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DYNAMIC ANALYSIS AND OPTIMAL CONTROL OF A TOXICANT-POPULATION MODEL WITH REACTION-DIFFUSION*

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Abstract In this paper, we study the threshold dynamics and optimal control of a toxicant-population model with reaction-diffusion to understand how toxicant affect populations. In order to obtain the extinction and persistent conditions of the toxicant, the basic reproduction number of the model is considered, when $R_0 < 1$, the toxicant-free equilibrium is globally attractive, when $R_0 > 1$, the solution to the system is uniformly persistent. We also introduce the optimal control strategy, with the method of dynamic programming, the Hamilton-Jacobi-Bellman (HJB) equation is constructed and the optimal control is obtained. Finally, we conduct numerical simulations to verify the theoretical analysis.

Keywords Toxicant-population model, reaction-diffusion, basic reproduction number, optimal control, Hamilton-Jacobi-Bellman equation.

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1. Introduction

In recent years, with the rapid development of industry and agriculture, the environmental pollution problem has become more and more serious [1, 23, 24], and many ecological problems have appeared one after another (e.g., the decrease of species diversity and the extinction of some species). Although people all over the world have been fighting against environmental pollution, a large number of pollution problems are still emerging. Therefore, it is extremely important to reduce the impact of toxicant on the population and the environment. Many scholars have studied the dynamic behavior of population in a polluted environment by establishing mathematical models.

In the 1980s, Hallam et al. [3–5] proposed a classic deterministic system of toxicant-population and studied the persistence and extinction of populations. Since then, a large number of toxicant-population models considering different influencing factors have been proposed to analyze the effects of toxicant [11,13–16,20,21,34,36]. For example, for ordinary differential equations, Liu et al. [13] investigated the ef-

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fects of impulsive toxicant input on the population in a polluted environment, and showed that the population is extinct when the impulsive period is less than some critical value, otherwise the population is permanent. Liu et al. [15] studied the survival analysis of a stochastic single-species population model with jumps in a polluted environment, and they found that Lévy jumps have significant effects to the persistence and extinction results. For age-structured equations, Luo et al. [16] studied the optimal control of the age-dependent population hybrid system in a polluted environment, the existence of optimal control policy is carefully verified by means of Ekeland's variational principle. Li et al. [11] used the truncated Euler-Maruyama (EM) method to study a stochastic age-structured population model with Markovian switching in a polluted environment. However, the above mentioned references did not consider the spatial heterogeneity. In the real world, spatial factors affect the sex ratio, age composition, immigration rate and emigration rate of organisms, in addition, we also know that the populations and toxicant in the ecology drift randomly around media such as soil and water (e.g., the nuclear waste water will diffuse into the surrounding seas), that is, spatial diffusion also has practical significance to be considered. Many population models introduce spatial diffusion such as [12, 30, 31], but the effects of toxicant are not considered, only a few articles on toxicant-population models have studied it, for example, Kang et al. studied the diffusion mechanism of populations and toxicant in [9].

In addition, it is well known that the proliferation of toxicant will lead to huge economic costs, mainly including the reduction of crop yields and the expenditures related to pollution prevention. Therefore, from the perspective of ecotoxicology and socioeconomics, how to formulate the control strategies for toxicant is an important and meaningful issue. To solve these problems, the optimal control strategy for toxicant should be formulated. As far as we know, most of the existing literature on optimal control use the method of maximum principle, it only considers the control over a period of time with fixed initial time and state. However, the control of toxicant may start at any time, so the dynamic programming is considered in this paper, the basic idea of this method applied to optimal control is to consider a family of optimal control problems with different initial times and states. In this way, we can control toxicant at any time and achieve the goal of minimizing the concentration of toxicant and the cost of application control.

In this paper, we discuss the threshold dynamics before studying the optimal control, when the threshold parameter $R_0 < 1$, the toxicant will go extinct, there is no need to control the toxicant, when $R_0 > 1$, the toxicant is persistent, so we consider controlling the toxicant in this situation. The contributions of this article are listed as follows:

- We establish a toxicant-population model with reaction-diffusion and obtain the threshold parameter of toxicant extinction and toxicant persistence.
- With the dynamic programming method, we construct the Hamilton-Jocobi-Bellman equation and prove the existence and uniqueness of its viscosity solution, obtain the optimal control of toxicant.

The organization of this paper is as follows. In section 2, we present model formulation and some important information. In section 3, we investigate the dynamics of the model, first prove the well-posedness and define the basic reproduction number R_0, then we show that R_0 is a threshold parameter of toxicant extinction and the toxicant persistence. In section 4, the optimal control problem is studied, we first give the objective function and prove the uniqueness and existence of the viscosity solution of the HJB equation, then the optimal control is obtained through the Hamiltonian function. In section 5, we perform numerical simulations to verify the correctness of the threshold dynamics and optimal control strategy.

2. Model formulation and preliminaries

Luo and He [16] established a toxicant-population model with age-structure in a polluted environment which takes the form

$$\begin{cases} \dfrac{\partial P(a,t)}{\partial a} + \dfrac{\partial P(a,t)}{\partial t} = -\mu(a, C_0(t))P(a,t), \\ \dfrac{dC_0(t)}{dt} = kC_e(t) - gC_0(t) - mC_0(t), \\ \dfrac{dC_e(t)}{dt} = -k_1C_e(t)P(t) - hC_e(t) + g_1C_0(t)P(t) + u(t), \end{cases} \tag{2.1}$$

where $P(a,t)$ represents the density of the population of age a at time t, $C_0(t)$ represents the concentration of the toxicant in the organism at time t and $C_e(t)$ represents the concentration of the toxicant in the environment at time t, and $u(t)$ is the rate of external input of the toxicant into the environment. The coefficients μ, k, g, m, k_1, g_1 and h denote positive constants characterizing the functional interactions among the model components.

However, environment in reality typically varies with respect to space and time, and this heterogeneity may directly affect the viability of the population and the persistence of the toxicant, thus the space-dependent parameters should be used due to spatial heterogeneity. Furthermore, it is already mentioned that spatial diffusion is an important factor in the biological system. Therefore, inspired by model (2.1) and combined with the above ideas, we consider a spatiotemporal dependent population model in a closed polluted environment as follows

$$\begin{cases} \dfrac{\partial P(x,t)}{\partial t} = d_1\Delta P(x,t) + \Lambda(x) - r(x)P(x,t) - \alpha(x)C_0(x,t)P(x,t), \\ \qquad x \in \Omega, t > 0, \\ \dfrac{\partial C_0(x,t)}{\partial t} = k(x)C_e(x,t) + f(x)C_0(x,t)P(x,t) - (g(x) + m(x))C_0(x,t), \\ \qquad x \in \Omega, t > 0, \\ \dfrac{\partial C_e(x,t)}{\partial t} = d_2\Delta C_e(x,t) - k_1(x)C_e(x,t)P(x,t) - h(x)C_e(x,t) \\ \qquad + g_1(x)C_0(x,t)P(x,t), x \in \Omega, t > 0, \end{cases} \tag{2.2}$$

with boundary and initial condition

$$\begin{cases} \dfrac{\partial P(x,t)}{\partial v} = \dfrac{\partial C_0(x,t)}{\partial v} = \dfrac{\partial C_e(x,t)}{\partial v} = 0, \ x \in \partial\Omega, \ t > 0, \\ P(x,0) = P^0(x), \ C_0(x,0) = C_0^0(x), \ C_e(x,0) = C_e^0(x), \ x \in \overline{\Omega}, \end{cases} \tag{2.3}$$

where $P(x,t)$ represents the density of the population at position x and time t, while $C_0(x,t)$ and $C_e(x,t)$ represent the concentration of toxicant in the organism and environment at position x and time t, respectively. The notation $\frac{\partial}{\partial v}$ means the normal derivative along v to $\partial\Omega$. The biological meaning of the parameters in model (2.2) are listed in Table 1.

Table 1. The meaning of the parameters

Parameters	Description
$r(x)$	the mortality rate function of the population
$\Lambda(x)$	the recruitment of population
$\alpha(x)$	the decreasing of the growth associated with the uptake of the toxicant
$k(x)$	the net organismal uptake rate of toxicant from the environment
$g(x)$	the net organismal excretion rate of toxicant
$m(x)$	the depuration rate of toxicant due to metabolic process and other losses
$h(x)$	the total loss rate of toxicant from the environment
$f(x)$	the change in the amount of toxicant as reflected by newborn organisms
$k_1(x)$	the loss rate of the toxicant that is due to the uptake of toxicant by the population
$g_1(x)$	the increase of the toxicant coming from the egestion of the total population

Let $\mathbb{X} := C(\overline{\Omega}, \mathbb{R}^3)$ be the state space associated with the supremum norm $\|\phi\|_{\mathbb{X}}$, where $\|\phi\|_{\mathbb{X}} = \max\left\{\sup\limits_{x\in\overline{\Omega}}|\phi_1(\cdot)|, \sup\limits_{x\in\overline{\Omega}}|\phi_2(\cdot)|, \sup\limits_{x\in\overline{\Omega}}|\phi_3(\cdot)|\right\}$. Define $\mathbb{X}^+ := C(\overline{\Omega}, \mathbb{R}^3_+)$, it follows that $(\mathbb{X}, \mathbb{X}^+)$ is a strongly ordered Banach space.

Denote $\Gamma_i(t, x, y)(i = 1, 2)$ is the Green function associated with the operator $\frac{\partial v}{\partial t} = \Delta v$ in Ω subject to the Neumann boundary condition as (2.3). Let $\Xi_1(t) : C(\overline{\Omega}, R) \to C(\overline{\Omega}, R)$ be the C_0 semigroup associated with $d_1\Delta - r(\cdot)$ subject to (2.3), therefore, we obtain that

$$(\Xi_1(t)\phi)(\cdot) = e^{-r(\cdot)t}\int_{\Omega}\Gamma_1(t, \cdot, y)\phi(y)dy,\ \forall\phi \in C(\overline{\Omega}, \mathbb{R}),\ t \geq 0,$$

and $\Xi_3(t) : C(\overline{\Omega}, R) \to C(\overline{\Omega}, R)$ is the C_0 semigroup associated with $d_2\Delta - h(x)$ subject to (2.3), so

$$(\Xi_3(t)\phi)(\cdot) = e^{-h(\cdot)t}\int_{\Omega}\Gamma_2(t, \cdot, y)\phi(y)dy,\ \forall\phi \in C(\overline{\Omega}, \mathbb{R}),\ t \geq 0.$$

Follow the conclusion in [25, Section 7.1], for $\forall t \geq 0$, $\Xi_i(t) : C(\overline{\Omega}, \mathbb{R}) \to (\overline{\Omega}, \mathbb{R})(i = 1, 3)$ is a strongly positive and compact semigroup. Denote

$$(\Xi_2(t)\phi)(\cdot) = e^{-(g(\cdot)+m(\cdot))t}\phi(\cdot).$$

Through the above setting, we obtain $X(t) = (\Xi_1(t), \Xi_2(t), \Xi_3(t)) : \mathbb{X} \to \mathbb{X}, t \geq 0$, is a C_0 semigroup(see, [22]).

Denote $\mathcal{G} = (G_1, G_2, G_3) : \mathbb{X}^+ \to \mathbb{X}$, and $G_i (i = 1, 2, 3)$ is defined by

$$\begin{cases} G_1(\phi)(\cdot) = \Lambda(\cdot) - \alpha(\cdot)\phi_1(\cdot)\phi_2(\cdot), \\ G_2(\phi)(\cdot) = f(\cdot)\phi_1(\cdot)\phi_2(\cdot) + k(\cdot)\phi_3(\cdot), \\ G_3(\phi)(\cdot) = g_1(\cdot)\phi_1(\cdot)\phi_2(\cdot) - k_1(\cdot)\phi_1(\cdot)\phi_3, \end{cases}$$

where $x \in \overline{\Omega}$ and $\phi := (\phi_1, \phi_2, \phi_3) = (P^0, C_0^0, C_e^0) \in \mathbb{X}^+$. Then (2.2) can be written as the following equation

$$u(\cdot, t) = X(t)\phi + \int_0^t X(t-s)\mathcal{G}(u(\cdot, s))ds,$$

where $u(\cdot, t) = (P(\cdot, t), C_0(\cdot, t), C_e(\cdot, t))$.

For convenience, we simply set

$$\bar{q}(x) := \max_{x\in\Omega}\{q(x)\}, \tilde{q}(x) := \min_{x\in\Omega}\{q(x)\},$$

where $q(x) = r(x), \alpha(x), k(x), g(x), m(x), k_1(x), g_1(x), h(x)$.

3. Threshold dynamics

In this section, we study the threshold dynamics of model (2.2). We first prove the well-posedness and define the basic reproduction number, and then show that R_0 is a threshold parameter for the extinction and persistence of toxicant.

3.1. Well-posedness

The following lemma considers the local solution of the system (2.2) on $\mathbb{X}^+$, which depends on the initial date ϕ.

Lemma 3.1. *For any initial date $\phi := (\phi_1, \phi_2, \phi_3) \in \mathbb{X}^+$, system (2.2) exists a unique solution $u(\cdot, t; \phi) = (P(\cdot, t), C_0(\cdot, t), C_e(\cdot, t))$ on $[0, \tau_{\max})$ with $u(\cdot, 0; \phi) = \phi$, where $\tau_{\max} \leq \infty$. Furthermore, if $t \in [0, \tau_{\max})$, $u(\cdot, t; \phi) \in \mathbb{X}^+$.*

Proof. Since X is generated by the linear homogeneous part of (2.2) and denoted by $\mathcal{B}$ with the domain as

$$D(\mathcal{B}) = \left\{\phi : \frac{\partial\phi}{\partial v} = 0 \text{ on } \partial\Omega, \mathcal{B}\phi \in \mathbb{X}\right\}.$$

In fact, for any $\phi \in \mathbb{X}^+$ and $\theta \geq 0$, we have

$$\phi + \theta\mathcal{G}(\phi) = \begin{bmatrix} \phi_1(\cdot) + \theta[\Lambda(\cdot) - \alpha(\cdot)\phi_1(\cdot)\phi_2(\cdot)] \\ \phi_2(\cdot) + \theta[f(\cdot)\phi_1(\cdot)\phi_2(\cdot) + k(\cdot)\phi_3(\cdot)] \\ \phi_3(\cdot) + \theta[g_1(\cdot)\phi_1(\cdot)\phi_2(\cdot) - k_1(\cdot)\phi_1(\cdot)\phi_3] \end{bmatrix} \geq \begin{bmatrix} \phi_1(\cdot)[1 - \theta\bar{\alpha}\phi_2(\cdot)] \\ \phi_2(\cdot) \\ \phi_3(\cdot)[1 - \theta\bar{k}_1\phi_1(\cdot)] \end{bmatrix}.$$

Therefore, we have the following equation

$$\lim_{\theta\to 0^+} \frac{1}{\theta}\text{dist}(\phi + \theta\mathcal{G}(\phi), \mathbb{X}^+) = 0, \ \forall\phi \in \mathbb{X}^+.$$

By [25, Corollary 4], we know that system (2.2) exists a unique positive solution $u(\cdot,t)$ on $[0,\tau_{\max})$, where $0 \le \tau_{\max} \le \infty$. Furthermore, $u(\cdot,t) \in \mathbb{X}^+$ is a classical solution of system (2.2). □

Lemma 3.2. [35, Lemma 1] *For any $d_W > 0$, $W^0(x) \not\equiv 0$, the following reaction-diffusion equation*

$$\begin{cases} \dfrac{\partial \omega}{\partial t} = d_1 \Delta \omega + \Lambda(x) - r(x)\omega, \ x \in \Omega, \ t > 0, \\ \dfrac{\partial \omega}{\partial v} = 0, \ x \in \partial\Omega, \ t > 0, \\ \omega(\cdot,0) = \omega^0(\cdot), \ x \in \Omega, \end{cases} \tag{3.1}$$

exists a unique globally asymptotically stable positive steady state $P^*(x)$. Furthermore, if $\Lambda(x) = \Lambda, r(x) = r$, then $P^* = \frac{\Lambda}{r}$.

The following lemma shows that the local solution can be extended to a global one, that is, $\tau_{\max} = \infty$.

Lemma 3.3. *For $\forall \phi \in \mathbb{X}^+$, system (2.2) exists a unique solution $u(\cdot,t;\phi) = (P(\cdot,t), C_0(\cdot,t), C_e(\cdot,t))$ on $[0,\infty)$ with $u(\cdot,0;\phi) = \phi$ and the semiflow $\Phi(t) : \mathbb{X}^+ \to \mathbb{X}^+, t \ge 0$, generated by system (2.2) is defined by*

$$\Phi(t)\phi := u(\cdot,t;\phi) := (P(\cdot,t;\phi),\ C_0(\cdot,t;\phi),\ C_e(\cdot,t;\phi)),\ \forall x \in \overline{\Omega},\ t \ge 0.$$

Moreover, $\Phi(t)$ is a point dissipative(ultimately bounded).

Proof. We first establish the boundedness of $P(x,t)$. According to the first equation of system (2.2) and Lemma 3.1, it is evident that $P(x,t)$ satisfies the following inequality

$$\begin{cases} \dfrac{\partial P(x,t)}{\partial t} \le d_1 \Delta P(x,t) + \Lambda(x) - r(x)P(x,t), \ x \in \Omega, \ t > 0, \\ \dfrac{\partial P(x,t)}{\partial v} = 0, \ x \in \partial\Omega, \ t > 0. \end{cases} \tag{3.2}$$

It is easy to see that $P(x,t)$ is a subsolution of (3.2) according to (3.1). Through Lemma 3.2 and the standard parabolic comparison theorem, we have

$$\limsup_{t\to\infty} P(x,t) \le P^*(x), \ \text{uniformly for } x \in \overline{\Omega}. \tag{3.3}$$

Thus, the ultimately bounded of $P(x,t)$ is proved. Then there exists a positive constant M_0, such that

$$\limsup_{t\to\infty} \|P(x,t)\| \le M_0 \tag{3.4}$$

holds, where $M_0 := \|P^*(x)\|$.

From [6,17], it is easy to see that the solution of $C_0(x,t)$ satisfies the L_1 bounded estimate, then there exists M' such that $\limsup\limits_{t\to\infty} \|C_0(x,t)\|_1 \le M'$. Now we apply the divergence theorem and integrate the third equation of system (2.2) yield that

$$\begin{aligned}
&\frac{\partial}{\partial t}\int_\Omega C_e(x,t)dx\\
=&\int_\Omega d_2\Delta C_e(x,t)dx-\int_\Omega k_1(x)C_e(x,t)P(x,t)dx\\
&-\int_\Omega h(x)C_e(x,t)dx+\int_\Omega g_1(x)C_0(x,t)P(x,t)dx\\
\leq&[\bar{g}_1(M_0+1)(M'+1)]|\Omega|-\tilde{h}\int_\Omega C_e(x,t)dx.
\end{aligned}$$

It follows that

$$\limsup_{t\to\infty}(\|C_e(x,t)\|_1)\leq M'',\ \text{with}\ M''=[\bar{g}_1(M_0+1)(M'+1)]/\tilde{h}.$$

In summary, we have the result that there exists a positive constant M_1, such that

$$\limsup_{t\to\infty}(\|P(x,t)\|_1+\|C_0(x,t)\|_1+\|C_e(x,t)\|_1)\leq M_1. \tag{3.5}$$

Therefore, the solution of (2.2) satisfies the L^1 bounded estimate.

The following will prove the ultimate boundedness of solution $(C_0(x,t),C_e(x,t))$ of system (2.2). We first verify it satisfies the L^{2^n} bounded estimate, that is for $n\geq 0$, there exists a positive constant M_{2^n}, such that

$$\limsup_{t\to\infty}(\|C_0(\cdot,t)\|_{2^n}+\|C_e(\cdot,t)\|_{2^n})\leq M_{2^n},\ \forall t>T, \tag{3.6}$$

for some large time $T>0$. We will use induction to prove (3.6). The case for $n=0$ is valid in (3.5), then we assume that (3.6) is valid for $n-1$, that is,

$$\limsup_{t\to\infty}(\|C_0(\cdot,t)\|_{2^{n-1}}+\|C_e(\cdot,t)\|_{2^{n-1}})\leq M_{2^{n-1}},\ \forall t>T. \tag{3.7}$$

Multiply the both sides of the third equation of (2.2) by $C_e^{2^n-1}(x,t)$ and integrate over Ω, we have

$$\begin{aligned}
&\frac{1}{2^n}\frac{\partial}{\partial t}\int_\Omega C_e^{2^n}(x,t)dx\\
\leq&d_2\int_\Omega C_e^{2^n-1}(x,t)\Delta C_e(x,t)dx-\int_\Omega k_1(x)C_e^{2^n}(x,t)P(x,t)dx\\
&-\int_\Omega h(x)C_e^{2^n}(x,t)dx+\int_\Omega g_1(x)C_e^{2^n-1}(x,t)C_0(x,t)P(x,t)dx.
\end{aligned} \tag{3.8}$$

Since that

$$\begin{aligned}
&d_2\int_\Omega C_e^{2^n-1}(x,t)\Delta C_e(x,t)dx\\
\leq&-d_2\int_\Omega \nabla C_e(x,t)\cdot\nabla C_e^{2^n-1}(x,t)dx\\
=&-(2^n-1)d_2\int_\Omega(\nabla C_e(x,t)\cdot\nabla C_e(x,t))C_e^{2^n-2}(x,t)dx\\
=&-\frac{2^n-1}{2^{2n-2}}d_2\int_\Omega\mid\nabla C_e^{2^{n-1}}(x,t)\mid^2 dx.
\end{aligned}$$

Then (3.8) becomes

$$\begin{aligned}
&\frac{1}{2^n}\frac{\partial}{\partial t}\int_\Omega C_e^{2^n}(x,t)dx\\
\le& - D_n\int_\Omega \mid \nabla C_e^{2^{n-1}}(x,t)\mid^2 dx - \int_\Omega k_1(x)C_e^{2^n}(x,t)P(x,t)dx \\
&- \int_\Omega h(x)C_e^{2^n}(x,t)dx + \int_\Omega g_1(x)C_e^{2^n-1}(x,t)C_0(x,t)P(x,t)dx,
\end{aligned}\tag{3.9}$$

where $D_n = \frac{2^n-1}{2^{2n-2}}d_2$.

Applying Young's inequality: $ab \le \varepsilon a^p + \varepsilon^{-\frac{q}{p}}b^q$, where $a, b, \varepsilon > 0$, $p, q > 1$ and $\frac{1}{p}+\frac{1}{q}=1$. Take $p = 2^n$, $q = 2^n/(2^n-1)$, we estimate $\int_\Omega C_0(x,t)C_e^{2^n-1}(x,t)dx$ as follows

$$\begin{aligned}
&\int_\Omega C_0(x,t)C_e^{2^n-1}(x,t)dx\\
\le&\varepsilon_0\int_\Omega C_0^{2^n}(x,t)dx + G_{\varepsilon_0}\int_\Omega C_e^{2^n}(x,t)dx,\ \text{for } t\ge t_0,\ G_{\varepsilon_0} = \varepsilon_0^{-\frac{1}{2^n-1}}.
\end{aligned}$$

Thus (3.9) can be estimated by

$$\begin{aligned}
&\frac{1}{2^n}\frac{\partial}{\partial t}\int_\Omega C_e^{2^n}(x,t)dx\\
\le& - D_n\int_\Omega \mid \nabla C_e^{2^{n-1}}(x,t)\mid^2 dx + H_n\int_\Omega C_e^{2^n}(x,t)dx \\
&+ \bar{g}_1(M_0+1)\varepsilon_0\int_\Omega C_0^{2^n}(x,t)dx,\ \text{for } t\ge t_0,
\end{aligned}\tag{3.10}$$

where $H_n = \bar{g}_1(M_0+1)G_{\varepsilon_0}$.

Multiplying both sides of the second equation of (2.2) by $C_0^{2^n-1}(x,t)$ and integrate it over Ω, we have

$$\begin{aligned}
&\frac{1}{2^n}\frac{\partial}{\partial t}\int_\Omega C_0^{2^n}(x,t)dx\\
\le& \int_\Omega k(x)C_0^{2^n-1}(x,t)C_e(x,t)dx + \int_\Omega f(x)C_0^{2^n}(x,t)P(x,t)dx\\
&- \int_\Omega (g(x)+m(x))C_0^{2^n}(x,t)dx\\
\le&\bar{k}\int_\Omega C_0^{2^n-1}(x,t)C_e(x,t)dx - [\tilde{g}+\tilde{m}-(M_0+1)\bar{f}]\int_\Omega C_0^{2^n}(x,t)dx.
\end{aligned}\tag{3.11}$$

Applying Young's inequality again, by setting $p = 2^n$ and $q = 2^n/(2^n-1)$ as follows

$$\begin{aligned}
&\int_\Omega C_0^{2^n-1}(x,t)C_e(x,t)dx\\
\le&\varepsilon_2\int_\Omega C_e^{2^n}(x,t)dx + G_{\varepsilon_2}\int_\Omega C_0^{2^n}(x,t)dx,\ \text{for } t\ge t_0,\ G_{\varepsilon_2} = \varepsilon_2^{-\frac{1}{2^n-1}}.
\end{aligned}$$

Then (3.11) becomes

$$\frac{1}{2^n}\frac{\partial}{\partial t}\int_\Omega C_0^{2^n}(x,t)dx \le -H_n'\int_\Omega C_0^{2^n}(x,t)dx + \bar{k}\varepsilon_2\int_\Omega C_e^{2^n}(x,t)dx,\tag{3.12}$$

where $H_n' = [\tilde{g} + \tilde{m} - (M_0+1)\bar{f}] - \bar{k}G_{\varepsilon_2}$.

Therefore, from (3.10) and (3.12), we obtain that

$$
\begin{aligned}
&\frac{1}{2^n}\frac{\partial}{\partial t}\int_\Omega (C_0^{2^n}(x,t) + C_e^{2^n}(x,t))dx \\
=& - D_n \int_\Omega |\nabla C_e^{2^{n-1}}(x,t)|^2\, dx + X\int_\Omega C_e^{2^n}(x,t)dx \\
&- Y\int_\Omega C_0^{2^n}(x,t)dx, \text{ for } t \geq t_0,
\end{aligned}
\tag{3.13}
$$

where $X = H_n + \bar{k}\varepsilon_2$ and $Y = H_n' - \bar{g}_1(M_0+1)\varepsilon_0$.

Applying interpolation inequality: for $\forall \varepsilon > 0$, there exists $G_\varepsilon > 0$ such that

$$\|\zeta\|_2^2 \leq \varepsilon \|\nabla\zeta\|_2^2 + G_\varepsilon \|\zeta\|_1^2, \text{ where } \zeta \in W^{1,2}(\Omega).$$

Let $\varepsilon_3 = D_n/(2X)$, $\xi = C_e^{2^{n-1}}(x,t)$, then

$$-D_n\int_\Omega |\nabla C_e^{2^{n-1}}(x,t)|^2\, dx \leq -2X\int_\Omega C_e^{2^n}(x,t)dx + 2XG_{\varepsilon_3}\left(\int_\Omega C_e^{2^{n-1}}(x,t)dx\right)^2.$$

Therefore, (3.13) becomes

$$
\begin{aligned}
&\frac{1}{2^n}\frac{\partial}{\partial t}\int_\Omega (C_0^{2^n}(x,t) + C_e^{2^n}(x,t))dx \\
\leq& - X\int_\Omega C_e^{2^n}(x,t) + 2XG_{\varepsilon_3}\left(\int_\Omega C_e^{2^{n-1}}(x,t)dx\right)^2 - Y\int_\Omega C_0^{2^n}(x,t)dx \\
\leq& - Z\int_\Omega (C_0^{2^n}(x,t) + C_e^{2^n}(x,t))dx + 2XG_{\varepsilon_3}\left(\int_\Omega C_e^{2^{n-1}}(x,t)dx\right)^2,
\end{aligned}
$$

where $Z = \min\{X, Y\}$.

Then follows from (3.7) that $\limsup_{t\to\infty}\int_\Omega C_e^{2^{n-1}}(x,t)dx \leq M_{2^{n-1}}^{2^{n-1}}$, which implies that

$$\limsup_{t\to\infty}\|(C_0(\cdot,t) + C_e(\cdot,t))\|_{2^n} \leq M_{2^n},$$

with $M_{2^n} = \sqrt[2^n]{\frac{2XG_{\varepsilon_3}M_{2^{n-1}}^{2^n}}{Z}}$.

According to continuous embedding $L^q(\Omega) \subset L^P(\Omega), q \geq p \geq 1$, we can conclude that for $\forall p > 1$, there exists a positive constant M_p, such that

$$\limsup_{t\to\infty}\|(C_0(\cdot,t) + C_e(\cdot,t))\|_p \leq M_p.$$

Applying the general results in [32, Lemma 2.4], it is clear that there exists a positive constant M_∞ such that $\limsup_{t\to\infty}\|C_0(\cdot,t)\| \leq M_\infty$ and $\limsup_{t\to\infty}\|C_e(\cdot,t)\| \leq M_\infty$. It implies that $C_0(x,t)$ and $C_e(x,t)$ is ultimately bounded. Therefore, the solution exists globally for $t \in [0,\infty)$ and $\Phi(t) : \mathbb{X}^+ \to \mathbb{X}^+$ is point dissipative. $\square$

Since there is no diffusion term in the second equation of system (2.2), the asymptotic smoothness of the solution semiflow $\Phi(t)$ is considered. We introduce the Kuratowski measure of noncompactness and define $\kappa(\cdot)$ by

$$\kappa(E) := \inf\{r : E \text{ has a finite cover of diameter} < r\},$$

for any bounded set E. Thus E is precompact if and only if $\kappa(E) = 0$. And $\Phi(t)$ is a κ-contraction that there exists a continuous function $s(t) : \mathbb{R}^+ \to \mathbb{R}^+$, $0 \leq s(t) \leq 1$, such that for $\forall t > 0$ and bounded set E, $\{\Phi(z)E, 0 \leq z \leq t\}$ is bounded and $\kappa(\Phi(t)E) \leq s(t)\kappa(E)$.

Lemma 3.4. *$\Phi(t)$ is κ-contracting that*

$$\lim_{t\to\infty} \kappa(\Phi(t)E) = 0, \textit{for any bounded set } E \subset \mathbb{X}^+,$$

where $\Phi(t)$ is defined in Lemma 3.3.

Proof. The right hand of the second equation of system (2.2) can be denoted as

$$\sigma(C_0, C_e) = k(x)C_e(x,t) + f(x)C_0(x,t)P(x,t) - (g(x) + m(x))C_0(x,t).$$

Then there exists a $\delta > 0$, such that

$$\frac{\partial\sigma(C_0, C_e)}{\partial C_0} = -(g(x) + m(x) - f(x)P(x,t)) \leq -\delta, (C_0, C_e) \in \mathbb{X}^+, \tag{3.14}$$

where $\delta = \tilde{g} + \tilde{m} - \bar{f}M_0$.

In fact, $\Phi(t)$ can be decomposed as $\Phi(t) = \Phi_1(t) + \Phi_2(t), t \geq 0$, where

$$\begin{aligned}&\Phi_1(t)(\phi)\\ =&(P(\cdot,t;\phi), \int_0^t e^{-(g(x)+m(x)-f(x)P(x,t))(t-s)}k(x)C_e(\cdot,t;\phi)ds,\ C_e(\cdot,t;\phi)),\ t \geq 0,\end{aligned}$$

and

$$\Phi_2(t)(\phi) = (0, e^{-(g(x)+m(x)-f(x)P(x,t))t}\phi_2, 0),\ t \geq 0.$$

Similar to [32, Lemma 2.5], for $\forall t > 0$, $\Phi_1(t)E$ is precompact. Therefore, $\kappa(\Phi_1(t)E) = 0$. Furthermore, the operator norm of $\Phi_2(t)$ can be estimated as

$$\|\Phi_2(t)\| = \sup_{\phi\in\mathbb{X}} \frac{\|\Phi_2(t)\phi\|_{\mathbb{X}}}{\|\phi\|_{\mathbb{X}}} \leq e^{-\delta t} \sup_{\phi\in\mathbb{X}} \frac{\|\phi\|_{\mathbb{X}}}{\|\phi\|_{\mathbb{X}}} = e^{-\delta t},$$

which implies that for $t > 0$,

$$\kappa(\Phi(t)E) \leq \kappa(\Phi_1(t)E) + \kappa(\Phi_2(t)E) \leq 0 + \|\Phi_2(t)\|\,\kappa(E) \leq e^{-\delta t}\kappa(E).$$

Thus, $\Phi(t)$ is a $\kappa-$contraction on $\mathbb{X}^+$ with the contraction function $s(t) = e^{-\delta t}$. This completes the proof. □

Theorem 3.1. *$\Phi(t)$ admits a connected global attractor on $\mathbb{X}^+$.*

Proof. From Lemma 3.3, we know that the solution of (2.2) is globally existing, unique, and ultimately bounded. From Lemma 3.4, we know from the κ-contraction condition that $\Phi(t)$ is asymptotically smooth. Therefore, as direct consequence of [2, Theorem 2.4.6], system (2.2) has a connected global attractor. This completes the proof. □

3.2. Basic reproduction number

Apply similar results as in [29, Section 3], we define the basic reproduction number R_0 for system (2.2), which is closely related to the stability of E_0, where $E_0 = (P^*(x), 0, 0)$ is the unique toxicant-free equilibrium of system (2.2). Linearizing (2.2) with (2.3) at E_0, we get the following subsystem for $C_0(x,t)$ and $C_e(x,t)$ component

$$\begin{cases} \dfrac{\partial C_0(x,t)}{\partial t} =& k(x)C_e(x,t) + f(x)C_0(x,t)P^*(x) - (g(x)+m(x))C_0(x,t), \\ & x \in \Omega, t > 0, \\ \dfrac{\partial C_e(x,t)}{\partial t} =& d_2\Delta C_e(x,t) - k_1 C_e(x,t)P^*(x,t) - h(x)C_e(x,t) \\ & + g_1(x)C_0(x,t)P^*(x,t), x \in \Omega, t > 0. \end{cases} \tag{3.15}$$

Define $T(t)$ as the solution semigroup of (3.15) with generator

$$B = \begin{bmatrix} f(x)P^*(x) - (g(x)+m(x)) & k(x) \\ g_1(x)P^*(x) & d_2\Delta - k_1(x)P^*(x) - h(x) \end{bmatrix} := B_1 + B_2,$$

where

$$B_1 = \begin{bmatrix} f(x)P^*(x) - (g(x)+m(x)) & 0 \\ g_1(x)P^*(x) & d_2\Delta - k_1(x)P^*(x) - h(x) \end{bmatrix},$$

and

$$B_2 = \begin{bmatrix} 0 & k(x) \\ 0 & 0 \end{bmatrix}.$$

Define $T_1(t)$ as the C_0-semigroup generated by operator B_1. we know that B_1 is cooperative for any $x \in \Omega$, this indicates that $T_1(t)$ is a positive semigroup. According to [27, Theorem 3.12], we know that both B and B_1 are resolvent-positive operators. Then we define the operator $\mathcal{L} := -B_2B_1^{-1}$ which has the following form

$$\begin{aligned} \mathcal{L}(\phi)(x) &= \int_0^\infty B_2(x)T_1(t)\phi(x)dt \\ &= B_2(x)\int_0^\infty T_1(t)\phi(x)dt, \phi \in C(\overline{\Omega}, \mathbb{R}^2),\ x \in \overline{\Omega}, \end{aligned} \tag{3.16}$$

$\mathcal{L}$ is well-defined, continuous, and positive operator on $C(\overline{\Omega}, \mathbb{R}^2)$, which maps the initial toxicant distribution ϕ to the distribution of the total new toxicant produced. We then follow the procedure in [29] to define the spectral radius of $\mathcal{L}$ as the basic reproduction number

$$R_0 = r(\mathcal{L}) = r(-B_2B_1^{-1}) = \sup\{|\ \lambda\ |, \lambda \in \sigma(\mathcal{L})\},$$

where $\sigma(\mathcal{L})$ is the spectrum of $\mathcal{L}$. By [27, Theorem 3.5] and [29, Lemma 2.2], we can obtain the following results.

Lemma 3.5. *Let $s(B) = \sup\{Re\lambda, \lambda \in \sigma(B)\}$ be the spectral bound of B. Then $R_0 - 1$ has the same sign as $s(B)$.*

Lemma 3.6. *Let λ_0 be the principal eigenvalue of the following eigenvalue problem*

$$\begin{cases} d_2\Delta\Psi - (h(x) + k_1(x)P^*(x))\Psi + \lambda\mathbb{H}(x)\Psi = 0, \ x \in \Omega, \\ \dfrac{\partial\Psi}{\partial v} = 0, \ x \in \partial\Omega, \end{cases} \tag{3.17}$$

where

$$\mathbb{H}(x) = \frac{g_1(x)k(x)P^*(x)}{g(x) + m(x) - f(x)P^*(x)}, \tag{3.18}$$

then $R_0 = 1/\lambda_0$.

Proof. Similar arguments in [29, Theorem 3.3], we define

$$\hat{B}_1 = \begin{bmatrix} d_2\Delta - V_{11} & -V_{12} \\ 0 & -V_{22} \end{bmatrix} \text{ and } \hat{B}_2 = \begin{bmatrix} 0 & 0 \\ F_{21} & 0 \end{bmatrix},$$

where

$$\begin{aligned} &F_{11} := 0, \ F_{12} := 0, \ F_{21} := k(x), \ F_{22} := 0, \\ &V_{11} := k_1(x)P^*(x) + h(x), \ V_{12} := -g_1(x)P^*(x), \\ &V_{21} := 0, \ V_{22} := g(x) + m(x) - f(x)P^*(x). \end{aligned}$$

From $F_{12} = 0$ and $F_{22} = 0$, it then follows that $R_0 = r(-B_1^{-1}B_2) = r(\mathcal{B}_1^{-1}\mathcal{F}_1)$, where $\mathcal{B}_1 = d_2\Delta - (V_{11} - V_{12}V_{22}^{-1}V_{21}) = d_2\Delta - (k_1(x)P^*(x) + h(x))$, and

$$\begin{aligned} \mathcal{F}_1 :&= F_{11} - V_{12}V_{22}^{-1}F_{21} \\ &= \frac{g_1(x)k(x)P^*(x)}{g(x) + m(x) - f(x)P^*(x)}. \end{aligned}$$

Therefore, for $\Psi \in C(\overline{\Omega}, R^2)$,

$$-\mathcal{B}_1^{-1}\mathcal{F}_1\Psi = -[d_2\Delta - (k_1(x)P^*(x) + h(x))]^{-1}\mathbb{H}(x)\Psi,$$

where $\mathbb{H}(x)$ is defined as in (3.18). Then, R_0 satisfies

$$[-[d_2\Delta - (k_1(x)P^*(x) + h(x))]^{-1}\mathbb{H}(x)]\Psi = R_0\Psi, \Psi \in C(\overline{\Omega}, \mathbb{R}^2),$$

that is

$$d_2\Delta\Psi - (k_1(x)P^*(x) + h(x))\Psi + \mathbb{H}(x)\frac{1}{R_0}\Psi = 0, \Psi \in C(\overline{\Omega}, \mathbb{R}^2).$$

From elliptic problem (3.17), and apply similar results as in [28], we obtain that

$$R_0 = \frac{1}{\lambda} = \sup_{\Psi \in H^1(\Omega), \Psi \neq 0} \frac{\int_\Omega \mathbb{H}(x)\Psi^2 dx}{\int_\Omega [d_2 \mid \nabla\Psi \mid^2 + (k_1(x)P^*(x) + h(x))\Psi^2]dx}, \tag{3.19}$$

which completes the proof. $\square$

Remark 3.1. When all parameters in (2.2) are constants, we have $P^*(x) = \frac{\Lambda}{r}$, thus R_0 can be reduced to

$$R_0^{\text{const}} = \frac{1}{\tilde{\lambda}} = (\frac{g_1 k\Lambda}{r(g+m) - f\Lambda})/(h + \frac{k_1\Lambda}{r}). \tag{3.20}$$

From Remark 3.1, we can see that how R_0 depends on the model parameters. We immediately obtain the following statements.

(i) R_0 is a monotone decreasing and positive function of $d_2 > 0$.

(ii) $R_0 \to \max\{\frac{\mathbb{H}(x)}{h(x)+k_1(x)P^*(x)} : x \in \Omega\}$ as $d_2 \to 0$.

(iii) $R_0 \to \frac{\int_\Omega \mathbb{H}(x)dx}{\int_\Omega (h(x)+k_1(x)P^*(x))dx}$ as $d_2 \to \infty$.

(iv) If $\int_\Omega \mathbb{H}(x)dx < \int_\Omega (h(x) + k_1(x)P^*(x))dx$, then there exists $\tilde{d}_2 \in (0,\infty)$ such that $R_0 > 1$ for $d_2 < \tilde{d}_2$, and $R_0 < 1$ for $d_2 > \tilde{d}_2$.

(v) If $\int_\Omega \mathbb{H}(x)dx > \int_\Omega (h(x) + k_1(x)P^*(x))dx$, for all $d_2 > 0$, we have $R_0 > 1$.

3.3. Toxicant extinction

The following results show that R_0 is a threshold parameter of the model (2.2) for the toxicant extinction.

Theorem 3.2. *The toxicant-free steady state E_0 is globally attractive if $R_0 < 1$, then the following equation hold*

$$\lim_{t\to\infty} \|u(x,t;\phi) - E_0\|_{\mathbb{X}} = 0, \text{ uniformly for } \forall x \in \Omega.$$

Proof. Fix $\varepsilon > 0$, then follows from (3.3), we know that there exists $t_0 > 0$ such that

$$P^*(x) - \varepsilon \le P(x,t) \le P^*(x) + \varepsilon, x \in \overline{\Omega}, \forall t > t_0.$$

It follows from the comparison principal for cooperative systems (see, [19]), we have

$$(C_0(x,t), C_e(x,t)) \le (\hat{C}_0(x,t),\ \hat{C}_e(x,t)),\ x \in \overline{\Omega},\ t > t_0,$$

where $(\hat{C}_0(x,t), \hat{C}_e(x,t))$ satisfies

$$\begin{cases} \dfrac{\partial \hat{C}_0(x,t)}{\partial t} = & k(x)\hat{C}_e(x,t) + f(x)\hat{C}_0(x,t)(P^*(x)+\varepsilon) - (g(x)+m(x))\hat{C}_0(x,t), \\ & x \in \Omega,\ t > t_0, \\ \dfrac{\partial \hat{C}_e(x,t)}{\partial t} = & d_2\Delta\hat{C}_e(x,t) - k_1(x)\hat{C}_e(x,t)(P^*(x)-\varepsilon) - h(x)\hat{C}_e(x,t) \\ & + g_1(x)C_0(x,t)(P^*(x)+\varepsilon),\ x \in \Omega,\ t > t_0, \\ \dfrac{\partial C_0(x,t)}{\partial v} = & \dfrac{\partial C_e(x,t)}{\partial v} = 0,\ x \in \partial\Omega,\ t > t_0. \end{cases} \tag{3.21}$$

Since $s(B) < 0$, there is a $\varepsilon > 0$ such that $s(B_\varepsilon) < 0$ and it corresponded to an associated eigenvector $(\psi_2^\varepsilon, \psi_3^\varepsilon) \gg 0$. Suppose that for any initial date $\phi \in \mathbb{X}^+$, we can find some $M > 0$ such that $(C_0(x,t_0;\phi), C_e(x,t_0;\phi)) \le M(\psi_2^\varepsilon, \psi_3^\varepsilon), x \in \overline{\Omega}$.

Note that the linear system (3.21) provide a solution $Me^{s(B_\varepsilon)(t-t_0)}\psi^\varepsilon$, where $\psi^\varepsilon = (\psi_2^\varepsilon, \psi_3^\varepsilon)$. According to the comparison (see, [35]), we obtain the following

$$(C_0(x,t_0;\phi), C_e(x,t_0;\phi)) \le (\hat{C}_0(x,t;\phi), \hat{C}_e(x,t;\phi)) = Me^{s(B_\varepsilon)(t-t_0)}(\psi_2^\varepsilon, \psi_3^\varepsilon).$$

Therefore, we have $(\hat{C}_0(x,t), \hat{C}_e(x,t)) \to (0,0)$ as $t \to \infty$ uniformly for $x \in \overline{\Omega}$. Then, $(C_0(x,t), C_e(x,t)) \to (0,0)$ as $t \to \infty$ uniformly for $x \in \overline{\Omega}$. Furthermore, from the first equation (2.2) and Lemma 3.2, we obtain that $\lim_{t\to\infty} P(x,t) = P^*(x)$ uniformly for $x \in \overline{\Omega}$. This completes the proof. $\square$

3.4. Toxicant persistence

This subsection indicates that R_0 is a threshold index for toxicant persistence.

Theorem 3.3. *System* (2.2) *is uniformly persistent if* $R_0 > 1$, *that is, there exists* $\sigma > 0$ *such that for any* $\phi \in \mathbb{X}^+$ *with* $\phi_2(\cdot) \not\equiv 0$ *or* $\phi_3(\cdot) \not\equiv 0$,

$$\liminf_{t\to\infty} z(x,t;\phi) \ge \sigma, \ \textit{uniform for } \forall x \in \overline{\Omega}, \ z = P, C_0, C_e. \tag{3.22}$$

In addition, system (2.2) *with* (2.3) *exists at least one positive steady state* E_*.

Proof. To proceed further, we define the sets

$$\mathbb{W}_0 = \{\phi(\cdot) \in \mathbb{X}^+ : \phi_3(\cdot) \not\equiv 0\},$$

and

$$\partial\mathbb{W}_0 = \mathbb{X}^+ \setminus \mathbb{W}_0 = \{\phi(\cdot) \in \mathbb{X}^+ : \phi_3(\cdot) \equiv 0\}.$$

Then $\mathbb{X}^+ = \mathbb{W}_0 \cup \partial\mathbb{W}_0$, $\mathbb{W}_0$ being relatively open in $\mathbb{X}^+$.

Let $\omega(\phi)$ be the omega limit set of the orbit $\gamma^+ := \{\Phi(t)\phi : t \ge 0\}$. Set

$$M_\partial := \{\phi \in \partial\mathbb{W}_0 : \Phi(t)\phi \in \partial\mathbb{W}_0, \ \forall t \ge 0\}. \tag{3.23}$$

Note that for any $\phi \in \mathbb{W}_0$, as the same arguments in [28, Lemma 3.5], we can get that $C_e(x,t;\phi) > 0, \forall x \in \Omega, t > 0$, and $\mathbb{W}_0$ was positively invariant in terms of the solution semiflow $\Phi(t)$, that is, $\Phi(t)\mathbb{W}_0 \subseteq \mathbb{W}_0$.

Next, we prove the following claims.

Claim 1. $\omega(\phi) = \{E_0\}, \forall \phi \in M_\partial$.

For $\phi \in M_\partial$, we have $\Phi(t)\phi \in M_\partial$, $\forall t \ge 0$. It follows that $C_e(x,t;\phi) \equiv 0$, $\forall t \ge 0$, this together with the last two equations of system(2.2), we can get that $C_0(\cdot,t;\phi) \equiv 0$, $\forall t \ge 0$. Then it follows that the first equation of system (2.2) is asymptotic to system (3.1), we obtain that

$$\frac{\partial P(x,t)}{\partial t} = d_1\Delta P(x,t) + \Lambda(x) - r(x)P(x,t), \ x \in \Omega, \ t > 0. \tag{3.24}$$

From Lemma 3.2, we know that system (3.24) exists a positive steady state P^*, which was globally asymptotically stable in $C(\Omega, \mathbb{R}_+)$. Thus $\omega(\phi) = \{E_0\}$, for $\forall \phi \in M_\partial$. This proves Claim 1.

Claim 2. $\limsup_{t\to\infty} \|\Phi(t)\phi - E_0\| \ge \sigma, \ \forall \phi \in \mathbb{W}_0$.

Assume for the contrary that there exists $\phi_0 \in \mathbb{W}_0$ such that

$$\limsup_{t\to\infty} \|\Phi(t)\phi_0 - E_0\| < \sigma,$$

then there exists a $t_1 > 0$ such that

$$\begin{aligned}&P^*(x) - \sigma < P(x,t;\phi_0) < P^*(x) + \sigma,\ C_0(x,t;\phi_0) < \sigma,\\ &C_e(x,t;\phi_0) < \sigma,\ \forall t > t_1,\ x \in \overline{\Omega}.\end{aligned}$$

Thus $(C_0(x,t;\phi), C_e(x,t;\phi))$ satisfies

$$\begin{cases}\dfrac{\partial C_0(x,t)}{\partial t} \geq & k(x)C_e(x,t) + f(x)C_0(x,t)(P^*(x)-\sigma) - (g(x)+m(x))C_0(x,t),\\ & x \in \Omega,\ t > t_1,\\ \dfrac{\partial C_e(x,t)}{\partial t} \geq & d_2\Delta C_e(x,t) - k_1(x)C_e(x,t)(P^*(x)+\sigma) - h(x)C_e(x,t)\\ & + g_1(x)C_0(x,t)(P^*(x)-\sigma),\ x \in \Omega,\ t > t_1,\\ \dfrac{\partial C_0(x,t)}{\partial v} = & \dfrac{\partial C_e(x,t)}{\partial v} = 0,\ x \in \partial\Omega,\ t > t_1.\end{cases}$$

It follows that $C_0(x,t) > 0, C_e(x,t) > 0, \forall x \in \overline{\Omega}, t > 0$, recall that the linear system

$$\begin{cases}\dfrac{\partial \check{C}_0(x,t)}{\partial t} = & k(x)\check{C}_e(x,t) + f(x)(P^*(x)-\sigma)\check{C}_0(x,t) - (g(x)+m(x))\check{C}_0(x,t),\\ & x \in \overline{\Omega},\ t > t_1,\\ \dfrac{\partial \check{C}_e(x,t)}{\partial t} = & d_2\Delta \check{C}_e(x,t) - k_1(x)\check{C}_e(x,t)(P^*(x)+\sigma) - h(x)\check{C}_e(x,t)\\ & + g_1(x)\check{C}_0(x,t)(P^*(x)-\sigma),\ x \in \overline{\Omega},\ t > t_1,\\ \dfrac{\partial \check{C}_0(x,t)}{\partial v} = & \dfrac{\partial \check{C}_e(x,t)}{\partial v} = 0,\ x \in \partial\Omega,\ t > t_1,\end{cases}$$

admits a solution $\varepsilon_0 e^{s(B_\sigma)(t-t_1)}\psi^\sigma$ for some positive constant ε_0, where $\psi^\sigma = (\psi_2^\sigma, \psi_3^\sigma)$. According to the comparison principal, we have

$$(C_0(\cdot,t;\phi_0), C_e(\cdot,t;\phi_0)) \geq \varepsilon_0 e^{s(\beta_\sigma)(t-t_1)}(\psi_2^\sigma, \psi_3^\sigma), t \geq t_1, x \in \overline{\Omega}.$$

Then $C_0(\cdot,t;\phi_0)$ and $C_e(\cdot,t;\phi_0)$ are unbounded as $s(B_\sigma) > 0$, this leads a contradiction. This completes the proof of Claim 2.

According to the standard procedures in [26], define a continuous function $p(\cdot) : \mathbb{X}^+ \to [0,\infty)$

$$p(\phi) := \min\{\phi_3(\cdot)\}, \forall \phi \in \mathbb{X}^+,$$

obviously, $p^{-1}(0,\infty) \subseteq \mathbb{W}_0$, and if $p(\phi) = 0$ with $\phi \in \mathbb{W}_0$ or $p(\phi) > 0$, then $p(\Phi(t)\phi) > 0, \forall\ t > 0$. Then p is a generalized distance function for the semiflow $\Phi(t) : \mathbb{X}^+ \to \mathbb{X}^+$. This discussion demonstrated that $\gamma^+(\phi)$ in M_∂ converged to E_0 and was isolated in $\mathbb{W}_0$, in addition, for the stable subset of E_0, $W^s(E_0) \bigcap \mathbb{W}_0 = \varnothing$. There is no cycle from E_0 to E_0 in M_∂. Therefore, from [26, Theorem 3] and similar arguments in [8, Theorem 3.4], we find that there exists a $\sigma > 0$ such that

$$\liminf_{t\to\infty} p(\Phi(t)\psi) > \sigma, \forall \psi \in \mathbb{W}_0,$$

which implies that

$$\liminf_{t\to\infty} C_e(x,t;\phi) \geq \sigma, \forall \phi \in \mathbb{W}_0.$$

In summary, $\Phi(t)$ is uniformly persistent with respect to $(\mathbb{W}_0, \partial\mathbb{W}_0)$. By [18], it follows that $\Phi(t) : \mathbb{W}_0 \to \mathbb{W}_0$ has a global attractor E_0, and system (2.2) admits at least one steady state $\hat{u}(\cdot)$ in $\mathbb{W}_0$, (see, [32]), which is a positive steady state of (2.2). This completes the proof. $\square$

4. Optimal control strategy

4.1. Problem statement

In this section, we formulate an optimal problem on the basis of system (2.2). We assume that the natural growth of toxicant will be affected by decision makers, that is, decision makers can take measures to control the toxicant. Specifically, decision makers can control the toxicant by increasing the loss rate of toxicant in the environment, h is the total loss rate in the natural environment, and we suppose $h+u$ is the total loss rate of the environment after the decision makers implement governance, where u is the governance intensity, and $0 \leq u(x,t) \leq 1$.

Taking into account the above assumption, the control problem of the model is given by

$$\begin{cases} \dfrac{\partial P(x,t)}{\partial t} =& d_1 \Delta P(x,t) + \Lambda(x) - r(x)P(x,t) - \alpha(x)C_0(x,t)P(x,t), \\ & x \in \Omega,\ t>0, \\ \dfrac{\partial C_0(x,t)}{\partial t} =& k(x)C_e(x,t) + f(x)C_0(x,t)P(x,t) - (g(x)+m(x))C_0(x,t), \\ & x \in \Omega,\ t>0, \\ \dfrac{\partial C_e(x,t)}{\partial t} =& d_2 \Delta C_e(x,t) - k_1(x,t)C_e(x,t)P(x,t) - h(x)C_e(x,t) \\ & + g_1(x)C_0(x,t)P(x,t) - u(x,t)C_e(x,t),\ x \in \Omega,\ t>0. \end{cases} \tag{4.1}$$

Here the control $u(x,t)$ is in

$$\mathcal{V}[\Omega \times I] = \{u(x,t) : \Omega \times I \to U \mid u(x,t) \text{ is measureable}\}.$$

The control system can be written as follows

$$\begin{cases} \dfrac{\partial y(x,t)}{\partial t} = b(x,t,y(x,t),u(x,t)),\ x \in \Omega,\ t \in [s,T], \\ y(0,s) = y_0, \end{cases} \tag{4.2}$$

where $y(x,t) = (P(x,t), C_0(x,t), C_e(x,t))^\top \in \mathbb{X}^+$, and y_0 is an initial value at time s, $(s, y_0) \in [0,T) \times \mathbb{X}^+$. For convenience, we denote $I = [s,T]$.

The purpose of the optimal control problem is to implement the control strategy to reduce the concentration of toxicant and minimize the cost. Therefore, the objective function we constructed is as follows

$$\begin{aligned} &J(s, y_0; u(x,t)) \\ =& \int_s^T \int_\Omega [\tau_1 u^2(x,t) + \tau_2 C_0(x,t) + \tau_3 C_e(x,t)] dxdt + \int_\Omega h(y(x,T))dx, \end{aligned} \tag{4.3}$$

where $\tau_1, \tau_2, \tau_3 \in R_+$ and $\int_\Omega h(y(x,T))dx$ is the penalty function corresponding to the terminal state. The meaning of the objective functional $J(s, y_0; u(x,t))$ is described as follows

(i) The term $\int_s^T \int_\Omega [\tau_2 C_0(x,t) + \tau_3 C_e(x,t)] dxdt$ represents the total number of toxicant concentration in the organism and environment over the time period T.

(ii) The term $\int_s^T \int_\Omega [\tau_1 u^2(x,t)]dxdt$ gives the total cost of applying the control strategy.

Then we denote

$$L(x,t,y(x,t),u(x,t)) := \int_\Omega [\tau_1 u^2(x,t) + \tau_2 C_0(x,t) + \tau_3 C_e(x,t)]dx. \tag{4.4}$$

Our object is to design the optimal controller $u^*(x,t), t \in I$, which minimizes or nearly minimize the cost functional $J(s, y_0; u(x,t))$. The value function is as follows

$$\begin{cases} V(s,y_0) = \inf\limits_{u(x,t)\in\mathcal{V}[\Omega\times I]} J(s,y_0;u(x,t)),\ \forall (s,y_0) \in [0,T)\times\mathbb{X}^+, \\ V(T,y_0) = h(y_0),\ \forall y_0 \in \mathbb{X}^+. \end{cases} \tag{4.5}$$

Before further study, we first make the following assumptions:

Assumption 4.1. *(U, d) is a separable metric space and $T > 0$.*

Assumption 4.2. *The control set U is convex.*

Assumption 4.3. *Different controls correspond to the same terminal state.*

Then, by [33, Chapter 4], we propose the following result called Bellman's principle of optimality.

Theorem 4.1. *Let Assumption 4.1, Assumption 4.2 hold, then for any $(s, y_0) \in [0,T) \times \mathbb{X}^+$,*

$$\begin{aligned} V(s,y_0) = \inf_{u(x,t)\in\mathcal{V}[\Omega\times I]} \Big\{ &\int_s^{\hat{s}} \int_\Omega [\tau_1 u^2(x,t) + \tau_2 C_0(x,t) + \tau_3 C_e(x,t)]dxdt \\ &+ V(\hat{s}, y(\hat{x},\hat{s}; s, y_0, u(x,t)))\Big\},\ \forall 0 \le s \le \hat{s} \le T. \end{aligned} \tag{4.6}$$

Proof. We denote the right-hand side of (4.6) by $\overline{V}(s,y_0)$. According to (4.5), we have

$$\begin{aligned} V(s,y_0) \le& J(s,y_0;u(x,t)) \\ =& \int_s^{\hat{s}} \int_\Omega [\tau_1 u^2(x,t) + \tau_2 C_0(x,t) + \tau_3 C_e(x,t)]dxdt \\ &+ J(\hat{s}, y(\hat{x},\hat{s}); u(x,t)),\ \forall u(x,t) \in \mathcal{V}[\Omega\times I]. \end{aligned}$$

Therefore, taking the infimum over $u(x,t) \in V[\Omega \times I]$, we obtain

$$V(s,y_0) \le \overline{V}(s,y_0). \tag{4.7}$$

Conversely, for $\forall \varepsilon > 0$, there exists a $u_\varepsilon(x,t) \in \mathcal{V}[\Omega \times I]$, such that

$$\begin{aligned} V(s,y_0) + \varepsilon &\ge J(s,y_0;u_\varepsilon(x,t)) \\ &\ge \int_s^{\hat{s}} \int_\Omega [\tau_1 u_\varepsilon^2(x,t) + \tau_2 C_0(x,t) + \tau_3 C_e(x,t)]dxdt + V(\hat{s}, y_\varepsilon(\hat{x},\hat{s})) \\ &\ge \overline{V}(s,y_0), \end{aligned} \tag{4.8}$$

where $y_\varepsilon(x,t) = y(x,t;s,y_0,u_\varepsilon(x,t))$. Combining (4.7) and (4.8), we obtain (4.6). This completes the proof. □

Theorem 4.2. *Suppose Assumption 4.1, Assumption 4.2 hold, then $V(s,y_0)$ is a solution to the following terminal value problem of a first-order partial differential equation:*

$$\begin{cases} 0 = -V_t + \sup_{u\in U}\{\int_\Omega \{-V_P(t,y)[d_1\Delta P(x,t) + \Lambda(x) - r(x)P(x,t) \\ \quad -\alpha(x)C_0(x,t)P(x,t)] - V_{C_0}(t,y)[k(x)C_e(x,t) + f(x)C_0(x,t)P(x,t) \\ \quad -(g(x)+m(x))C_0(x,t)] - V_{C_e}(t,y)[d_2\Delta C_e(x,t) - k_1(x)C_e(x,t)P(x,t) \\ \quad -h(x)C_e(x,t) + g_1C_0(x,t)P(x,t) - u(x,t)C_e(x,t)]\}dx \\ \quad -\int_\Omega [\tau_1 u^2(x,t) + \tau_2 C_0(x,t) + \tau_3 C_e(x,t)]dx\}, (x,t,y)\in\Omega\times[0,T)\times\mathbb{X}^+, \\ V|_{t=T} = \int_\Omega h(y(x,T))dx, y\in\mathbb{X}^+, \end{cases} \tag{4.9}$$

we call (4.9) *the Hamilton-Jacobi-Bellman(HJB) equation associated with the value function* (4.5).

Proof. Fix a $u \in U$. Let $y(x,t)$ be the state trajectory corresponding to the control $u(x,t) \equiv u$. By (4.6) with $\hat{s} \downarrow s$, we obtain

$$\begin{aligned} 0 \geq & -\frac{V(\hat{s},y(\cdot,\hat{s})) - V(s,y_0)}{\hat{s}-s} - \frac{1}{\hat{s}-s}\int_s^{\hat{s}}\int_\Omega [\tau_1 u^2 + \tau_2 C_0(x,t) + \tau_3 C_e(x,t)]dxdt \\ \to & -V_t(s,y_0) - \{\int_\Omega \{V_P(t,y)[d_1\Delta P(x,t) + \Lambda(x) - r(x)P(x,t) \\ & -\alpha(x)C_0(x,t)P(x,t)] + V_{C_0}[k(x)C_e(x,t) + f(x)C_0(x,t)P(x,t) \\ & -(g(x)+m(x))C_0(x,t)] + V_{C_e}[d_2\Delta C_e(x,t) - k_1(x)C_e(x,t)P(x,t) \\ & -h(x)C_e(x,t) + g_1(x)C_0(x,t)P(x,t) - u(x,t)C_e(x,t)]\}dx\} \\ & -\int_\Omega [\tau_1 u^2 + \tau_2 C_0(x,t) + \tau_3 C_e(x,t)]dx, \end{aligned}$$

which results in

$$\begin{aligned} 0 \geq & -V_t(s,y_0) + \sup_{u\in U}\{\int_\Omega \{-V_P(t,y)[d_1\Delta P(x,t) + \Lambda(x) - r(x)P(x,t) \\ & -\alpha(x)C_0(x,t)P(x,t)] - V_{C_0}(t,y)[k(x)C_e(x,t) + f(x)C_0(x,t)P(x,t) \\ & -(g(x)+m(x))C_0(x,t)] - V_{C_e}(t,y)[d_2\Delta C_e(x,t) - k_1(x)C_e(x,t)P(x,t) \\ & -h(x)C_e(x,t) + g_1(x)C_0(x,t)P(x,t) - uC_e(x,t)]\}dx \\ & -\int_\Omega [\tau_1 u^2 + \tau_2 C_0(x,t) + \tau_3 C_e(x,t)]dx\}. \end{aligned} \tag{4.10}$$

On the other hand, for $\forall\varepsilon > 0, 0 \leq s \leq \hat{s} \leq T$ with $\hat{s} - s > 0$ small enough, there exists a $u \equiv u_{\varepsilon,\hat{s}}(x,t) \in \mathcal{V}[\Omega\times I]$ such that

$$V(s,y_0) + \varepsilon(\hat{s}-s) \geq \int_s^{\hat{s}}\int_\Omega [\tau_1 u^2(x,t) + \tau_2 C_0(x,t) + \tau_3 C_e(x,t)]dxdt + V(\hat{s},y(\hat{x},\hat{s})).$$

Therefore it follows that

$$-\varepsilon \leq -\frac{V(\hat{s},y(x,\hat{s})) - V(s,y_0)}{\hat{s}-s}$$

$$
\begin{aligned}
&-\frac{1}{\hat{s}-s}\int_s^{\hat{s}}\int_\Omega[\tau_1 u^2(x,t)+\tau_2 C_0(x,t)+\tau_3 C_e(x,t)]dxdt\\
=&\frac{1}{\hat{s}-s}\int_s^{\hat{s}}\{-V_t(t,y)-\int_\Omega\{V_P(t,y)[d_1\Delta P(x,t)+\Lambda(x)-r(x)P(x,t)\\
&-\alpha(x)C_0(x,t)P(x,t)]+V_{C_0}[k(x)C_e(x,t)+f(x)C_0(x,t)P(x,t)\\
&-(g(x)+m(x))C_0(x,t)]+V_{C_e}[d_2\Delta C_e(x,t)-k_1(x)C_e(x,t)P(x,t)\\
&-h(x)C_e(x,t)+g_1(x)C_0(x,t)P(x,t)-u(x,t)C_e(x,t)]\}dx\\
&-\int_\Omega[\tau_1 u^2(x,t)+\tau_2 C_0(x,t)+\tau_3 C_e(x,t)]dx\}dt\\
\le&\frac{1}{\hat{s}-s}\int_s^{\hat{s}}\{-V_t(t,y)+\sup_{u\in U}\{\int_\Omega\{-V_P(t,y)[d_1\Delta P(x,t)+\Lambda(x)-r(x)P(x,t)\\
&-\alpha(x)C_0(x,t)P(x,t)]-V_{C_0}(t,y)[k(x)C_e(x,t)+f(x)C_0(x,t)P(x,t)\\
&-(g(x)+m(x))C_0(x,t)]-V_{C_e}(t,y)[d_2\Delta C_e(x,t)-k_1(x)C_e(x,t)P(x,t)\\
&-h(x)C_e(x,t)+g_1(x)C_0(x,t)P(x,t)-uC_e(x,t)]\}dx\\
&-\int_\Omega[\tau_1 u^2+\tau_2 C_0(x,t)+\tau_3 C_e(x,t)]dx\}\}dt\\
\to&-V_t(s,y_0)+\sup_{u\in U}\{\int_\Omega\{-V_P(t,y)[d_1\Delta P(x,t)+\Lambda(x)-r(x)P(x,t)\\
&-\alpha(x)C_0(x,t)P(x,t)]-V_{C_0}(t,y)[k(x)C_e(x,t)+f(x)C_0(x,t)P(x,t)\\
&-(g(x)+m(x))C_0(x,t)]-V_{C_e}(t,y)[d_2\Delta C_e(x,t)-k_1(x)C_e(x,t)P(x,t)\\
&-h(x)C_e(x,t)+g_1(x)C_0(x,t)P(x,t)-uC_e(x,t)]\}dx\\
&-\int_\Omega[\tau_1 u^2+\tau_2 C_0(x,t)+\tau_3 C_e(x,t)]dx\}. \qquad (4.11)
\end{aligned}
$$

Combining (4.10) and (4.11), we obtain our results. □

Next, by [33, Chapter 4], we define the viscosity solution of (4.9).

Definition 4.1. A function $V \in C([0,T]\times\mathbb{X}^+)$ is called a viscosity subsolution(or supersolution) of (4.9) if

$$V(T,y)\le\int_\Omega h(y)dx \text{ (or } V(T,y)\ge\int_\Omega h(y)dx),\forall y\in\mathbb{X}^+,$$

and for any $\varphi\in C^1([0,T]\times\mathbb{X}^+)$, whenever $V-\varphi$ attains a local maximum(or minimum) at $(t,y)\in[0,T]\times\mathbb{X}^+$, we have

$$
\begin{aligned}
&-\varphi_t(t,y)+\sup_{u\in U}H(t,y,u,-\varphi_y(t,y))\le 0,\\
&(\text{or } -\varphi_t(t,y)+\sup_{u\in U}H(t,y,u,-\varphi_y(t,y))\ge 0).
\end{aligned}
$$

In the case that V is both a viscosity subsolution and supersolution of (4.9), it is called a viscosity solution of (4.9).

Theorem 4.3. *Let Assumption 4.1, Assumption 4.2 and Assumption 4.3 hold. Then the value function $V(\cdot,\cdot)$ satisfies*

$$|V(s,y_0)-V(\bar{s},\bar{y}_0)|\le K|s-\bar{s}|,\forall(s,y_0),(\bar{s},\bar{y}_0)\in[0,T]\times\mathbb{X}^+, \qquad (4.12)$$

for some $K > 0$. Moreover, V is the only viscosity solution of (4.9) *in the class* $C([0,T] \times \mathbb{X}^+)$.

Proof. Since

$$\begin{aligned}
&|V(s,y_0) - V(\bar{s},\bar{y}_0)| \\
=&|\inf_{u(x,t)\in\mathcal{V}[\Omega\times I]} \int_s^T \int_\Omega [\tau_1 u^2(x,t) + \tau_2 C_0(x,t) + \tau_3 C_e(x,t)]dxdt + \int_\Omega h(y(x,T))dx \\
&- \inf_{u(x,t)\in\mathcal{V}[\Omega\times I]} \int_{\bar{s}}^T \int_\Omega [\tau_1 u^2(x,t) + \tau_2 C_0(x,t) + \tau_3 C_e(x,t)]dxdt + \int_\Omega h(y(x,T))dx| \\
\leq&|\int_s^{\bar{s}} \int_\Omega [\tau_1 u^2(x,t) + \tau_2 C_0(x,t) + \tau_3 C_e(x,t)]dxdt| \\
=&K|s-\bar{s}|,
\end{aligned} \tag{4.13}$$

where $K = \max_{t\in I} \int_\Omega [\tau_1 u^2(x,t) + \tau_2 C_0(x,t) + \tau_3 C_e(x,t)]dx$. Therefore, (4.12) is valid. Then apply the general results in [33, Chapter 4], we know that $V(\cdot,\cdot)$ is a viscosity solution of the HJB equation (4.9). □

4.2. Optimal control

In this subsection, we discuss the existence of the optimal control for the system (4.1) and construct the Hamiltonian $H(t,y,u,p)$ to solve the optimal control problem.

Theorem 4.4. *There exists an optimal control $u^*(x,t) \in U$ and a corresponding optimal state $(P^*(x,t), C_0^*(x,t), C_e^*(x,t))$ such that*

$$V(s,y_0) = \inf_{u(x,t)\in U} J(s,y_0;u^*),$$

subject to the control system (4.1).

Proof. We can complete the proof in a similar way as in [10, Theorem 4.1]. □

Theorem 4.5. *Let $u^*(x,t)$ be optimal control variable, $P^*(x,t), C_0^*(x,t)$ and $C_e^*(x,t)$ are corresponding optimal state variables, then we have the following optimal control:*

$$u^*(x,t) = \min\left\{1, \max\left\{0, -\frac{1}{2}\tau_1^{-1} C_e^*(x,t) p_3(x,t)\right\}\right\}, \tag{4.14}$$

where

$$\begin{cases}
\partial p_1(x,t) =\{[-d_1\Delta + r(x) + \alpha(x)C_0(x,t)]p_1(x,t) - f(x)C_0(x,t)p_2(x,t) \\
\qquad\qquad + [k_1(x)C_e(x,t) - g_1(x)C_0(x,t)]p_3(x,t)\}\partial t, \\
\partial p_2(x,t) =\{\alpha(x)P(x,t)p_1(x,t) + [(g(x)+m(x)) - fP(x,t)]p_2(x,t) \\
\qquad\qquad - g_1(x)P(x,t)p_3(x,t) - \tau_2\}\partial t, \\
\partial p_3(x,t) =\{-k(x)p_2(x,t) + [-d_2\Delta + k_1(x)P(x,t) + h(x) + u(x,t)]p_3(x,t) \\
\qquad\qquad - \tau_3\}\partial t.
\end{cases}$$

Proof. The Hamiltonian function $H(t, y, u, p)$ is given by

$$
\begin{aligned}
&H(t, y, u, p) \\
:=&p_1(x,t)[d_1\Delta P(x,t) + \Lambda(x) - r(x)P(x,t) - \alpha(x)C_0(x,t)P(x,t)] \\
&+ p_2(x,t)[k(x)C_e(x,t) + f(x)C_0(x,t)P(x,t) - (g(x) + m(x))C_0(x,t)] \\
&+ p_3(x,t)[d_2\Delta C_e(x,t) - k_1(x)C_e(x,t)P(x,t) - h(x)C_e(x,t) \\
&+ g_1(x)C_0(x,t)P(x,t) - u(x,t)C_e(x,t)] \\
&- \tau_1 u^2(x,t) + \tau_2 C_0(x,t) + \tau_3 C_e(x,t), \forall (t, y, u, p) = [0, T] \times \mathbb{X}^+ \times U \times \mathbb{R}^3,
\end{aligned}
\tag{4.15}
$$

where $p(x,t)$ is equivalent to $V_y(t, y)$.

Applying the general results in [33], let $\frac{\partial H(t,y,u,p)}{\partial u} = 0$, we obtain the optimal control $u^*(x,t)$ as follows

$$
u^*(x,t) = -\frac{1}{2}\tau_1^{-1}C_e^*(x,t)p_3(x,t). \tag{4.16}
$$

Therefore, according to the properties of control variables, (4.14) holds. This completes the proof. □

Remark 4.1. For different initial value, we can construct an optimal control problem to obtain the optimal pair. The main steps are as follows. Firstly, solve the HJB equation (4.9) to find the $V(T, y(x,t))$; Secondly, find $u^*(x,t)$ through Hamiltonian function (4.15); Finally, combine the $u^*(x,t)$ to solve model (4.1) to get the optimal pair $(y^*(x,t), u^*(x,t))$.

5. Numerical simulations

This section aims to illustrate the effectiveness of our theoretical results that obtained in previous sections. To simulate the threshold dynamics of persistence and extinction, we selected two sets of parameters, which are obtained from [9, 11, 14] and listed in Table 2. For both sets of parameters, we calculate $R_0 = 0.17$ and $R_0 = 1.8$ through (3.20) (i.e., $R_0 = (\frac{g_1 k\Lambda}{r(g+m)-f\Lambda})/(h + \frac{k_1\Lambda}{r})$).

Table 2. Parameter values of numerical experiments for model (2.2)

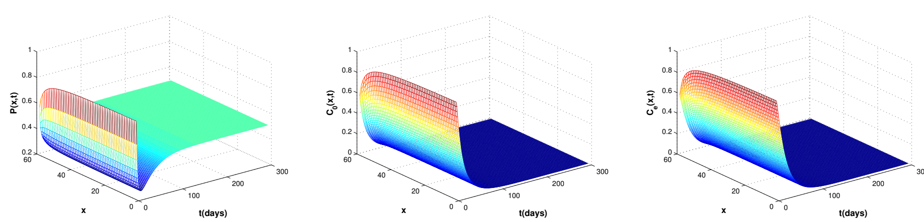
Parameters	d_1	d_2	r	k	g	m	h	α	Λ	f	k_1	g_1
Value($R_0 = 0.17$)	0.005	0.05	0.2	0.1	0.2	0.08	0.1	0.4	0.1	0.1	0.05	0.1
Value($R_0 = 1.8$)	0.005	0.05	0.1	0.1	0.08	0.12	0.1	0.2	0.1	0.1	0.01	0.2

5.1. Numerical simulation of threshold dynamics

We use the Milstein method [7] and Matlab software to perform numerical simulations in this subsection. We first discretize the model (2.2) to obtain the following form

$$
\begin{cases}
P(\eta,\theta) = P(\eta,\theta-1) + [(d_1(P(\eta+1,\theta-1) - 2P(\eta,\theta-1) \\
\qquad + P(\eta-1,\theta-1)))/h^2 + \Lambda - rP(\eta,\theta-1) \\
\qquad - \alpha C_0(\eta,\theta-1)P(\eta,\theta-1)]\Delta t, \\
C_0(\eta,\theta) = C_0(\eta,\theta-1) + [kC_e(\eta,\theta-1) + fC_0(\eta,\theta-1)P(\eta,\theta-1) \\
\qquad - (g+m)C_0(\eta,\theta-1)]\Delta t, \\
C_e(\eta,\theta) = C_e(\eta,\theta-1) + [(d_2(C_e(\eta+1,\theta-1) - 2C_e(\eta,\theta-1) \\
\qquad + C_e(\eta-1,\theta-1)))/h^2 - k_1C_e(\eta,\theta-1)P(\eta,\theta-1) - hC_e(\eta,\theta-1) \\
\qquad + g_1C_0(\eta,\theta-1)P(\eta,\theta-1)]\Delta t.
\end{cases}
$$

Figure 1 is a simulation of $R_0 < 1$, simple calculation from the data in the table to get $\mathcal{R}_0 = 0.17$, as shown in Fig 1(b) and Fig 1(c), the concentration of the toxicant converges to 0 over time, that is to say, the toxicant eventually goes extinct. This is the same conclusion as given by Theorem 3.2, and $E_0 \in \mathbb{X}^+$ is globally attractive.



(a) (b) (c)

Figure 1. The evolution of P, C_0 and C_e of system (2.2) for $\mathcal{R}_0 = 0.17 < 1$.

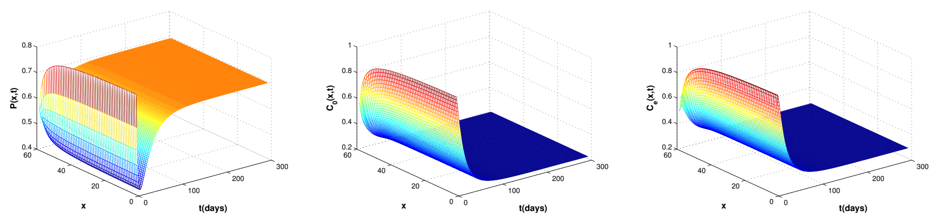
The case of $R_0 > 1$ is shown in Figure 2. It can be seen that the concentration of toxicant tend to be a positive constant distribution over time, in other words, the toxicant is persistent. This is the same conclusion as Theorem 3.3, and system (2.2) with (2.3) exists a positive steady state E_*.

5.2. Numerical simulation of optimal control

In order to obtain the discrete optimal control problem, we assume the step size is $\triangle > 0$, and $T = n\triangle$, where n is a positive integers. Then, time interval $[0, T]$ can be divided as

$$t_0 = 0 < t_1 < \cdots < t_n = T.$$

In the following, we give the algorithm for optimal control in Table 3.



(a) (b) (c)

Figure 2. The evolution of P, C_0 and C_e of system (2.2) for $\mathcal{R}_0 = 1.8 > 1$.

Table 3. Algorithm 1

Step 1: for $\lambda = 0$ **do**
$P^{\lambda} = P(0)$; $C_0^{\lambda} = C_0(0)$; $C_e^{\lambda} = C_e(0)$
end for
for $\lambda = n$ **do**
$p_1^{\lambda} = p_1(0)$; $p_2^{\lambda} = p_2(0)$; $p_3^{\lambda} = p_3(0)$
end for
Step 2: for $\lambda = 0, 1, \cdots, n-1$ **do**
$P^{\lambda+1} = P^{\lambda} + \triangle[d_1 \Delta P^{\lambda} + \Lambda - rP^{\lambda} - \alpha C_0^{\lambda} P^{\lambda}]$
$C_0^{\lambda+1} = C_0^{\lambda} + \triangle[kC_e^{\lambda} + fC_0^{\lambda} - (g+m)C_0^{\lambda}]$
$C_e^{\lambda+1} = C_e^{\lambda} + \triangle[d_1 \Delta C_e^{\lambda} - k_1 C_e^{\lambda} P^{\lambda} - hC_e^{\lambda} + g_1 C_0^{\lambda} P^{\lambda} - u^{\lambda} C_e^{\lambda}]$
for $j = 1, 2, 3$ **do**
$p_j^{n-\lambda-1} = p_j^{n-\lambda} - \triangle \times \mathrm{Temp}_j$
end for
$D^{\lambda+1} = -\frac{1}{2}\tau_1^{-1} C_e^{\lambda} p_3^{n-\lambda}$
$u^{\lambda+1} = \min\{1, \max\{0, D^{\lambda+1}\}\}$
end for
Step 3: for $\lambda = 1, 2, \cdots, n$ **do**
$P^*(t_{\lambda}) = P^{\lambda}$; $C_0^*(t_{\lambda}) = C_0^{\lambda}$; $C_e^*(t_{\lambda}) = C_e^{\lambda}$
$u^*(t_{\lambda}) = u^{\lambda}$
end for

Where

$$\mathrm{Temp}p_1 = -d_1 \Delta p_1^{n-\lambda} + (r + \alpha C_0^{\lambda}) p_1^{n-\lambda} - fC_0^{\lambda} p_2^{n-\lambda} + (k_1 C_e^{\lambda} - g_1 C_0^{\lambda}) p_3^{n-\lambda},$$

$$\mathrm{Temp}p_2 = \alpha P^{\lambda} p_1^{n-\lambda} + (g + m - fP^{\lambda}) p_2^{n-\lambda} - g_1 P^{\lambda} p_3^{n-\lambda} - \tau_2,$$

$$\mathrm{Temp}p_3 = -kp_2^{n-\lambda} - d_2 \Delta p_3^{n-\lambda} + (k_1 P^{\lambda} + h + u^{\lambda}) p_3^{n-\lambda} - \tau_3.$$

With the help of **Algorithm 1**, we can define the values of P, C_0, C_e, p_i and u at nodal points by P^k, C_0^k, C_e^k, p_i^k and u^k, respectively, where $0 \le k \le n$ and $i = 1, 2, 3$. The initial value is selected as $(P^0, C_0^0, C_e^0) = (1, 1, 0.8)$, and the wight constants in objective function (4.3) are set as $\tau_1 = 1.5 \times 10^{-3}, \tau_2 = 1 \times 10^{-3}, \tau_3 = 2 \times 10^{-3}$.

In Fig 3, we compare the trajectories of state variables with no control and optimal control, and present the trajectory of optimal control variable. As shown in Fig 3(a) and Fig 3(b), the concentration of toxicant decrease more rapidly and significantly after the control is applied. More precisely, the decrease in toxicant concentration is the fastest in the initial stage, especially in the first 50 days after implementation of control. Afterwards, the decline rate slows down and the concentration of the toxicant gradually approaches zero. The corresponding control intensity is shown in Fig 3(c), it can be observed that the control intensity continues to increase in an initial short period of time, which helps to lower the peak concentration of toxicant in the environment and reduce the duration of toxicant. Eventually, the intensity of control stabilizes with the stability of the toxicant concentration. This shows the effectiveness of our control strategy.

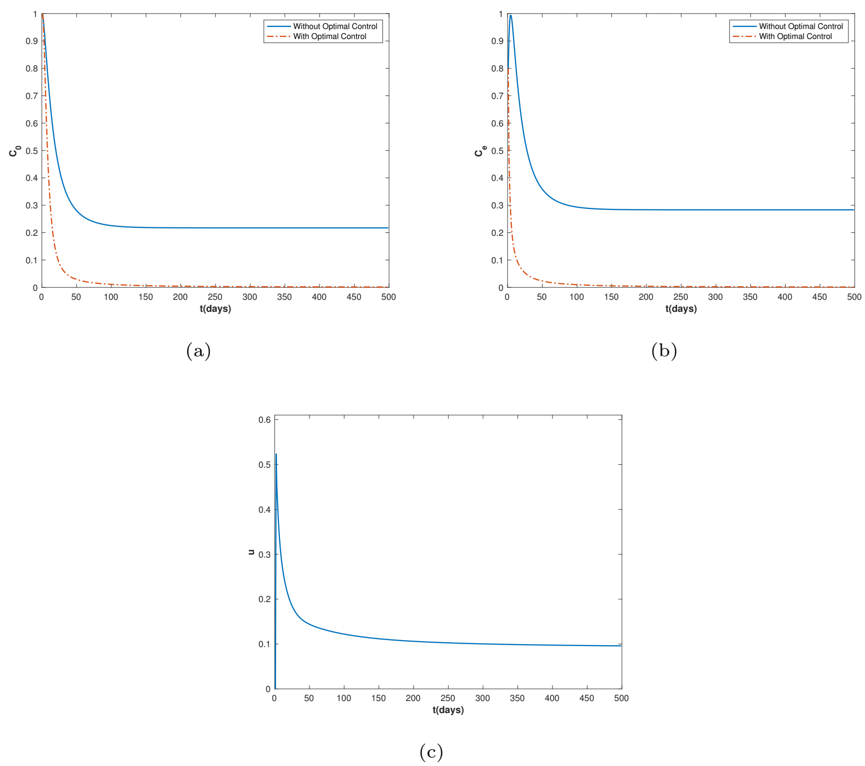


Figure 3. The paths of the optimal state variables C_0, C_e, and the optimal control variable u.

6. Conclusions

In this work, we establish a toxicant-population model with reaction-diffusion, and study its dynamic behavior and optimal control problems respectively. By defining the basic reproduction number R_0, we discuss the threshold dynamics, which shows that R_0 is a threshold parameter for the extinction (Theorem 3.2) and persistence (Theorem 3.3) of toxicant. Due to the persistence of toxicant when $R_0 > 1$, the optimal control of toxicant is considered, with the goal of minimizing the concentration of toxicant while minimizing the cost of control. By means of dynamic programming, the HJB equation is constructed, and the existence and uniqueness of the viscosity solution of the HJB equation is proved. Through Hamiltonian function, the optimal control of toxicant is obtained. Finally, several numerical examples are provided to illustrate the theoretical results.

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