Journal of Applied Analysis and Computation
Volume 14, Number 2, April 2024, 642–656
Website:http://www.jaac-online.com
DOI:10.11948/20220402

A SINGULAR FRACTIONAL DIFFERENTIAL EQUATION WITH RIESZ-CAPUTO DERIVATIVE*

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Abstract In this paper, we obtain the existence results for positive solutions of a class of boundary value problems for fractional differential equations with Riesz-Caputo derivative by using of the theory of Leray-Schauder degree. The interesting point is the nonlinear term $f(t, u)$ may be singular at $u = 0$. An example is also given to demonstrate the validity of the main result.

Keywords Positive solution, singular boundary value problem, Riesz-Caputo derivative, Leray-Schauder degree.

MSC(2010) 34A08.

1. Introduction

During the past few decades, fractional calculus has attracted the attention of many researchers in different fields [3, 12, 14–16, 18]. For example, in many dynamic processes, even if the factors affecting the process have disappeared, the influence of memory is often persistent, only by applying the fractional derivative can we describe this process accurately. In addition, fractional calculus also is applied in physics, chemistry, mechanics, economics, etc. Most of the work done so far discusses Riemann-Liouville or Caputo derivative, which is one-side fractional operators only reflected the past or future memory effect. In order to describe many processes which started at the past states, also relying on its development in the future, we introduce the Riesz fractional derivative. Some recent applications of this derivative were given in [4, 5, 8, 11, 17, 20, 21, 24], but we can see all the Riesz fractional derivative appeared in these literatures are in the framework of the Riemann-Liouville fractional derivative. In contrast to the Riemann-Liouville fractional derivative, the Caputo fractional derivative was shown to possess a suitable generalization of the extremum principle [13]. So, it makes more sense to study the Riesz-Caputo fractional derivative, but to our knowledge, there are few results have been seen in literature about existence results of fractional differential equation with Riesz-Caputo derivative.

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We now present the basic definitions and Lemmas about Riesz-Caputo derivative so that readers can understand the introduction better with R-C derivatives.

Let $\beta > 0$ and $n-1 < \beta \leq n,\ n \in N$ and $n = [v]$, $[\cdot]$ the ceiling of a number.

Definition 1.1. [9] (Riemann-Liouville fractional integrals) Let f be a continuous function in [0,T]. For $\tau \in [0,T]$, the left Riemann-Liouville fractional integral ${}_0I_\tau^\beta f(\tau)$ and the right Riemann-Liouville fractional integral ${}_\tau I_T^\beta f(\tau)$ of order $\beta > 0$ are given by

$$ {}_0I_\tau^\beta f(\tau) = \frac{1}{\Gamma(\beta)} \int_0^\tau (\tau - s)^{\beta-1} f(s)ds, $$

$$ {}_\tau I_T^\beta f(\tau) = \frac{1}{\Gamma(\beta)} \int_\tau^T (s - \tau)^{\beta-1} f(s)ds. $$

Definition 1.2. [9] (Riesz fractional integral) Let f be a continuous function in [0,T]. For $\tau \in [0,T]$, the Riesz fractional integral ${}_0^RI_T^\beta f(\tau)$ of order $\beta > 0$ is given by

$$ {}_0^RI_T^\beta f(\tau) = \frac{1}{2\Gamma(\beta)} \int_0^T |\tau - s|^{\beta-1} f(s)ds. $$

Remark 1.1. [9] From Definition 1.1 and 1.2, we have

$$ {}_0^RI_T^\beta f(\tau) = \frac{1}{2}({}_0I_\tau^\beta f(\tau) +_\tau I_T^\beta f(\tau)). $$

Definition 1.3. [9] (Fractional derivative in the sense of Caputo) Let f be a continuous function in [0,T]. For $\tau \in [0,T]$, the left Caputo fractional derivative ${}_0^CD_\tau^\beta f(\tau)$ and the right Caputo fractional derivative ${}_\tau^CD_T^\beta f(\tau)$ of order $\beta > 0$ are given by

$$ {}_0^CD_\tau^\beta f(\tau) ={}_0 I_\tau^{n-\beta} D^n f(\tau) = \frac{1}{\Gamma(n-\beta)} \int_0^\tau \frac{f^{(n)}(s)}{(\tau - s)^{\beta+1-n}} ds, $$

$$ {}_\tau^CD_T^\beta f(\tau) ={}_\tau I_T^{n-\beta} (-D)^n f(\tau) = \frac{(-1)^n}{\Gamma(n-\beta)} \int_\tau^T \frac{f^{(n)}(s)}{(s-\tau)^{\beta+1-n}} ds. $$

Definition 1.4. [9] (Fractional derivative in the sense of Riesz-Caputo) Let f be a continuous function in [0,T]. For $\tau \in [0,T]$, the Riesz-Caputo fractional derivative ${}_0^{RC}D_T^\beta f(\tau)$ of order $\beta > 0$ is given by

$$ {}_0^{RC}D_T^\beta f(\tau) ={}_0^R I_\tau^{n-\beta} D^n f(\tau) = \frac{1}{\Gamma(n-\beta)} \int_0^T \frac{f^{(n)}(s)}{|\tau - s|^{\beta+1-n}} ds. $$

Remark 1.2. [9] From Definition 1.3 and 1.4, we have

$$ {}_0^{RC}D_T^\beta f(\tau) = \frac{1}{2}\big({}_0^CD_\tau^\beta f(\tau) + (-1)^n\ {}_\tau^CD_T^\beta f(\tau)\big). $$

Lemma 1.1. [12,19] *If $f(\tau) \in C^n[0,T]$, then*

$$ {}_0I_\tau^\beta\ {}_0^CD_\tau^\beta f(\tau) = f(\tau) - \sum_{l=0}^{n-1} \frac{f^{(l)}(0)}{l!} (\tau - 0)^l $$

and

$$ {}_\tau I_T^\beta\ {}_\tau^CD_T^\beta f(\tau) = (-1)^n \big[f(\tau) - \sum_{l=0}^{n-1} \frac{(-1)^l f^{(l)}(T)}{l!} (T-\tau)^l\big]. $$

From above, thus we have

$$\begin{aligned}&{}_0^RI_T^\beta\,{}_0^{RC}D_T^\beta f(\tau)\\&=\frac{1}{2}\big({}_0I_\tau^\beta\,{}_0^CD_\tau^\beta+{}_\tau I_T^\beta\,{}_0^CD_\tau^\beta\big)f(\tau)+(-1)^n\frac{1}{2}\big({}_0I_\tau^\beta\,{}_\tau^CD_T^\beta+{}_\tau I_T^\beta\,{}_\tau^CD_T^\beta\big)f(\tau)\\&=\frac{1}{2}\big({}_0I_\tau^\beta\,{}_0^CD_\tau^\beta+(-1)^n_\tau I_T^\beta\,{}_\tau^CD_T^\beta\big)f(\tau).\end{aligned}$$

In particular, if $0<\beta\le 1$ and $f(\tau)\in C^1[0,T]$, then

$$ {}_0I_T^\beta\,{}_0^{RC}D_T^\beta f(\tau)=f(\tau)-\frac{1}{2}(f(0)+f(T)). \tag{1.1}$$

Chen et al. [6] discussed a class of boundary value problems for fractional differential equations with the Riesz-Caputo derivative

$$\begin{aligned}&{}_0^{RC}D_T^\gamma y(\tau)=g(\tau,y(\tau)),\quad \tau\in[0,T],\quad 0<\gamma\le 1,\\&y(0)=y_0,\quad y(T)=y_T,\end{aligned}$$

where ${}_0^{RC}D_T^\gamma$ is the Riesz-Caputo derivative. By means of a new fractional Gronwall inequalities and some fixed point theorems, the authors obtained some existence results of the above problems.

Chen et al. [7] studied the anti-periodic fractional boundary value problems with Riesz-Caputo derivative

$$\begin{aligned}&{}_0^{RC}D_T^\gamma y(\tau)=g(\tau,y(\tau)),\quad \tau\in J,\quad J=[0,T],\quad 1<\gamma\le 2,\\&y(0)+y(T)=0,\quad y'(0)+y'(T)=0,\end{aligned}$$

where ${}_0^{RC}D_T^\gamma$ is the Riesz-Caputo derivative.

Gu et al. [10] presented the existence results for a class of fractional differential equations with the Riesz-Caputo derivative

$$\begin{aligned}&{}_0^{RC}D_1^\alpha x(\xi)=h(\xi,x(\xi)),\quad \xi\in[0,1],\quad 0<\alpha\le 1,\\&x(0)=x_0,\quad x(1)=x_1,\end{aligned}$$

where ${}_0^{RC}D_1^\alpha$ is the Riesz-Caputo derivative. By use of Leray-Schauder and Krasnosel'skii fixed point theorems, the authors obtained the positive solutions for the above problems.

However, as far as the authors know, there are few papers on the existence of solutions for fractional differential equation with Riesz-Caputo derivative and no work has been reported on the singular Riesz-Caputo fractional equation. Thus, motivated by the above documents, this paper will pay attention to the following singular four-point fractional boundary value problems

$$ {}_0^{RC}D_1^\gamma y(t)=f(t,y(t)),\quad t\in[0,1],\quad 0<\gamma\le 1, \tag{1.2}$$

$$ y(0)=ay(\xi),\quad y(1)=by(\eta), \tag{1.3}$$

where ${}_0^{RC}D_1^\gamma$ is the Riesz-Caputo derivative.

Now we list some conditions for convenience.

(H_1) $a\ge 0,\ b\ge 0,\ a+b<2,\ 0<\xi,\eta<1$;

(H_2) $f:[0,1]\times(0,\infty)\to[0,\infty)$ is continuous, i.e. $f(t,u)$ is singular at $u=0$.

The work presented in this paper has the following new features. First, different from [6, 7, 10],the nonlinear term $f(t,u)$ may be singular at $u=0$ in this paper. Second, the boundary condition (1.3) is a generalization of document [6, 7, 10].

2. The preliminary lemmas

Lemma 2.1. *Let $h \in C([0,1],R), \delta$ is a fixed positive constant. A function $y \in C^1[0,1]$ is a solution of the fractional boundary value problem*

$$
\begin{aligned}
&{}_0^{RC}D_1^{\gamma} y(t) = h(t), \quad t \in [0,1], \quad 0 < \gamma \le 1, && (2.1)\\
&y(0) - ay(\xi) = \delta, \quad y(1) - by(\eta) = \delta, && (2.2)
\end{aligned}
$$

if and only if $y(t)$ is given by

$$
\begin{aligned}
y(t) = &\frac{2\delta}{2-a-b} + \frac{a}{(2-a-b)\Gamma(\gamma)} \int_0^{\xi} (\xi - u)^{\gamma-1} h(u)du \\
&+ \frac{a}{(2-a-b)\Gamma(\gamma)} \int_{\xi}^{1} (u-\xi)^{\gamma-1} h(u)du \\
&+ \frac{b}{(2-a-b)\Gamma(\gamma)} \int_0^{\eta} (\eta - u)^{\gamma-1} h(u)du \\
&+ \frac{b}{(2-a-b)\Gamma(\gamma)} \int_{\eta}^{1} (u-\eta)^{\gamma-1} h(u)du \\
&+ \frac{1}{\Gamma(\gamma)} \int_0^{t} (t-u)^{\gamma-1} h(u)du + \frac{1}{\Gamma(\gamma)} \int_t^1 (u-t)^{\gamma-1} h(u)du.
\end{aligned} \tag{2.3}
$$

Proof. Applying Lemma 2.1 to the equation (2.1), we have

$$
\begin{aligned}
y(t) &= \frac{1}{2}y(0) + \frac{1}{2}y(1) + \frac{1}{\Gamma(\gamma)} \int_0^1 |t-u|^{\gamma-1} h(u)du \\
&= \frac{1}{2}y(0) + \frac{1}{2}y(1) + \frac{1}{\Gamma(\gamma)} \int_0^t (t-u)^{\gamma-1} h(u)du + \frac{1}{\Gamma(\gamma)} \int_t^1 (u-t)^{\gamma-1} h(u)du.
\end{aligned} \tag{2.4}
$$

It follows that

$$
y(\xi) = \frac{1}{2}y(0) + \frac{1}{2}y(1) + \frac{1}{\Gamma(\gamma)} \int_0^{\xi} (\xi-u)^{\gamma-1} h(u)du + \frac{1}{\Gamma(\gamma)} \int_{\xi}^1 (u-\xi)^{\gamma-1} h(u)du,
$$

$$
y(\eta) = \frac{1}{2}y(0) + \frac{1}{2}y(1) + \frac{1}{\Gamma(\gamma)} \int_0^{\eta} (\eta-u)^{\gamma-1} h(u)du + \frac{1}{\Gamma(\gamma)} \int_{\eta}^1 (u-\eta)^{\gamma-1} h(u)du.
$$

Considering the boundary condition (2.2), we obtain

$$
y(0) = \delta + \frac{a}{2}y(0) + \frac{a}{2}y(1) + \frac{a}{\Gamma(\gamma)} \int_0^{\xi} (\xi-u)^{\gamma-1} h(u)du + \frac{a}{\Gamma(\gamma)} \int_{\xi}^1 (u-\xi)^{\gamma-1} h(u)du, \tag{2.5}
$$

$$
y(1) = \delta + \frac{b}{2}y(0) + \frac{b}{2}y(1) + \frac{b}{\Gamma(\gamma)} \int_0^{\eta} (\eta-u)^{\gamma-1} h(u)du + \frac{b}{\Gamma(\gamma)} \int_{\eta}^1 (u-\eta)^{\gamma-1} h(u)du. \tag{2.6}
$$

By (2.5), we get

$$
\begin{aligned}
y(0) = &\frac{\delta}{1-\frac{a}{2}} + \frac{a}{2-a}y(1) + \frac{a}{(1-\frac{a}{2})\Gamma(\gamma)} \int_0^{\xi} (\xi-u)^{\gamma-1} h(u)du \\
&+ \frac{a}{(1-\frac{a}{2})\Gamma(\gamma)} \int_{\xi}^1 (u-\xi)^{\gamma-1} h(u)du.
\end{aligned} \tag{2.7}
$$

From (2.6), (2.7), we have

$$\begin{aligned}(1-\tfrac{b}{2})y(1)=\delta+\tfrac{b}{2}\Big[&\frac{\delta}{1-\frac{a}{2}}+\frac{a}{2-a}y(1)+\frac{a}{(1-\frac{a}{2})\Gamma(\gamma)}\int_0^{\xi}(\xi-u)^{\gamma-1}h(u)du\\&+\frac{a}{(1-\frac{a}{2})\Gamma(\gamma)}\int_{\xi}^1(u-\xi)^{\gamma-1}h(u)du\Big]\\&+\frac{b}{\Gamma(\gamma)}\int_0^{\eta}(\eta-u)^{\gamma-1}h(u)du+\frac{b}{\Gamma(\gamma)}\int_{\eta}^1(u-\eta)^{\gamma-1}h(u)du.\end{aligned}\tag{2.8}$$

Therefore,

$$\begin{aligned}y(1)=&\frac{\delta(2-a)}{2-a-b}+\frac{b\delta}{2-a-b}+\frac{ab}{(2-a-b)\Gamma(\gamma)}\int_0^{\xi}(\xi-u)^{\gamma-1}h(u)du\\&+\frac{ab}{(2-a-b)\Gamma(\gamma)}\int_{\xi}^1(u-\xi)^{\gamma-1}h(u)du\\&+\frac{b(2-a)}{(2-a-b)\Gamma(\gamma)}\int_0^{\eta}(\eta-u)^{\gamma-1}h(u)du\\&+\frac{b(2-a)}{(2-a-b)\Gamma(\gamma)}\int_{\eta}^1(u-\eta)^{\gamma-1}h(u)du,\end{aligned}\tag{2.9}$$

$$\begin{aligned}y(0)=&\frac{\delta}{1-\frac{a}{2}}+\frac{a\delta}{2-a-b}+\frac{ab\delta}{(2-a)(2-a-b)}\\&+\frac{a^2b+2a(2-a-b)}{(2-a)(2-a-b)\Gamma(\gamma)}\int_0^{\xi}(\xi-u)^{\gamma-1}h(u)du\\&+\frac{a^2b+2a(2-a-b)}{(2-a)(2-a-b)\Gamma(\gamma)}\int_{\xi}^1(u-\xi)^{\gamma-1}h(u)du\\&+\frac{ab}{(2-a-b)\Gamma(\gamma)}\int_0^{\eta}(\eta-u)^{\gamma-1}h(u)du\\&+\frac{ab}{(2-a-b)\Gamma(\gamma)}\int_{\eta}^1(u-\eta)^{\gamma-1}h(u)du.\end{aligned}\tag{2.10}$$

From (2.4), (2.9), (2.10), we complete the proof. □

3. Positive solutions of the singular problem (1.2), (1.3)

Let the space $X=C[0,1]$ be endowed with the maximum norm $\|y\|=\max_{0\le t\le 1}|y(t)|$. It is well known that X is a Banach space.

Let $F:[0,1]\times R\to[0,+\infty)$ is continuous.

We now transform the following fractional boundary value problem

$$ {}_0^{RC}D_1^{\gamma}y(t)=F(t,y(t)),\quad t\in[0,1],\quad 0<\gamma\le 1, \tag{3.1}$$
$$ y(0)-ay(\xi)=\delta,\quad y(1)-by(\eta)=\delta, \tag{3.2}$$

into a fixed point problem. Define an integral operator $T : X \to X$ by

$$
\begin{aligned}
Ty(t) = &\frac{2\delta}{2-a-b} + \frac{a}{(2-a-b)\Gamma(\gamma)}\Bigg[\int_0^\xi (\xi-u)^{\gamma-1}F(u,y(u))du \\
&+ \textstyle\int_\xi^1 (u-\xi)^{\gamma-1}F(u,y(u))du\Bigg] \\
&+ \frac{b}{(2-a-b)\Gamma(\gamma)}\Bigg[\int_0^\eta (\eta-u)^{\gamma-1}F(u,y(u))du \\
&+ \int_\eta^1 (u-\eta)^{\gamma-1}F(u,y(u))du\Bigg] \\
&+ \frac{1}{\Gamma(\gamma)}\int_0^t (t-u)^{\gamma-1}F(u,y(u))du + \frac{1}{\Gamma(\gamma)}\int_t^1 (u-t)^{\gamma-1}F(u,y(u))du.
\end{aligned}
\tag{3.3}
$$

Lemma 3.1. *The operator T is a completely continuous operator.*

Proof. Now, for the sake of proving that T is continuous, we have to show that $\|Ty_n - Ty\| \to 0$ as $n \to \infty$,

$$
\begin{aligned}
|Ty_n(t) - Ty(t)| \le &\frac{a}{(2-a-b)\Gamma(\gamma)}\Bigg[\int_0^\xi (\xi-u)^{\gamma-1}|F(u,y_n(u)) - F(u,y(u))|du \\
&+ \int_\xi^1 (u-\xi)^{\gamma-1}|F(u,y_n(u)) - F(u,y(u))|du\Bigg] \\
&+ \frac{b}{(2-a-b)\Gamma(\gamma)}\Bigg[\int_0^\eta (\eta-u)^{\gamma-1}|F(u,y_n(u)) - F(u,y(u))|du \\
&+ \int_\eta^1 (u-\eta)^{\gamma-1}|F(u,y_n(u)) - F(u,y(u))|du\Bigg] \\
&+ \frac{1}{\Gamma(\gamma)}\int_0^t (t-u)^{\gamma-1}|F(u,y_n(u)) - F(u,y(u))|du \\
&+ \frac{1}{\Gamma(\gamma)}\int_t^1 (u-t)^{\gamma-1}|F(u,y_n(u)) - F(u,y(u))|du \\
\le &\left[\frac{a[\xi^\gamma + (1-\xi)^\gamma] + b[\eta^\gamma + (1-\eta)^\gamma] + 2(2-a-b)}{(2-a-b)\Gamma(\gamma+1)}\right] \\
&\|F(\cdot, y_n(\cdot)) - F(\cdot, y(\cdot))\|.
\end{aligned}
$$

Since F is continuous, we have

$$\|Ty_n - Ty\| \to 0 \qquad \text{as} \quad n \to \infty.$$

Now, we prove T is compact. Let $\Omega_R = \{y \in C[0,1] | \|y\| \le R\}$. For any $y \in \Omega_R$, we have $F(u, y(u)) \le \max_{u\in[0,1], Y\in[-R,R]} F(u, Y) =: M$.

Therefore,

$$\|Ty\| = \frac{2\delta}{2-a-b} + \frac{a}{(2-a-b)\Gamma(\gamma)}\Bigg[\int_0^\xi (\xi-u)^{\gamma-1}F(u,y(u))du$$

$$
\begin{aligned}
&+\int_{\xi}^{1}(u-\xi)^{\gamma-1}F(u,y(u))du\Big]\\
&+\frac{b}{(2-a-b)\Gamma(\gamma)}\Big[\int_{0}^{\eta}(\eta-u)^{\gamma-1}F(u,y(u))du\\
&+\int_{\eta}^{1}(u-\eta)^{\gamma-1}F(u,y(u))du\Big]\\
&+\frac{1}{\Gamma(\gamma)}\int_{0}^{t}(t-u)^{\gamma-1}F(u,y(u))du+\frac{1}{\Gamma(\gamma)}\int_{t}^{1}(u-t)^{\gamma-1}F(u,y(u))du\\
\leq&\frac{2\delta}{2-a-b}+\frac{Ma}{(2-a-b)\Gamma(\gamma+1)}(\xi^{r}+(1-\xi)^{r})\\
&+\frac{Mb}{(2-a-b)\Gamma(\gamma+1)}(\eta^{r}+(1-\eta)^{r})+\frac{M}{(\Gamma(\gamma+1)}(t^{r}+(1-t)^{r})\\
\leq&\frac{2\delta}{2-a-b}+\frac{Ma}{(2-a-b)\Gamma(\gamma+1)}(\xi^{r}+(1-\xi)^{r})\\
&+\frac{Mb}{(2-a-b)\Gamma(\gamma+1)}(\eta^{r}+(1-\eta)^{r})+\frac{2M}{(\Gamma(\gamma+1)}.
\end{aligned}
$$

Further, we know for $0\leq t_1\leq t_2\leq 1$,

$$
\begin{aligned}
&\|(Ty)(t_2)-(Ty)(t_1)\|\\
=&\frac{1}{\Gamma(\gamma)}\int_{0}^{t_2}(t_2-u)^{\gamma-1}F(u,y(u))du+\frac{1}{\Gamma(\gamma)}\int_{t_2}^{1}(u-t_2)^{\gamma-1}F(u,y(u))du\\
&-\frac{1}{\Gamma(\gamma)}\int_{0}^{t_1}(t_1-u)^{\gamma-1}F(u,y(u))du-\frac{1}{\Gamma(\gamma)}\int_{t_1}^{1}(u-t_1)^{\gamma-1}F(u,y(u))du\\
=&\frac{1}{\Gamma(\gamma)}\int_{0}^{t_1}(t_2-u)^{\gamma-1}F(u,y(u))du+\frac{1}{\Gamma(\gamma)}\int_{t_1}^{t_2}(t_2-u)^{\gamma-1}F(u,y(u))du\\
&+\frac{1}{\Gamma(\gamma)}\int_{t_2}^{1}(u-t_2)^{\gamma-1}F(u,y(u))du-\frac{1}{\Gamma(\gamma)}\int_{0}^{t_1}(t_1-u)^{\gamma-1}F(u,y(u))du\\
&-\frac{1}{\Gamma(\gamma)}\int_{t_1}^{t_2}(u-t_1)^{\gamma-1}F(u,y(u))du-\frac{1}{\Gamma(\gamma)}\int_{t_2}^{1}(u-t_1)^{\gamma-1}F(u,y(u))du\\
\leq&\frac{M}{\Gamma(\gamma+1)}(t_2^{\gamma}-t_1^{\gamma}+(1-t_2)^{\gamma}-(1-t_1)^{\gamma}).
\end{aligned}
$$

So, we get $\|(Ty)(t_2)-(Ty)(t_1)\|\to 0$ as $t_2\to t_1$, thus, the operator T is equicontinuous. The Arzela-Ascoli theorem guarantees that T is compact. □

Lemma 3.2. *Assume that $(H_1),(H_2)$ hold. If $y(t)$ is a solution of the fractional boundary value problem (3.1), (3.2), then $y(t)\geq\dfrac{2\delta}{2-a-b}$.*

Proof. As we all know, the solution of the problem (3.1), (3.2) is a fixed point of the integral operator T, the conditions $a, b \geq 0$, $a + b < 2$, $F : [0,1] \times R \to [0, +\infty)$, $0 < \xi, \eta < 1$ imply that $y(t) \geq \dfrac{2\delta}{2-a-b}$. □

Lemma 3.3. *Assume that there exists a constant* $M > \dfrac{2\delta}{2-a-b}$ *independent of* λ *such that for* $\lambda \in [0,1]$, $\|y\| \neq M$, *where* $y(t)$ *satisfies*

$$\begin{cases} {}_0^{RC}D_1^{\gamma} y(t) = \lambda F(t, y(t)), \quad t \in [0,1], \quad 0 < \gamma \leq 1, \\ y(0) - ay(\xi) = \delta, \quad y(1) - by(\eta) = \delta. \end{cases} \tag{3.4}$$

Then problem $(3.4)_1$ *has at least one solution* $y(t)$ *with* $\|y\| \leq M$, *where* $(3.4)_1$ *denotes the problem of* (3.4) *when* $\lambda = 1$.

Proof. For any $\lambda \in [0,1]$, define $T_\lambda : X \to X$ by

$$\begin{aligned} T_\lambda y(t) = {} & \frac{2\delta}{2-a-b} + \frac{a}{(2-a-b)\Gamma(\gamma)} \Bigg[\int_0^{\xi} (\xi - u)^{\gamma-1} \lambda F(u, y(u)) du \\ & + \int_{\xi}^{1} (u - \xi)^{\gamma-1} \lambda F(u, y(u)) du \Bigg] \\ & + \frac{b}{(2-a-b)\Gamma(\gamma)} \Bigg[\int_0^{\eta} (\eta - u)^{\gamma-1} \lambda F(u, y(u)) du \\ & + \int_{\eta}^{1} (u - \eta)^{\gamma-1} \lambda F(u, y(u)) du \Bigg] \\ & + \frac{1}{\Gamma(\gamma)} \int_0^t (t-u)^{\gamma-1} \lambda F(u, y(u)) du + \frac{1}{\Gamma(\gamma)} \int_t^1 (u-t)^{\gamma-1} \lambda F(u, y(u)) du. \end{aligned}$$

Hence, the solution $y(t)$ of (3.4) is nothing but the fixed point of T_λ, i.e., $y(t) = T_\lambda y(t)$. From Lemma 3.1, T_λ is completely continuous. Let $\Omega = \{y \in X : \|y\| < M\}$ is an open set in E. Suppose that there exists $y \in \partial\Omega$ such that $T_1 y = y$, then $y(t)$ is a solution of $(3.4)_1$ with $\|y\| \leq M$. So the proof is completed. Otherwise, for any $y \in \partial\Omega$, $T_1 y \neq y$. If $\lambda = 0$, for $y \in \partial\Omega$,

$$(I - T_0)y(t) = y(t) - T_0 y(t) = y(t) - \frac{2\delta}{2-a-b} \neq 0 \quad \text{for} \quad \|y\| = M > \frac{2\delta}{2-a-b}.$$

For $\lambda \in (0,1)$, if the fractional boundary value problem (3.4) has a solution $y(t)$, then we get $\|y\| \neq M$, which is a contradiction to $y \in \partial\Omega$. So, for any $y \in \partial\Omega$ and $\lambda \in [0,1]$, $T_\lambda y \neq y$. By the homotopy invariance of Leray-Schauder degree, we obtain

$$Deg\{I - T_1, \Omega, 0\} = Deg\{I - T_0, \Omega, 0\} = 1.$$

Therefore, we deduce that T_1 has a fixed point $y \in \Omega$ which is a solution $y(t)$ with $\|y\| \leq M$ of the problem $(3.4)_1$. The proof is completed. □

Let

$$\begin{aligned} \Theta(t) = {} & \frac{a}{(2-a-b)\Gamma(\gamma)} \left[\int_0^{\xi} (\xi - u)^{\gamma-1} \omega_M(u) du + \int_{\xi}^{1} (u - \xi)^{\gamma-1} \omega_M(u) du \right] \\ & + \frac{b}{(2-a-b)\Gamma(\gamma)} \left[\int_0^{\eta} (\eta - u)^{\gamma-1} \omega_M(u) du + \int_{\eta}^{1} (u - \eta)^{\gamma-1} \omega_M(u) du \right] \\ & + \frac{1}{\Gamma(\gamma)} \int_0^t (t-u)^{\gamma-1} \omega_M(u) du + \frac{1}{\Gamma(\gamma)} \int_t^1 (u-t)^{\gamma-1} \omega_M(u) du. \end{aligned}$$

Theorem 3.1. *Assume that $(H_1), (H_2)$ hold. In addition, the following conditions hold:*

(H_3) *for each $K > 0$, there exists a function ω_K which is continuous on $[0,1]$ and positive on $(0,1)$ satisfying $f(t,y) \geq \omega_K(t)$ on $(0,1) \times (0,K]$;*

(H_4) *for each $\delta > 0$, there exist nonnegative continuous function $\varphi(t)$ and nonnegative nondecreasing continuous function $\psi(y)$ such that $0 \leq f(t,y) \leq \varphi(t)\psi(y)$ for $(t,y) \in [0,1] \times [\frac{2\delta}{2-a-b}, \infty)$;*

(H_5) *there exists $M > 0$ such that*

$$\left(\frac{a[\xi^\gamma + (1-\xi)^\gamma] + b[\eta^\gamma + (1-\eta)^\gamma] + 2(2-a-b)}{(2-a-b)\Gamma(\gamma+1)}\right)\varphi^*\psi(M) < M,$$

where $\varphi^* = \sup\{\varphi(t) : t \in [0,1]\}$. Then the singular fractional boundary value problem (1.2) (1.3) has a positive solution $y(t)$ with $\|y\| \leq M$.

Proof. From (H_5), we select $M > 0$ and $0 < \varepsilon < M$ such that

$$\left(\frac{a[\xi^\gamma + (1-\xi)^\gamma] + b[\eta^\gamma + (1-\eta)^\gamma] + 2(2-a-b)}{(2-a-b)\Gamma(\gamma+1)}\right)\varphi^*\psi(M) + \varepsilon < M. \tag{3.5}$$

Choose $n_0 \in \{1,2,3,\cdots\}$ to satisfy $\dfrac{2}{n_0(2-a-b)} \leq \varepsilon$, let $N_0 = \{n_0, n_0+1, n_0+2, n_0+3, \cdots\}$.

In the following, we demonstrate the following problem

$$\begin{cases} {}^{RC}_{0}D^{\gamma}_{1} y(t) = f(t, y(t)), \quad t \in [0,1], \quad 0 < \gamma \leq 1, \\ y(0) - ay(\xi) = \frac{1}{m}, \quad y(1) - by(\eta) = \frac{1}{m} \end{cases} \tag{3.6}$$

has a solution for each $m \in N_0$.

For the sake of getting a solution of the problem (3.6) for each $m \in N_0$, we study the following problem

$$\begin{cases} {}^{RC}_{0}D^{\gamma}_{1} y(t) = f^*(t, y(t)), \quad t \in [0,1], \quad 0 < \gamma \leq 1, \\ y(0) - ay(\xi) = \frac{1}{m}, \quad y(1) - by(\eta) = \frac{1}{m}, \end{cases} \tag{3.7}$$

where

$$f^*(t,y) = \begin{cases} f(t,y), & y \geq \dfrac{2}{m(2-a-b)}, \\ f(t, \dfrac{2}{m(2-a-b)}), & y < \dfrac{2}{m(2-a-b)}. \end{cases}$$

Obviously, $f^* \in C([0,1] \times R, [0,+\infty))$. In order to obtain a solution of problem (3.7)for each $m \in N_0$, we discuss the following family of problems

$$\begin{cases} {}^{RC}_{0}D^{\gamma}_{1} y(t) = \lambda f^*(t, y(t)), \quad t \in [0,1], \quad 0 < \gamma \leq 1, \\ y(0) - ay(\xi) = \frac{1}{m}, \quad y(1) - by(\eta) = \frac{1}{m}. \end{cases} \tag{3.8}$$

For $\forall \lambda \in [0,1]$, we claim that any solution $y(t)$ of (3.8) must satisfy $\|y\| \neq M$. Otherwise, for some $\lambda \in [0,1]$, let $y(t)$ be a solution of (3.8) such that $\|y\| = M$. From Lemma 3.2,

$$y(t) \geq \frac{2}{m(2-a-b)} \quad \text{for} \quad t \in [0,1]. \tag{3.9}$$

By (3.9) and (H_4), we have

$$
\begin{aligned}
M =&\|y\| \\
\leq&\frac{2}{m(2-a-b)}+\frac{a}{(2-a-b)\Gamma(\gamma)}\left[\int_0^{\xi}(\xi-u)^{\gamma-1}\lambda f^*(u,y(u))du\right.\\
&\left.+\int_{\xi}^{1}(u-\xi)^{\gamma-1}\lambda f^*(u,y(u))du\right]\\
&+\frac{b}{(2-a-b)\Gamma(\gamma)}\left[\int_0^{\eta}(\eta-u)^{\gamma-1}\lambda f^*(u,y(u))du\right.\\
&\left.+\int_{\eta}^{1}(u-\eta)^{\gamma-1}\lambda f^*(u,y(u))du\right]\\
&+\frac{1}{\Gamma(\gamma)}\int_0^t(t-u)^{\gamma-1}\lambda f^*(u,y(u))du+\frac{1}{\Gamma(\gamma)}\int_t^1(u-t)^{\gamma-1}\lambda f^*(u,y(u))du\\
\leq&\frac{2}{n_0(2-a-b)}+\frac{a}{(2-a-b)\Gamma(\gamma)}\left[\int_0^{\xi}(\xi-u)^{\gamma-1}f(u,y(u))du\right.\\
&\left.+\int_{\xi}^{1}(u-\xi)^{\gamma-1}f(u,y(u))du\right]\\
&+\frac{b}{(2-a-b)\Gamma(\gamma)}\left[\int_0^{\eta}(\eta-u)^{\gamma-1}f(u,y(u))du+\int_{\eta}^{1}(u-\eta)^{\gamma-1}f(u,y(u))du\right]\\
&+\frac{1}{\Gamma(\gamma)}\int_0^t(t-u)^{\gamma-1}f(u,y(u))du+\frac{1}{\Gamma(\gamma)}\int_t^1(u-t)^{\gamma-1}f(u,y(u))du.\\
\leq&\varepsilon+\frac{a}{(2-a-b)\Gamma(\gamma)}\left[\int_0^{\xi}(\xi-u)^{\gamma-1}\varphi(u)\psi(y(u))du\right.\\
&\left.+\int_{\xi}^{1}(u-\xi)^{\gamma-1}\varphi(u)\psi(y(u))du\right]\\
&+\frac{b}{(2-a-b)\Gamma(\gamma)}\left[\int_0^{\eta}(\eta-u)^{\gamma-1}\varphi(u)\psi(y(u))du\right.\\
&\left.+\int_{\eta}^{1}(u-\eta)^{\gamma-1}\varphi(u)\psi(y(u))du\right]\\
&+\frac{1}{\Gamma(\gamma)}\int_0^t(t-u)^{\gamma-1}\varphi(u)\psi(y(u))du+\frac{1}{\Gamma(\gamma)}\int_t^1(u-t)^{\gamma-1}\varphi(u)\psi(y(u))du\\
\leq&\varepsilon+\left(\frac{a}{(2-a-b)\Gamma(\gamma)}\left[\frac{\xi^{\gamma}}{\gamma}+\frac{(1-\xi)^{\gamma}}{\gamma}\right]+\frac{b}{(2-a-b)\Gamma(\gamma)}\left[\frac{\eta^{\gamma}}{\gamma}+\frac{(1-\eta)^{\gamma}}{\gamma}\right]\right.\\
&\left.+\frac{1}{\Gamma(\gamma)}\left[\frac{t^{\gamma}}{\gamma}+\frac{(1-t)^{\gamma}}{\gamma}\right]\right)\varphi^*\psi(\|y\|)
\end{aligned}
$$

$$\leq \varepsilon + \left(\frac{a[\xi^\gamma + (1-\xi)^\gamma] + b[\eta^\gamma + (1-\eta)^\gamma] + 2(2-a-b)}{(2-a-b)\Gamma(\gamma+1)}\right)\varphi^*\psi(M).$$

Thus,

$$M = \|y\| \leq \varepsilon + \left(\frac{a[\xi^\gamma + (1-\xi)^\gamma] + b[\eta^\gamma + (1-\eta)^\gamma] + 2(2-a-b)}{(2-a-b)\Gamma(\gamma+1)}\right)\varphi^*\psi(M) < M.$$

This is a contradiction. So the claim is proved. Thus from Lemma 3.3, we have (3.7) has at least a solution $y^m(t)$ with $\|y^m(t)\| \leq M$ for any fixed m. Lemma 3.2 guarantees $y^m(t) \geq \frac{2}{m(2-a-b)}$, so $f^*(t, y^m(t)) = f(t, y^m(t))$. Consequently, $y^m(t)$ is a solution of the fractional boundary value problem (3.6).

Next, we claim that $y^m(t)$ has a uniform sharper lower bound, i.e., there exists a function $\Theta(t)$ which is continuous on $[0,1]$ and positive on $(0,1)$ such that

$$y^m(t) \geq \Theta(t), \qquad t \in [0,1]$$

for all $m \in N_0$. Considering $0 < \frac{2}{m(2-a-b)} \leq y^m(t) \leq M$. (H_3) guarantees there exists a continuous function $\omega_M : (0,1) \to (0, +\infty)$ satisfying

$$f(t, y^m(t)) \geq \omega_M(t), \quad t \in (0,1).$$

So, we have

$$\begin{aligned}
y^m(t) =& \frac{2}{m(2-a-b)} + \frac{a}{(2-a-b)\Gamma(\gamma)}\left[\int_0^\xi (\xi-u)^{\gamma-1} f(u, y^m(u))du\right.\\
&\left. + \int_\xi^1 (u-\xi)^{\gamma-1} f(u, y^m(u))du\right]\\
&+ \frac{b}{(2-a-b)\Gamma(\gamma)}\left[\int_0^\eta (\eta-u)^{\gamma-1} f(u, y^m(u))du\right.\\
&\left. + \int_\eta^1 (u-\eta)^{\gamma-1} f(u, y^m(u))du\right]\\
&+ \frac{1}{\Gamma(\gamma)}\int_0^t (t-u)^{\gamma-1} f(u, y^m(u))du + \frac{1}{\Gamma(\gamma)}\int_t^1 (u-t)^{\gamma-1} f(u, y^m(u)))du\\
\geq& \frac{a}{(2-a-b)\Gamma(\gamma)}\left[\int_0^\xi (\xi-u)^{\gamma-1}\omega_M(u)du + \int_\xi^1 (u-\xi)^{\gamma-1}\omega_M(u)du\right]\\
&+ \frac{b}{(2-a-b)\Gamma(\gamma)}\left[\int_0^\eta (\eta-u)^{\gamma-1}\omega_M(u)du + \int_\eta^1 (u-\eta)^{\gamma-1}\omega_M(u)du\right]\\
&+ \frac{1}{\Gamma(\gamma)}\int_0^t (t-u)^{\gamma-1}\omega_M(u)du + \frac{1}{\Gamma(\gamma)}\int_t^1 (u-t)^{\gamma-1}\omega_M(u)du.
\end{aligned}$$

Thus, we have for any $m \in N_0$,

$$y^m(t) \geq \Theta(t), \qquad t \in [0,1].$$

For $t, \tau \in [0,1], \tau < t$, we can get

$$
\begin{aligned}
&|y^m(t) - y^m(\tau)| \\
=&\Big|\frac{1}{\Gamma(\gamma)}\int_0^t (t-u)^{\gamma-1} f(u, y^m(u))du + \frac{1}{\Gamma(\gamma)}\int_t^1 (u-t)^{\gamma-1} f(u, y^m(u))du \\
&- \frac{1}{\Gamma(\gamma)}\int_0^\tau (\tau-u)^{\gamma-1} f(u, y^m(u))du - \frac{1}{\Gamma(\gamma)}\int_\tau^1 (u-\tau)^{\gamma-1} f(u, y^m(u))du\Big| \\
\leq&\Big|\frac{1}{\Gamma(\gamma)}\int_0^t (t-u)^{\gamma-1} f(u, y^m(u))du - \frac{1}{\Gamma(\gamma)}\int_0^\tau (\tau-u)^{\gamma-1} f(u, y^m(u))du\Big| \\
&+ \Big|\frac{1}{\Gamma(\gamma)}\int_t^1 (u-t)^{\gamma-1} f(u, y^m(u)))du - \frac{1}{\Gamma(\gamma)}\int_\tau^1 (u-\tau)^{\gamma-1} f(u, y^m(u))du\Big| \\
=&\Big|\frac{1}{\Gamma(\gamma)}\int_0^\tau (t-u)^{\gamma-1} f(u, y^m(u))du + \frac{1}{\Gamma(\gamma)}\int_\tau^t (t-u)^{\gamma-1} f(u, y^m(u))du \\
&- \frac{1}{\Gamma(\gamma)}\int_0^\tau (\tau-u)^{\gamma-1} f(u, y^m(u))du\Big| \\
&+ \Big|\frac{1}{\Gamma(\gamma)}\int_t^1 (u-t)^{\gamma-1} f(u, y^m(u)))du - \frac{1}{\Gamma(\gamma)}\int_\tau^t (u-\tau)^{\gamma-1} f(u, y^m(u))du \\
&- \frac{1}{\Gamma(\gamma)}\int_t^1 (u-\tau)^{\gamma-1} f(u, y^m(u)))du\Big| \\
\leq&\Big|\frac{1}{\Gamma(\gamma)}\int_0^\tau \Big[(t-u)^{\gamma-1} - (\tau-u)^{\gamma-1}\Big] f(u, y^m(u))du\Big| \\
&+ \Big|\frac{1}{\Gamma(\gamma)}\int_\tau^t (t-u)^{\gamma-1} f(u, y^m(u))du\Big| \\
&+ \Big|\frac{1}{\Gamma(\gamma)}\int_t^1 \Big[(u-t)^{\gamma-1} - (u-\tau)^{\gamma-1}\Big] f(u, y^m(u))du\Big| \\
&+ \Big|\frac{1}{\Gamma(\gamma)}\int_\tau^t (u-\tau)^{\gamma-1} f(u, y^m(u))du\Big| \\
\leq&\frac{1}{\Gamma(\gamma)}\varphi^*\psi(M)\Big|\frac{t^\gamma - (t-\tau)^\gamma - \tau^\gamma}{\gamma}\Big| + \frac{1}{\Gamma(\gamma)}\varphi^*\psi(M)\Big|\frac{(t-\tau)^\gamma}{\gamma}\Big| \\
&+ \frac{1}{\Gamma(\gamma)}\varphi^*\psi(M)\Big|\frac{(1-t)^\gamma - (1-\tau)^\gamma + (t-\tau)^\gamma}{\gamma}\Big| + \frac{1}{\Gamma(\gamma)}\varphi^*\psi(M)\Big|\frac{(t-\tau)^\gamma}{\gamma}\Big|.
\end{aligned}
$$

Therefore,

$$|y^m(t) - y^m(\tau)| \to 0, \qquad |t-\tau| \to 0.$$

So, we have $\{y^m(t)\}_{m\in N_0}$ is equicontinuous on $[0,1]$. On the other hand, $0 < y^m(t) \leq M$ implies that $\{y^m(t)\}_{m\in N_0}$ is uniformly bounded on $[0,1]$. Using the Arzela-Ascoli Theorem, there is a subsequence $N_1 \subset N_0$ and a function $y(t)$, besides $\{y^m(t)\}_{m\in N_1}$ converges uniformly on $[0,1]$ to $y(t)$. For

$$\begin{aligned} y^m(t) =& \frac{2}{m(2-a-b)} + \frac{a}{(2-a-b)\Gamma(\gamma)}\left[\int_0^{\xi}(\xi-u)^{\gamma-1}f(u,y^m(u))du \right.\\ &\left. + \int_{\xi}^{1}(u-\xi)^{\gamma-1}f(u,y^m(u))du\right] \\ &+ \frac{b}{(2-a-b)\Gamma(\gamma)}\left[\int_0^{\eta}(\eta-u)^{\gamma-1}f(u,y^m(u))du \right.\\ &\left. + \int_{\eta}^{1}(u-\eta)^{\gamma-1}f(u,y^m(u))du\right] \\ &+ \frac{1}{\Gamma(\gamma)}\int_0^t (t-u)^{\gamma-1}f(u,y^m(u))du + \frac{1}{\Gamma(\gamma)}\int_t^1 (u-t)^{\gamma-1}f(u,y^m(u)))du, \end{aligned} \tag{3.10}$$

let $m \to +\infty$ in (3.10), considering the Lebegue's dominated convergence theorem, we get

$$\begin{aligned} y(t) =& \frac{a}{(2-a-b)\Gamma(\gamma)}\left[\int_0^{\xi}(\xi-u)^{\gamma-1}f(u,y(u))du + \int_{\xi}^{1}(u-\xi)^{\gamma-1}f(u,y(u))du\right] \\ &+\frac{b}{(2-a-b)\Gamma(\gamma)}\left[\int_0^{\eta}(\eta-u)^{\gamma-1}f(u,y(u))du + \int_{\eta}^{1}(u-\eta)^{\gamma-1}f(u,y(u))du\right] \\ &+\frac{1}{\Gamma(\gamma)}\int_0^t (t-u)^{\gamma-1}f(u,y(u))du + \frac{1}{\Gamma(\gamma)}\int_t^1 (u-t)^{\gamma-1}f(u,y(u)))du, \end{aligned}$$

which implies that

$$\begin{aligned} &{}_0^{RC}D_1^{\gamma}y(t) = f(t,y(t)), \quad t\in[0,1], \quad 0<\gamma\le 1, \\ &y(0) = ay(\xi), \quad y(1) = by(\eta), \end{aligned}$$

this combing with $0 < \|y\| \le M$ implies that $y(t)$ is a positive solution of the singular fractional boundary value problem with Riesz-Caputo derivative (1.2),(1.3). □

4. Example

Example 4.1. Consider the following BVP:

$$ {}_0^{RC}D_1^{\frac{1}{2}}y(t) = f(t,y(t)), \quad t\in[0,1], \tag{4.1}$$

$$ y(0) = y(\frac{1}{4}), \quad y(1) = \frac{1}{2}y(\frac{1}{2}), \tag{4.2}$$

where

$$f(t,y) = \frac{y^{\frac{1}{3}}}{(1+e^t)y}.$$

Then the fractional boundary value problem (4.1), (4.2) has at least one positive solution.

Proof. Here $a = 1 \geq 0, b = \frac{1}{2} \geq 0, a + b = \frac{3}{2} < 2, \xi = \frac{1}{4}, \eta = \frac{1}{2}, \gamma = \frac{1}{2}$.
(H_3) for each positive constant K,

$$f(t, y) = \frac{y^{\frac{1}{3}}}{(1 + e^t)y} \geq \frac{1}{(1 + e^t)K^{\frac{2}{3}}} = \omega_K(t) \text{ on } (0, 1) \times (0, K];$$

(H_4) for each positive constant δ, let $\varphi(t) = \dfrac{1}{1 + e^t}$, $\psi(y) = y^{\frac{1}{3}\frac{2-a-b}{2\delta}}$, Obviously, $0 < f(t, y) \leq \varphi(t)\psi(y)$, for $(t, y) \in [0, 1] \times [\frac{2\delta}{2-a-b}, \infty), \varphi^* = \frac{1}{2}$, $\psi(M) = M^{\frac{1}{3}\frac{2-a-b}{2\delta}}$.
(H_5) $\Gamma(\gamma + 1) = \Gamma(\frac{3}{2}) \approx 0.886$, $\xi^\gamma = (\frac{1}{4})^{\frac{1}{2}} = 0.5$, $(1 - \xi)^\gamma = (\frac{3}{4})^{\frac{1}{2}} = \frac{1.732}{2}$, $\eta^\gamma = (\frac{1}{2})^{\frac{1}{2}} = 0.707$, $(1 - \eta)^\gamma = (\frac{1}{2})^{\frac{1}{2}} = 0.707$, thus

$$\frac{a[\xi^\gamma + (1 - \xi)^\gamma] + b[\eta^\gamma + (1 - \eta)^\gamma] + 2(2 - a - b)}{(2 - a - b)\Gamma(\gamma + 1)} \approx 6.937,$$

for each positive constant δ, we can always choose the appropriate $M > 0$ to satisfy $\frac{6.937}{2}\psi(M) < M$. According to Theorem 3.1, BVP (4.1), (4.2) has at least one positive solution on $[0, 1]$. □

Acknowledgements

This work is supported by the Natural Science Foundation of Tianjin (No. (19JCYBJC30700)).

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