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# STATISTICAL ENSEMBLES IN INTEGRABLE HAMILTONIAN SYSTEMS WITH PERIODIC FORCED TERMS

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**Abstract** The aim of this study was to explore the statistical ensembles problem of integrable Hamiltonian systems with periodic forced terms. The findings indicated that, over an extended time period, the average value of the system's observations converges to the initial average value within a single cycle, for a given observation function G. This effect weakens the convergence conditions. We also established the weak convergence of a measure induced by a one-parameter flow, considering the time average, and made an inference corresponding to the system discussed in this article.

**Keywords** Fourier analysis, periodic forced terms, statistical ensembles.

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## 1. Introduction

The central focus of ensemble research, also referred to as the Gibbs ensemble [12], is to investigate the conditions under which a system converges from a non-equilibrium state to an equilibrium state, as well as identifying the equilibrium state to which it converges. In mathematical context, the aim of the research is to explore the conditions that must be satisfied for the mathematical expectation of the observation function to approach infinity as time progresses under the evolution of the system, considering an observable function $G(q, p)$ and a probability density function $\rho(q, p, t)$ for $(q, p) \in \mathbb{R}^{2n}$. When the mathematical expectation of $G$ is independent of time $t$, the system is said to be in equilibrium; otherwise, it is said to be in a non-equilibrium state. Furthermore, a key research area in this field is to determine the form that the expectation converges.

Several studies and applications have been conducted in the field of ensemble research. Chan [6] developed a non-equilibrium ensemble method for gas dilution using the non-equilibrium grand canonical ensemble distribution function. The findings indicated that the non-equilibrium grand canonical partition function can be used to deduce all macroscopic non-equilibrium variables for any system that exhibits deviations from equilibrium. Zhukov and Cao (2006) [17] established a non-equilibrium statistical operator method applicable to particle ensembles in classical phase space, which are governed by equations of motion other than Hamiltonian dynamics. The method can be used for molecular dynamics simulations and quasi-

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classical or non-equilibrium approximations in quantum-classical dynamics. Bi and Liu (2011) [1] explored the formalism associated with the theory of non-equilibrium statistical ensembles, which was based on the foundations laid by the Brussels-Austin School and incorporating further developments in subdynamic equations proposed by other scholars. These scholars evaluated a spin boson model with strong coupling using this methodology. Identifying the relevant non-decoherence properties and obtaining the density reduction operator for a standard ensemble are straightforward procedures. Next year, Subsequently, N Buri et al. (2012) analyzed the general statistical ensemble of mixed quantum classical systems using the Hamiltonian formula [4]. This perspective considers any probability density in the mixed phase space as a potential physically distinguishable statistical set of the mixed system. The statistical operators of the mixed state of the mixed system and its quantum subsystems are obtained by derivation and analysis of the dynamic equations. These equations exhibit an intrinsic dependence on the overall probability density within the mixed phase space. N Buri et al. [5] analyzed the statistical ensemble representation of quantum states on the quantum phase space as outlined by the Hamiltonian formulation of quantum mechanics. The discussion focused on the lack of uniqueness of probability density within quantum phase space concerning quantum mixed states, the Liouville dynamics of probability density, and the potential use of edge distributions to represent reduced states in binary systems. Hahn and Fine (2016) introduced the stability criteria for characterizing quantum statistical ensembles of macroscopic systems [8]. The findings indicated that statistical ensembles with restricted energy distributions, such as canonical or microcanonical ensembles, can be used in the context of the fundamentals of quantum statistical physics. From a mathematical perspective, statistical ensembles are frequently simplified to the evolution of the initial random variable distribution along specific Hamiltonian systems over time. Yuzbashyan [16] established the concept of long-term ensemble for Hamiltonian systems including quantum systems. In addition, Chad Mitchell explored the long-term behavior of bounded orbits associated with the initial ensemble in a nondegenerate integrable Hamiltonian system. He provided the conditions for the ensemble's convergence, expressed in terms of action angle, towards the equilibrium state [11]. Liu and Li (2023) extended the investigation of statistical ensembles and weak convergence of measures for integrable Hamiltonian systems to include cases of integrable Hamiltonian systems with almost periodic transitions [9].

Although integrable Hamiltonian systems hold significance in mathematical and physical research, real-world scenarios typically involve various disturbances affecting the system. In the current study, we primarily investigate a specific type of perturbed integrable Hamiltonian system namely, an integrable Hamiltonian system with periodic forced terms. Several scholars have conducted research on perturbed integrable Hamiltonian systems and obtained several intriguing results. Long (1990) evaluated the presence and multiplicity of periodic solutions for a Hamiltonian system with bounded forcing terms [10]. Yu and Zheng [15] used the three critical point theorem and derived adequate conditions for the presence of three periodic solutions in second-order discrete Hamiltonian systems with small strong forcing terms. Bin and Huang (2009) presented findings on boundary value problems involving forced terms in discrete Hamiltonian systems [3] based on the Saddle point theorem and the minimum action principle. Furthermore, the scholars developed a novel action functional to implement the variational method; which differed from the conventional action functional used when periodic boundary conditions are present.

The aim of this study was to explore the specific conditions required to ensure that a statistical ensemble of orbits gradually converges to an invariant steady-state under a smooth flow. In addition, we define and illustrate the concept of weak convergence of probability measures under a one-parameter flow. The system evaluated in this study is an integrable Hamiltonian system that undergoes evolution with the influence of forced terms, characterized by periodic and continuous behavior over time. Most previous studies mainly focused on the types and properties of solutions, whereas in the current study, we explored the convergence behavior of the statistical ensemble of the system under the influence of the system. The action space of probability measures undergoes changes in response to the flow and to the best of our knowledge, this field is currently unexplored. We demonstrated that under certain specific assumptions, after time averaging, the ensemble of the system converges to a fixed periodic average case.

The subsequent sections of article are organized as follows: the fundamental definitions, lemmas, theorems, and the symbols for the application of the article are presented in the Second section. The main results and the experimental validations are presented in Section 3.

# 2. Preliminaries

This section is divided into two parts. In the first part, we provide the essential explanations and assumptions for the perturbed Hamiltonian system under consideration. Simultaneously, we provide the corresponding one-parameter flow of the system to illustrate the problem under study; in the second part, we provide the notations, definitions, lemmas, and theorems required to demonstrate the results of this paper.

## 2.1. The system and the problems studied in this paper

Consider an integrable Hamiltonian system with forced terms

$$\begin{aligned} \dot{I}\left(t\right) &= \varepsilon p_1\left(t\right), \\ \dot{\theta}\left(t\right) &= \omega\left(I\right) + \varepsilon p_2\left(t\right). \end{aligned} \tag{2.1}$$

The coordinates $(I, \theta)$ are action-angle variables in which the action variable $I \in \Omega$ and $\theta \in \mathbb{T}^N$. $\Omega \subset \mathbb{R}^N$ is a nonempty and open set and $\mathbb{T}^N = \mathbb{R}^N/\mathbb{Z}^N$ is the $N-$dimensional torus. $p_1(\cdot) : \mathbb{R} \to \mathbb{R}^N$ and $p_2(\cdot) : \mathbb{R} \to \mathbb{R}^N$ are both periodic functions with period $T$, and their 0-order Fourier coefficients are zero, which ensures that their indefinite integrals are still periodic functions, that is to say $P_1(t) = \int_0^t p_1(s)\, ds$, $P_2(t) = \int_0^t p_2(s)\, ds$, are both periodic functions with period $T$

The parameter $\varepsilon$ is considered as a sufficient small constant and it is easy to know that under the perturbations, the set $\Omega$ will have a slight deformatio. We note the $\Omega$ under perturbations at time $t$ as $\Omega_t^\varepsilon$ and $\Omega_\infty^\varepsilon = \bigcup_{t=0}^{\infty} \Omega_t^\varepsilon$ is a bounded and open set since the perturbations are periodic.

We refer to the map $\omega(\cdot) : \Omega_\infty^\varepsilon \to \mathbb{R}^N$ as the frequency map of the system (2.1). $D\omega(I) = \{\frac{\partial \omega_i(I)}{\partial I_j}\}_{1 \le i,j \le N}$ is the Jaccobian matrix of $\omega(\cdot)$ at $I$. We say $I \in \Omega_\infty^\varepsilon$ is the regular point of $\omega(\cdot)$ if and only if $rank\,[D\omega(I)] = N$. Otherwise $I$

is called critical point. In order to provide results for this article, we must assume the following assumptions for the frequency mapping $\omega(\cdot) : \Omega_\infty^\varepsilon \to \mathbb{R}^N$:

**(A1)** $\omega(\cdot) \in C^4(\Omega_\infty^\varepsilon)$;

**(A2)** $D\omega, D^2\omega, D^3\omega$ are all bounded with $M_{D\omega}, M_{D^2\omega}, M_{D^3\omega}$.

The system (2.1) defines a one-parameter flow which is given by

$$\phi_t(I,\theta) = \left(I + \varepsilon \int_0^t p_1(s)\, ds, \theta + \int_0^t \omega\left(I + \varepsilon \int_0^s p_1(\tau)\, d\tau\right) ds + \varepsilon \int_0^t p_2(s)\, ds\right).$$

The Jaccobian matrix of $\phi_t(I,\theta)$,

$$D\phi_t(I,\theta) = \begin{pmatrix} \mathbf{1}_N & 0 \\ * & \mathbf{1}_N \end{pmatrix},$$

satisfies $det[D\phi_t(I,\theta)] \equiv 1$. Thus the flow is volume preserving and $det\left(D\phi_t^{-1}(I,\theta)\right) \equiv 1$. To simplify the symbols in the following, we set

$$\begin{aligned} I_\varepsilon(I,t) &= I + \varepsilon \int_0^t p_1(s)\, ds, \\ \theta_\varepsilon(I,\theta,t) &= \theta + \int_0^t \omega\left(I + \varepsilon \int_0^s p_1(\tau)\, d\tau\right) ds + \varepsilon \int_0^t p_2(s)\, ds, \\ \Delta\theta_t(I) &= \int_0^t \omega\left(I + \varepsilon \int_0^s p_1(\tau)\, d\tau\right) ds + \varepsilon \int_0^t p_2(s)\, ds. \end{aligned}$$

In this paper, we treat the system 2.1 as an initial value problem. Suppose the probability density function $\rho_0(I_0,\theta_0)$ of the initial condition $(I(0),\theta(0)) = (I_0,\theta_0)$ is in $L^1\left(\Omega \times \mathbb{T}^N\right)$. And the point $\phi_t(I_0,\theta_0) \in \Omega_t^\varepsilon$ is described by

$$\rho_t(I,\theta) = \rho_0\left(\phi_t^{-1}(I,\theta)\right).$$

The central topic of this paper will be introduced following.

Given an observable function $G : \Omega_\infty^\varepsilon \times \mathbb{T}^N \to \mathbb{R}$, we define the expectation of $G$ under the flow $\phi_t$ as

$$\begin{aligned} < G >_t = E_t[G(I,\theta)] &= \int_{\Omega_t^\varepsilon \times \mathbb{T}^N} G(I,\theta)\, \rho_t(I,\theta)\, dId\theta \\ &= \int_{\Omega \times \mathbb{T}^N} G(\phi_t(I,\theta))\, \rho_t(\phi_t(I,\theta))\, det(D\phi_t(I,\theta))\, dId\theta \\ &= \int_{\Omega \times \mathbb{T}^N} G(\phi_t(I,\theta))\, \rho_0(I,\theta)\, dId\theta. \end{aligned}$$

One of our aim is to explore the long time behavior of $\mathbf{M_t}[< G >_s]$, where

$$\mathbf{M_t}[f(s)] = \frac{1}{t} \int_0^t f(s)\, ds.$$

And we will also consider the probability measures $P_t$ defined by:

$$P_t(A) = \int_A \rho_t(I,\theta)\, dId\theta, \quad A \subset \Omega_t^\varepsilon \times \mathbb{T}^N.$$

## 2.2. Some necessary reviews

Let's begin by reviewing the Riemann Lebesgue Lemma given in article [14], which differs from the traditional one. In fact, their primary distinction depends on whether or not the exponential position of exp is a function of $x$.

**Lemma 2.1** (VIII 2.1 Proposition 4, [14])**.** *Let $\Omega \subset \mathbb{R}^n$ be open. Suppose that $a \in L^1(\Omega)$, and that $\phi \in C^2(\Omega)$ is a real-valued function with $\nabla\phi \neq 0$. Then for $\lambda \in \mathbb{R}$,*

$$I(\lambda) = \int_\Omega a(x)\exp\left[\sqrt{-1}\lambda\phi(x)\right]dx \to 0 \qquad as \qquad |\lambda| \to \infty.$$

In article [9], Xinyu Liu and Yong Li have generalized the lemma in a more general situation.

During the proof process, the following two theorems are necessary.

**Theorem 2.1** (Chapter 12 Theorem 19, [13])**.** *Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ be two measure spaces and $\nu$ be complete. Let $f$ be integrable over $X \times Y$ with respect to the product measure $\mu \times \nu$. Then for almost all $x \in X$, the $x-$section of $f$, $f(x, \cdot)$, is integrable over $Y$ with respect to $\nu$ and*

$$\int_{X\times Y} f d(\mu \times \nu) = \int_X \left[\int_Y f(x,y)\,d\nu(y)\right] d\mu(x).$$

**Theorem 2.2** (Chapter 1 Theorem 3, [7])**.** *Let $f_i$ be a sequence of complex-valued measurable functions on a measure space $(S, \Sigma, \mu)$. Suppose that the sequence converges pointwise to a function $f$ and is dominated by some integrable function $g$ in the sense that*

$$|f_i(x)| \leq g(x)$$

*for all numbers $i$ in the index set of the sequence and all points $x \in S$. Then $f$ is integrable (in the Lebesgue sense) and*

$$\lim_{i\to\infty}\int_S |f_i(x) - f(x)|d\mu = 0,$$

*which also implies*

$$\lim_{i\to\infty}\int_S f_i(x)\,d\mu = \int_S f(x)\,d\mu.$$

Parseval's theorem is required to verify that the research object can satisfy the conditions of the preceding two theorems. This theorem is fundamental and can be seen in many textbooks

**Theorem 2.3.** *Suppose $f(x)$ is a square-integrable function over $[-\pi, \pi]$ (i.e. $f(x)$ and $f^2(x)$ are integrable on that interval), with the Fourier series*

$$f(x) \simeq \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos(nx) + b_n\sin(nx)\right).$$

*Then*

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f^2(x)\,dx = \frac{a_0^2}{2} + \sum_{i=1}^{\infty}\left(a_i^2 + b_i^2\right).$$

Next, let's provide some definitions. It is worth noting that here we provide a measure of weak convergence in the sense of time average and apply it in the subsequent proof.

**Definition 2.1** (Chapter1 Definition 1, [2])**.** Let $X$ be a metric space and let $\mathcal{B}(X)$ denote the $\sigma$- algebra of Borel subsets of $X$. A sequence $\{P_n\}$ of probability measures defined on the measurable space $(X, \mathcal{B}(X))$ is said to converge weakly to a probability measure $P$ , also defined on $(X, \mathcal{B}(X))$, if for any $g \in C_b(X)$ we have:

$$\lim_{n\to+\infty} \int_X g dP_n = \int_X g dP.$$

In this case, we write $P_n \Rightarrow P$.

Under the influence of the corresponding flow in system (2.1), the $\rho_0$-induced probability measure will vary. Unlike article [11], this change is not only predicated on the probability measure, but also on its action space. Consequently, we offer the definition *weakly convergence of the probability measure under the influence of flow.*

**Definition 2.2.** Let $\mathcal{B}(\Omega \times \mathbb{T}^n)$ denote the $\sigma$- algebra of Borel subsets of $\Omega \times \mathbb{T}^n$ equiped with probability measure $P_0$. Under the influence of one-parameter flow $\phi_t$, $P_0$ is transformed into $P_t$, and $\Omega_t \times \mathbb{T}^N = \phi_t\left(\Omega \times \mathbb{T}^N\right)$, $\Omega_\infty = \bigcup_{t=0}^{\infty} \Omega_t$. $P_t$ which is dependent on $t$ is said to converge weakly to a probability measure $P$ in time-average sense, if for any $g \in C_b(\Omega_\infty \times \mathbb{T}^n)$, we have:

$$\lim_{t\to+\infty} \mathbf{M_t}\left[\int_{\Omega_t\times\mathbb{T}^n} g dP_s\right] = \mathbf{M_T}\left[\int_{\Omega_t} g dP_0\left(\phi_t^{-1}(I,\theta)\right)\right].$$

In this case, we write $P_t \Rightarrow P_0$.

Moreover, we set

$$\bar{f} = \frac{1}{(2\pi)^N}\int_{\mathbb{T}^N} f d\theta,$$
$$\hat{f}(\vec{n}) = \frac{1}{(2\pi)^N}\int_{\mathbb{T}^N} f(\theta)\exp\left(-\sqrt{-1} < \vec{n}, \theta >\right) d\theta.$$

# 3. Main results

This section contains two distinct parts. In the first part, we present several lemmas and preliminary theorems, along with their proofs, ; in the second part, we present the key findings and provide supporting evidence.

## 3.1. Lemmas and preliminary theorems

Firstly, we provide the following two lemmas, whose proofs are relatively simple.

**Lemma 3.1.** *Assume $M(t)$ is a bounded and locally integrable function defined on $\mathbb{R}^+$ with bound $M \gg 1$ and*

$$\lim_{t\to+\infty} M(t) = 0.$$

*Then*

$$\lim_{l\to+\infty} \frac{1}{l}\int_0^l M(t)\,dt = 0.$$

**Proof.** By the assumptions of the $M(t)$ : $\forall \epsilon > 0$

$$\exists\, t_0'(\epsilon) > 0, \quad s.t.\ \forall t > t_0'(\epsilon)\,,\ |M(t)| < \frac{\epsilon}{M}.$$

Taking $t_0(\epsilon) = max\left\{t_0'(\epsilon)\,,\left(M^2-\epsilon\right)t_0'(\epsilon)/\left[(M-1)\,\epsilon\right]\right\}$, and we get: $\forall l > t_0(\epsilon)$,

$$\begin{aligned}\frac{1}{l}\Big|\int_0^l M(t)\,dt\ \Big| &\le \frac{1}{l}\Big|\left(\int_0^{t_0(\epsilon)} + \int_{t_0(\epsilon)}^l\right) M(t)\,dt\ \Big| \\ &\le \frac{1}{l}\int_0^{t_0(\epsilon)} \big|M(t)\big|dt + \frac{1}{l}\int_{t_0(\epsilon)}^l \big|M(t)\big|dt \\ &\le M t_0(\epsilon)/l + (\epsilon/M)\left[l - t_0(\epsilon)\right]/l \\ &< \epsilon.\end{aligned}$$

□

**Lemma 3.2.** *Suppose the $f : \mathbb{R} \to \mathbb{R}$ is a periodic funtion with period $T > 0$. Then*

$$\lim_{t\to+\infty} \mathbf{M_t}[f] = \mathbf{M_T}[f]\,.$$

**Proof.** Actually, for any $t \in \mathbb{R}^+$, we can write

$$t = nT + \alpha_t,$$

where $n \in \mathbb{N}$ and $0 \le \alpha_t < T$, and

$$t \to +\infty \iff n \to +\infty, \tag{3.1}$$

$$\begin{aligned}\mathbf{M_t}[f] &= \frac{1}{t}\int_0^t f(s)\,ds \\ &= \frac{1}{nT+\alpha_t}\int_0^{nT+\alpha_t} f(s)\,ds \\ &= \frac{1}{nT+\alpha_t}\left(\int_0^{nT} + \int_{nT}^{nT+\alpha_t}\right) f(s)\,ds \\ &= \frac{1}{nT+\alpha_t}\left(n\int_0^{T} + \int_0^{\alpha_t}\right) f(s)\,ds \\ &= \frac{n}{nT+\alpha_t}\int_0^T f(s)\,ds + \frac{1}{nT+\alpha_t}\int_0^{\alpha_t} f(s)\,ds \\ &= \frac{1}{T+\frac{\alpha_t}{n}}\int_0^T f(s)\,ds + \frac{1}{nT+\alpha_t}\int_0^{\alpha_t} f(s)\,ds,\end{aligned}$$

$$\begin{aligned}\lim_{t\to+\infty}\mathbf{M_t}[f] &= \lim_{n\to+\infty}\left\{\frac{1}{T+\frac{\alpha_t}{n}}\int_0^T f(s)\,ds + \frac{1}{nT+\alpha_t}\int_0^{\alpha_t} f(s)\,ds\right\} \\ &= \frac{1}{T}\int_0^T f(s)\,ds \\ &= \mathbf{M_T}[f]\,.\end{aligned}$$

□

Next, two preliminary Theorems are presented. They guarantee that theorems 2.1 and 2.2 will be utilized in the subsequent proof procedure.

**Theorem 3.1.** *Given* $G \in C\left(\Omega_\infty^\varepsilon \times \mathbb{T}^N\right)$,

$$\hat{G}\left(I_\varepsilon\left(I,t\right),\vec{n}\right) = \frac{1}{\left(2\pi\right)^N}\int_{\mathbb{T}^N} G\left(I_\varepsilon\left(I,t\right),\theta\right)\exp\left(-\sqrt{-1} < \vec{n},\theta >\right)d\theta, \quad \vec{n} \in \mathbb{Z}^N.$$

*For any fixed* $\vec{n} \in \mathbb{Z}^N$, $\hat{G}\left(I_\varepsilon\left(I,t\right),\vec{n}\right)$ *is a contiunous function w.r.t* $I$*, and this continuity is not affected by* $t$*, which implies* $\forall \varepsilon > 0,\ \forall I \in \Omega,\ \exists\ \delta\left(\varepsilon, I\right) > 0,\ \forall I' \in \Omega,\ \left|I - I'\right| < \delta\left(\varepsilon, I\right),$

$$|\hat{G}\left(I_\varepsilon\left(I',t\right),\vec{n}\right) - \hat{G}\left(I_\varepsilon\left(I,t\right),\vec{n}\right)| < \varepsilon.$$

**Proof.** For any fixed $\vec{n} \in \mathbb{Z}^N$,

$$\begin{aligned}
|\hat{G}\left(I_\varepsilon\left(I,t\right),\vec{n}\right)| &= \frac{1}{\left(2\pi\right)^N}\left|\int_{\mathbb{T}^N} G\left(I_\varepsilon\left(I,t\right),\theta\right)\exp\left(-\sqrt{-1} < \vec{n},\theta >\right)d\theta\right| \\
&\leq \frac{1}{\left(2\pi\right)^N}\int_{\mathbb{T}^N}\left|G\left(I_\varepsilon\left(I,t\right),\theta\right)\exp\left(-\sqrt{-1} < \vec{n},\theta >\right)\right|d\theta \\
&\leq \frac{1}{\left(2\pi\right)^N}\int_{\mathbb{T}^N}\left|G\left(I_\varepsilon\left(I,t\right),\theta\right)\right|d\theta \\
&\leq ||G||_\infty.
\end{aligned}$$

And $\forall \varepsilon > 0,\ \forall I \in \Omega_\infty^\varepsilon,\ \exists\ \delta\left(I,\varepsilon\right) > 0,\ \forall I' \in \Omega_\infty^\varepsilon,\ \left|I' - I\right| < \delta\left(I,\varepsilon\right),$

$$|G\left(I',\theta\right) - G\left(I,\theta\right)| < \varepsilon.$$

Moreover, $\forall I, I' \in \Omega,\ \left|I' - I\right| < \delta\left(I,\varepsilon\right),\ |I_\varepsilon\left(I',t\right) - I_\varepsilon\left(I,t\right)| < \delta\left(I,\varepsilon\right),$

$$\begin{aligned}
&|\hat{G}\left(I_\varepsilon\left(I',t\right),\vec{n}\right) - \hat{G}\left(I_\varepsilon\left(I,t\right),\vec{n}\right)| \\
\leq&\frac{1}{\left(2\pi\right)^N}\left|\int_{\mathbb{T}^N}\left[G\left(I_\varepsilon\left(I,t\right),\vec{n}\right) - G\left(I_\varepsilon\left(I',t\right),\vec{n}\right)\right]\exp\left(-\sqrt{-1} < \vec{n},\theta >\right)d\theta\right| \\
\leq&\frac{1}{\left(2\pi\right)^N}\int_{\mathbb{T}^N}\left|G\left(I_\varepsilon\left(I,t\right),\vec{n}\right) - G\left(I_\varepsilon\left(I',t\right),\vec{n}\right)\right|\left|\exp\left(-\sqrt{-1} < \vec{n},\theta >\right)\right|d\theta \\
\leq&\frac{1}{\left(2\pi\right)^N}\int_{\mathbb{T}^N}\left|G\left(I_\varepsilon\left(I,t\right),\vec{n}\right) - G\left(I_\varepsilon\left(I',t\right),\vec{n}\right)\right|d\theta \\
<&\varepsilon.
\end{aligned}$$

$\square$

**Theorem 3.2.** *Suppose* $G \in C\left(\Omega_\infty^\varepsilon\right),\ \rho_0 \in C_c\left(\Omega \times \mathbb{T}^N\right)$*, then*

$$\int_\Omega \sum_{\vec{n}\in\mathbb{Z}^N} |\hat{G}\left(I_\varepsilon\left(I,t\right),\vec{n}\right)\hat{\rho}_0\left(I,-\vec{n}\right)|dI < \infty.$$

**Proof.** By Theorem 2.3, we can get for fixed $I \in \Omega$

$$\begin{aligned}
\sum_{\vec{n}\in\mathbb{Z}^N} |\hat{G}\left(I_\varepsilon\left(I,t\right),\vec{n}\right)| &\equiv \frac{1}{\left(2\pi\right)^N}\int_{\mathbb{T}^N} |G\left(I_\varepsilon\left(\left(I,t\right),\cdot\right)\right)|^2 d\theta \\
&= \|G\left(I_\varepsilon\left(\left(I,t\right),\cdot\right)\right)\|_{L^2}^2 < \infty.
\end{aligned}$$

In a similar way,

$$\|\hat{\rho}_0 (I, \cdot)\|_{L^2}^2 < \infty.$$

By Cauchy-Schwarz inequality,

$$\begin{aligned}
&\sum_{\vec{n}\in\mathbb{Z}^N} |\hat{G}(I_\varepsilon (I,t), \vec{n})\, \hat{\rho}_0 (I, -\vec{n})| \\
\leq &\left(\sum_{\vec{n}\in\mathbb{Z}^N} |\hat{G}(I_\varepsilon (I,t), \vec{n})|^2\right)^{\frac{1}{2}} \left(\sum_{\vec{n}\in\mathbb{Z}^N} |\hat{\rho}_0 (I, -\vec{n})|^2\right)^{\frac{1}{2}} \\
\leq &\|G(I_\varepsilon (I,t), \cdot)\|_{L^2} \|\rho_0 (I, \cdot)\|_{L^2}.
\end{aligned}$$

We denote the support set of $\rho_0$ as $K_{\rho_0}$ and have

$$\begin{aligned}
\int_\Omega \sum_{\vec{n}\in\mathbb{Z}^N} |\hat{G}(I_\varepsilon (I,t), \vec{n})\, \hat{\rho}_0 (I, -\vec{n})|\, dI &\leq \int_\Omega \|G(I_\varepsilon (I,t), \cdot)\|_{L^2} \|\rho_0 (I, \cdot)\|_{L^2} dI \\
&\leq \int_{K_{f_0}} \|G(I_\varepsilon (I,t), \cdot)\|_{L^2} \|\rho_0 (I, \cdot)\|_{L^2} dI \\
&< \infty.
\end{aligned}$$

□

## 3.2. Main results and their proofs

Now we can present our main findings and provide proofs of them.

**Theorem 3.3.** *In system (2.1), suppose the frequency map $\omega$ satisfies the assumptions* **(A1)**, **(A2)**, *and the probability density function $\rho_0 \in C_c\left(\Omega \times \mathbb{T}^N\right)$. Then for any $G \in C\left(\Omega_\infty^\varepsilon \times \mathbb{T}^N\right)$ and some $\varepsilon > 0$ fixed,*

$$\lim_{t\to\infty} \mathbf{M_t}\left[< G >_s\right] =< \mathbf{M_T}\left[\bar{G}\left[I + \varepsilon P_1 (t)\right]\right] >_0 . \tag{3.2}$$

**Proof.** As previously stated, we have

$$\begin{aligned}
< G >_t = E_t\left[G(I,\theta)\right] &= \int_{\Omega_t^\varepsilon \times \mathbb{T}^N} G(I,\theta)\, \rho_t (I,\theta)\, dI d\theta \\
&= \int_{\Omega \times \mathbb{T}^N} G(\phi_t (I,\theta))\, \rho_t (\phi_t (I,\theta))\, det\left(D\phi_t (I,\theta)\right) dI d\theta \\
&= \int_{\Omega \times \mathbb{T}^N} G(\phi_t (I,\theta))\, \rho_0 (I,\theta)\, dI d\theta \\
&= \int_{\Omega \times \mathbb{T}^N} G(I_\varepsilon (I,t), \theta_\varepsilon (I,\theta,t))\, \rho_0 (I,\theta)\, dI d\theta.
\end{aligned}$$

On the one hand, the Fourier coefficients of $G(I_\varepsilon (I,t), \theta_\varepsilon (I,\theta,t))$

$$\begin{aligned}
&\frac{1}{(2\pi)^2} \int_{\mathbb{T}^N} G(I_\varepsilon (I,t), \theta_\varepsilon (I,\theta,t)) \exp\left(-\sqrt{-1} < \vec{n}, \theta >\right) d\theta \\
=&\frac{1}{(2\pi)^2} \int_{\mathbb{T}^N} G(I_\varepsilon (I,t), \theta + \Delta\theta_t (I)) \exp\left(-\sqrt{-1} < \vec{n}, \theta >\right) d\theta
\end{aligned}$$

$$
\begin{aligned}
&=\frac{1}{(2\pi)^2}\int_{\mathbb{T}^N} G\left(I_\varepsilon(I,t),\theta+\Delta\theta_t(I)\right)\exp\left(-\sqrt{-1}<\vec{n},\theta+\Delta\theta_t(I)>\right)\\
&\quad\times\exp\left(\sqrt{-1}<\vec{n},\Delta\theta_t(I)>\right)d\theta\\
&=\frac{1}{(2\pi)^2}\int_{\mathbb{T}^N} G\left(I_\varepsilon(I,t),\theta+\Delta\theta_t(I)\right)\exp\left(-\sqrt{-1}<\vec{n},\theta+\Delta\theta_t(I)>\right)d\theta\\
&\quad\times\exp\left(\sqrt{-1}<\vec{n},\Delta\theta_t(I)>\right)\\
&=\frac{1}{(2\pi)^2}\int_{\mathbb{T}^N} G\left(I_\varepsilon(I,t),\tilde{\theta}\right)\exp\left(-\sqrt{-1}<\vec{n},\tilde{\theta}>\right)d\theta\exp\left(\sqrt{-1}<\vec{n},\Delta\theta_t(I)>\right)\\
&=\hat{G}\left(I_\varepsilon(I,t),\vec{n}\right)\exp\left(\sqrt{-1}<\vec{n},\Delta\theta_t(I)>\right),
\end{aligned}
$$

on the other hand, the Fourier coefficients of $\rho_0^*(I,\theta)=\rho_0(I,\theta)$,

$$
\begin{aligned}
\hat{\rho}_0^*(I,\vec{n}) &= \frac{1}{(2\pi)^2}\int_{\mathbb{T}^N}\rho_0^*(I,\theta)\exp\left(-\sqrt{-1}<-\vec{n},\theta>\right)d\theta\\
&= \frac{1}{(2\pi)^2}\int_{\mathbb{T}^N}\rho_0(I,\theta)\exp\left(-\sqrt{-1}<-\vec{n},\theta>\right)d\theta\\
&= \hat{\rho}_0(I,-\vec{n}),
\end{aligned}
$$

where $*$ denotes complex conjungation. Theorem 2.3 implies that

$$
\begin{aligned}
&\frac{1}{(2\pi)^N}\int_{\mathbb{T}^N} G\left(I_\varepsilon(I,t),\theta_\varepsilon(I,\theta,t)\right)\rho_0(I,\theta)\,d\theta\\
=&\frac{1}{(2\pi)^N}\int_{\mathbb{T}^N} G\left(I_\varepsilon(I,t),\theta_\varepsilon(I,\theta,t)\right)\rho_0^*(I,\theta)\,d\theta\\
=&\sum_{\vec{n}\in\mathbb{Z}^N}\hat{G}\left(I_\varepsilon(I,t),\vec{n}\right)\hat{\rho}_0^*(I,\vec{n})\exp\left(\sqrt{-1}<\vec{n},\Delta\theta_t(I)>\right)\\
=&\sum_{\vec{n}\in\mathbb{Z}^N}\hat{G}\left(I_\varepsilon(I,t),\vec{n}\right)\hat{\rho}_0(I,\vec{n})\exp\left(\sqrt{-1}<\vec{n},\Delta\theta_t(I)>\right)\\
=&\sum_{\vec{n}\in\mathbb{N}^N}\hat{G}\left[I_\varepsilon(I,t),\vec{n}\right]\hat{\rho}_0(I,-\vec{n})\exp\left[\sqrt{-1}t<\vec{n},\frac{1}{t}\Delta\theta_t(I)>\right]\\
=&\sum_{\vec{n}\in\mathbb{N}^N}a_{\vec{n}}(I,t)\exp\left[\sqrt{-1}t\Phi_{\vec{n}}(I,t)\right]\exp\left[\varepsilon\sqrt{-1}<\vec{n},\mathbf{M_t}[p_2]>\right],
\end{aligned}
$$

where

$$
\begin{aligned}
&a_{\vec{n}}(I,t)=\hat{G}\left[I_\varepsilon(I,t),\vec{n}\right]\hat{\rho}_0(I,-\vec{n}),\\
&\Phi_{\vec{n}}(I,t)=<\vec{n},\mathbf{M_t}\left[\omega\left(I_\varepsilon(I,s)\right)\right]>,\\
&<G>_t=(2\pi)^N\sum_{\vec{n}\in\mathbb{N}^N}\int_\Omega a_{\vec{n}}(I,t)\exp\left[\sqrt{-1}t\Phi_{\vec{n}}(I,t)\right]\exp\left[\varepsilon\sqrt{-1}<\vec{n},M_t[p_2]>\right]dI.
\end{aligned}
$$

It is easy to know, $\forall\vec{n}\in\mathbb{N}^n,\ a_{\vec{n}}\in C_b(\Omega\times\mathbb{R},\mathcal{T})$,

$$
\begin{aligned}
\Phi_{\vec{n}}(I,t) &= <\vec{n},\frac{1}{t}\int_0^t\omega\left[I_\varepsilon(s)\,ds\right]>\\
&=\frac{1}{t}\int_0^t <\vec{n},\omega(I)+\varepsilon<D\omega(I),P_1(s)>
\end{aligned}
$$

$$+\varepsilon^2 \int_0^1 (1-\alpha) < D^2\omega\left(I_{\alpha,s}\right) P_1(s), P_1(s) > d\alpha > ds,$$

where $I_{\alpha,s} = I + \alpha\varepsilon P_1(s)$.

We analyze $\nabla_I \Phi_{\vec{n}}(I,t)$ and $\Delta_I \Phi_{\vec{n}}(I,t)$ separately. Note that

$$\begin{aligned}\nabla_I \Phi_{\vec{n}}(I,t) = &< \vec{n}, D\omega(I) > + \frac{1}{t}\int_0^t \varepsilon < \vec{n}, < D^2\omega(I), P_1 >> \\ &+ \varepsilon^2 < \vec{n}, \int_0^1 (1-\alpha) < D^3\omega\left(I_{\alpha,\cdot}\right) P_1, P_1 > d\alpha > (s)\, ds.\end{aligned}$$

If we set

$$\varepsilon \ll \frac{\delta}{2\left(M_{D^2\omega}M_{P_1} + M_{D^3\omega}M_{P_1}\right)},$$

then by trigonometric inequality,

$$\begin{aligned}|\nabla_I \Phi_{\vec{n}}(I,t)| \geq &|| < \vec{n}, D\omega > | - |\frac{1}{t}\int_0^t \varepsilon < \vec{n}, < D^2\omega, P_1 >> \\ &+ \varepsilon^2 < \vec{n}, \int_0^1 (1-\alpha) < D^3\omega P_1, P_1 > d\alpha > ds|| \\ \geq &|| < \vec{n}, D\omega > | - [\frac{1}{t}\int_0^t \varepsilon \mid < \vec{n}, < D^2\omega, P_1 >> | \\ &+ \varepsilon^2 | < \vec{n}, \int_0^1 (1-\alpha) < D^3\omega P_1, P_1 > d\alpha| > ds]| \\ \geq &\frac{|\vec{n}|\delta}{2} \\ > &0,\end{aligned}$$

$$\begin{aligned}\Delta_I \Phi_{\vec{n}}(I,t) = &< \vec{n}, D^2\omega(I) > + \frac{1}{t}\int_0^t \varepsilon < \vec{n}, < D^3\omega(I), P_1 >> \\ &+ \varepsilon^2 < \vec{n}, \int_0^1 (1-\alpha) < D^4\omega\left(I_{\alpha,\cdot}\right) P_1, P_1 > d\alpha > (s)\, ds.\end{aligned}$$

For any closed set $K \subset \tilde{\Omega}$,

$$|\Delta_I \Phi_{\vec{n}}(I,t)| \leq |\vec{n}| \left( M_{D^2\omega,K} + M_{D^3\omega,K} M_{P_1} + \frac{1}{2} M_{D^4\omega,K} M_{P_1} \right).$$

We have,

$$\forall \vec{n} \neq \vec{0}, \lim_{t\to+\infty} (2\pi)^N \int_\Omega a_{\vec{n}}(I,t) \exp\left[\sqrt{-1}t\phi_{\vec{n}}(I,t)\right] \exp\left[\varepsilon\sqrt{-1} < \vec{n}P_2(t) >\right] dI = 0.$$

Then

$$\begin{aligned}\lim_{t\to\infty} \frac{1}{t}\int_0^t < G >_s ds &= \lim_{t\to\infty} \frac{1}{t}\int_0^t (2\pi)^N \int_\Omega \hat{G}\left[I_\varepsilon(s), \vec{0}\right] \hat{\rho}_0\left(I, \vec{0}\right) dI ds \\ &= \lim_{t\to\infty} \frac{1}{t}\int_0^t \left\{ \int_{\Omega\times\mathbb{T}^N} \bar{G}\left[I_\varepsilon(s)\right] \rho_0(I,\theta)\, dI d\theta \right\} ds\end{aligned}$$

$$
\begin{aligned}
&= \int_{\Omega\times\mathbb{T}^N} \left\{ \lim_{t\to\infty} \frac{1}{t} \int_0^t \bar{G}\left[I+\varepsilon P_1(s)\right] ds \right\} \rho_0(I,\theta)\, dId\theta \\
&=< \mathbf{M_T}\left[\bar{G}\left[I+\varepsilon P_1(t)\right]\right] >_0 .
\end{aligned}
$$

□

**Remark 3.1.** When $\varepsilon = 0$, the result above is the same as the one in article [11].

**Corollary 3.1.** *In system (2.1), suppose the frequency map $\omega$ satisfies assumptions* **(A1)**, **(A2)**. *And the probability density function $\rho_0 \in L^1\left(\Omega\times\mathbb{T}^N\right)$. For any function $G \in C_b\left(\Omega^\varepsilon_\infty \times \mathbb{T}^N\right)$, the conclusion in Theorem 3.3 still holds.*

**Proof.** This is quite simple by a density argument. Suppose $\rho_0 \in L^1\left(\Omega\times\mathbb{T}^N\right)$, we choose a sequence $\{\rho_0^n\}_{n=1}^\infty$, such that

$$
\|\rho_0 - \rho_0^n\|_{L^1} \to 0, \quad \text{as} \quad n\to\infty.
$$

Then

$$
\begin{aligned}
&\mathbf{M_t}\left(< G >_s\right) - \mathbf{M_t}^n\left(< G >_s\right) \\
=&\frac{1}{t}\int_0^t \int_{\Omega\times\mathbb{T}^N} G\left(\phi_s(I,\theta)\right) f_0(I,\theta)\, dId\theta - \frac{1}{t}\int_0^t \int_{\Omega\times\mathbb{T}^N} G\left(\phi_s(I,\theta)\right) f_0^n(I,\theta)\, dId\theta ds \\
=&\frac{1}{t}\int_0^t \int_{\Omega\times\mathbb{T}^N} G\left(\phi_s(I,\theta)\right)\left[f_0(I,\theta) - f_0^n(I,\theta)\right] dId\theta dt
\end{aligned}
$$

implies that

$$
\begin{aligned}
&\left|\mathbf{M_t}\left(< G >_s\right) - \mathbf{M_t}^n\left(< G >_s\right)\right| \\
\le&\frac{1}{t}\int_0^t \int_{\Omega\times\mathbb{T}^N} \left|G\left(\phi_s(I,\theta)\right)\right|\left|\left[\rho_0(I,\theta) - \rho_0^n(I,\theta)\right]\right| dId\theta ds \\
\le&\|\rho_0 - \rho_0^n\|_{L^1}|G| \to 0, \quad \text{as} \quad n\to\infty.
\end{aligned}
$$

□

In the above corollary, we relaxed the requirement for the $\rho_0$, but as a sacrifice, the requirement for $G$ was stricter. Next, we will relax the restrictions on $G$.

**Corollary 3.2.** *In system (2.1), suppose that the frequency map $\omega$ satisfies assumptions* (**A1**), (**A2**). *And the probability density function $\rho_0 \in L^1\left(\Omega\times\mathbb{T}^N\right)$. For any function $G \in C\left(\Omega^\varepsilon_\infty \times \mathbb{T}^N\right)$, if there exists $h \in C\left(\Omega^\varepsilon_\infty\right),\ h > 0$, such that*

$$
\begin{aligned}
&\int_{\Omega\times\mathbb{T}^N} h(I)\,\rho_0(I,\theta)\, dId\theta = R < \infty, \\
&G'\left(\phi_t(I,\theta)\right) := \frac{G\left(\phi_t(I,\theta)\right)}{h(I)} < \infty \quad \forall (I,\theta) \in \Omega\times\mathbb{T}^N,
\end{aligned}
$$

*then the conclusion in Theorem 3.3 still holds.*

**Proof.** Define

$$
\rho_0'(I,\theta) = \frac{h(I)\,\rho_0(I,\theta)}{R}.
$$

Since $\rho_0' \geq 0$ and

$$\int_{\Omega\times\mathbb{T}^N} \rho_0'(I,\theta)\, dId\theta = \frac{1}{R}\int_{\Omega\times\mathbb{T}^N} h(I)\,\rho_0(I,\theta)\, dId\theta = 1,$$

$\rho_0'(I,\theta)$ defines a probability density.

$$\begin{aligned}
&\lim_{t\to\infty}\int_{\Omega\times\mathbb{T}^N} G'(\phi_s(I,\theta))\,\rho_0'(I,\theta)\, dId\theta\\
=&\lim_{t\to\infty}\int_{\Omega\times\mathbb{T}^N} \mathbf{M_t}\left[G'(\phi_t(I,\theta))\right]\rho_0'(I,\theta)\, dId\theta\\
=&\int_{\Omega\times\mathbb{T}^N} \mathbf{M_t}\left[\frac{\bar{G}(\phi_s(I,\theta))}{\bar{h}(I)}\right]\frac{h(I)\,\rho_0(I,\theta)}{R} dId\theta\\
=&\int_{\Omega\times\mathbb{T}^N} \frac{\mathbf{M_t}\left[\bar{G}(\phi_s(I,\theta))\right]}{h(I)}\frac{h(I)\,\rho_0(I,\theta)}{R} dId\theta\\
=&\frac{1}{R}\int_{\Omega\times\mathbb{T}^N} \mathbf{M_t}\left[\bar{G}(\phi_s(I,\theta))\right]\rho_0(I,\theta)\, dId\theta,
\end{aligned}$$

so

$$\begin{aligned}
\lim_{t\to\infty}\mathbf{M_t}\left[< G >_s\right] &= R\lim_{t\to\infty}\int_0^t\int_{\Omega\times\mathbb{T}^N} G'(\phi_s(I,\theta))\,\rho_0'(I,\theta)\, dId\theta ds\\
&=< \mathbf{M_T}\left[\bar{G}(\phi_s(I.\theta))\right] >_0 .
\end{aligned}$$

□

**Remark 3.2.** In contrast to reference [11], Corollary 3.2 imposes more stringent criteria on functions G and h. As the action variable fluctuates over time, it is unfeasible to reduce the requirement of G to the degree illustrated in [11]. Unfortunately, the requirement for frequency mapping $\omega$ cannot be modified.

**Theorem 3.4.** *In system (2.1), let the frequency map $\omega$ satisfy assumptions* **(A1)**, **(A2)**. *Suppose the probability density function $\rho_0 \in L^1\left(\Omega\times\mathbb{T}^N\right)$, $\rho_t$ is the probability density function of $(I,\theta)\in\Omega_t^\varepsilon\times\mathbb{T}^N$ under the one-parameter flow $\phi_t$, and $P_0$ and $P_t$ are their induced probability measures, respectively. Then*

$$\mathbf{M_t}\left[P_s\right]\Rightarrow\mathbf{M_T}\left[\bar{P}_0\left(\phi_s^{-1}\right)\right], \quad \textit{as } t\to\infty.$$

**Proof.** Following the equation 3.2 in Theorem 3.3, we have

$$\begin{aligned}
&\lim_{t\to\infty}\frac{1}{t}\int_0^t\int_{\Omega_s^\varepsilon\times\mathbb{T}^N} G(I,\theta)\, dP_s ds\\
=&\lim_{t\to\infty}\frac{1}{t}\int_0^t\int_{\Omega_s\times\mathbb{T}^N} G(I,\theta)\,\rho_s(I,\theta)\, dId\theta ds\\
=&\lim_{t\to\infty}\frac{1}{t}\int_0^t\int_{\Omega_s^\varepsilon\times\mathbb{T}^N} G(I,\theta)\,\rho_0\left(\phi_s^{-1}(I,\theta)\right) dId\theta ds\\
=&\lim_{t\to\infty}\mathbf{M_t}[< G >_s]\\
=&< \mathbf{M_T}[\bar{G}[I+\varepsilon P_1(t)]] >_0
\end{aligned}$$

$$
\begin{aligned}
&= \int_{\Omega\times\mathbb{T}^N} \frac{1}{T}\int_0^T \frac{1}{(2\pi)^N}\int_{\mathbb{T}^N} G\left(\phi_t\left(I,\theta\right)\right) d\theta dt \rho_0\left(I,\theta\right) dI d\theta \\
&= \frac{1}{T}\int_0^T \int_{\Omega\times\mathbb{T}^N} G\left(\phi_s\left(I,\theta\right)\right) \frac{1}{(2\pi)^N}\int_{\mathbb{T}^N} \rho_0\left(I,\theta\right) d\theta dI d\theta ds \\
&= \frac{1}{T}\int_0^T \int_{\Omega_s^\epsilon\times\mathbb{T}^N} G\left(I,\theta\right) \frac{1}{(2\pi)^N}\int_{\mathbb{T}^N} \rho\left(\phi_s^{-1}\left(I,\theta\right)\right) d\theta dI d\theta ds \\
&= \frac{1}{T}\int_0^T \int_{\Omega_s^\epsilon\times\mathbb{T}^N} G\left(I,\theta\right) d\bar{P}_0\left(\phi_s^{-1}\right) ds.
\end{aligned}
$$

□

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