{4.8}$$

Let $\sigma|_{(4.8)\text{ holds}} \equiv \hat{\sigma}_0$. Then, by (4.2) we obtain

$$c_{2m-2,1}(\hat{\sigma}_0) = \frac{4}{3}\,\lambda^{[m-1]}_{0,2m-2}\sigma_{0,2m-2}.$$

And by (4.1), (4.6), (4.7), (3.30) and (3.31), we obtain

$$b_0^{[1]}(\hat{\sigma}_0) = \sqrt{2}\,\pi\left(-\frac{1}{4}\right)^{m-1}\lambda^{[m-1]}_{0,2m-2}\sigma_{0,2m-2}.$$

Similar to the analysis in Section 3, we can not find simple zeros of $M_1(h,\hat{\sigma}_0)$ in $(-\frac{1}{4},0)$. If $\sigma_{0,2m-2} \neq 0$, by Lemma 1.1, system (1.1) has at least $5m-5$ limit cycles near L_0 for some (ε,σ) near $(0,\hat{\sigma}_0)$, of which $m-1$ limit cycles surround L_0.

References

[1] V. I. Arnold, *Loss of stability of self-induced oscillations near resonance, and versal deformations of equivariant vector fields*, Funct. Anal. Appl., 1977, 11, 85–92.

[2] W. Geng, M. Han, Y. Tian and A. Ke, *Heteroclinic bifurcation of limit cycles in perturbed cubic Hamiltonian systems by higher-order analysis*, J. Differential Equations, 2023, 357, 412–435.

[3] W. Geng and Y. Tian, *Bifurcation of limit cycles near heteroclinic loops in near-Hamiltonian systems*, Commun. Nonlinear Sci. Numer. Simul., 2021, 95(12), 105666.

[4] M. Han, *On Hopf cyclicity of planar systems*, J. Math. Anal. Appl., 2000, 245(2), 404–422.

[5] M. Han, *Asymptotic expansions of Melnikov functions and limit cycle bifurcations*, Internat. J. Bifur. Chaos Appl. Sci. Engrg., 2012, 22(12), 1250296, 30pp.

[6] M. Han and J. Chen, *On the number of limit cycles in double homoclinic bifurcations*, Sci. China Ser. A, 2000, 43(9), 914–928.

[7] M. Han, J. Yang and J. Li, *General study on limit cycle bifurcation near a double homoclinic loop*, J. Differential Equations, 2023, 347, 1–23.

[8] M. Han, J. Yang, A.-A. Tarţa and Y. Gao, *Limit cycles near homoclinic and heteroclinic loops*, J. Dynam. Differential Equations, 2008, 20, 923–944.

[9] M. Han, J. Yang and D. Xiao, *Limit cycle bifurcations near a double homoclinic loop with a nilpotent saddle*, Internat. J. Bifur. Chaos Appl. Sci. Engrg., 2012, 22(8), 1250189, 33pp.

[10] M. Han and Y. Ye, *On the coefficients appearing in the expansion of Melnikov functions in homoclinic bifurcations*, Ann. Differential Equations, 1998, 14(2), 156–162.

[11] M. Han and P. Yu, *Normal forms, Melnikov Functions and Bifurcations of Limit Cycles*, Vol. 181 of Applied Mathematical Sciences, Springer, London, 2012.

[12] M. Han, H. Zang and J. Yang, *Limit cycle bifurcations by perturbing a cuspidal loop in a Hamiltonian system*, J. Differential Equations, 2009, 246(1), 129–163.

[13] H. He and D. Xiao, *On the global center of planar polynomial differential systems and the related problems*, J. Appl. Anal. Comput., 2022, 12(3), 1141–1157.

[14] D. Hilbert, *Mathematical problems*, Bull. Amer. Math. Soc., 1902, 8(10), 437–479.

[15] W. Hou and S. Liu, *Melnikov functions for a class of piecewise Hamiltonian systems*, J. Nonlinear Model. and Anal., 2023, 5(1), 123–145.

[16] I. D. Iliev, C. Li and J. Yu, *On the cubic perturbations of the symmetric 8-loop Hamiltonian*, J. Differential Equations, 2020, 269(4), 3387–3413.

[17] C. Li, *Abelian integrals and limit cycles*, Qual. Theory Dyn. Syst., 2012, 11(1), 111–128.

[18] J. Li, *Hilbert's 16th problem and bifurcations of planar polynomial vector fields*, Internat. J. Bifur. Chaos Appl. Sci. Engrg., 2003, 13(1), 47–106.

[19] J. Llibre and C. Valls, *The 16th Hilbert problem for discontinuous piecewise linear differential systems separated by the algebraic curve* $y = x^n$, Math. Phys. Anal. Geom., 2023, 26(4), Paper No. 25, 9pp.

[20] M. Mousavi and H. R. Z. Zangeneh, *Limit cycle bifurcations by perturbing a Hamiltonian system with a 3-polycycle having a cusp of order one or two*, Internat. J. Bifur. Chaos Appl. Sci. Engrg., 2018, 28(3), 1850038, 20pp.

[21] R. Roussarie, *On the number of limit cycles which appear by perturbation of separatrix loop of planar vector fields*, Bol. Soc. Brasil. Mat., 1986, 17(2), 67–101.

[22] Y. Tian and M. Han, *Hopf and homoclinic bifurcations for near-Hamiltonian systems*, J. Differential Equations, 2017, 262(4), 3214–3234.

[23] L. Wei, *Limit cycle bifurcations near generalized homoclinic loop in piecewise smooth systems with a cusp*, Nonlinear Anal. Real World Appl., 2017, 38, 306–326.

[24] L. Wei, F. Liang and S. Lu, *Limit cycle bifurcations near a generalized homoclinic loop in piecewise smooth systems with a hyperbolic saddle on a switch line*, Appl. Math. Comput., 2014, 243, 298–310.

[25] J. Yang and M. Han, *Limit cycles near a double homoclinic loop*, Ann. Differential Equations, 2007, 23(4), 536–545.

[26] J. Yang and L. Zhao, *The perturbation of a class of hyper-elliptic Hamilton system with a double eight figure loop*, Qual. Theory Dyn. Syst., 2017, 16(2), 317–360.