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EXISTENCE, UNIQUENESS AND REGULARITY OF SOLUTIONS FOR FRACTIONAL INTEGRO-DIFFERENTIAL EQUATIONS WITH STATE-DEPENDENT DELAY

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Abstract Within this paper, we consider the existence and uniqueness of solutions for fractional integro-differential equations with state-dependent delay on the Lipschitz continuous function space. Our results are obtained by using the resolvent operator theory and the generalized Banach contraction mapping principle. The regularity of solutions of fractional integro-differential equations with state-dependent delay is also discussed. Finally, an example is provided as an application.

Keywords Fractional integro-differential equations, state-dependent delay, resolvent theory, generalized Banach contraction mapping principle.

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1. Introduction

In the process of modeling, the variation of things is not only related to the current moment, but also to the past moment. Therefore, it is necessary to introduce the time delay term, that is, the representation of the past state, which can reveal important features concerning the evolution of the modeled phenomenon. This kind of equation is called delay differential equation. In some applications, the time delay term is affected by time variation, and the equation is called time-varying delay differential equation. Besides that, if the delay term depends on unknown variables, we call it state-dependent delay differential equation.

In recent years, the differential equation with state-dependent delay is one of the research hotspot of functional differential equations, such as the maturity of biological population, the incubation period of virus, the lag effect of drug efficacy and so on. We can refer to [3, 5, 8, 15, 17, 18, 26, 27, 33] and references therein. The earliest research on differential equations with state-dependent delay can be traced back to an 1806 paper of Poisson in [29], but what really attracts attention is the two body problem in electric power science, which discussed by Driver in the 1960s. For more details we refer the reader to [9–11].

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Compared with the ordinary delay differential equation, the differential equation with state-dependent delay does not have smoothness. But it can further accurately describe some properties of the system, which makes the research results more valuable in application. See literatures [1, 21]. The time-varying delay differential equation is usually regarded as a linear approximation of the differential equation with state-dependent delay. Therefore, the dynamics and analytical properties of the latter are more complicated.

In practical application, the following model with state-dependent delay was often used in population dynamics, see [4, 7, 13, 14],

$$\begin{aligned}
&z'(t,\varsigma) = \triangle z(t,\varsigma) + \mathcal{L}\left(t, \int_0^t k(t,s) z\left(s - \xi(s, z(s,\cdot),\varsigma)\right) \mathrm{d}s\right), \quad t \in [0,a],\\
&z(t,\cdot)\mid_{\partial\Omega} = 0,\\
&z(s,\varsigma) = \varphi(s,\varsigma), \quad s \in [-p,0],
\end{aligned}$$

where $\Omega \subset \mathbb{R}^n$ is open, bounded and has smooth boundary. $\triangle$ is the Laplacian operator in the sense of distributions. $\mathcal{L}$ is continuous function. $\varsigma \in \Omega$ and $k \in C([0,a] \times [0,a]; \mathbb{R})$.

In [6], Cooke and Huang interpreted the local stability of the single bottleneck network by applying the formal linearization result for equation with state-dependent delay,

$$\mathrm{d}u(t)/\mathrm{d}t = f(u_t, \int_{-r_0}^0 \mathrm{d}\eta(s) g(u_t(-\tau(u_t) + s))),$$

where $f : C([-r,0];\mathbb{R}^n) \times \mathbb{R}^n \to \mathbb{R}^n$, $g : \mathbb{R}^n \to \mathbb{R}^n$ are continuously differentiable. $\tau : C([-r,0];\mathbb{R}^n) \to [0,r_1](r_0+r_1 \leq r)$ is continuous, and η is the bounded variation on $[-r_0,0]$.

Recently, E. Hernández and J. Wu in [20] discussed the existence and uniqueness of $C^{1+\alpha}$ strict solutions for the following abstract integro-differential equations with state-dependent delay by means of the Banach contraction principle in $C_{Lip}([-p,a];X)$ space,

$$\begin{aligned}
&u'(t) = Au(t) + F(t, u(t), \int_0^t K(t,\tau) u(\sigma(\tau, u(\tau))) \mathrm{d}\tau), \quad t \in [0,a],\\
&u_0 = \varphi \in C([-p,0];X),
\end{aligned}$$

where X is a Banach space, $-A : D(A) \subset X \to X$ is the generator of the analytic semigroup of the bounded linear operator $\{T(t) : t \geq 0\}$. K is an operator valued map and F, σ are suitable functions.

Fractional calculus models are of universal application in population dynamics, electrical dynamics of the composite medium, memory and genetic properties of many materials. For further account we refer to [23, 25, 31, 32, 35]. It is of great significance in real life to investigate the fractional equations with state-dependent delay. Examples include oil exploration, 3D printing and so on. Therefore, the fractional equation with state-dependent delay has become the focus of many researchers.

However, so far, we have not found the results of fractional integro-differential equations with state-dependent delay in Lipschitz function space. To close the gap, motivated by the above works, the purpose of this paper is to extend the idea of the literature [22] to the following fractional integro-differential system with state-dependent delay,

$$ {}^cD^q x(t) + Ax(t) = Z\left(t, x_{\mu(t,x_t)}, (Gx)(t)\right), \quad 0 \le t \le T, \tag{1.1} $$
$$ x_0 = \phi \in \mathfrak{B}_X = C([-r,0];X), \quad -r \le t \le 0, \tag{1.2} $$

where ${}^cD^q$ is the generalized fractional derivative of order $q \in (0,1)$ in Caputo sense. $-A : D(A) \subset X \to X$ is the generator of the analytic semigroup of the bounded linear operator $\{T(t) : t \ge 0\}$ on Banach space X. G is defined as $(Gx)(t) = \int_0^t a(t,s,x_{\mu(s,x_s)})\mathrm{d}s$, a is continuous function. μ and Z are continuous functions to be specified later. For any continuous function x, we employ x_t to represent the element in $\mathfrak{B}_X$, which is defined as $x_t(\theta) = x(t+\theta), \theta \in [-r,0]$, $t \in [0,T]$. Here x_t shows that the historical state until time t.

The mapping $x \longmapsto x_{\mu(\cdot,x_{(\cdot)})}$ is generally not Lipschitz, which is the main obstruction to get the results of form (1.1)-(1.2) in space $C([-r,T];X)$. The innovation of [22] is to give the results of the existence and uniqueness of the solutions on Lipschitz function space for a highly nontrivial problem in the framework of semigroups. The main approach is Banach contraction mapping principle that forms the core of the proof of Theorem 2.1 in [22].

Compared with the integer order in [22], this article mainly highlights two aspects. On the one hand, the integer order is extended to the fractional order by the theory of resolvent operator. In the process of proving Theorem 3.1, similarly to [22], we would like to get $Q_q(\cdot)\phi(0)$ is Lipschitz continuous on the whole interval $[0,T]$ but that is beyond our reach at this point. To deal with this problem, we first obtain the internally closed Lipschitz continuity, and then $Q_q(\cdot)\phi(0)$ is Lipschitz continuous if the initial value $\phi(0)$ is sufficiently smooth. On the other hand, the global solutions are achieved and the relevant coefficient conditions are weakened by applying the generalized Banach contraction mapping principle, rather than the local solutions. In addition, we prove the existence of the strict solution of the system (1.1)-(1.2) by using a necessary and sufficient condition of the strict solution.

The structure of the paper is as follows. Section 2 reviews some symbols, definitions and lemmas. Section 3 proves the existence and uniqueness of the mild solution of the system (1.1)-(1.2) on $C_{Lip}([-r,T];X)$ by the principle of generalized Banach contraction mapping. Section 4 gives the results of the strict solution of (1.1)-(1.2). The last section illustrates the feasibility of the results through an example.

2. Preliminaries

Let E and F be Banach spaces, $\mathbb{R}$ and $\mathbb{N}$ be the sets of real numbers and positive integers. $C([0,T];E)$ and $C_{Lip}([0,T];E)$ are function spaces, consisting respectively of the continuous functions and Lipschitz continuous functions. $C([0,T];E)$ is equipped with the supremum norm $\|z\|_{C([0,T];E)} = \sup_{s\in[0,T]} \|z(s)\|$. $C_{Lip}([0,T];E)$ is endowed with the norm $\|z\|_{C_{Lip}([0,T];E)} = \|z\|_{C([0,T];E)} + [z]_{C_{Lip}([0,T];E)}$, where

$$ [z]_{C_{Lip}([0,T];E)} = \sup_{t,s\in[0,T],t\neq s} \frac{\|z(t)-z(s)\|}{|\, t-s \,|} < \infty. $$

Similarly, $C_{Lip}([0,T]\times E;F)$ is endowed with the norm

$$\|z\|_{C_{Lip}([0,T]\times E;F)} = \|z\|_{C([0,T]\times E;F)} + [z]_{C_{Lip}([0,T]\times E;F)},$$

where $[z]_{C_{Lip}([0,T]\times E;F)} = \left\{\sup \frac{|z(t,x)-z(s,y)|}{|t-s|+\|x-y\|_E}, (t,x),(s,y)\in[0,T]\times E,(t,x)\neq(s,y)\right\}$. Let $\mathcal{L}(E;F)$ be the space which is bounded linear operators from E to F, $\mathcal{L}(E) = \mathcal{L}(E;E)$ for short. The set $B_r(\phi;E) := \{\xi \in C([0,T];E); \|\xi-\phi\| \leq r\}$, where ϕ is the origin and r is the radius.

Throughout the rest of the article, let $-A : D(A) \subset X \to X$ be the generator of the analytic semigroup of the bounded linear operator $\{T(t) : t \geq 0\}$ on Banach space X. Assume that $0 \in \rho(A)$, where $\rho(A)$ is the resolvent of A. The space $X_1 = D(A)$ is endowed with the norm $\|x\|_{X_1} = \|Ax\|$. Moreover, the fractional power A^k, $0 < k \leq 1$ can be defined as a closed linear operator on the domain $D(A^k)$. The semigroup $\{T(t) : t \geq 0\}$ has the following properties:
(i) There exists a constant $M \geq 1$ such that $M := \sup\limits_{0\leq t<\infty} | T(t) | < \infty$.
(ii) For any $0 < t \leq T$ and $0 < k \leq 1$, there is a constant M_k such that $\|A^kT(t)\| \leq \frac{M_k}{t^k}$.

During the following discuss, we give some concepts needed in this text.

Definition 2.1. ([28]) Let $g \in L^1([0,a];X)$. The Riemann-Liouville integral of the order $\gamma \in (0,+\infty)$ is characterized as

$$J_0^\gamma g(t) = \int_0^t \frac{(t-\tau)^{\gamma-1}g(\tau)}{\Gamma(\gamma)}\mathrm{d}\tau, \quad t\in[0,a],$$

where Γ is the Gamma function.

Definition 2.2. ([28]) Let $m-1<\gamma<m$, $m\in\mathbb{N}$, and $g \in C^m([0,a];X)$. The Caputo fractional derivative of order γ is given by

$$^cD_0^\gamma g(t) = \int_0^t \frac{(t-\tau)^{m-\gamma-1}g^{(m)}(\tau)}{\Gamma(m-\gamma)}\mathrm{d}\tau, \quad t\in[0,a].$$

The integrals in the above definitions can be understood in the sense of Bochner.

Definition 2.3. A function $x \in C([-r,T];X)$ is said to be a mild solution of the system (1.1)-(1.2) if

$$x(t) = \begin{cases} Q_q(t)\phi(0) + \displaystyle\int_0^t (t-s)^{q-1}R_q(t-s)Z\left(s, x_{\mu(s,x_s)}, (Gx)(s)\right)\mathrm{d}s, \ t\in[0,T], \\ \phi(t), \quad t\in[-r,0], \end{cases} \tag{2.1}$$

where

$$Q_q(t) = \int_0^\infty \xi_q(\theta)T(t^q\theta)\mathrm{d}\theta, \quad R_q(t) = q\int_0^\infty \theta\xi_q(\theta)T(t^q\theta)\mathrm{d}\theta,$$
$$\xi_q(\theta) = \frac{1}{q}\theta^{-1-\frac{1}{q}}\varpi_q(\theta^{-\frac{1}{q}}), \quad \varpi_q(\theta) = \frac{1}{\pi}\sum_{n=1}^\infty(-1)^{n-1}\theta^{-nq-1}\frac{\Gamma(nq+1)}{n!}\sin(n\pi q),$$

ξ_q is the probability density function defined on $(0,\infty)$, i.e., $0<\theta<\infty$, $\xi_q(\theta)\geq 0$, and $\int_0^\infty \xi_q(\theta)\mathrm{d}\theta = 1$.

Remark 2.1. It is checked that

$$\int_0^\infty \theta^v \xi_q(\theta)\mathrm{d}\theta = \int_0^\infty \theta^{-qv} \varpi_q(\theta)\mathrm{d}\theta = \frac{\Gamma(1+v)}{\Gamma(1+qv)}, \text{ for all } v \in [0,1].$$

Remark 2.2. In the process of solving the fractional equations, we obtain the well-posedness of the homogeneous equations by using the resolvent operators theory. In general, the solution given in terms of the probability density is a special form of the solution that given by a family of resolvent operators. Therefore, the above mentioned operators $\{Q_q(t); t \geq 0\}$ and $\{R_q(t); t \geq 0\}$ are also a family of resolvent operators in [2] and [24].

Lemma 2.1. ([34]) *Assume the bounded linear operator $\{T(t), t \geq 0\}$ is the analytic semigroup and has a generator A on $(X, \|\cdot\|_X)$. The following statements are valid:*
(i) for any fixed $t \geq 0$, the operators $Q_q(t)$ and $R_q(t)$ are bounded linear,

$$\|Q_q(t)x\| \leq M\|x\|, \qquad \|R_q(t)x\| \leq \frac{qM}{\Gamma(1+q)}\|x\|, \quad \forall x \in X;$$

(ii) for any fixed $t \geq 0$, the operators $Q_q(t)$ and $R_q(t)$ are strongly continuous;
(iii) for any $0 \leq t \leq T$, $\|AQ_q(t)x\| \leq M\|Ax\|$, $\forall x \in D(A)$;
(iv) for any $0 < t \leq T$, $\|AR_q(t)x\| \leq qM_1 t^{-q}\|x\|$, $\forall x \in X$.

3. Existence and uniqueness of mild solutions

In this section, we prove the existence and uniqueness of the solutions of (1.1)-(1.2), based on assumptions and lemmas as follows.

Now, we assume that $(E, \|\cdot\|_E)$ and $(F, \|\cdot\|_F)$ are Banach spaces. $(E, \|\cdot\|_E) \hookrightarrow (F, \|\cdot\|_F) \hookrightarrow (X, \|\cdot\|_X)$, where the notation $(E, \|\cdot\|_E) \hookrightarrow (X, \|\cdot\|_X)$ is used to indicate that $(E, \|\cdot\|_E)$ is continuously embedded into $(X, \|\cdot\|_X)$.

Lemma 3.1. ([19]) *Let x, $y \in C_{Lip}([-r,T];E)$, $x_0 = y_0 = \phi$, and $\mu \in C_{Lip}([0,T] \times \mathfrak{B}_E; [0,T])$. If for any $0 \leq t \leq T$, $\mu(t, x_t) \leq T$ and $\mu(t, y_t) \leq T$, one has*
(i) $x_{(\cdot)}$, $x_{\mu(\cdot, x_{(\cdot)})} \in C_{Lip}([0,T]; \mathfrak{B}_E)$;
(ii) $[x_{(\cdot)}]_{C_{Lip}([0,T];\mathfrak{B}_E)} \leq \max\{[x]_{C_{Lip}([0,T];E)}; [\phi]_{C_{Lip}([-r,0];E)}\}$;
(iii) $[x_{\mu(\cdot, x_{(\cdot)})}]_{C_{Lip}([0,T];\mathfrak{B}_E)}$
$\leq [x_{(\cdot)}]_{C_{Lip}([0,T];\mathfrak{B}_E)} [\mu]_{C_{Lip}([0,T]\times\mathfrak{B}_E;[0,T])} \left(1 + [x_{(\cdot)}]_{C_{Lip}([0,T];\mathfrak{B}_E)}\right)$;
(iv) $\|x_{\mu(\cdot, x_{(\cdot)})} - y_{\mu(\cdot, y_{(\cdot)})}\|_{C([0,T];E)}$
$\leq \left(1 + [y_{(\cdot)}]_{C_{Lip}([0,T];E)} [\mu]_{C_{Lip}([0,T]\times\mathfrak{B}_E;[0,T])}\right) \times \|x - y\|_{C([0,T];E)}$.

Lemma 3.2. *Suppose that $x \in E$. For any $\varepsilon > 0$, $Q_q(\cdot)x \in C_{Lip}([\varepsilon, T]; E)$.*

Proof. Since $x \in E$ and $\{T(t), t \geq 0\}$ is the analytic semigroup, for any $0 < \varepsilon \leq t \leq T$, the result shows that

$$\begin{aligned}
\|Q_q(t+h)x - Q_q(t)x\| &= \left\| \int_0^\infty \xi_q(\theta)[T((t+h)^q\theta) - T(t^q\theta)]x\mathrm{d}\theta \right\| \\
&= \left\| \int_0^\infty \xi_q(\theta)[\int_t^{t+h} AT(\tau^q\theta)x\mathrm{d}\tau^q\theta]\mathrm{d}\theta \right\|
\end{aligned}$$

$$
\begin{aligned}
&= \left\| \int_t^{t+h} \tau^{q-1} A R_q(\tau) x \mathrm{d}\tau \right\| \\
&\leq q M_1 \int_t^{t+h} \tau^{q-1} \tau^{-q} \mathrm{d}\tau \cdot \|x\| \\
&\leq q M_1 \|x\| t^{-1} h \\
&\leq q M_1 \varepsilon^{-1} \|x\| h.
\end{aligned}
$$

Thus $Q_q(\cdot)x \in C_{Lip}([\varepsilon, T]; E)$. The proof is completed. □

Remark 3.1. By [[24], Lemma 4.5], if $\phi(0)$ is sufficiently smooth, that is, $\phi(0) \in D(A^n)$ with $nq \geq 1$, we have $Q_q(\cdot)\phi(0) \in C_{Lip}([0, T]; X)$. In fact, let A generates a q-times resolvent family Q_q, then for any $x \in D(A^n), nq \geq 1$, one has

$$
\begin{aligned}
Q_q(t)x &= x + (g_q * Q_q)(t)Ax \\
&= x + (g_q * 1)(t)Ax + (g_q * (g_q * Q_q))(t)A^2 x \\
&= \cdots + \cdots \\
&= x + g_{q+1}(t)Ax + \cdots + g_{(n-1)q+1}(t)A^{n-1}x + (g_{nq} * Q_q)(t)A^n x.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
Q_q'(t)x &= g_q(t)Ax + \cdots + g_{(n-1)q}(t)A^{n-1}x + (g_{nq-1} * Q_q)(t)A^n x \\
&= \sum_{j=1}^{n-1} g_{jq}(t)A^j x + (g_{nq-1} * Q_q)(t)A^n x.
\end{aligned}
$$

By continuous differentiability of $Q_q(t)x$, we can deduce that $Q_q(\cdot)x$ is Lipschitz continuous on $[0, T]$.

Remark 3.2. Recently, in [17], the authors studied the existence and uniqueness of solutions as well as the local well-posedness for fractional differential equations with state-dependent delay on the Lipschitz continuous space. The core point of [17] and the present paper is that $Q_q(\cdot)x \in C_{Lip}([0, T]; X)$, for any $x \in D(A^n)$ ($Q_q(\cdot)x$ means the same as the $S_\alpha(\cdot)x$ in [17]). However, the approaches of the two papers are different. In [17], assuming that A is an almost sectorial operator, the authors used resolvent operators of growth β with the Mittag-Leffler function to obtain $S_\alpha(\cdot)x \in C_{Lip}([0, a]; X)$ for all $x \in D(A^n)$, n is sufficiently large. But in this paper, we only need to assume that A is an analytic semigroup to directly prove that $Q_q(\cdot)x \in C_{Lip}([0, T]; X)$, for any $x \in D(A^n)$ with $nq \geq 1$.

The specific conclusions are deduced in this paper under the following assumptions:

(H1) The function $a : D := \{(t, s) \in [0, T] \times [0, T] : 0 \leq s \leq t \leq T\} \times \mathfrak{B}_E \to E$ is continuous and there are two constants $L_a > 0$ and $L_a^* > 0$ such that for any $(t_i, s_i) \in D$, $i = 1, 2$, $x, y \in \mathfrak{B}_E$, we have

$$
\|a(t_1, s_1, x) - a(t_2, s_2, y)\|_E \leq L_a \left(| t_1 - t_2 | + | s_1 - s_2 | + \|x - y\|_{\mathfrak{B}_E} \right)
$$

and

$$
L_a^* = \max_{0 \leq s \leq t \leq T} \|a(t, s, 0)\|_E.
$$

(H2) $R_q \in L^1([0,T];\mathcal{L}(F,E))$, $Z \in C([0,T]\times\mathfrak{B}_E\times E;F)$ and there is a non-decreasing positive function $P_z \in C([0,T];\mathbb{R}^+)$ such that

$$\|Z(t,x_1,y_1)-Z(s,x_2,y_2)\| \le P_z(\delta)(\mid t-s\mid + \|x_1-x_2\|_{\mathfrak{B}_E} + \|y_1-y_2\|_E),$$

where for any $t,s\in[0,T]$, $x_i \in B_\delta(\phi;\mathfrak{B}_E)$, $y_i\in B_\delta(0;E)$, $i=1,2$.

(H3) The function $\mu \in C_{Lip}([0,T]\times\mathfrak{B}_E;[0,T])$ and $\mu(0,\phi)=0$. There is a constant $r^*>0$ such that $0\le\mu(t,\psi)\le t$, for any $t\in[0,T]$ and $\psi\in B_{r^*}(\phi;\mathfrak{B}_E)\subset\mathfrak{B}_E$.

For notational convenience, denote $\upsilon_1 := [\mu]_{C_{Lip}([0,T]\times\mathfrak{B}_E;[0,T])}$, $\upsilon_2 := L_aT$, $\upsilon_3 := L_a\|\phi\|_{\mathfrak{B}_E} + L_a^*$.

(H4) There exists $\zeta>0$ such that

$$P_z\left(\zeta(1+\zeta)\upsilon_1T\right)\left(\Lambda_1+\Lambda_2\zeta(1+\zeta)\right) + [Q_q(\cdot)\phi(0)]_{C_{Lip}([0,T];E)} \le \zeta,$$

where

$$\Lambda_1 = \frac{(2+q)MT^q(1+\upsilon_2+\upsilon_3)}{\Gamma(2+q)},\quad \Lambda_2 = \frac{(2+q)MT^q(1+\upsilon_2)\upsilon_1}{\Gamma(2+q)},\ \text{and } \phi(0)\in E.$$

Lemma 3.3. *If the hypotheses (H1) and (H3) hold, for any $x\in C_{Lip}([-r,T];E)$, we have $(Gx)(\cdot)\in C_{Lip}([0,T];E)$, where the operator G defined by*

$$(Gx)(t) = \int_0^t a(t,s,x_{\mu(s,x_s)})\mathrm{d}s, \qquad a\in C(D\times\mathfrak{B}_E;E).$$

Proof. According to Lemma 3.1 (i)-(iii) and the condition (H3), for any $x\in C_{Lip}([-r,T];E)$ and $0\le t\le T$, we know

$$\begin{aligned}
&\|x_{\mu(t,x_t)}-\phi\|_{\mathfrak{B}_E}\\
\le&[x]_{C_{Lip}([-r,T];E)}\mid\mu(t,x_t)-\mu(0,x_0)\mid\\
\le&[x]_{C_{Lip}([-r,T];E)}[\mu]_{C_{Lip}([0,T]\times\mathfrak{B}_E;[0,T])}\left(t+\|x_t-x_0\|\right)\\
\le&[x]_{C_{Lip}([-r,T];E)}[\mu]_{C_{Lip}([0,T]\times\mathfrak{B}_E;[0,T])}\left(1+[x]_{C_{Lip}([-r,T];E)}\right)T.
\end{aligned}$$

Hence, we have

$$\begin{aligned}
&\|x_{\mu(t,x_t)}\|_{\mathfrak{B}_E}\\
\le&\|x_{\mu(t,x_t)}-\phi\|_{\mathfrak{B}_E}+\|\phi\|_{\mathfrak{B}_E}\\
\le&[x]_{C_{Lip}([-r,T];E)}[\mu]_{C_{Lip}([0,T]\times\mathfrak{B}_E;[0,T])}\left(1+[x]_{C_{Lip}([-r,T];E)}\right)T+\|\phi\|_{\mathfrak{B}_E}.
\end{aligned}$$

In addition, for $0\le t<T$ and $h>0$ with $t+h\in[0,T]$, we find

$$\begin{aligned}
&\|(Gx)(t+h)-(Gx)(t)\|_E\\
\le&\int_0^t\left\|a(t+h,s,x_{\mu(s,x_s)})-a(t,s,x_{\mu(s,x_s)})\right\|_E\mathrm{d}s\\
&+\int_t^{t+h}\left\|a(t+h,s,x_{\mu(s,x_s)})\right\|_E\mathrm{d}s\\
\le&L_aTh+\int_t^{t+h}\left\|a(t+h,s,x_{\mu(s,x_s)})-a(t+h,s,0)\right\|_E\mathrm{d}s
\end{aligned}$$

$$
\begin{aligned}
&+\int_t^{t+h}\|a(t+h,s,0)\|_E\,\mathrm{d}s\\
\leq &L_aTh+L_a\left([x]_{C_{Lip}([-r,T];E)}[\mu]_{C_{Lip}([0,T]\times\mathfrak{B}_E;[0,T])}\right.\\
&\left.\times(1+[x]_{C_{Lip}([-r,T];E)})T+\|\phi\|_{\mathfrak{B}_E}\right)h+L_a^*h\\
\leq &\left(\upsilon_1\upsilon_2[x]_{C_{Lip}([-r,T];E)}(1+[x]_{C_{Lip}([-r,T];E)})+\upsilon_2+\upsilon_3\right)h.
\end{aligned}
$$

Therefore,

$$
[Gx]_{C_{Lip}([0,T];E)}\leq\upsilon_1\upsilon_2[x]_{C_{Lip}([-r,T];E)}(1+[x]_{C_{Lip}([-r,T];E)})+\upsilon_2+\upsilon_3.
$$

The proof is completed. □

Lemma 3.4. *Assume that the conditions (H1)-(H3) are satisfied, $x\in C_{Lip}([-r,T];E)$, then $Z\left(\cdot,x_{\mu(\cdot,x_{(\cdot)})},(Gx)(\cdot)\right)\in C_{Lip}([0,T];F)$.*

Proof. We know that

$$
\|x_{\mu(t,x_t)}-\phi\|_{\mathfrak{B}_E}\leq[x]_{C_{Lip}([-r,T];E)}[\mu]_{C_{Lip}([0,T]\times\mathfrak{B}_E;[0,T])}\left(1+[x]_{C_{Lip}([-r,T];E)}\right)T,
$$

for $t\in[0,T]$. By condition (H2) and Lemma 3.3, for $0\leq t<T$, $h>0$, and $t+h\in[0,T]$, we get

$$
\begin{aligned}
&\left\|Z(t+h,x_{\mu(t+h,x_{t+h})},(Gx)(t+h))-Z(t,x_{\mu(t,x_t)},(Gx)(t))\right\|_F\\
\leq &P_z\left([x]_{C_{Lip}([-r,T];E)}[\mu]_{C_{Lip}([0,T]\times\mathfrak{B}_E;[0,T])}(1+[x]_{C_{Lip}([-r,T];E)})T\right)\\
&\times\left(h+\|x_{\mu(t+h,x_{t+h})}-x_{\mu(t,x_t)}\|+\|(Gx)(t+h)-(Gx)(t)\|\right)\\
\leq &P_z\left([x]_{C_{Lip}([-r,T];E)}(1+[x]_{C_{Lip}([-r,T];E)})\upsilon_1T\right)\left(h+[x]_{C_{Lip}([-r,T];E)}\right.\\
&\times(1+[x]_{C_{Lip}([-r,T];E)})\upsilon_1h\\
&\left.+\upsilon_1\upsilon_2[x]_{C_{Lip}([-r,T];E)}(1+[x]_{C_{Lip}([-r,T];E)})h+\upsilon_2h+\upsilon_3h\right)\\
\leq &P_z\left([x]_{C_{Lip}([-r,T];E)}(1+[x]_{C_{Lip}([-r,T];E)})\upsilon_1T\right)\left([x]_{C_{Lip}([-r,T];E)}\right.\\
&\left.\times(1+[x]_{C_{Lip}([-r,T];E)})(1+\upsilon_2)\upsilon_1+1+\upsilon_2+\upsilon_3\right)h.
\end{aligned}
$$

Therefore, we can conclude that $Z\left(\cdot,x_{\mu_{(\cdot,x_{(\cdot)})}},(Gx)(\cdot)\right)\in C_{Lip}([0,T];F)$, and

$$
\begin{aligned}
&\left[Z(\cdot,x_{\mu_{(\cdot,x_{(\cdot)})}},(Gx)(\cdot))\right]_{C_{Lip}([0,T];F)}\\
\leq &P_z\left([x]_{C_{Lip}([-r,T];E)}(1+[x]_{C_{Lip}([-r,T];E)})\upsilon_1T\right)\left([x]_{C_{Lip}([-r,T];E)}\right.\\
&\left.\times(1+[x]_{C_{Lip}([-r,T];E)})(1+\upsilon_2)\upsilon_1+1+\upsilon_2+\upsilon_3\right).
\end{aligned}
$$

The proof is completed. □

Theorem 3.1. *Suppose that the conditions (H1)-(H4) hold, and $\phi\in C_{Lip}([-r,0];E)$. If $\phi(0)\in D(A^n)$ with $nq\geq 1$, and $Z(0,\phi,0)=0$, the system (1.1)-(1.2) has a unique mild solution $x\in C_{Lip}([-r,T];E)$.*

Proof. Let

$$
K(T;\zeta)=\{x\in C([-r,T];E):x_0=\phi,x\in C_{Lip}([-r,T];E),[x]_{C_{Lip}([-r,T];E)}\leq\zeta\},
$$

where ζ is in condition (H4). Consider the operator $\Gamma : K(T;\zeta) \to C([-r,T];E)$ defined by

$$\Gamma x(t) = \begin{cases} Q_q(t)\phi(0) + \displaystyle\int_0^t (t-s)^{q-1} R_q(t-s) Z\left(s, x_{\mu(s,x_s)}, (Gx)(s)\right) \mathrm{d}s, \ t \in [0,T]; \\ \phi(t), \quad t \in [-r,0]. \end{cases}$$

For $x \in K(T;\zeta)$, $\Gamma x \in C([-r,T];E)$, as is easily verified.

We divide our proof in two distinct parts.

Step 1. Prove that Γ transforms $K(T;\zeta)$ into itself. Let $x \in K(T;\zeta)$, according to Lemma 3.4, we have

$$\begin{aligned} &[Z(\cdot, x_{\mu_{(\cdot,x_{(\cdot)})}}, (Gx)(\cdot))]_{C_{Lip}([0,T];F)} \\ \le & P_z\left([x]_{C_{Lip}([-r,T];E)}(1+[x]_{C_{Lip}([-r,T];E)})\upsilon_1 T\right)\left([x]_{C_{Lip}([-r,T];E)}\right. \\ & \left.\times(1+[x]_{C_{Lip}([-r,T];E)})(1+\upsilon_2)\upsilon_1 + 1 + \upsilon_2 + \upsilon_3\right). \end{aligned}$$

By Lemma 3.2, we have $Q_q(\cdot)\phi(0) \in C_{Lip}([\varepsilon,T];E)$, for any $\phi(0) \in E$. Since $\phi(0) \in D(A^n)$, $nq \ge 1$, it follows that $Q_q(\cdot)\phi(0) \in C_{Lip}([0,T];E)$. Thus, for $t \in [0,T)$, $h > 0$ and $t+h \in [0,T]$, we obtain the estimate

$$\begin{aligned} &\|\Gamma x(t+h) - \Gamma x(t)\|_E \\ \le & [Q_q(\cdot)\phi(0)]_{C_{Lip}([0,T];E)}\, h + \Big\|\int_{-h}^t (t-s)^{q-1} R_q(t-s) Z\left(s+h, x_{\mu(s+h,x_{s+h})},\right. \\ & \left.(Gx)(s+h)\right) \mathrm{d}s - \int_0^t (t-s)^{q-1} R_q\ (t-s) Z\left(s, x_{\mu(s,x_s)}, (Gx)(s)\right) \mathrm{d}s\Big\| \\ \le & [Q_q(\cdot)\phi(0)]_{C_{Lip}([0,T];E)}\, h + \int_0^h (t+h-s)^{q-1} \left\|R_q(t+h-s) Z\left(s, x_{\mu(s,x_s)},\right.\right. \\ & \left.\left.(Gx)(s)\right)\right\| \mathrm{d}s + \int_0^t (t-s)^{q-1} \left\|R_q(t-s)\left[Z\left(s+h, x_{\mu(s+h,x_{s+h})}, (Gx)(s+h)\right)\right.\right. \\ & \left.\left. - Z\left(s, x_{\mu(s,x_s)}, (Gx)(s)\right)\right]\right\| \mathrm{d}s \\ \le & [Q_q(\cdot)\phi(0)]_{C_{Lip}([0,T];E)}\, h + \int_0^h (t+h-s)^{q-1} \| R_q(t+h-s)\left[Z\left(s, x_{\mu(s,x_s)},\right.\right. \\ & \left.\left.(Gx)(s)\right) - Z\left(0,\phi,0\right)\right]\| \mathrm{d}s + \frac{MT^q}{\Gamma(1+q)}[Z(\cdot, x_{\mu_{(\cdot,x_{(\cdot)})}}, (Gx)(\cdot))]_{C_{Lip}([0,T];F)} h \\ \le & [Q_q(\cdot)\phi(0)]_{C_{Lip}([0,T];E)}\, h + \int_0^h (t+h-s)^{q-1} \| R_q(t+h-s)\left[Z\left(s, x_{\mu(s,x_s)},\right.\right. \\ & \left.\left.(Gx)(s)\right) - Z\left(0, x_{\mu(0,x_0)}, (Gx)(0)\right)\right]\| \mathrm{d}s + \frac{MT^q}{\Gamma(1+q)}[Z(\cdot, x_{\mu_{(\cdot,x_{(\cdot)})}}, \\ & (Gx)(\cdot))]_{C_{Lip}([0,T];F)} h \\ \le & [Q_q(\cdot)\phi(0)]_{C_{Lip}([0,T];E)}\, h + \frac{Mq}{\Gamma(1+q)}[Z(\cdot, x_{\mu_{(\cdot,x_{(\cdot)})}}, (Gx)(\cdot))]_{C_{Lip}([0,T];F)} \\ & \times \int_0^h (t+h-s)^{q-1} s \mathrm{d}s + \frac{MT^q}{\Gamma(1+q)}[Z(\cdot, x_{\mu_{(\cdot,x_{(\cdot)})}}, (Gx)(\cdot))]_{C_{Lip}([0,T];F)} h \\ \le & [Q_q(\cdot)\phi(0)]_{C_{Lip}([0,T];E)}\, h + \frac{Mq}{\Gamma(1+q)}[Z(\cdot, x_{\mu_{(\cdot,x_{(\cdot)})}}, (Gx)(\cdot))]_{C_{Lip}([0,T];F)} \end{aligned}$$

$$
\begin{aligned}
&\times \int_0^1 (1-z)^{q-1} z \mathrm{d}z \cdot h^{q+1} + \frac{MT^q}{\Gamma(1+q)} [Z(\cdot, x_{\mu_{(\cdot,x_{(\cdot)})}}, (Gx)(\cdot))]_{C_{Lip}([0,T];F)} h \\
\leq &\left[Q_q(\cdot)\phi(0)\right]_{C_{Lip}([0,T];E)} h + \frac{MT^q}{\Gamma(2+q)} [Z(\cdot, x_{\mu_{(\cdot,x_{(\cdot)})}}, (Gx)(\cdot))]_{C_{Lip}([0,T];F)} h \\
&+ \frac{MT^q}{\Gamma(1+q)} [Z(\cdot, x_{\mu_{(\cdot,x_{(\cdot)})}}, (Gx)(\cdot))]_{C_{Lip}([0,T];F)} h \\
\leq &\left[Q_q(\cdot)\phi(0)\right]_{C_{Lip}([0,T];E)} h + \frac{MT^q P_z\left(\zeta(1+\zeta)\upsilon_1 T\right)}{\Gamma(2+q)} \left(\zeta(1+\zeta)(1+\upsilon_2)\upsilon_1 + 1\right. \\
&\left.+\upsilon_2 + \upsilon_3\right) h + \frac{MT^q P_z\left(\zeta(1+\zeta)\upsilon_1 T\right)}{\Gamma(1+q)} \left(\zeta(1+\zeta)(1+\upsilon_2)\upsilon_1 + 1 + \upsilon_2 + \upsilon_3\right) h \\
\leq &\left[Q_q(\cdot)\phi(0)\right]_{C_{Lip}([0,T];E)} h + \frac{(2+q)MT^q P_z\left(\zeta(1+\zeta)\upsilon_1 T\right)}{\Gamma(2+q)} \left(\zeta(1+\zeta)(1+\upsilon_2)\right. \\
&\left.\times \upsilon_1 + 1 + \upsilon_2 + \upsilon_3\right) h.
\end{aligned}
$$

Consequently, from (H4), we can conclude that

$$
\begin{aligned}
&[\Gamma x]_{C_{Lip}([0,T];E)} \\
\leq &\left[Q_q(\cdot)\phi(0)\right]_{C_{Lip}([0,T];E)} + \frac{(2+q)MT^q P_z\left(\zeta(1+\zeta)\upsilon_1 T\right)}{\Gamma(2+q)} \left(\zeta(1+\zeta)(1+\upsilon_2)\upsilon_1\right. \\
&\left.+1 + \upsilon_2 + \upsilon_3\right) \\
\leq &\left[Q_q(\cdot)\phi(0)\right]_{C_{Lip}([0,T];E)} + P_z\left(\zeta(1+\zeta)\upsilon_1 T\right)\left(\Lambda_1 + \Lambda_2 \zeta(1+\zeta)\right) \\
\leq &\zeta,
\end{aligned}
$$

where $\Lambda_1 = \frac{(2+q)MT^q(1+\upsilon_2+\upsilon_3)}{\Gamma(2+q)}$, $\Lambda_2 = \frac{(2+q)MT^q(1+\upsilon_2)\upsilon_1}{\Gamma(2+q)}$.

On the other hand, $(\Gamma x)_0 = \phi \in C_{Lip}([-r,0];E)$, for all $t \in [-r,0]$. Thus, from Lemma 3.1, we know

$$
[\Gamma x]_{C_{Lip}([-r,T];E)} \leq \max\{[\Gamma x]_{C_{Lip}([0,T];E)}, [\phi]_{C_{Lip}([-r,0];E)}\} \leq \zeta.
$$

In summary, we deduce that $\Gamma x \in C_{Lip}([-r,T];E)$.

Step 2. To prove that Γ is a contraction mapping on $K(T;\zeta)$. Let $x, y \in K(T;\zeta)$. From the hypothesis (H2), Lemma 3.1 (iv) and Lemma 3.4, we find

$$
\begin{aligned}
&\|\Gamma x(t) - \Gamma y(t)\| \\
\leq &\int_0^t (t-s)^{q-1} \|R_q(t-s)\| \left\| Z\left(s, x_{\mu(s,x_s)}, (Gx)(s)\right) - Z\left(s, y_{\mu(s,y_s)}, (Gy)(s)\right) \right\| \mathrm{d}s \\
\leq &\frac{qM}{\Gamma(1+q)} P_z\left(\zeta(1+\zeta)\upsilon_1 T\right) \int_0^t (t-s)^{q-1} \left(\|x_{\mu(s,x_s)} - y_{\mu(s,y_s)}\| \right. \\
&\left. + \int_0^s \|a(s,\tau,x_{\mu(\tau,x_\tau)}) - a(s,\tau,y_{\mu(\tau,y_\tau)})\| \mathrm{d}\tau \right) \mathrm{d}s \\
\leq &\frac{qM}{\Gamma(1+q)} P_z(\zeta(1+\zeta)\upsilon_1 T) \int_0^t (t-s)^{q-1} \\
&\times \left((1 + [y]_{C_{Lip}([-r,T];E)} [\mu]_{C_{Lip}([0,T]\times \mathfrak{B}_E;[0,T])}) \|x(s) - y(s)\| \mathrm{d}s \right. \\
&\left. + L_a \int_0^s (1 + [y]_{C_{Lip}([-r,T];E)} [\mu]_{C_{Lip}([0,T]\times \mathfrak{B}_E;[0,T])}) \|x(\tau) - y(\tau)\| \mathrm{d}\tau \right) \mathrm{d}s
\end{aligned}
$$

$$
\begin{aligned}
&\leq \frac{qM}{\Gamma(1+q)} P_z\left(\zeta(1+\zeta)\upsilon_1 T\right)(1+\zeta\upsilon_1)\int_0^t (t-s)^{q-1}\|x(s)-y(s)\|\mathrm{d}s \\
&\quad + \frac{qM}{\Gamma(1+q)} P_z\left(\zeta(1+\zeta)\upsilon_1 T\right)(1+\zeta\upsilon_1)L_a\int_0^t (t-s)^{q-1}\int_0^s \|x(\tau)-y(\tau)\|\mathrm{d}\tau\mathrm{d}s \\
&\leq \frac{M\Phi_\zeta(1+\upsilon_2)t^q}{\Gamma(1+q)}\|x-y\|,
\end{aligned}
$$

where Φ_ζ is denoted by $\Phi_\zeta := P_z\left(\zeta(1+\zeta)\upsilon_1 T\right)(1+\zeta\upsilon_1)$.

Moreover,

$$
\begin{aligned}
&\|\Gamma^2 x(t)-\Gamma^2 y(t)\| \\
\leq&\frac{qM\Phi_\zeta}{\Gamma(1+q)}\int_0^t (t-s)^{q-1}\|\Gamma x(s)-\Gamma y(s)\|\,\mathrm{d}s + \frac{qM\Phi_\zeta}{\Gamma(1+q)}L_a\int_0^t (t-s)^{q-1}\int_0^s \|\Gamma x(\tau) \\
&-\Gamma y(\tau)\|\,\mathrm{d}\tau\mathrm{d}s \\
\leq&\left(\frac{qM\Phi_\zeta}{\Gamma(1+q)}\right)^2(1+\upsilon_2)\int_0^t (t-s)^{q-1}\int_0^s (s-s_1)^{q-1}\|x(s_1)-y(s_1)\|\mathrm{d}s_1\mathrm{d}s \\
&+\left(\frac{qM\Phi_\zeta}{\Gamma(1+q)}\right)^2(\upsilon_2+\upsilon_2^2)\int_0^t (t-s)^{q-1}\int_0^s (s-s_1)^{q-1}\mathrm{d}s_1\mathrm{d}s\|x-y\| \\
\leq&\left(\frac{qM\Phi_\zeta}{\Gamma(1+q)}\right)^2(1+\upsilon_2)^2\frac{1}{q}\int_0^t (t-s)^{q-1}s^q\mathrm{d}s\|x-y\| \\
\leq&\left(\frac{qM\Phi_\zeta}{\Gamma(1+q)}\right)^2(1+\upsilon_2)^2\frac{1}{q}\frac{\Gamma(q)\Gamma(1+q)}{\Gamma(1+2q)}t^{2q}\|x-y\| \\
\leq&\frac{M^2\Phi_\zeta^2(1+\upsilon_2)^2t^{2q}}{\Gamma(1+2q)}\|x-y\|.
\end{aligned}
$$

Suppose that for any $j\in\mathbb{N}$, the following inequality is established

$$
\|\Gamma^j x(t)-\Gamma^j y(t)\| \leq \frac{M^j\Phi_\zeta^j(1+\upsilon_2)^jt^{jq}}{\Gamma(1+jq)}\|x-y\|.
$$

Then, we obtain

$$
\begin{aligned}
&\|\Gamma^{j+1}x(t)-\Gamma^{j+1}y(t)\| \\
\leq&\frac{qM\Phi_\zeta}{\Gamma(1+q)}\int_0^t (t-s)^{q-1}\left\|\Gamma^j x(s)-\Gamma^j y(s)\right\|\mathrm{d}s + \frac{qM\Phi_\zeta}{\Gamma(1+q)}L_a\int_0^t (t-s)^{q-1} \\
&\times\int_0^s \left\|\Gamma^j x(\tau)-\Gamma^j y(\tau)\right\|\mathrm{d}\tau\mathrm{d}s \\
\leq&\frac{qM\Phi_\zeta}{\Gamma(1+q)}\frac{M^j\Phi_\zeta^j(1+\upsilon_2)^j}{\Gamma(1+jq)}\int_0^t (t-s)^{q-1}s^{jq}\mathrm{d}s\|x-y\| \\
&+\frac{qM\Phi_\zeta L_a}{\Gamma(1+q)}\frac{M^j\Phi_\zeta^j(1+\upsilon_2)^j}{\Gamma(1+jq)}\int_0^t (t-s)^{q-1}\int_0^s \tau^{jq}\mathrm{d}\tau\mathrm{d}s\|x-y\| \\
\leq&\frac{qM\Phi_\zeta}{\Gamma(1+q)}\frac{M^j\Phi_\zeta^j(1+\upsilon_2)^j}{\Gamma(1+jq)}\int_0^1 (1-z)^{q-1}z^{jq}\mathrm{d}z\cdot t^{(j+1)q}\|x-y\| \\
&+\frac{qM\Phi_\zeta L_a}{\Gamma(1+q)}\frac{M^j\Phi_\zeta^j(1+\upsilon_2)^j}{\Gamma(1+jq)}\frac{1}{1+jq}\int_0^1 (1-z)^{q-1}z^{jq+1}\mathrm{d}z\cdot t^{(j+1)q+1}
\end{aligned}
$$

$$\begin{aligned}
&\le \frac{qM\Phi_\zeta}{\Gamma(1+q)} \frac{M^j\Phi_\zeta^j(1+\upsilon_2)^j}{\Gamma(1+jq)} \frac{\Gamma(q)\Gamma(1+jq)}{\Gamma(1+(j+1)q)} t^{(j+1)q}\|x-y\| \\
&\quad + \frac{qM\Phi_\zeta\upsilon_2}{\Gamma(1+q)} \frac{M^j\Phi_\zeta^j(1+\upsilon_2)^j}{\Gamma(1+jq)} \frac{\Gamma(q)\Gamma(2+jq)}{\Gamma(2+(j+1)q)} \frac{1}{1+jq} t^{(j+1)q}\|x-y\| \\
&\le \frac{M^{j+1}\Phi_\zeta^{j+1}(1+\upsilon_2)^j}{\Gamma(1+(j+1)q)} t^{(j+1)q}\|x-y\| + \frac{1}{1+jq} \frac{M^{j+1}\Phi_\zeta^{j+1}\upsilon_2(1+\upsilon_2)^j}{\Gamma(1+(j+1)q)} t^{(j+1)q} \\
&\quad \times \|x-y\| \\
&\le \frac{M^{j+1}\Phi_\zeta^{j+1}(1+\upsilon_2)^{j+1}t^{(j+1)q}}{\Gamma(1+(j+1)q)}\|x-y\|.
\end{aligned}$$

Through mathematical induction, for any $n = 1, 2, \cdots$, one has

$$\begin{aligned}
\|(\Gamma^n x)(t) - (\Gamma^n y)(t)\| &\le \frac{M^n\Phi_\zeta^n(1+\upsilon_2)^n t^{nq}}{\Gamma(1+nq)}\|x-y\| \\
&\le \frac{M^n\Phi_\zeta^n(1+\upsilon_2)^n T^{nq}}{\Gamma(1+nq)}\|x-y\|,
\end{aligned}$$

which implies that Γ^n is a contraction for sufficiently large n. By generalized Banach contraction mapping principle, the operator Γ has a unique fixed point $x \in C_{Lip}([-r,T];E)$, which is the mild solution of the system (1.1)-(1.2). The proof is completed. □

If the function Z in the assumption (H2) satisfies the uniform Lipschitz condition, the similar result holds.

(H2)′ $R_q \in L^1([0,T];\mathcal{L}(F,E))$, $Z \in C([0,T]\times\mathfrak{B}_E\times E;F)$, and there exists a Lipschitz constant $P > 0$ such that

$$\|Z(t,x_1,y_1) - Z(s,x_2,y_2)\| \le P(\mid t-s\mid + \|x_1-x_2\|_{\mathfrak{B}_E} + \|y_1-y_2\|_E),$$

where for any $t, s \in [0,T]$, $x_i \in \mathfrak{B}_E$, $y_i \in E$, $i = 1, 2$.

(H4)′ There exists $\zeta' > 0$ such that

$$\Lambda_1' + \Lambda_2'\zeta'(1+\zeta') \le \zeta',$$

where

$$\begin{aligned}
\Lambda_1' &= [Q_q(\cdot)\phi(0)]_{C_{Lip}([0,T];E)} + \frac{(2+q)MT^qP(1+\upsilon_2+\upsilon_3)}{\Gamma(2+q)}, \\
\Lambda_2' &= \frac{(2+q)MT^qP(1+\upsilon_2)\upsilon_1}{\Gamma(2+q)} \text{ and } \phi(0) \in E.
\end{aligned}$$

Theorem 3.2. *Assume that the conditions (H1), (H2)′, (H3), (H4)′ are true, and $\phi \in C_{Lip}([-r,0];E)$. If $\phi(0) \in D(A^n)$ with $nq \ge 1$, and $Z(0,\phi,0) = 0$, the system (1.1)-(1.2) has a unique mild solution $x \in C_{Lip}([-r,T];E)$.*

Proof. Let

$$K(T;\zeta') = \{x \in C([-r,T];E) : x_0 = \phi, x \in C_{Lip}([-r,T];E), [x]_{C_{Lip}([-r,T];E)} \le \zeta'\},$$

where ζ' is in condition (H4)′, and Γ be defined as in the proof of Theorem 3.1.

Step 1. According to the condition (H2)′ and Lemma 3.3, for any $x \in C_{Lip}([-r,T];E)$, we find

$$Z\left(\cdot, x_{\mu(\cdot,x_{(\cdot)})}, (Gx)(\cdot)\right) \in C_{Lip}([0,T];F),$$

and

$$
\begin{aligned}
&[Z(\cdot, x_{\mu_{(\cdot, x(\cdot))}}, (Gx)(\cdot))]_{C_{Lip}([0,T];F)} \\
\leq &P\left([x]_{C_{Lip}([-r,T];E)}(1+[x]_{C_{Lip}([-r,T];E)})(1+\upsilon_2)\upsilon_1+1+\upsilon_2+\upsilon_3\right).
\end{aligned}
$$

Step 2. To establish $\Gamma(K(T;\zeta')) \subset K(T;\zeta')$. Similar to the proof of Theorem 3.1, from the condition (H4)$'$, we can get that

$$
\begin{aligned}
&[\Gamma x]_{C_{Lip}([0,T];E)} \\
\leq &[Q_q(\cdot)\phi(0)]_{C_{Lip}([0,T];E)} + \frac{(2+q)MT^qP}{\Gamma(2+q)}(\zeta'(1+\zeta')(1+\upsilon_2)\upsilon_1+1+\upsilon_2+\upsilon_3) \\
\leq &\Lambda_1' + \Lambda_2'\zeta'(1+\zeta') \\
\leq &\zeta',
\end{aligned}
$$

where $\Lambda_1' = [Q_q(\cdot)\phi(0)]_{C_{Lip}([0,T];E)} + \frac{(2+q)MT^qP(1+\upsilon_2+\upsilon_3)}{\Gamma(2+q)}$, $\Lambda_2' = \frac{(2+q)MT^qP(1+\upsilon_2)\upsilon_1}{\Gamma(2+q)}$. Therefore, $\Gamma x \in C_{Lip}([-r,T];E)$.

Step 3. The mapping Γ is a contraction on $K(T;\zeta')$ in the same way as the proof of Theorem 3.1.

For any $n = 1, 2, \cdots$, we have

$$
\begin{aligned}
\|(\Gamma^n x)(t) - (\Gamma^n y)(t)\| &\leq \frac{M^nP^n(1+\zeta'\upsilon_1)^n(1+\upsilon_2)^n t^{nq}}{\Gamma(1+nq)}\|x-y\| \\
&\leq \frac{M^nP^n(1+\zeta'\upsilon_1)^n(1+\upsilon_2)^n T^{nq}}{\Gamma(1+nq)}\|x-y\|.
\end{aligned}
$$

Thus, Γ^n is a contraction on $K(T;\zeta')$, as $n \to \infty$. From the generalized Banach contraction mapping principle, we know that the system (1.1)-(1.2) has the mild solution $x \in C_{Lip}([-r,T];E)$. The proof is completed. □

Remark 3.3. In the proof of Theorem 3.1 and Theorem 3.2, the conditions (H4) and (H4)$'$ are satisfied if some conditions are imposed on $\upsilon_1 := [\mu]_{C_{Lip}([0,T]\times\mathfrak{B}_E;[0,T])}$ or T.

Remark 3.4. In [16, 18–22], the existence and uniqueness of local solutions for integer order systems can be obtained by using the Banach contraction mapping principle. In this paper, we use the principle of generalized Banach contraction mapping to solve the existence and uniqueness of global solutions for fractional order systems, rather than local solutions. In fact, we can also prove the existence and uniqueness of local solutions for integer order systems by using the principle of generalized Banach contraction mapping. In some sense, the conditions in [16,18–22] can be optimized.

4. Existence of strict solutions

The remainder of this article focuses on how to obtain the existence of strict solutions for the equations (1.1) and (1.2).

Definition 4.1. The function $x \in C([-r,T];X)$ is called strict solutions of (1.1)-(1.2) if ${}^cD^q x \in C([0,T];X)$, $x\mid_{[0,T]} \in C([0,T];X) \cap C([0,T];D(A))$, $x\mid_{[-r,0]} = \phi$, and x satisfies the equation (1.1).

Now, we introduce the necessary and sufficient conditions for the strict solutions of (1.1)-(1.2).

Lemma 4.1. ([24]) *Let A generates a q-resolvent family $\{Q_q(t) : t \geq 0\}$, $0 < q < 1$, $\phi(0) \in D(A)$ and $Z \in C([0,T] \times \mathfrak{B}_E \times E; F)$. Then the following three conclusions are equivalent:*
(i) for any $0 \leq t \leq T$, the system (1.1)-(1.2) has a strict solution;
(ii) for any $0 \leq t \leq T$, $H(t) \in D(A)$, and $H \in C([0,T]; D(A))$, where

$$H(t) = \int_0^t (t-s)^{q-1} R_q(t-s) Z\left(s, x_{\mu(s,x_s)}, (Gx)(s)\right) \mathrm{d}s;$$

(iii) for any $0 \leq t \leq T$, $Q_q * Z$ is differentiable.

Theorem 4.1. *The assumptions of Theorem 3.1 remain valid and $x \in C_{Lip}([-r,T]; E)$ is the mild solution of (1.1)-(1.2) on $[-r,T]$. Then $x(\cdot)$ is the strict solution of (1.1)-(1.2).*

Proof. According to Lemma 4.1, we need to show that $H(t) \in D(A)$ and $H \in C([0,T]; D(A))$. For $0 \leq t \leq T$, we know

$$\begin{aligned} H(t) = &\int_0^t (t-s)^{q-1} R_q(t-s) \left[Z\left(s, x_{\mu(s,x_s)}, (Gx)(s)\right) - Z\left(t, x_{\mu(t,x_t)}, \right.\right. \\ &\left.\left. (Gx)(t)\right)\right] \mathrm{d}s + \int_0^t (t-s)^{q-1} R_q(t-s) Z\left(t, x_{\mu(t,x_t)}, (Gx)(t)\right) \mathrm{d}s \\ =: &\, m_1(t) + m_2(t). \end{aligned}$$

First, we prove $m_1(t) \in D(A)$. From Lemma 2.1 (iv), Lemma 3.4 and the closedness of A, for $0 \leq t \leq T$, we have

$$\begin{aligned} &\left\| (t-s)^{q-1} A R_q(t-s) \left[Z\left(s, x_{\mu(s,x_s)}, (Gx)(s)\right) - Z\left(t, x_{\mu(t,x_t)}, (Gx)(t)\right)\right] \right\| \\ \leq & q M_1 [Z(\cdot, x_{\mu_{(\cdot, x_{(\cdot)})}}, (Gx)(\cdot))]_{C_{Lip}([0,T];F)} (t-s)^{q-1} (t-s)^{-q} (t-s) \\ \leq & q M_1 [Z(\cdot, x_{\mu_{(\cdot, x_{(\cdot)})}}, (Gx)(\cdot))]_{C_{Lip}([0,T];F)}. \end{aligned}$$

Consequently, the function

$$s \to (t-s)^{q-1} A R_q(t-s) \left[Z\left(s, x_{\mu(s,x_s)}, (Gx)(s)\right) - Z\left(t, x_{\mu(t,x_t)}, (Gx)(t)\right)\right]$$

is integrable on $[0,t]$.

Obviously, the function

$$s \to (t-s)^{q-1} A R_q(t-s) Z\left(t, x_{\mu(t,x_t)}, (Gx)(t)\right)$$

is also integrable, for $0 \leq s \leq t$. This indicates that $H(t) \in D(A)$ for all $t \in [0,T]$.

Next, we prove $AH(t)$ is continuous.

$$\begin{aligned} AH(t) = &\int_0^t (t-s)^{q-1} A R_q(t-s) \left[Z\left(s, x_{\mu(s,x_s)}, (Gx)(s)\right) - Z\left(t, x_{\mu(t,x_t)}, \right.\right. \\ &\left.\left. (Gx)(t)\right)\right] \mathrm{d}s + \int_0^t (t-s)^{q-1} A R_q(t-s) Z\left(t, x_{\mu(t,x_t)}, (Gx)(t)\right) \mathrm{d}s \\ =: &\, I_1(t) + I_2(t). \end{aligned}$$

By the proof of Lemma 3.2, we know

$$I_2(t) = -(Q_q(t) - I)Z\left(t, x_{\mu(t,x_t)}, (Gx)(t)\right).$$

Therefore, from Lemma 2.1 (ii) and (H2), it is easy to know that $I_2(t)$ is continuous, for all $0 \le t \le T$.

Now, we estimate $I_1(t)$. Let $0 < h \le T - t$, one has

$$\begin{aligned}
&\|I_1(t+h) - I_1(t)\| \\
\le&\Big\|\int_{-h}^{t}(t-s)^{q-1}AR_q(t-s)\left[Z\left(s+h, x_{\mu(s+h,x_{s+h})}, (Gx)(s+h)\right) - Z\left(t+h,\right.\right. \\
&\left.\left. x_{\mu(t+h,x_{t+h})}, (Gx)(t+h)\right)\right]\mathrm{d}s - \int_0^t (t-s)^{q-1}AR_q(t-s)\left[Z\left(s, x_{\mu(s,x_s)},\right.\right. \\
&\left.\left.(Gx)(s)\right) - Z\left(t, x_{\mu(t,x_t)}, (Gx)(t)\right)\right]\mathrm{d}s\Big\| \\
\le&\Big\|\int_0^h (t+h-s)^{q-1}AR_q(t+h-s)\left[Z\left(s, x_{\mu(s,x_s)}, (Gx)(s)\right) - Z\left(t+h,\right.\right. \\
&\left.\left. x_{\mu(t+h,x_{t+h})}, (Gx)(t+h)\right)\right]\mathrm{d}s\Big\| + \Big\|\int_0^t (t-s)^{q-1}AR_q(t-s)\left[Z\left(s+h,\right.\right. \\
&\left.\left. x_{\mu(s+h,x_{s+h})}, (Gx)(s+h)\right) - Z\left(s, x_{\mu(s,x_s)}, (Gx)(s)\right)\right]\mathrm{d}s\Big\| + \Big\|\int_0^t (t-s)^{q-1} \\
&AR_q(t-s)\left[Z\left(t+h, x_{\mu(t+h,x_{t+h})}, (Gx)(t+h)\right) - Z\left(t, x_{\mu(t,x_t)}, (Gx)(t)\right)\right]\mathrm{d}s\Big\| \\
=:&k_1^t(h) + k_2^t(h) + k_3^t(h).
\end{aligned}$$

For $k_1^t(h)$, using Lemma 2.1 (iv) and Lemma 3.4, we get

$$\begin{aligned}
&\|k_1^t(h)\| \\
=&\Big\|\int_0^h (t+h-s)^{q-1}AR_q(t+h-s)\left[Z\left(s, x_{\mu(s,x_s)}, (Gx)(s)\right)\right. \\
&\left.-Z\left(t+h, x_{\mu(t+h,x_{t+h})}, (Gx)(t+h)\right)\right]\mathrm{d}s\Big\| \\
\le&\Big\|\int_0^h (t+h-s)^{q-1}AR_q(t+h-s)[Z(\cdot, x_{\mu_{(\cdot,x_{(\cdot)})}}, (Gx)(\cdot))]_{C_{Lip}([0,T];F)} \\
&\times(t+h-s)\mathrm{d}s\Big\| \\
\le& qM_1[Z(\cdot, x_{\mu_{(\cdot,x_{(\cdot)})}}, (Gx)(\cdot))]_{C_{Lip}([0,T];F)}h,
\end{aligned}$$

which implies that $k_1^t(h) \to 0$ as $h \to 0^+$.

Moreover, $k_2^t(h) \to 0$ as $h \to 0^+$ by the following estimate

$$\begin{aligned}
\|k_2^t(h)\| =&\Big\|\int_0^t (t-s)^{q-1}AR_q(t-s)\left[Z\left(s+h, x_{\mu(s+h,x_{s+h})}, (Gx)(s+h)\right)\right. \\
&\left.-Z\left(s, x_{\mu(s,x_s)}, (Gx)(s)\right)\right]\mathrm{d}s\Big\| \\
\le&[Z(\cdot, x_{\mu_{(\cdot,x_{(\cdot)})}}, (Gx)(\cdot))]_{C_{Lip}([0,T];F)}\Big\|\int_0^t (t-s)^{q-1}AR_q(t-s)h\mathrm{d}s\Big\|.
\end{aligned}$$

Similarly, $k_3^t(h) \to 0$ as $h \to 0^+$. Consequently, $AH(t)$ is continuous, which implies that $H \in C([0,T]; D(A))$.

To sum up, $x(\cdot)$ is a strict solution of (1.1)-(1.2). The proof is completed. $\square$

Theorem 4.2. *The assumptions of Theorem 3.2 remain valid and $x \in C_{Lip}([-r,T];E)$ is the mild solution of (1.1)-(1.2) on $[-r,T]$. Then $x(\cdot)$ is the strict solution of (1.1)-(1.2).*

The proof is analogous to that in Theorem 4.1. We omit further details.

5. Application

In this section, we present an example motivated from population dynamics, which indicates how our results can be applied to concrete problems.

We consider

$$\begin{cases} {}^cD^{\frac{1}{2}}\psi(t,\varsigma) + \dfrac{\partial^2}{\partial\varsigma^2}\psi(t,\varsigma) = \displaystyle\int_{-r}^{0} e^{2s}\psi(s+\mu_1(t)\mu_2(\|\psi(t)\|),\varsigma)\mathrm{d}s + \int_0^t \sin(t-s) \\ \qquad\qquad\qquad \times \displaystyle\int_{-r}^{0} e^{2\tau}\left[\psi(\tau+\mu_1(t)\mu_2(\|\psi(t)\|),\varsigma)+1\right]\mathrm{d}\tau\mathrm{d}s, \\ \psi(t,0)=\psi(t,\pi)=0, \\ \psi(\theta,\varsigma)=\phi(\theta,\varsigma), \end{cases} \tag{5.1}$$

where $t \in [0,1]$, $\theta \in [-r,0]$, $\varsigma \in [0,\pi]$, $\mu_i \in C([0,1];[0,1])$, $i=1,2$. Let $\mathfrak{B}_X = C([-r,0];X)$, $\phi \in \mathfrak{B}_X$.

Let $X = L^2[0,\pi]$ and $-A : D(A) \subset X \to X$ be the operator $A\omega = \omega''$ with domain $D(A) := \{\omega \in X : \omega(0)=\omega(\pi)=0, \omega'' \in X\}$. The operator A is given by

$$A\omega = \sum_{n=1}^{\infty} -n^2\langle\omega,\omega_n\rangle\omega_n,$$

where $\omega_n(t) = \sqrt{\frac{2}{\pi}}\sin nt$, $n=1,2,\cdots$, is the orthogonal set of eigenvectors of A. A is the infinitesimal generator of an analytic semigroup $\{T(t) : t \geq 0\}$ on X. For every $t>0$, $\omega \in X$,

$$T(t)\omega = \sum_{n=1}^{\infty} e^{-n^2 t}\langle\omega,\omega_n\rangle\omega_n.$$

Moreover, we hypothesize that $x(t)(\varsigma) = \psi(t,\varsigma)$, $\mu(t,x_t) = \mu_1(t)\mu_2(\|\psi(t)\|)$ and $Z : [0,1]\times\mathfrak{B}_X\times X \to X$ is given by

$$Z(t,\varphi,(G\varphi)(t))(\varsigma) = \int_{-r}^{0} e^{2\theta}\varphi(\theta)(\varsigma)\mathrm{d}\theta + \int_0^t \sin(t-s)\int_{-r}^{0} e^{2\theta}\left[\varphi(\theta)(\varsigma)+1\right]\mathrm{d}\theta\mathrm{d}s.$$

Then, with these settings, the equation (5.1) can be written the abstract form of system (1.1)-(1.2).

Therefore, for $0 \leq t_1 \leq t_2 \leq 1$ and $\varphi, \upsilon \in B_\delta(\phi;\mathfrak{B}_X)$, we have

$$\begin{aligned} &\|Z(t_2,\varphi,(G\varphi)(t_2)) - Z(t_1,\upsilon,(G\upsilon)(t_1))\| \\ =&\left\|\int_{-r}^{0} e^{2\theta}\varphi(\theta)\mathrm{d}\theta + \int_0^{t_2}\sin(t_2-s)\int_{-r}^{0} e^{2\theta}\left[\varphi(\theta)+1\right]\mathrm{d}\theta\mathrm{d}s - \int_{-r}^{0} e^{2\theta}\upsilon(\theta)\mathrm{d}\theta \right.\\ &\left. - \int_0^{t_1}\sin(t_1-s)\int_{-r}^{0} e^{2\theta}\left[\upsilon(\theta)+1\right]\mathrm{d}\theta\mathrm{d}s\right\| \end{aligned}$$

$$
\begin{aligned}
\leq & \frac{1}{2}(1-e^{-2r})\left\|\varphi-\upsilon\right\|_{\mathfrak{B}_X}+\left\|\int_0^{t_2}\sin(t_2-s)\int_{-r}^0 e^{2\theta}\varphi(\theta)\mathrm{d}\theta\mathrm{d}s\right.\\
&\left.-\int_0^{t_2}\sin(t_2-s)\int_{-r}^0 e^{2\theta}\upsilon(\theta)\mathrm{d}\theta\mathrm{d}s\right\|+\left\|\int_0^{t_2}\sin(t_2-s)\int_{-r}^0 e^{2\theta}\upsilon(\theta)\mathrm{d}\theta\mathrm{d}s\right.\\
&\left.-\int_0^{t_1}\sin(t_1-s)\int_{-r}^0 e^{2\theta}\upsilon(\theta)\mathrm{d}\theta\mathrm{d}s\right\|+\frac{(1-e^{-2r})}{2}(\cos t_1-\cos t_2)\\
\leq & (1-e^{-2r})\left\|\varphi-\upsilon\right\|_{\mathfrak{B}_X}+\frac{1}{2}(1-e^{-2r})\left(1+\|\phi\|_{\mathfrak{B}_X}+\delta\right)|t_2-t_1|,
\end{aligned}
$$

which implies that the condition (H2) is satisfied with

$$
P_z(\delta)=\frac{1-e^{-2r}}{2}\left(3+\|\phi\|_{\mathfrak{B}_X}+\delta\right).
$$

In addition, through calculation, we get that condition (H1) is satisfied, where $L_a=\frac{1}{2}(1-e^{-2r})$ and $L_a^*=1$. We can choose the appropriate expressions of $\mu_1(t)$ and $\mu_2(t)$ such that $\mu(t,x_t)\leq t$.

Suppose that $\phi(0)\in D(A^n)$ with $n\geq 2$ and some conditions are imposed on $\upsilon_1:=[\mu_1(\cdot)\mu_2(\|\psi(\cdot)\|)]_{C_{Lip}([0,1]\times\mathfrak{B}_X;[0,1])}$ such that there exists a $\zeta>0$, such that the following inequality holds

$$
\begin{aligned}
&\left(\frac{1-e^{-2r}}{2}\left(3+\|\phi\|_{\mathfrak{B}_X}\right)+\frac{1-e^{-2r}}{2}\zeta(1+\zeta)\upsilon_1\right)\left(\Lambda_1+\Lambda_2\zeta(1+\zeta)\right)\\
&+\left[Q_q(\cdot)\phi(0)\right]_{C_{Lip}([0,1];E)}\\
\leq&\zeta,
\end{aligned}
$$

where

$$
\begin{aligned}
\Lambda_1&=\frac{5M\left(4+(1-e^{-2r})(1+\|\phi\|_{\mathfrak{B}_X})\right)}{3\sqrt{\pi}},\\
\Lambda_2&=\frac{5M\upsilon_1(2+(1-e^{-2r}))}{3\sqrt{\pi}}.
\end{aligned}
$$

Thus, the assumptions in Theorem 3.1 and Theorem 4.1 are satisfied. It follows that there exist unique mild and strict solutions to problem (5.1) defined on $[0,1]$.

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