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PARTIAL PERMANENCE AND STATIONARY DISTRIBUTION OF A DELAYED STOCHASTIC FACULTATIVE MUTUALISM MODEL WITH FEEDBACK CONTROLS*

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Abstract This paper characterizes a facultative mutualism model with feedback controls by using delayed stochastic differential equations, in which each interspecific mutualism term contains saturation effects and distributed delays with strong kernels. Firstly, we transform the stochastic facultative mutualism model with strong kernels into an equivalent eight-dimensional stochastic model by a linear chain technique. After that, sufficient criteria for partial permanence of both species and the existence of a unique stationary distribution are established, respectively. Finally, illustrative examples and corresponding numerical simulations are carried out to support our theoretical results.

Keywords Delayed stochastic mutualism model, feedback controls, partial permanence, stationary distribution.

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1. Introduction

As a ubiquitous phenomenon in nature, mutualism is a biological interaction between two/many species that benefits both/each other [35]. Practical examples of mutualism are various, including associations between pollinators and flowering plants [16, 42], seed dispersers and plants [33, 34], sea anemones and anemone fishes [30, 37], sphagnum and cyanobacteria [2, 5]. By degree of dependence between species, mutualism may be classified as obligate or facultative, where a facultative mutualist is one which benefits in some way from the interactions with another species but can also survive on its own [36]. In the past few decades, starting from the classical Lotka-Volterra models [27, 43], many population models have been proposed to describe facultative mutualist interactions [26, 31, 32, 40]. Note that Qi et al. [38] recently introduced the following two-species facultative mutualism model with saturation effects which was motivated from a corresponding competi-

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tive model proposed by Gopalsamy [11]

$$\begin{cases} dz_1(t) = z_1(t)[r_1 - \mu_1 z_1(t) + \dfrac{\alpha_1 z_2(t)}{1+z_2(t)}]dt, \\ dz_2(t) = z_2(t)[r_2 - \mu_2 z_2(t) + \dfrac{\alpha_2 z_1(t)}{1+z_1(t)}]dt, \end{cases} \tag{1.1}$$

where the biological significance of the variable $z_i(t)$ and the parameters μ_i, r_i, α_i can be found in Table 1, and all parameters are positive constants. The nonlinear term $\alpha_1 z_1/(1+z_1)$ (or $\alpha_2 z_2/(1+z_2)$) has a saturation value α_1 (or α_2) for sufficiently large z_1 (or z_2), because in the real world, finite resources lead to the fact that as the increase of one cooperator's density, its cooperative capacity does not tend to infinity and may be upper-bounded (see [22, 38]).

Table 1. The biological significance of variables and parameters of models (1.1) and (1.2).

Natations	Biological meanings
$z_i(t)$	The number of individual species i at time t $(i = 1, 2)$
r_i	The intrinsic growth rates
μ_i	The intraspecific competition rates
α_i	The interspecific mutualism rates
$u_i(t)$	The 'indirect control' variables
k_i	The suppression rates of control variables u_i to species z_i
m_i	The inhibition rates of control variables u_i
n_i	The controllable rates

It is worth noting that many species are at risk of extinction due to over-fishing by humans. One of the important issues facing humans is how to regulate the ecosystem rationally and protect the endangered species to ensure the sustainable development of the ecosystem. Xiao et al. [44] showed that searching for certain schemes (such as harvesting or culling procedures) to save species from extinction would allow species to reach a desired state. To this end, feedback control variables (or 'indirect control' variables [1,21]) are introduced into biomathematical modeling to describe such certain schemes [12, 13, 44]. Later, this novel idea has been further investigated in some more complex mutualism models [7, 14, 46]. Obviously, with the idea of feedback controls, we can establish a new model based on model (1.1)

$$\begin{cases} dz_1(t) = z_1(t)[r_1 - \mu_1 z_1(t) + \dfrac{\alpha_1 z_2(t)}{1+z_2(t)} - k_1 u_1(t)]dt, \\ dz_2(t) = z_2(t)[r_2 - \mu_2 z_2(t) + \dfrac{\alpha_2 z_1(t)}{1+z_1(t)} - k_2 u_2(t)]dt, \\ du_1(t) = (-m_1 u_1(t) + n_1 z_1(t))dt, \\ du_2(t) = (-m_2 u_2(t) + n_2 z_2(t))dt, \end{cases} \tag{1.2}$$

where the biological significance of $u_i(t), k_i, m_i$ and n_i is listed in Table 1.

Furthermore, for a realistic situation, the present state of species may be affected by the cumulative effects of past history. Hence, it is natural to introduce the distributed delays into biomathematical modeling [6,8,18,20,25]. For the distributed delay, one typical form of the kernel can be chosen as a Gamma distribution delay kernel [28]. The mathematical expression of the kernel is governed by $G(t) = \frac{t^h \omega^{h+1} e^{-\omega t}}{h!}$, where ω is a positive constant standing for the decay rate of past memory effect, and h is a nonnegative integer. In particular, the strong kernel case $G(t) = t\omega^2 e^{-\omega t}$ implies that the maximum effect on growth rate response at any time comes from species density at the previous time (see [38,39]). Recalling model (1.2), we incorporate distributed delays with strong kernels into model (1.2) and obtain

$$\begin{cases} dz_1(t) = z_1(t)[r_1 - \mu_1 z_1(t) + \alpha_1 \displaystyle\int_{-\infty}^{t} \frac{(t-s)\omega_2^2 e^{-\omega_2(t-s)} z_2(s)}{1+z_2(s)} ds - k_1 u_1(t)]dt, \\ dz_2(t) = z_2(t)[r_2 - \mu_2 z_2(t) + \alpha_2 \displaystyle\int_{-\infty}^{t} \frac{(t-s)\omega_1^2 e^{-\omega_1(t-s)} z_1(s)}{1+z_1(s)} ds - k_2 u_2(t)]dt, \\ du_1(t) = (-m_1 u_1(t) + n_1 z_1(t))dt, \\ du_2(t) = (-m_2 u_2(t) + n_2 z_2(t))dt. \end{cases} \tag{1.3}$$

However, most species growth phenomena in the real world are not simply deterministic and are often influenced by environmental noises, it is more rational to construct stochastic models than deterministic models that are fully determined by the parameter values and the initial conditions. Since random perturbations are ubiquitous, May [32] pointed out that the growth rates in population models should be stochastic in his big book, and this topic has been extensively developed and studied in some mutualism models [17,23,38,47]. Along with this idea, we adopt the perturbation approach used by [15,38] to introduce two coupling noises, and assume that the intrinsic growth rates of each species in model (1.3) are stochastically perturbed with $r_i \to r_i + \beta_{i1} dW_1(t) + \beta_{i2} dW_2(t)$ $(i = 1, 2)$. Then a delayed stochastic facultative mutualism model with feedback controls is derived as follows

$$\begin{cases} dz_1(t) = z_1(t)[r_1 - \mu_1 z_1(t) + \alpha_1 \displaystyle\int_{-\infty}^{t} \frac{(t-s)\omega_2^2 e^{-\omega_2(t-s)} z_2(s)}{1+z_2(s)} ds - k_1 u_1(t)]dt \\ \qquad + \sum_{i=1}^{2} \beta_{1i} z_1(t) dW_i(t), \\ dz_2(t) = z_2(t)[r_2 - \mu_2 z_2(t) + \alpha_2 \displaystyle\int_{-\infty}^{t} \frac{(t-s)\omega_1^2 e^{-\omega_1(t-s)} z_1(s)}{1+z_1(s)} ds - k_2 u_2(t)]dt \\ \qquad + \sum_{i=1}^{2} \beta_{2i} z_2(t) dW_i(t), \\ du_1(t) = (-m_1 u_1(t) + n_1 z_1(t))dt, \\ du_2(t) = (-m_2 u_2(t) + n_2 z_2(t))dt, \end{cases} \tag{1.4}$$

where $W_i(t)$ $(i = 1, 2)$ are considered to be standard and mutually independent Brownian motions defined on this probability space $(\Omega, \mathcal{F}, \{\mathcal{F}\}_{t\geq 0}, \mathbb{P})$ obeying the usual conditions, and β_{ij}^2 $(i, j = 1, 2)$ are the intensities of white noises.

Assign

$$f_i(t) = \int_{-\infty}^{t} (t-s)\omega_i^2 e^{-\omega_i(t-s)} \frac{z_i(s)}{1+z_i(s)} ds, \quad v_i(t) = \int_{-\infty}^{t} \omega_i e^{-\omega_i(t-s)} \frac{z_i(s)}{1+z_i(s)} ds. \tag{1.5}$$

Next, similar to [10, 28, 38], with the help of chain techniques we can transform the above stochastic model (1.4) with strong kernel delays into a undelayed eight-dimensional model

$$\begin{cases} dz_1(t) = z_1(t)[r_1 - \mu_1 z_1(t) + \alpha_1 f_2(t) - k_1 u_1(t)]dt + \sum_{i=1}^{2} \beta_{1i} z_1(t) dW_i(t), \\ dz_2(t) = z_2(t)[r_2 - \mu_2 z_2(t) + \alpha_2 f_1(t) - k_2 u_2(t)]dt + \sum_{i=1}^{2} \beta_{2i} z_2(t) dW_i(t), \\ df_1(t) = \omega_1(v_1(t) - f_1(t))dt, \\ df_2(t) = \omega_2(v_2(t) - f_2(t))dt, \\ dv_1(t) = \omega_1(\frac{z_1(t)}{1+z_1(t)} - v_1(t))dt, \\ dv_2(t) = \omega_2(\frac{z_2(t)}{1+z_2(t)} - v_2(t))dt, \\ du_1(t) = (-m_1 u_1(t) + n_1 z_1(t))dt, \\ du_2(t) = (-m_2 u_2(t) + n_2 z_2(t))dt. \end{cases} \tag{1.6}$$

Similar to [41], we understand that the relationship between models (1.4) and (1.6) is so-called equivalence from the following explanation: If $(z_1(t), z_2(t), u_1(t), u_2(t)) \in \mathbb{R}_+^4$ is the solution of model (1.4) corresponding to continuous and bounded initial function $(\varphi_1(t), \varphi_2(t), \psi_1(t), \psi_2(t)) : (-\infty, 0] \to \mathbb{R}^4$, then $(z_1(t), z_2(t), f_1(t), f_2(t), v_1(t), v_2(t), u_1(t), u_2(t)) \in \mathbb{R}_+^8$ is a solution of model (1.6) with $z_i(0) = \varphi_i(0)$, $u_i(0) = \psi_i(0)$, and

$$f_i(0) = -\int_{-\infty}^{0} s\omega_i^2 e^{\omega_i s} \frac{\varphi_i(s)}{1+\varphi_i(s)} ds, \ \ v_i(0) = \int_{-\infty}^{0} \omega_i e^{\omega_i s} \frac{\varphi_i(s)}{1+\varphi_i(s)} ds.$$

Conversely, if $(z_1(t), z_2(t), f_1(t), f_2(t), v_1(t), v_2(t), u_1(t), u_2(t))$ is any solution of model (1.6) defined on entire real line and bounded on $(-\infty, 0]$, then $f_i(t)$ and $v_i(t)$ are given by (1.5), and so $(z_1(t), z_2(t), u_1(t), u_2(t))$ satisfies (1.4).

Due to the relationship between the solutions to models (1.4) and (1.6), then we have focus on model (1.6) in the subsequent sections. As a continuation of previous work [38], this paper aims to investigate the role of feedback controls and noise perturbations on partial permanence and stationary distribution. The present investigation is organized as follows. Two necessary lemmas and some notations are provided in Section 2. Sufficient conditions for partial permanence of both species are established in Section 3. Section 4 discusses the existence of a unique stationary distribution. Some numerical examples and their corresponding simulation figures are given in Section 5. A brief discussion section comes to the end of the paper.

2. Fundamental preliminaries

For the convenience of subsequent proofs, we list below some notations and lemmas.

Let $\mathbb{R}_+^d = \{x \in \mathbb{R}^d : x_i > 0, 1 \le i \le d\}$. For the continuous and bounded function $g(t)$ on $[0, +\infty)$, we define $\langle g(t)\rangle = t^{-1}\int_0^t g(s)ds$, $g_* = \liminf_{t\to+\infty} g(t)$, and $g^* = \limsup_{t\to+\infty} g(t)$.

Lemma 2.1 ([38]). *For all $t > 0$, we have $f_i(t) \le 1, v_i(t) \le 1$ and $\lim_{t\to\infty} f_i(t)/t = \lim_{t\to\infty} v_i(t)/t = 0$, $i = 1, 2$.*

Lemma 2.2. *For any initial value $Z(0) = (z_1(0), z_2(0), f_1(0), f_2(0), v_1(0), v_2(0), u_1(0), u_2(0)) \in \mathbb{R}_+^8$, there exists a unique solution $Z(t) = (z_1(t), z_2(t), f_1(t), f_2(t), v_1(t), v_2(t), u_1(t), u_2(t))$ to model (1.6) for all $t \ge 0$ and $Z(t)$ remains in $\mathbb{R}_+^8$ with probability one.*

The proof of Lemma 2.2 is postponed to **Appendix A**.

3. Partial permanence

This section is devoted to establishing partial permanence of both species, and detailed proofs are listed in **Appendix B**.

Assign

$$\delta_i = (\beta_{i1}^2 + \beta_{i2}^2)/2,\ \lambda_i = (m_i\mu_i - k_i n_i)(r_i - \delta_i)/(m_i\mu_i),\ i = 1, 2.$$

Theorem 3.1. *Suppose that $r_2 - \delta_2 + \alpha_2 < 0$, $r_1 - \delta_1 > 0$ and $\lambda_1 > 0$, then species z_2 is exponentially extinct (denoted by* **EE***) while species z_1 is permanent in time average (denoted by* **PTA***) and $\lambda_1/\mu_1 \le \langle z_1(t)\rangle_* \le \langle z_1(t)\rangle^* \le (r_1 - \delta_1)/\mu_1$ a.s.*

Theorem 3.2. *If $r_1 - \delta_1 + \alpha_1 < 0$, $r_2 - \delta_2 > 0$ and $\lambda_2 > 0$, then species z_1 is* **EE** *while species z_2 is* **PTA** *and $\lambda_2/\mu_2 \le \langle z_2(t)\rangle_* \le \langle z_2(t)\rangle^* \le (r_2 - \delta_2)/\mu_2$ a.s.*

4. Stationary distribution

For model (1.6), this section explores the existence of a unique stationary distribution by using the theory of Has'minskill [19]. Three assumptions and preliminary Lemmas 4.1-4.3 are needed later and listed in the following.

Assumption (H_1). $m_1 > n_1$, $m_2 > n_2$.

Assumption (H_2). $\zeta_i = r_i - \alpha_i - k_i - \delta_i > 0,\ i = 1, 2$.

Assumption (H_3). $(\mu_i + \alpha_i - k_i)/\xi > 0$, *where* $\xi = \mu_1\mu_2 + k_1k_2 - \mu_1k_2 - \mu_2k_1 - \alpha_1\alpha_2$.

Consider the following integral equation

$$X(t) = X(t_0) + \int_{t_0}^t b(s, X(s))ds + \sum_{l=1}^k \int_{t_0}^t \varrho_l(s, X(s))dB_l(s). \tag{4.1}$$

Lemma 4.1 ([19]). *Let the vectors $b(s, x), \varrho_1(s, x), \ldots, \varrho_k(s, x)$ be continuous functions of (s, x) and the coefficients of Eq. (4.1) are independent of t, such that the* *following conditions are satisfied on $O_R \in \mathbb{R}^d_+$ for every $R > 0$*

$$\begin{aligned} &|b(s,x_1) - b(s,x_2)| + \sum_{l=1}^{k} |\varrho_l(s,x_1) - \varrho_l(s,x_2)| \le \mathcal{D}|x_1 - x_2|, \\ &|b(s,x)| + \sum_{l=1}^{k} |\varrho_l(s,x)| \le \mathcal{D}(1 + |x|), \end{aligned} \tag{4.2}$$

where $\mathcal{D}$ is a constant. Furthermore, there exists a twice continuously differentiable C^2- function $V(x)$ in $\mathbb{R}^d_+$ satisfying $LV(x) \le -1$ outside some compact set. Then Eq. (4.1) admits a solution, which is a stationary distribution.

Remark 4.1 ([47]). The condition (4.2) of Lemma 4.1 can be replaced by the global existence of the solution to Eq. (4.1) in view of Remark 5 of Xu et al. [45].

Lemma 4.2. *If Assumption (H_1) holds and let $Z(t)$ be a solution to model (1.6) with initial condition $Z(0) > 0$, then there exists a constant $K_q > 0$ such that for any $q > 0$, $\mathbb{E}[z_i^q] \le K_q, \mathbb{E}[f_i^q] \le K_q, \mathbb{E}[v_i^q] \le K_q, \mathbb{E}[u_i^q] \le K_q, i = 1, 2$.*

Lemma 4.3. *Suppose $Z(t) = (z_1(t), z_2(t), f_1(t), f_2(t), v_1(t), v_2(t), u_1(t), u_2(t))$ is a solution of model (1.6) with $Z(0) \in \mathbb{R}^8_+$. Then almost every path $Z(t)$ of model (1.6) is uniformly continuous on $t \ge 0$.*

Theorem 4.1. *If Assumptions (H_1)-(H_3) hold, there exists a positive solution $Z(t)$ of model (1.6) which is a stationary Markov process. Moreover, this solution is globally attractive. That is, model (1.6) has a unique stationary distribution.*

The proofs of Lemmas 4.2-4.3 and Theorem 4.1 are presented in **Appendices C-E.**

5. Illustrative examples and simulations

In the previous sections, we have presented the main results (Theorems 3.1, 3.2 and 4.1) of model (1.6). To further support our analytical results, we will perform numerical simulations by using MATLAB.

Since the present study is not a case study, there is no real data available, and hence the parameters of model (1.6) are estimated data. We first fix the initial value $(z_1(0), z_2(0), f_1(0), f_2(0), v_1(0), v_2(0), u_1(0), u_2(0)) = (0.13, 0.22, 0.16, 0.15, 0.21, 0.12, 0.18, 0.14)$ and partial parameter values (see Table 2). The noise intensities β_{ij}^2 $(i, j = 1, 2)$ and the suppression rates k_i $(i = 1, 2)$ are varied to verify the analytical results.

Table 2. Parameter values used in model (1.6).

Parameters	r_1	r_2	μ_1	μ_2	α_1	α_2	ω_1	ω_2	m_1	m_2	n_1	n_2
Values	0.48	0.59	0.57	0.68	0.15	0.13	0.17	0.27	4.1	4.5	3.2	3.5

Example 5.1. To visually analyze the role of noise intensities β_{ij}^2 and suppression rates k_i (i.e., the suppression intensities of the feedback control variables u_i to

species z_i) on the partial permanence of both species, we will discuss two cases.

Case 1. For species z_1, we choose relatively small suppression rates ($k_1 = 0.08, k_2 = 0.09$) and noise intensities ($\beta_{11}^2 = 0.037^2, \beta_{12}^2 = 0.042^2$) while for species z_2, choose relatively large suppression rates ($k_1 = 0.31, k_2 = 0.29$) and noise intensities ($\beta_{21}^2 = 0.84^2, \beta_{22}^2 = 0.87^2$). A calculation shows that $r_1 - \delta_1 = 0.4784 > 0, \lambda_1 = 0.426 > 0$ and $r_2 - \delta_2 + \alpha_2 = -0.0112 < 0$. It follows from Theorem 3.1 that species z_1 is PTA while z_2 is EE (see Figure 1).

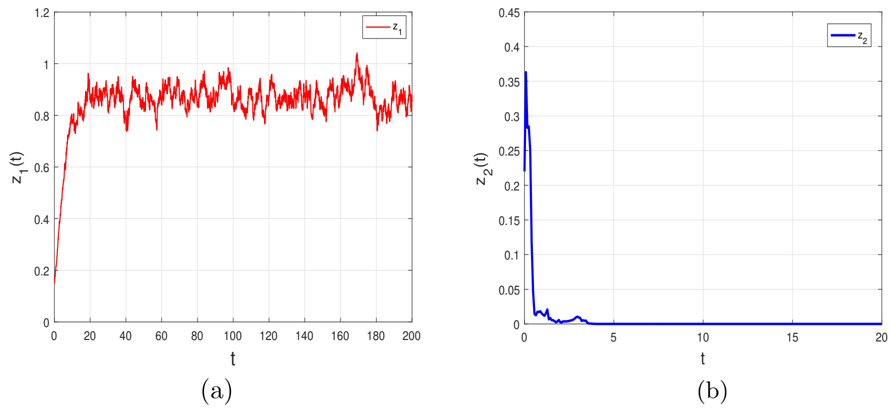


Figure 1. **(a)** Permanence in time average of species z_1; **(b)** Exponential extinction of species z_2.

Case 2. In contrast to Case 1, large suppression rates ($k_1 = 0.31, k_2 = 0.29$) and noise intensities ($\beta_{11}^2 = 0.82^2, \beta_{12}^2 = 0.85^2$) are chosen for species z_1 while relatively small suppression rates ($k_1 = 0.08, k_2 = 0.09$) and noise intensities ($\beta_{21}^2 = 0.032^2, \beta_{22}^2 = 0.046^2$) are designed for species z_2, then $r_1 - \delta_1 + \alpha_1 = -0.0674 < 0, r_2 - \delta_2 = 0.5884 > 0$ and $\lambda_2 = 0.5279 > 0$. By Theorem 3.2 we can obtain that species z_1 is EE while z_2 is PTA (see Figure 2).

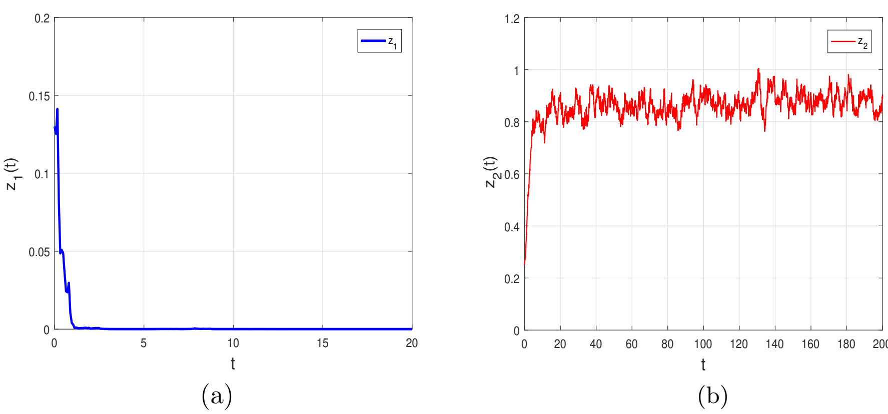
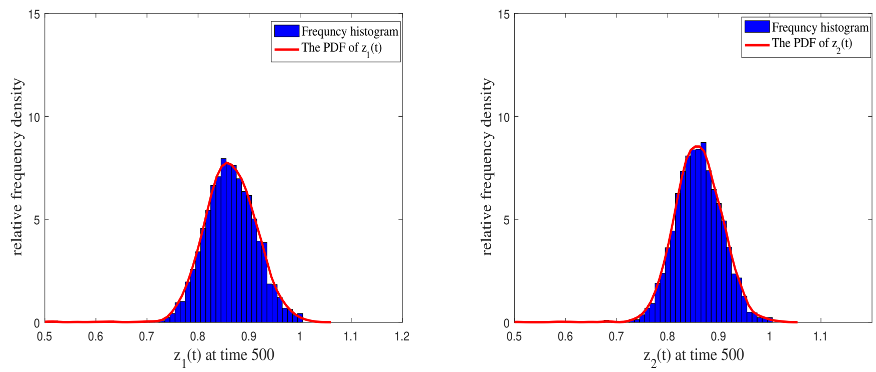


Figure 2. **(a)** Exponential extinction of species z_1; **(b)** Permanence in time average of species z_2.

Example 5.2. We fix a set of suitably small noise intensities ($\beta_{11}^2 = 0.037^2, \beta_{12}^2 = 0.042^2, \beta_{21}^2 = 0.032^2, \beta_{22}^2 = 0.046^2$) and vary suppression rates k_i to simulate the stationary distribution of model (1.6). This example will be performed by the following two cases.

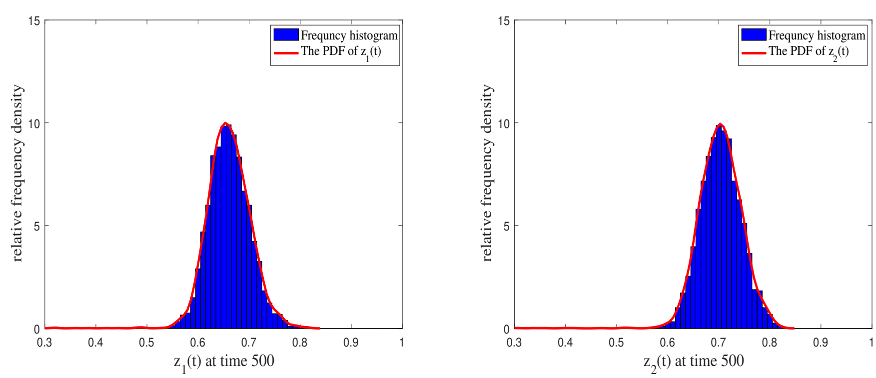
Case $1'$. When we select small suppression rates $k_1 = 0.08, k_2 = 0.09$, a direct calculation shows that $\zeta_1 = 0.2484 > 0, \zeta_2 = 0.3684 > 0, (\mu_1 + \alpha_1 - k_1)/\xi = 2.3739 > 0$, and $(\mu_2 + \alpha_2 - k_2)/\xi = 2.6706 > 0$ and then Theorem 4.1 holds. So model (1.6) has a unique stationary distribution (see Figure 3).



(a) (b)

Figure 3. (a)-(b) Frequency histograms of species z_1 and z_2, the curves are the probability density function (PDF) of species z_1 and z_2 when $k_1 = 0.08, k_2 = 0.09$.

Case $2'$. We choose relatively large suppression rates $k_1 = 0.31, k_2 = 0.29$ camparied with Case $1'$, we get $\zeta_1 = 0.0184 > 0, \zeta_2 = 0.1684 > 0, (\mu_1 + \alpha_1 - k_1)/\xi = 5.0061 > 0$, and $(\mu_2 + \alpha_2 - k_2)/\xi = 6.3492 > 0$. It follows from Theorem 4.1 that model (1.6) owns a unique stationary distribution and the level of this distribution is smaller than that of in Case $1'$ (see Figure 4).



(a) (b)

Figure 4. (a)-(b) Frequency histograms of species z_1 and z_2, the curves are the probability density function (PDF) of species z_1 and z_2 when $k_1 = 0.31, k_2 = 0.29$.

6. Discussions

This section is concerned with the biological discussions of model (1.6). Let us recall Theorems 3.1-4.1 and corresponding numerical examples, we can find the following interesting facts:

• It follows from the conditions ($r_i - \delta_i > 0$) of Theorem 3.1 (or Theorem 3.2) that relatively small noise intensities are helpful to PTA (permanence in time average) of species z_i. At the same time, the conditions $\lambda_i = [1 - k_i n_i/(m_i \mu_i)](r_i - \delta_i) > 0$ show that relatively small k_i (i.e., the suppression rates of feedback control variables u_i to species z_i) are also important. However, the conditions $r_i - \delta_i + \alpha_i < 0$ indicate that the other species $z_j (i \neq j)$ will be exponentially extinct when noise intensities are relatively large. The above results are also verified by Figures 1 and 2.

• It follows from the conditions $r_i - \alpha_i - k_i - \delta_i > 0$ and $(\mu_i + \alpha_i - k_i)/\xi > 0$ of Theorem 4.1 that relatively small noise intensities and suppression rates are helpful for the existence of the unique stationary distribution. Moreover, comparing Figure 3 (a) and Figure 4 (a) (or comparing Figure 3 (b) and Figure 4 (b)), we observe that if suppression rates are smaller, then the distribution level of species will be larger when other parameters are unchanged. Especially, letting $k_i \to 0$, then $r_i - \alpha_i - k_i - \delta_i > 0$ are approximate to $r_i - \alpha_i - \delta_i > 0$ and $(\mu_i + \alpha_i - k_i)/\xi > 0, \xi = \mu_1\mu_2 + k_1k_2 - \mu_1k_2 - \mu_2k_1 - \alpha_1\alpha_2$ becomes $\alpha_1\alpha_2 < \mu_1\mu_2$. These degenerate results are same as those in Theorem 4.1 in [38]. Also, we can conclude from the inequality $\alpha_1\alpha_2 < \mu_1\mu_2$ that the influence of interspecific mutualism is weaker than intraspecific competition.

Taking notice of the interesting experiment that parameter values of model may follow Gamma distribution (see an insightful work in Ref. [9]), we will try to consider a similar topic in the future.

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Appendices

A. The proof of Lemma 2.2

Proof. We construct a C^2-function $V_1 : \mathbb{R}^8_+ \to \mathbb{R}_+$ in such a way that

$$V_1(Z(t)) = \sum_{i=1}^{2}(z_i - 1 - \ln z_i + f_i - 1 - \ln f_i + v_i - 1 - \ln v_i + u_i - 1 - \ln u_i + \frac{k_i u_i}{m_i}).$$

Note that $V_1(Z(t))$ is a nonnegative function and it can be verified from the fact $0 \le x - 1 - \ln x$ for any $x > 0$. Applying Itô's formula to $V_1(Z(t))$, one derives that

$$dV_1(Z(t)) = LV_1(Z(t))dt + \sum_{i=1}^{2}(\beta_{1i}z_1 dW_i - \beta_{1i}dW_i + \beta_{2i}z_2 dW_i - \beta_{2i}dW_i),$$

where $LV_1(Z(t))$ is given by

$$
\begin{aligned}
LV_1(Z(t)) = & -\mu_1 z_1^2 - \mu_2 z_2^2 + \alpha_1 f_2 z_1 + \alpha_2 f_1 z_2 + r_1 z_1 + r_2 z_2 + \mu_1 z_1 + \mu_2 z_2 \\
& + 2\omega_1 + 2\omega_2 + n_1 z_1 + n_2 z_2 + m_1 + m_2 - r_1 - r_2 - \alpha_1 f_2 - \alpha_2 f_1 \\
& - \omega_1 f_1 - \omega_2 f_2 - m_1 u_1 - m_2 u_2 - k_1 u_1 z_1 - k_2 u_2 z_2 - \frac{\omega_1 v_1}{f_1} \\
& - \frac{\omega_2 v_2}{f_2} - \frac{\omega_1 z_1}{v_1(1+z_1)} - \frac{\omega_2 z_2}{v_2(1+z_2)} - \frac{n_1 z_1}{u_1} - \frac{n_2 z_2}{u_2} + \frac{\omega_1 z_1}{1+z_1} \\
& + \frac{\omega_2 z_2}{1+z_2} + \frac{\beta_{11}^2 + \beta_{12}^2}{2} + \frac{\beta_{21}^2 + \beta_{22}^2}{2} + \frac{k_2 n_2 z_2}{m_2} + \frac{k_1 n_1 z_1}{m_1} \\
\le & \lambda + 3\omega_1 + 3\omega_2 + m_1 + m_2 + \frac{\beta_{11}^2 + \beta_{12}^2}{2} + \frac{\beta_{21}^2 + \beta_{22}^2}{2},
\end{aligned}
$$

and

$$
\begin{aligned}
\lambda = & r_1 z_1 + r_2 z_2 - \mu_1 z_1^2 - \mu_2 z_2^2 + \alpha_1 f_2 z_1 + \alpha_2 f_1 z_2 + \mu_1 z_1 + \mu_2 z_2 \\
& + n_1 z_1 + n_2 z_2 + \frac{k_1 n_1 z_1}{m_1} + \frac{k_2 n_2 z_2}{m_2}.
\end{aligned}
$$

We can know from Lemma 2.1 that $f_1 \le 1$ and $f_2 \le 1$, and then λ is bounded when $z_1, z_2 \in (0, +\infty)$. Therefore, $LV_1(Z(t))$ is upper bounded. The rest proof is similar to that of Theorem 2.1 in [3] and hence we omit it. □

B. The proofs of Theorems 3.1 and 3.2

B.1. The proof of Theorem 3.1

Proof. We first give the species z_2 is EE (exponentially extinct). Using Itô's formula to the first two equations of model (1.6), one has

$$
d\ln z_i(t) = (r_i - \delta_i - \mu_i z_i(t) + \alpha_i f_j(t) - k_i u_i(t))dt + \sum_{j=1}^{2} \beta_{ij} dW_j(t), \ i, j = 1, 2, \ i \ne j. \tag{B.1}
$$

An integration from 0 to t on both sides of (B.1) leads to

$$
\ln \frac{z_i(t)}{z_i(0)} = (r_i - \delta_i)t - \mu_i \int_0^t z_i(s)ds + \alpha_i \int_0^t f_j(s)ds - k_i \int_0^t u_i(s)ds + \sum_{j=1}^{2} \beta_{ij} W_j(t). \tag{B.2}
$$

Dividing both sides of (B.2) by t and using Lemma 2.1, we have

$$
t^{-1} \ln \frac{z_i(t)}{z_i(0)} \le r_i - \delta_i + \alpha_i - t^{-1}\mu_1 \int_0^t z_i(s)ds - t^{-1} k_i \int_0^t u_i(s)ds + t^{-1} \sum_{j=1}^{2} \beta_{ij} W_j(t). \tag{B.3}
$$

In light of the strong law of large numbers for local martingales [29] we obtain that $\lim_{t\to+\infty} t^{-1} W_i(t) = 0$, which together with (B.3) and the condition $r_2 - \delta_2 + \alpha_2 < 0$ of Theorem 3.1 yields

$$
\limsup_{t\to+\infty} t^{-1} \ln z_2(t) \le r_2 - \delta_2 + \alpha_2 < 0,
$$

which means that species z_2 is EE. Furthermore, we have

$$\lim_{t\to+\infty} z_2(t) = 0 \quad \text{a.s.} \tag{B.4}$$

Next, we show that the species z_1 is PTA (permanent in time average). We integrate the fourth and sixth equations of model (1.6) over the interval $[0, t]$ and obtain

$$f_2(t) - f_2(0) = \omega_2(\int_0^t v_2(s)ds - \int_0^t f_2(s)ds),$$
$$v_2(t) - v_2(0) = \omega_2(\int_0^t \frac{z_2(s)}{1+z_2(s)}ds - \int_0^t v_2(s)ds).$$

Consequently,

$$\lim_{t\to+\infty} t^{-1}f_2(t) = \lim_{t\to+\infty} t^{-1}f_2(0) + \omega_2 \lim_{t\to+\infty}\langle v_2(t)\rangle - \omega_2 \lim_{t\to+\infty}\langle f_2(t)\rangle,$$
$$\lim_{t\to+\infty} t^{-1}v_2(t) = \lim_{t\to+\infty} t^{-1}v_2(0) + \omega_2 \lim_{t\to+\infty}\langle \frac{z_2(t)}{1+z_2(t)}\rangle - \omega_2 \lim_{t\to+\infty}\langle v_2(t)\rangle.$$

Additionally, a direct application of Lemma 2.1 shows $\lim_{t\to+\infty} t^{-1}f_2(t) = 0$ and $\lim_{t\to+\infty} t^{-1}v_2(t) = 0$. And combining $\lim_{t\to+\infty} t^{-1}f_2(0) = 0$, $\lim_{t\to+\infty} t^{-1}v_2(0) = 0$, we get

$$\lim_{t\to+\infty}\langle f_2(t)\rangle = \lim_{t\to+\infty}\langle v_2(t)\rangle = \lim_{t\to+\infty}\langle \frac{z_2(t)}{1+z_2(t)}\rangle. \tag{B.5}$$

According to (B.4), for a arbitrarily small $\varepsilon > 0$, choose $T > 0$ such that for $t > T$,

$$0 < \langle \frac{z_2(t)}{1+z_2(t)}\rangle < \frac{\varepsilon}{2\alpha_1}$$

is satisfied, which together with (B.5) gives that

$$0 < \langle f_2(t)\rangle < \varepsilon/(2\alpha_1). \tag{B.6}$$

Also, a sufficiently small $\varepsilon > 0$ satisfies $-\varepsilon < t^{-1}\ln z_1(0) < \varepsilon/2$. Accordingly, one obtains from (B.2) that

$$\ln z_1(t) \le (r_1 - \delta_1 + \varepsilon)t - \mu_1 \int_0^t z_1(s)ds + \sum_{i=1}^{2} \beta_{1i}W_i(t).$$

Since $r_1 - \delta_1 > 0$, applying Lemma 4 in [24] leads to

$$\langle z_1(t)\rangle^* \le (r_1 - \delta_1)/\mu_1 \ a.s. \tag{B.7}$$

Note that the seventh equation of model (1.6) implies

$$u_1(t) = u_1(0)\exp\{-m_1 t\} + \exp\{-m_1 t\}n_1 \int_0^t \exp\{m_1 s\}z_1(s)ds. \tag{B.8}$$

In view of (B.7) and (B.8), we have

$$\langle u_1(t)\rangle^* \le n_1(r_1 - \delta_1)/(m_1\mu_1) \ a.s. \tag{B.9}$$

Reusing $-\varepsilon < t^{-1}\ln z_1(0) < \varepsilon/2$, by substituting (B.6) and (B.9) into (B.2), we derive

$$\begin{aligned}\ln z_1(t) &\geq [r_1-\delta_1-\frac{k_1n_1(r_1-\delta_1)}{m_1\mu_1}-\varepsilon]t-\mu_1\int_0^t z_1(s)ds+\sum_{i=1}^2\beta_{1i}W_i(t)\\ &=(\lambda_1-\varepsilon)t-\mu_1\int_0^t z_1(s)ds+\sum_{i=1}^2\beta_{1i}W_i(t),\end{aligned}$$

where $\lambda_1=(m_1\mu_1-k_1n_1)(r_1-\delta_1)/(m_1\mu_1)$. Since $\lambda_1>0$, and $\varepsilon>0$ is arbitrarily small, it follows from Lemma 4 in [24] that

$$\langle z_1(t)\rangle_*\geq\lambda_1/\mu_1\ a.s. \tag{B.10}$$

By combining (B.7) and (B.10) we can get

$$\lambda_1/\mu_1\leq\langle z_1(t)\rangle_*\leq\langle z_1(t)\rangle^*\leq(r_1-\delta_1)/\mu_1\ a.s.$$

This completes the proof. □

B.2. The proof of Theorem 3.2

Proof. Similar to Theorem 3.1, Theorem 3.2 is valid and the details are omitted. □

C. The proof of Lemma 4.2

Proof. Let

$$V_2(Z(t))=\frac{z_1^q}{q}+\frac{z_2^q}{q}+\frac{\mu_1f_1^{q+1}}{2\omega_1}+\frac{\mu_2f_2^{q+1}}{2\omega_2}+\frac{\mu_1v_1^{q+1}}{2\omega_1}+\frac{\mu_2v_2^{q+1}}{2\omega_2}+\frac{\mu_1u_1^{q+1}}{4n_1}+\frac{\mu_2u_2^{q+1}}{4n_2}. \tag{C.1}$$

In light of Itô's formula, one gets

$$\begin{aligned}dV_2(Z(t)) &= z_1^{q-1}dz_1+\frac{q-1}{2}z_1^{q-2}(dz_1)^2+z_2^{q-1}dz_2+\frac{q-1}{2}z_2^{q-2}(dz_2)^2\\ &\quad+\frac{\mu_1(q+1)}{2\omega_1}f_1^qdf_1+\frac{\mu_2(q+1)}{2\omega_2}f_2^qdf_2+\frac{\mu_1(q+1)}{2\omega_1}v_1^qdv_1\\ &\quad+\frac{\mu_2(q+1)}{2\omega_2}v_2^qdv_2+\frac{\mu_1(q+1)}{4n_1}u_1^qdu_1+\frac{\mu_2(q+1)}{4n_2}u_2^qdu_2\\ &= LV_2(Z(t))dt+z_1^q\sum_{i=1}^2\beta_{1i}dW_i(t)+z_2^q\sum_{i=1}^2\beta_{2i}dW_i(t),\end{aligned}$$

in which

$$\begin{aligned}LV_2(Z(t)) =&[r_1-\mu_1z_1+\alpha_1f_2-k_1u_1+\frac{q-1}{2}(\beta_{11}^2+\beta_{12}^2)]z_1^q\\ &+[r_2-\mu_2z_2+\alpha_2f_1-k_2u_2+\frac{q-1}{2}(\beta_{21}^2+\beta_{22}^2)]z_2^q\\ &+\frac{\mu_1(q+1)}{2}(v_1f_1^q-f_1^{q+1})+\frac{\mu_2(q+1)}{2}(v_2f_2^q-f_2^{q+1})\end{aligned}$$

$$+\frac{\mu_1(q+1)}{2}(\frac{z_1 v_1^q}{1+z_1}-v_1^{q+1})+\frac{\mu_2(q+1)}{2}(\frac{z_2 v_2^q}{1+z_2}-v_2^{q+1})$$
$$+\frac{\mu_1(q+1)}{4}(z_1 u_1^q-\frac{m_1}{n_1}u_1^{q+1})+\frac{\mu_2(q+1)}{4}(z_2 u_2^q-\frac{m_2}{n_2}u_2^{q+1}). \quad (C.2)$$

In order to estimate (C.2), we use Young's inequality to obtain that

$$\begin{aligned}&\frac{\mu_i(q+1)}{2}(v_i f_i^q-f_i^{q+1})\\ &\le \frac{\mu_i(q+1)}{2}[\frac{1}{q+1}v_i^{q+1}+\frac{q}{q+1}f_i^{q+1}-f_i^{q+1}]\\ &=\frac{\mu_i}{2}v_i^{q+1}-\frac{\mu_i}{2}f_i^{q+1}.\end{aligned} \quad (C.3)$$

Using the same technique as (C.3), we have

$$\begin{aligned}&\frac{\mu_i(q+1)}{2}(\frac{z_i v_i^q}{1+z_i}-v_i^{q+1})\\ &\le \frac{\mu_i(q+1)}{2}[\frac{1}{q+1}(\frac{z_i}{1+z_i})^{q+1}+\frac{q}{q+1}v_i^{q+1}-v_i^{q+1}]\\ &\le \frac{\mu_i}{2}z_i^{q+1}-\frac{\mu_i}{2}v_i^{q+1}.\end{aligned} \quad (C.4)$$

Under Assumption (H_1), we derive

$$\begin{aligned}&\frac{\mu_i}{4}(q+1)(z_i u_i^q-\frac{m_i}{n_i}u_i^{q+1})\\ &\le \frac{\mu_i}{4}(q+1)[\frac{1}{q+1}z_i^{q+1}+\frac{q}{q+1}u_i^{q+1}-\frac{m_i}{n_i}u_i^{q+1}]\\ &\le \frac{\mu_i}{4}(q+1)[\frac{1}{q+1}z_i^{q+1}+\frac{qm_i}{(q+1)n_i}u_i^{q+1}-\frac{m_i}{n_i}u_i^{q+1}]\\ &=\frac{\mu_i}{4}z_i^{q+1}-\frac{\mu_i m_i}{4n_i}u_i^{q+1}.\end{aligned} \quad (C.5)$$

It follows from Lemma 2.1 that $f_i \le 1$. By combining (C.3)-(C.5), one can derive from (C.2) that

$$\begin{aligned}LV_2(Z(t)) \le &-\frac{\mu_1}{4}z_1^{q+1}+[r_1+\alpha_1+\frac{q-1}{2}(\beta_{11}^2+\beta_{12}^2)]z_1^q-\frac{\mu_1}{2}f_1^{q+1}-\frac{\mu_1 m_1}{4n_1}u_1^{q+1}\\ &-\frac{\mu_2}{4}z_2^{q+1}+[r_2+\alpha_2+\frac{q-1}{2}(\beta_{21}^2+\beta_{22}^2)]z_2^q-\frac{\mu_2}{2}f_2^{q+1}-\frac{\mu_2 m_2}{4n_2}u_2^{q+1}.\end{aligned}$$

Choosing a positive constant η, we have

$$\begin{aligned}L[e^{\eta t}V_2(Z(t))] &= \eta e^{\eta t}V_2(Z(t))+e^{\eta t}LV_2(Z(t))\\ &\le e^{\eta t}\{-\frac{\mu_1}{4}z_1^{q+1}+[r_1+\alpha_1+\frac{q-1}{2}(\beta_{11}^2+\beta_{12}^2)+\frac{\eta}{q}]z_1^q\\ &\quad+(\frac{\mu_1\eta}{2\omega_1}-\frac{\mu_1}{2})f_1^{q+1}+\frac{\mu_1\eta}{2\omega_1}v_1^{q+1}+(\frac{\mu_1\eta}{4n_1}-\frac{\mu_1 m_1}{4n_1})u_1^{q+1}\\ &\quad-\frac{\mu_2}{4}z_2^{q+1}+[r_2+\alpha_2+\frac{q-1}{2}(\beta_{21}^2+\beta_{22}^2)+\frac{\eta}{q}]z_2^q\\ &\quad+(\frac{\mu_2\eta}{2\omega_2}-\frac{\mu_2}{2})f_2^{q+1}+\frac{\mu_2\eta}{2\omega_2}v_2^{q+1}+(\frac{\mu_2\eta}{4n_2}-\frac{\mu_2 m_2}{4n_2})u_2^{q+1}\}.\end{aligned}$$

Suppose the above constant η is sufficiently small such that $0 < \eta < \min\{\omega_i, m_i\}$, and note that $\mu_i \eta v_i^{q+1}/2\omega_i \leq \mu_i \eta/2\omega_i$, one further gets

$$L(e^{\eta t} V_2(Z(t))) \leq S_1 e^{\eta t}, \tag{C.6}$$

where

$$S_1 = \max_{z_1, z_2 \in \mathbb{R}_+^2} \{-\frac{\mu_1}{4} z_1^{q+1} + [r_1 + \alpha_1 + \frac{q-1}{2}(\beta_{11}^2 + \beta_{12}^2) + \frac{\eta}{q}] z_1^q + \frac{\mu_1 \eta}{2\omega_1} - \frac{\mu_2}{4} z_2^{q+1} + [r_2 + \alpha_2 + \frac{q-1}{2}(\beta_{21}^2 + \beta_{22}^2) + \frac{\eta}{q}] z_2^q + \frac{\mu_2 \eta}{2\omega_2}\}.$$

Integrating both sides of (C.6) from 0 to t and taking the expectation, we get

$$\mathbb{E}[V_2(Z(t))] \leq V_2(Z(0)) e^{-\eta t} + S_1/\eta, \quad t \geq 0.$$

Based on the continuity of $V_2(Z(t))$ and the boundedness of $V_2(Z(0))e^{-\eta t}$, we know that there exists a positive constant S_2 such that

$$\mathbb{E}[V_2(Z(t))] \leq S_2, \quad t \geq 0.$$

We further obtain from (C.1) that $\mathbb{E}[z_i^q/q] \leq \mathbb{E}[V_2(Z(t))] \leq S_2$, that is

$$\mathbb{E}[z_i^q] \leq q S_2, \ i = 1, 2.$$

Meanwhile, it follows from (C.1) that $\mathbb{E}[f_i^{q+1}] \leq 2\omega_i S_2/\mu_i$. Then applying the Cauchy-Schwarz inequality [29], we obtain positive constants ρ_i such that

$$\mathbb{E}[f_i^q] \leq \rho_i \mathbb{E}[f_i^{q+1}]^{\frac{q}{q+1}} \leq \rho_i (2\omega_i S_2/\mu_i)^{\frac{q}{q+1}}, \quad i = 1, 2.$$

Similarly, there exist positive constants σ_i such that

$$\mathbb{E}[v_i^q] \leq \sigma_i (2\omega_i S_2/\mu_i)^{\frac{q}{q+1}}, \quad i = 1, 2,$$

and provide positive constants γ_i such that

$$\mathbb{E}[u_i^q] \leq \gamma_i (4 n_i S_2/\mu_i)^{\frac{q}{q+1}}, \quad i = 1, 2.$$

To sum up, we let

$$K_q = \max\{q S_2, \rho_i (2\omega_i S_2/\mu_i)^{\frac{q}{q+1}}, \sigma_i (2\omega_i S_2/\mu_i)^{\frac{q}{q+1}}, \gamma_i (4 n_i S_2/\mu_i)^{\frac{q}{q+1}}, i = 1, 2\},$$

which confirms Lemma 4.2. □

D. The proof of Lemma 4.3

Proof. Firstly, we consider $z_1(t)$. Integrating the first equation of model (1.6) over the interval $[t_1, t_2]$, one has

$$z_1(t_2) - z_1(t_1) = \int_{t_1}^{t_2} z_1(s)(r_1 - \mu_1 z_1(s) + \alpha_1 f_2(s) - k_1 u_1(s)) ds + \sum_{i=1}^{2} \beta_{1i} \int_{t_1}^{t_2} z_1(s) dW_i(s).$$

Let $q > 2$, with the help of the inequality $|a+b+c|^q \le 3^{q-1}(|a|^q + |b|^q + |c|^q)$, we have

$$\begin{aligned}\mathbb{E}[|z_1(t_2) - z_1(t_1)|^q] &= \mathbb{E}[|\int_{t_1}^{t_2} z_1(s)(r_1 - \mu_1 z_1(s) + \alpha_1 f_2(s) - k_1 u_1(s))ds \\ &\quad + \int_{t_1}^{t_2} \beta_{11} z_1(s) dW_1(s) + \int_{t_1}^{t_2} \beta_{12} z_1(s) dW_2(s)|] \\ &\le 3^{q-1}\{\mathbb{E}[|\int_{t_1}^{t_2} z_1(s)(r_1 - \mu_1 z_1(s) + \alpha_1 f_2(s) - k_1 u_1(s))ds|^q] \\ &\quad + \mathbb{E}[|\int_{t_1}^{t_2} \beta_{11} z_1(s) dW_1(s)|^q] + \mathbb{E}[|\int_{t_1}^{t_2} \beta_{12} z_1(s) dW_2(s)|^q]\}.\end{aligned} \tag{D.1}$$

Recalling Lemma 4.2 and using Hölder inequality [29], we get

$$\begin{aligned}&\mathbb{E}[|\int_{t_1}^{t_2} z_1(s)(r_1 - \mu_1 z_1(s) + \alpha_1 f_2(s) - k_1 u_1(s))ds|^q] \\ \le&\mathbb{E}[|(\int_{t_1}^{t_2} 1^{\frac{q}{q-1}} ds)^{\frac{q-1}{q}} (\int_{t_1}^{t_2} z_1(s)^q (r_1 - \mu_1 z_1(s) + \alpha_1 f_2(s) - k_1 u_1(s))^q ds)^{\frac{1}{q}}|^q] \\ \le&(t_2 - t_1)^{q-1}\mathbb{E}[\int_{t_1}^{t_2} |z_1(s)(r_1 - \mu_1 z_1(s) + \alpha_1 f_2(s) - k_1 u_1(s))|^q ds] \\ \le&(t_2 - t_1)^{q-1}\int_{t_1}^{t_2} \frac{1}{2}(\mathbb{E}[|z_1(s)|^{2q}] + \mathbb{E}[|r_1 - \mu_1 z_1(s) + \alpha_1 f_2(s) - k_1 u_1(s)|^{2q}])ds\end{aligned} \tag{D.2}$$

$$\begin{aligned}\le&(t_2 - t_1)^{q-1}\int_{t_1}^{t_2} \frac{1}{2}(\mathbb{E}[|z_1(s)|^{2q}] + 4^{2q-1}(r_1^{2q} + \mu_1^{2q}\mathbb{E}[|z_1(s)|^{2q}] + \alpha_1^{2q}\mathbb{E}[|f_2(s)|^{2q}] \\ &+ k_1^{2q}\mathbb{E}[|u_1(s)|^{2q}]))ds \\ \le&(t_2 - t_1)^{q-1}\int_{t_1}^{t_2} \frac{1}{2}[K_{2q} + 4^{2q-1}(r_1^{2q} + \mu_1^{2q} K_{2q} + \alpha_1^{2q} K_{2q} + k_1^{2q} K_{2q})]ds \\ =&\frac{(t_2 - t_1)^q}{2}[K_{2q} + 4^{2q-1}(r_1^{2q} + \mu_1^{2q} K_{2q} + \alpha_1^{2q} K_{2q} + k_1^{2q} K_{2q})].\end{aligned}$$

Moreover, by Moment inequality for stochastic integral [29], one has

$$\begin{aligned}&\mathbb{E}[|\int_{t_1}^{t_2} \beta_{11} z_1(s) dW_1(s)|^q] + \mathbb{E}[|\int_{t_1}^{t_2} \beta_{12} z_1(s) dW_2(s)|^q] \\ &\le (\beta_{11}^q + \beta_{12}^q)(\frac{q(q-1)}{2})^{\frac{q}{2}}(t_2 - t_1)^{\frac{q-2}{2}}\int_{t_1}^{t_2} \mathbb{E}[|z_1(s)|^q]ds \\ &= (\beta_{11}^q + \beta_{12}^q)[\frac{q(q-1)}{2}(t_2 - t_1)]^{\frac{q}{2}} K_q.\end{aligned} \tag{D.3}$$

Then substituting (D.2) and (D.3) into (D.1), we can derive that

$$\begin{aligned}&\mathbb{E}[|z_1(t_2) - z_1(t_1)|^q] \\ \le& \frac{3^{q-1}(t_2 - t_1)^q}{2}[K_{2q} + 4^{2q-1}(r_1^{2q} + \mu_1^{2q} K_{2q} + \alpha_1^{2q} K_{2q} + k_1^{2q} K_{2q})] \\ &+ 3^{q-1}(\beta_{11}^q + \beta_{12}^q)[\frac{q(q-1)}{2}(t_2 - t_1)]^{\frac{q}{2}} K_q \\ =& M_1(t_2 - t_1)^{\frac{q}{2}},\end{aligned} \tag{D.4}$$

where

$$M_1 = 3^{q-1}[\frac{(t_2-t_1)^{\frac{q}{2}}}{2}(K_{2q} + 4^{2q-1}(r_1^{2q} + \mu_1^{2q}K_{2q} + \alpha_1^{2q}K_{2q} + k_1^{2q}K_{2q})) \\ + (\beta_{11}^q + \beta_{12}^q)(\frac{q(q-1)}{2})^{\frac{q}{2}}K_q].$$

Secondly, we consider $f_1(t)$. Integrating the third equation of model (1.6) from t_1 to t_2 leads to

$$f_1(t_2) - f_1(t_1) = \int_{t_1}^{t_2} \omega_1(v_1(s) - f_1(s))ds.$$

Similar to (D.2), it follows from Hölder inequality [29] and Lemma 4.2 that

$$\begin{aligned} &\mathbb{E}[|f_1(t_2) - f_1(t_1)|^q] \\ \leq &\mathbb{E}[|\int_{t_1}^{t_2} \omega_1(v_1(s) - f_1(s)ds)|^q] \\ \leq &\mathbb{E}[|(\int_{t_1}^{t_2} 1^{\frac{q}{q-1}}ds)^{\frac{q-1}{q}}(\int_{t_1}^{t_2} \omega_1^q(v_1(s) - f_1(s))^q ds)^{\frac{1}{q}}|^q] \\ \leq &(t_2-t_1)^{q-1}\int_{t_1}^{t_2} \mathbb{E}[|\omega_1(v_1(s) - f_1(s))|^q]ds \\ \leq &(t_2-t_1)^{q-1}\int_{t_1}^{t_2} 2^{q-1}(\omega_1^q\mathbb{E}[|v_1(s)|^q] + \omega_1^q\mathbb{E}[|f_1(s)|^q])ds \\ \leq &M_2(t_2-t_1)^{\frac{q}{2}}, \end{aligned} \tag{D.5}$$

where $M_2 = 2^q(t_2-t_1)^{\frac{q}{2}}\omega_1^q K_q$.

Thirdly, we consider $v_1(t)$. For any $0 \leq t_1 \leq t_2$, a direct integration of the fifth equation of model (1.6) shows

$$v_1(t_2) - v_1(t_1) = \int_{t_1}^{t_2} \omega_1(\frac{z_1(s)}{1+z_1(s)} - f_1(s))ds.$$

Similar to (D.5), one obtains

$$\begin{aligned} &\mathbb{E}[|v_1(t_2) - v_1(t_1)|^q] \\ = &\mathbb{E}[|\int_{t_1}^{t_2} \omega_1(\frac{z_1(s)}{1+z_1(s)} - v_1(s))ds|^q] \\ \leq &\mathbb{E}[|(\int_{t_1}^{t_2} 1^{\frac{q}{q-1}}ds)^{\frac{q-1}{q}}(\int_{t_1}^{t_2} \omega_1^q(\frac{z_1(s)}{1+z_1(s)} - v_1(s))^q ds)^{\frac{1}{q}}|^q] \\ \leq &(t_2-t_1)^{q-1}\int_{t_1}^{t_2} \mathbb{E}[|\omega_1(\frac{z_1(s)}{1+z_1(s)} - v_1(s))|^q]ds \\ \leq &(t_2-t_1)^{q-1}\int_{t_1}^{t_2} 2^{q-1}(\omega_1^q\mathbb{E}[|\frac{z_1(s)}{1+z_1(s)}|^q] + \omega_1^q E[|v_1(s)|^q])ds \\ \leq &(t_2-t_1)^{q-1}\int_{t_1}^{t_2} 2^{q-1}(\omega_1^q\mathbb{E}[|z_1(s)|^q] + \omega_1^q\mathbb{E}[|v_1(s)|^q])ds \\ \leq &M_2(t_2-t_1)^{\frac{q}{2}}. \end{aligned} \tag{D.6}$$

Finally, we consider $u_1(t)$. Integrating the seventh equation of model (1.6) over the interval $[t_1, t_2]$ gives that

$$u_1(t_2) - u_1(t_1) = \int_{t_1}^{t_2} (n_1 z_1(s) - m_1 u_1(s))ds.$$

Similar to (D.6), we have

$$\begin{aligned}
&\mathbb{E}[|u_1(t_2) - u_1(t_1)|^q] \\
&= \mathbb{E}[|(\int_{t_1}^{t_2} 1^{\frac{q}{q-1}})^{\frac{q-1}{q}} (\int_{t_1}^{t_2} (n_1 z_1(s) - m_1 u_1(s))^q ds)^{\frac{1}{q}}|^q] \\
&\leq (t_2 - t_1)^{q-1} \int_{t_1}^{t_2} \mathbb{E}[|n_1 z_1(s) - m_1 u_1(s)|^q] ds \\
&\leq (t_2 - t_1)^{q-1} \int_{t_1}^{t_2} 2^{q-1}(n_1^q \mathbb{E}[|z_1(s)|^q] + m_1^q \mathbb{E}[|u_1(s)|^q]) ds \\
&\leq M_3 (t_2 - t_1)^{\frac{q}{2}},
\end{aligned} \tag{D.7}$$

where $M_3 = 2^{q-1}(t_2 - t_1)^{\frac{q}{2}}(n_1^q + m_1^q)K_q$.

By repeating the same analysis method as above, one gets that $z_2(t), f_2(t), v_2(t)$, and $u_2(t)$ own similar results as those of (D.4)-(D.7), respectively. Thus, we know from Lemma 3.4 in [45] that almost every sample path $Z(t)$ of model (1.6) is uniformly continuous on $t \geq 0$. □

E. The proof of Theorem 4.1

Proof. We will prove Theorem 4.1 through the following two steps.

Step 1 (existence of a smooth Markov process). According to Lemma 4.1 and Remark 4.1, we only need to develop a nonnegative C^2-function $V(Z(t))$ and a closed set $\Gamma \subset \mathbb{R}^8_+$ satisfying $LV(Z(t)) \leq -1$ for any $Z(t) \in \mathbb{R}^8_+ \setminus \Gamma$.

We first define

$$V_3(Z(t)) = QV_4(Z(t)) + V_5(Z(t)) + V_6(Z(t)),$$

where

$$\begin{aligned}
V_4(Z(t)) &= \frac{\alpha_1}{\omega_2} v_2 + \frac{\alpha_2}{\omega_1} v_1 - \frac{k_1}{m_1} \ln u_1 - \frac{k_2}{m_2} \ln u_2 - \ln z_1 - \ln z_2, \\
V_5(Z(t)) &= -\ln f_1 - \ln f_2 - \ln v_1 - \ln v_2 - \ln u_1 - \ln u_2, \\
V_6(Z(t)) &= \frac{1}{\theta + 2}(z_1 + z_2 + f_1 + f_2 + 2v_1 + 2v_2 + u_1 + u_2)^{\theta+2},
\end{aligned}$$

with θ is a positive constant and Q will be given later. In view of the continuity of the function $V_3(Z(t))$, it is not difficult to see that there exists a point $\tilde{Z}(t) = (z_1^{\min}, z_2^{\min}, f_1^{\min}, f_2^{\min}, v_1^{\min}, v_2^{\min}, u_1^{\min}, u_2^{\min})$ in the interior of $\mathbb{R}^8_+$, at which $V_3(Z(t))$ will be minimized, then we construct a nonnegative C^2-function $V : \mathbb{R}^8_+ \to \mathbb{R}_+ \cup \{0\}$

$$V(Z(t)) = V_3(Z(t)) - V_3(\tilde{Z}(t)).$$

Employing Itô's formula to $V_4(Z(t))$, we get

$$\begin{aligned} LV_4(Z(t)) &= -[r_1 - \mu_1 z_1 + \alpha_1 f_2 - k_1 u_1 - \delta_1] + \alpha_1(\frac{z_2}{1+z_2} - v_2) - \frac{k_1 n_1 z_1}{m_1 u_1} + k_1 \\ &\quad - [r_2 - \mu_2 z_2 + \alpha_2 f_1 - k_2 u_2 - \delta_2] + \alpha_2(\frac{z_1}{1+z_1} - v_1) - \frac{k_2 n_2 z_2}{m_2 u_2} + k_2 \\ &\le -\zeta_1 + \mu_1 z_1 + k_1 u_1 - \zeta_2 + \mu_2 z_2 + k_2 u_2, \end{aligned} \tag{E.1}$$

where $\zeta_i = r_i - \alpha_i - k_i - \delta_i,\ i = 1, 2$. A calculation for $V_5(Z(t))$ shows that

$$\begin{aligned} LV_5(Z(t)) = &-\frac{\omega_1 z_1}{(1+z_1)v_1} - \frac{\omega_2 z_2}{(1+z_2)v_2} - \frac{\omega_1 v_1}{f_1} - \frac{\omega_2 v_2}{f_2} \\ &- \frac{n_1 z_1}{u_1} - \frac{n_2 z_2}{u_2} + 2\omega_1 + 2\omega_2 + m_1 + m_2. \end{aligned} \tag{E.2}$$

Using Itô's formula to $V_6(Z(t))$, we obtain

$$\begin{aligned} LV_6(Z(t)) &= (z_1 + z_2 + f_1 + f_2 + 2v_1 + 2v_2 + u_1 + u_2)^{\theta+1}(r_1 z_1 + r_2 z_2 - \mu_1 z_1^2 \\ &\quad - \mu_2 z_2^2 + \alpha_1 f_2 z_1 + \alpha_2 f_1 z_2 - k_1 u_1 z_1 - k_2 u_2 z_2 - \omega_1 f_1 - \omega_2 f_2 - \omega_1 v_1 \\ &\quad - \omega_2 v_2 - m_1 u_1 - m_2 u_2 + n_1 z_1 + n_2 z_2 + \frac{2\omega_1 z_1}{1+z_1} + \frac{2\omega_2 z_2}{1+z_2}) \\ &\quad + (\theta+1)(z_1 + z_2 + f_1 + f_2 + 2v_1 + 2v_2 + u_1 + u_2)^{\theta}(\delta_1 z_1^2 + \delta_2 z_2^2) \\ &\le (z_1 + z_2 + f_1 + f_2 + 2v_1 + 2v_2 + u_1 + u_2)^{\theta+1}(r_1 z_1 + r_2 z_2 + \alpha_1 z_1 \\ &\quad + \alpha_2 z_2 + n_1 z_1 + n_2 z_2 + 2\omega_1 + 2\omega_2) - \mu_1 z_1^{\theta+3} - \mu_2 z_2^{\theta+3} - \omega_1 f_1^{\theta+2} \\ &\quad - \omega_2 f_2^{\theta+2} - \omega_1 v_1^{\theta+2} - \omega_2 v_2^{\theta+2} - m_1 u_1^{\theta+2} - m_2 u_2^{\theta+2} + (\theta+1)(z_1 + z_2 \\ &\quad + f_1 + f_2 + 2v_1 + 2v_2 + u_1 + u_2)^{\theta}(\delta_1 z_1^2 + \delta_2 z_2^2) \\ &\le -\frac{\mu_1 z_1^{\theta+3}}{2} - \frac{\mu_2 z_2^{\theta+3}}{2} - \frac{\omega_1 f_1^{\theta+2}}{2} - \frac{\omega_2 f_2^{\theta+2}}{2} - \frac{\omega_1 v_1^{\theta+2}}{2} - \frac{\omega_2 v_2^{\theta+2}}{2} \\ &\quad - \frac{m_1 u_1^{\theta+2}}{2} - \frac{m_2 u_2^{\theta+2}}{2} + F_1, \end{aligned} \tag{E.3}$$

where

$$\begin{aligned} F_1 = &\sup_{Z(t)\in\mathbb{R}^8_+}\{(z_1 + z_2 + f_1 + f_2 + 2v_1 + 2v_2 + u_1 + u_2)^{\theta+1}(r_1 z_1 + r_2 z_2 + \alpha_1 z_1 \\ &+ \alpha_2 z_2 + n_1 z_1 + n_2 z_2 + 2\omega_1 + 2\omega_2) - \frac{\mu_1 z_1^{\theta+3}}{2} - \frac{\mu_2 z_2^{\theta+3}}{2} - \frac{\omega_1 f_1^{\theta+2}}{2} - \frac{\omega_2 f_2^{\theta+2}}{2} \\ &- \frac{\omega_1 v_1^{\theta+2}}{2} - \frac{\omega_2 v_2^{\theta+2}}{2} - \frac{m_1 u_1^{\theta+2}}{2} - \frac{m_2 u_2^{\theta+2}}{2} + (\theta+1)(z_1 + z_2 + f_1 + f_2 + 2v_1 \\ &+ 2v_2 + u_1 + u_2)^{\theta}(\delta_1 z_1^2 + \delta_2 z_2^2)\} \\ < &+\infty. \end{aligned}$$

From (E.1)-(E.3), we derive

$$
\begin{aligned}
LV(Z(t)) \le & -Q(\zeta_1+\zeta_2)+F_2-\frac{\omega_1 z_1}{(1+z_1)v_1}-\frac{\omega_2 z_2}{(1+z_2)v_2}-\frac{\omega_1 v_1}{f_1}-\frac{\omega_2 v_2}{f_2}-\frac{n_1 z_1}{u_1}\\
& -\frac{n_2 z_2}{u_2}-\frac{\mu_1 z_1^{\theta+3}}{4}-\frac{\mu_2 z_2^{\theta+3}}{4}-\frac{\omega_1 f_1^{\theta+2}}{2}-\frac{\omega_2 f_2^{\theta+2}}{2}-\frac{\omega_1 v_1^{\theta+2}}{2}-\frac{\omega_2 v_2^{\theta+2}}{2}\\
& -\frac{m_1 u_1^{\theta+2}}{4}-\frac{m_2 u_2^{\theta+2}}{4},
\end{aligned}
$$

where

$$
\begin{aligned}
F_2 = & \sup_{(z_1,z_2,u_1,u_2)\in\mathbb{R}_+^4}\{Q\mu_1 z_1-\frac{\mu_1 z_1^{\theta+3}}{4}+Q\mu_2 z_2-\frac{\mu_2 z_2^{\theta+3}}{4}+Qk_1 u_1-\frac{m_1 u_1^{\theta+2}}{4}\\
& +Qk_2 u_2-\frac{m_2 u_2^{\theta+2}}{4}+2\omega_1+2\omega_2+m_1+m_2+F_1\}.
\end{aligned}
$$

We choose a large enough $Q>0$ satisfying $-Q(\zeta_1+\zeta_2)+F_2\le -2$. Then

$$
LV(Z(t)) \le \begin{cases}
-Q(\zeta_1+\zeta_2)+F_2\le -2, & \text{as} \quad z_i\to 0^+,\\
-Q(\zeta_1+\zeta_2)+F_2-\dfrac{\omega_i v_i}{f_i}\to-\infty, & \text{as} \quad f_i\to 0^+ \text{ and } v_i\nrightarrow 0^+,\\
-Q(\zeta_1+\zeta_2)+F_2-\dfrac{\omega_i z_i}{(1+z_i)v_i}\to-\infty, & \text{as} \quad v_i\to 0^+ \text{ and } z_i\nrightarrow 0^+,\\
-Q(\zeta_1+\zeta_2)+F_2-\dfrac{n_i z_i}{u_i}\to-\infty, & \text{as} \quad u_i\to 0^+ \text{ and } z_i\nrightarrow 0^+,\\
-Q(\zeta_1+\zeta_2)+F_2-\dfrac{\mu_i z_i^{\theta+3}}{4}\to-\infty, & \text{as} \quad z_i\to+\infty,\\
-Q(\zeta_1+\zeta_2)+F_2-\dfrac{\omega_i f_i^{\theta+2}}{2}\le -2, & \text{as} \quad f_i\to 1,\\
-Q(\zeta_1+\zeta_2)+F_2-\dfrac{\omega_i v_i^{\theta+2}}{2}\le -2, & \text{as} \quad v_i\to 1,\\
-Q(\zeta_1+\zeta_2)+F_2-\dfrac{m_i u_i^{\theta+2}}{4}\to-\infty, & \text{as} \quad u_i\to+\infty.
\end{cases}
$$

It is straightforward to see that for a sufficient small $\epsilon>0$ such that $LV(Z(t))\le -1$ for any $(Z(t))\in\mathbb{R}_+^8\setminus\Gamma$, where $\Gamma=[\epsilon,\frac{1}{\epsilon}]\times[\epsilon,\frac{1}{\epsilon}]\times[\epsilon^3,\frac{1}{1+\epsilon^3}]\times[\epsilon^3,\frac{1}{1+\epsilon^3}]\times[\epsilon^2,\frac{1}{1+\epsilon^2}]\times[\epsilon^2,\frac{1}{1+\epsilon^2}]\times[\epsilon^2,\frac{1}{\epsilon^2}]\times[\epsilon^2,\frac{1}{\epsilon^2}]$.

Step 2 (global attractivity). Let $\bar{Z}(t)=(\bar{z}_1(t),\bar{z}_2(t),\bar{f}_1(t),\bar{f}_2(t),\bar{v}_1(t),\bar{v}_2(t),\bar{u}_1(t),\bar{u}_2(t))$ be any positive solution to model (1.6) with $\bar{Z}(0)>0$. By the fifth and sixth equations of model (1.6), we derive

$$
d(v_i(t)-\bar{v}_i(t))=[\omega_i(\frac{z_i(t)}{1+z_i(t)}-\frac{\bar{z}_i(t)}{1+\bar{z}_i(t)})-\omega_i(v_i(t)-\bar{v}_i(t))]dt,\ i=1,2. \quad \text{(E.4)}
$$

We integrate both sides of (E.4) and have

$$
v_i(t)-\bar{v}_i(t)=(v_i(0)-\bar{v}_i(0))e^{-\omega_i t}+\omega_i e^{-\omega_i t}\int_0^t e^{\omega_i s}(\frac{z_i(s)}{1+z_i(s)}-\frac{\bar{z}_i(s)}{1+\bar{z}_i(s)})ds. \quad \text{(E.5)}
$$

Consequently,

$$|v_i(t)-\bar{v}_i(t)| \le |v_i(0)-\bar{v}_i(0)|e^{-\omega_i t}+\omega_i e^{-\omega_i t}\int_0^t e^{\omega_i s}\Big|\frac{z_i(s)}{1+z_i(s)}-\frac{\bar{z}_i(s)}{1+\bar{z}_i(s)}\Big|ds.$$

Since

$$\Big|\frac{z_i(t)}{1+z_i(t)}-\frac{\bar{z}_i(t)}{1+\bar{z}_i(t)}\Big| = \Big|\frac{z_i(t)-\bar{z}_i(t)}{(1+z_i(t))(1+\bar{z}_i(t))}\Big| \le |z_i(t)-\bar{z}_i(t)|,$$

we have

$$|v_i(t)-\bar{v}_i(t)| \le |v_i(0)-\bar{v}_i(0)|e^{-\omega_i t}+\omega_i e^{-\omega_i t}\int_0^t e^{\omega_i s}|z_i(s)-\bar{z}_i(s)|ds. \tag{E.6}$$

An integration of both sides of (E.6) over $[0,t]$ leads to

$$\begin{aligned}
&\int_0^t |v_i(s)-\bar{v}_i(s)|ds\\
&\le -\frac{1}{\omega_i}(e^{-\omega_i t}-1)|v_i(0)-\bar{v}_i(0)|+\omega_i\int_0^t d\nu\int_0^\nu e^{\omega_i(s-\nu)}|z_i(s)-\bar{z}_i(s)|ds\\
&=\frac{1}{\omega_i}(1-e^{-\omega_i t})|v_i(0)-\bar{v}_i(0)|+\omega_i\int_0^t e^{\omega_i s}|z_i(s)-\bar{z}_i(s)|ds\int_s^t e^{-\omega_i\nu}d\nu\\
&=\frac{1}{\omega_i}(1-e^{-\omega_i t})|v_i(0)-\bar{v}_i(0)|+\int_0^t |z_i(s)-\bar{z}_i(s)|(1-e^{\omega_i(s-t)})ds\\
&\le \frac{1}{\omega_i}|v_i(0)-\bar{v}_i(0)|+\int_0^t |z_i(s)-\bar{z}_i(s)|ds.
\end{aligned} \tag{E.7}$$

Similarly, from the third and fourth equations of model (1.6) we have

$$d(f_i(t)-\bar{f}_i(t)) = [\omega_i(v_i(t)-\bar{v}_i(t))-\omega_i(f_i(t)-\bar{f}_i(t))]dt,\ i=1,2.$$

Corresponding to (E.5), we can get

$$f_i(t)-\bar{f}_i(t) = (f_i(0)-\bar{f}_i(0))e^{-\omega_i t}+\omega_i e^{-\omega_i t}\int_0^t e^{\omega_i s}(v_i(s)-\bar{v}_i(s))ds.$$

Furthermore, one has

$$\begin{aligned}
&\int_0^t |f_i(s)-\bar{f}_i(s)|ds\\
&\le -\frac{1}{\omega_i}(e^{-\omega_i t}-1)|f_i(0)-\bar{f}_i(0)|+\omega_i\int_0^t d\nu\int_0^\nu e^{\omega_i(s-\nu)}|v_i(s)-\bar{v}_i(s)|ds\\
&=\frac{1}{\omega_i}(1-e^{-\omega_i t})|f_i(0)-\bar{f}_i(0)|+\omega_i\int_0^t e^{\omega_i s}|v_i(s)-\bar{v}_i(s)|ds\int_s^t e^{-\omega_i\nu}d\nu\\
&=\frac{1}{\omega_i}(1-e^{-\omega_i t})|f_i(0)-\bar{f}_i(0)|+\int_0^t |v_i(s)-\bar{v}_i(s)|(1-e^{\omega_i(s-t)})ds\\
&\le \frac{1}{\omega_i}|f_i(0)-\bar{f}_i(0)|+\int_0^t |v_i(s)-\bar{v}_i(s)|ds.
\end{aligned} \tag{E.8}$$

Similarly, we can conclude from the last two equations of model (1.6) that

$$d(u_i(t)-\bar{u}_i(t)) = [n_i(z_i(t)-\bar{z}_i(t))-m_i(u_i(t)-\bar{u}_i(t))]dt,\ i=1,2.$$

Thus

$$u_i(t)-\bar{u}_i(t)=(u_i(0)-\bar{u}_i(0))e^{-m_it}+n_ie^{-m_it}\int_0^t e^{m_is}(z_i(s)-\bar{z}_i(s))ds,$$

from which we can derive that

$$\begin{aligned}&\int_0^t|u_i(s)-\bar{u}_i(s)|ds\\ &\leq-\frac{1}{m_i}(e^{-m_it}-1)|u_i(0)-\bar{u}_i(0)|+n_i\int_0^t d\nu\int_0^\nu e^{m_i(s-\nu)}|z_i(s)-\bar{z}_i(s)|ds\\ &=\frac{1}{m_i}(1-e^{-m_it})|u_i(0)-\bar{u}_i(0)|+n_i\int_0^t e^{m_is}|z_i(s)-\bar{z}_i(s)|ds\int_s^t e^{-m_i\nu}d\nu \qquad (E.9)\\ &=\frac{1}{m_i}(1-e^{-m_it})|u_i(0)-\bar{u}_i(0)|+\frac{n_i}{m_i}\int_0^t|z_i(s)-\bar{z}_i(s)|(1-e^{m_i(s-t)})ds\\ &\leq\frac{1}{m_i}|u_i(0)-\bar{u}_i(0)|+\int_0^t|z_i(s)-\bar{z}_i(s)|ds.\end{aligned}$$

Assign

$$V_7(t)=\frac{\mu_2+\alpha_2-k_2}{\xi}|\ln z_1(t)-\ln\bar{z}_1(t)|+\frac{\mu_1+\alpha_1-k_1}{\xi}|\ln z_2(t)-\ln\bar{z}_2(t)|.$$

Evaluating the right differential $D^+V_7(t)$ of $V_7(t)$, we obtain that

$$\begin{aligned}D^+V_7(t)=&\frac{\mu_2+\alpha_2-k_2}{\xi}\mathrm{sgn}\{z_1(t)-\bar{z}_1(t)\}d(\ln z_1(t)-\ln\bar{z}_1(t))\\ &+\frac{\mu_1+\alpha_1-k_1}{\xi}\mathrm{sgn}\{z_2(t)-\bar{z}_2(t)\}d(\ln z_2(t)-\ln\bar{z}_2(t))\\ \leq&\frac{\mu_2+\alpha_2-k_2}{\xi}[-\mu_1|z_1(t)-\bar{z}_1(t)|+\alpha_1|f_2(t)-\bar{f}_2(t)|\\ &+k_1|u_1(t)-\bar{u}_1(t)|]dt+\frac{\mu_1+\alpha_1-k_1}{\xi}[-\mu_2|z_2(t)-\bar{z}_2(t)|\\ &+\alpha_2|f_1(t)-\bar{f}_1(t)|+k_2|u_2(t)-\bar{u}_2(t)|]dt,\end{aligned}$$

which togerther with (E.7)-(E.9), yields that

$$\begin{aligned}&V_7(t)-V_7(0)\\ \leq&\frac{\mu_2+\alpha_2-k_2}{\xi}[-\mu_1\int_0^t|z_1(s)-\bar{z}_1(s)|ds+\frac{\alpha_1}{\omega_2}|f_2(0)-\bar{f}_2(0)|\\ &+\alpha_1\int_0^t|v_2(s)-\bar{v}_2(s)|ds+\frac{k_1}{m_1}|u_1(0)-\bar{u}_1(0)|+k_1\int_0^t|z_1(s)-\bar{z}_1(s)|ds]\\ &+\frac{\mu_1+\alpha_1-k_1}{\xi}[-\mu_2\int_0^t|z_2(s)-\bar{z}_2(s)|ds+\frac{\alpha_2}{\omega_1}|f_1(0)-\bar{f}_1(0)|\\ &+\alpha_2\int_0^t|v_1(s)-\bar{v}_1(s)|ds+\frac{k_2}{m_2}|u_2(0)-\bar{u}_2(0)|+k_2\int_0^t|z_2(s)-\bar{z}_2(s)|ds]\\ \leq&\frac{\alpha_1(\mu_2+\alpha_2-k_2)}{\omega_2\xi}|f_2(0)-\bar{f}_2(0)|+\frac{\alpha_1(\mu_2+\alpha_2-k_2)}{\omega_2\xi}|v_2(0)-\bar{v}_2(0)|\end{aligned}$$

$$+ \frac{\alpha_2(\mu_1+\alpha_1-k_1)}{\omega_1\xi}|f_1(0)-\bar{f}_1(0)| + \frac{\alpha_2(\mu_1+\alpha_1-k_1)}{\omega_1\xi}|v_1(0)-\bar{v}_1(0)|$$

$$+ \frac{k_1(\mu_2+\alpha_2-k_2)}{m_1\xi}|u_1(0)-\bar{u}_1(0)| + \frac{k_2(\mu_1+\alpha_1-k_1)}{m_2\xi}|u_2(0)-\bar{u}_2(0)|$$

$$+ (\frac{\alpha_2(\mu_1+\alpha_1-k_1)}{\xi} + \frac{k_1(\mu_2+\alpha_2-k_2)}{\xi} - \frac{\mu_1(\mu_2+\alpha_2-k_2)}{\xi})$$

$$\times \int_0^t |z_1(s)-\bar{z}_1(s)|ds$$

$$+ (\frac{\alpha_1(\mu_2+\alpha_2-k_2)}{\xi} + \frac{k_2(\mu_1+\alpha_1-k_1)}{\xi} - \frac{\mu_2(\mu_1+\alpha_1-k_1)}{\xi})$$

$$\times \int_0^t |z_2(s)-\bar{z}_2(s)|ds$$

$$= \frac{\alpha_1(\mu_2+\alpha_2-k_2)}{\omega_2\xi}|f_2(0)-\bar{f}_2(0)| + \frac{\alpha_1(\mu_2+\alpha_2-k_2)}{\omega_2\xi}|v_2(0)-\bar{v}_2(0)|$$

$$+ \frac{\alpha_2(\mu_1+\alpha_1-k_1)}{\omega_1\xi}|f_1(0)-\bar{f}_1(0)| + \frac{\alpha_2(\mu_1+\alpha_1-k_1)}{\omega_1\xi}|v_1(0)-\bar{v}_1(0)|$$

$$+ \frac{k_1(\mu_2+\alpha_2-k_2)}{m_1\xi}|u_1(0)-\bar{u}_1(0)| + \frac{k_2(\mu_1+\alpha_1-k_1)}{m_2\xi}|u_2(0)-\bar{u}_2(0)|$$

$$- \int_0^t |z_1(s)-\bar{z}_1(s)|ds - \int_0^t |z_2(s)-\bar{z}_2(s)|ds.$$

Rearranging the above inequality, one gets

$$\begin{aligned}
& V_7(t) + \int_0^t |z_1(s)-\bar{z}_1(s)|ds + \int_0^t |z_2(s)-\bar{z}_2(s)|ds \\
\le\ & V_7(0) + \frac{\alpha_1(\mu_2+\alpha_2-k_2)}{\omega_2\xi}|f_2(0)-\bar{f}_2(0)| + \frac{\alpha_1(\mu_2+\alpha_2-k_2)}{\omega_2\xi}|v_2(0)-\bar{v}_2(0)| \\
& + \frac{k_1(\mu_2+\alpha_2-k_2)}{m_1\xi}|u_1(0)-\bar{u}_1(0)| + \frac{\alpha_2(\mu_1+\alpha_1-k_1)}{\omega_1\xi}|f_1(0)-\bar{f}_1(0)| \\
& + \frac{\alpha_2(\mu_1+\alpha_1-k_1)}{\omega_1\xi}|v_1(0)-\bar{v}_1(0)| + \frac{k_2(\mu_1+\alpha_1-k_1)}{m_2\xi}|u_2(0)-\bar{u}_2(0)| \\
\le\ & +\infty,
\end{aligned}$$

from which we obtain that $|z_i(t)-\bar{z}_i(t)| \in L^1[0,+\infty)$. By a similar deduction, it follows from (E.7)-(E.9) that $|f_i(t)-\bar{f}_i(t)|$, $|v_i(t)-\bar{v}_i(t)|$ and $|u_i(t)-\bar{u}_i(t)| \in L^1[0,+\infty)$. Thus, by Barhalat's Lemma [4] and Lemma 4.3, we get that

$$\begin{aligned}
\lim_{t\to+\infty}|z_i(t)-\bar{z}_i(t)| = \lim_{t\to+\infty}|f_i(t)-\bar{f}_i(t)| &= \lim_{t\to+\infty}|v_i(t)-\bar{v}_i(t)| \\
&= \lim_{t\to+\infty}|u_i(t)-\bar{u}_i(t)| = 0.
\end{aligned}$$

Combining **Step 1** with **Step 2**, the proof of Theorem 4.1 is completed. □

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