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NEW INEQUALITIES FOR POSITIVE CONVEX FUNCTIONS

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Abstract The main goal of this paper is to present new inequalities for convex and log-convex functions. The significance of these inequalities follows from the way they extend many known results in the literature concerning convex functions, log-convex functions, means comparisons and matrix inequalities.

Keywords Jensen's inequality, log-convexity, mean inequalities, matrix inequalities.

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1. Introduction

Different types of convex functions have played a major role in the field of Mathematical inequalities. In this paper, we will be interested in convex and log-convex functions.

Recall that a function $f : I \to \mathbb{R}$ is said to be convex on the interval I if

$$f(\alpha x+\beta y)\leq \alpha f(x)+\beta f(y), \tag{1.1}$$

for all $x, y \in I$ and $\alpha, \beta > 0$ with $\alpha + \beta = 1$. If $f > 0$ and $\log f$ is convex, then f is called log-convex.

Accordingly, a log-convex function is a positive function satisfying

$$f(\alpha x+\beta y)\leq f(x)^{\alpha}f(y)^{\beta}, \tag{1.2}$$

for the same parameters as in (1.1).

Recalling Young's inequality that asserts the following inequality for the positive numbers a, b, α and β with $\alpha + \beta = 1$,

$$a^{\alpha}b^{\beta}\leq \alpha a+\beta b, \tag{1.3}$$

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the inequality (1.2) implies that a log-convex function is necessarily convex.

The celebrated Jensen inequality extends (1.1) to n parameters as follows.

$$f\Big(\sum_{i=1}^{n}\mu_i x_i\Big) \leq \sum_{i=1}^{n}\mu_i f(x_i), \tag{1.4}$$

where $f : I \to \mathbb{R}$ is convex, $x_1, \cdots, x_n \in I$ and $\mu_1, \cdots \mu_n \geq 0$ are such that $\sum_{i=1}^{n}\mu_i = 1$. The inequality (1.4) can be found in any standard book about convex functions, see [24] for example. We refer the reader to [7,8,17,18,20–22,24–26] as a sample of the rich literature treating convex functions and some of their applications.

Applying Jensen's inequality (1.4) to the function $\log f$ implies the inequality

$$f\Big(\sum_{i=1}^{n}\mu_i x_i\Big) \leq \prod_{i=1}^{n} f^{\mu_i}(x_i), \tag{1.5}$$

for the same parameters as in (1.4), when f is log-convex.

A considerable attention has been paid in the literature to refine or reverse (1.4), and hence (1.5). For example, in [22] the following refinement of (1.4) was shown

$$f\left(\sum_{i=1}^{n}\mu_i x_i\right) + n\mu_{\min}\left(\frac{1}{n}\sum_{i=1}^{n} f(x_i) - f\left(\sum_{i=1}^{n}\frac{x_i}{n}\right)\right) \leq \sum_{i=1}^{n}\mu_i f(x_i), \tag{1.6}$$

where $\mu_{\min} = \min\{\mu_1, \ldots, \mu_n\}$. This inequality was reversed in the same reference by the inequality

$$f\left(\sum_{i=1}^{n}\mu_i x_i\right) + n\mu_{\max}\left(\frac{1}{n}\sum_{i=1}^{n} f(x_i) - f\left(\sum_{i=1}^{n}\frac{x_i}{n}\right)\right) \geq \sum_{i=1}^{n}\mu_i f(x_i), \tag{1.7}$$

where $\mu_{\max} = \max\{\mu_1, \cdots, \mu_n\}$.

Both inequalities were treated later by adding as many as refining terms in [25].

The inequalities (1.6) and (1.7) can be written in the form

$$\begin{aligned} &\min\left\{\frac{\mu_i}{1/n}\right\}\left(\frac{1}{n}\sum_{i=1}^{n} f(x_i) - f\left(\sum_{i=1}^{n}\frac{x_i}{n}\right)\right) \\ &\leq \sum_{i=1}^{n}\mu_i f(x_i) - f\left(\sum_{i=1}^{n}\mu_i x_i\right) \\ &\leq \max\left\{\frac{\mu_i}{1/n}\right\}\left(\frac{1}{n}\sum_{i=1}^{n} f(x_i) - f\left(\sum_{i=1}^{n}\frac{x_i}{n}\right)\right). \end{aligned} \tag{1.8}$$

We notice here that the factor $\frac{1}{n}$ satisfies $\sum_{i=1}^{n}\frac{1}{n} = 1$. Extending (1.8), Aldaz [1] proved the following more general inequality, which yields (1.8) upon letting $\tau_i = \frac{1}{n}$.

Theorem 1.1. *Let $f : I \longrightarrow \mathbb{R}$ be convex, $\{x_1, \ldots, x_n\} \subset I$, $\{\mu_1, \ldots, \mu_n\} \subset (0,1)$ and $\{\tau_1, \ldots, \tau_n\} \subset (0,1)$ be such that $\sum_{i=1}^{n}\mu_i = \sum_{i=1}^{n}\tau_i = 1$. Then*

$$\min_{i=1,\ldots,n}\left\{\frac{\mu_i}{\tau_i}\right\} \leq \frac{\sum_{i=1}^{n}\mu_i f(x_i) - f\Big(\sum_{i=1}^{n}\mu_i x_i\Big)}{\sum_{i=1}^{n}\tau_i f(x_i) - f\Big(\sum_{i=1}^{n}\tau_i x_i\Big)} \leq \max_{i=1,\ldots,n}\left\{\frac{\mu_i}{\tau_i}\right\}. \tag{1.9}$$

If $f : I \to [0, \infty)$ is convex, then f^λ is convex, for any $\lambda \geq 1$. Applying Theorem 1.1 to the function f^λ implies

$$\min_{i=1,\ldots,n}\left\{\frac{\mu_i}{\tau_i}\right\} \leq \frac{\sum_{i=1}^n \mu_i f^\lambda(x_i) - f^\lambda\Big(\sum_{i=1}^n \mu_i x_i\Big)}{\sum_{i=1}^n \tau_i f^\lambda(x_i) - f^\lambda\Big(\sum_{i=1}^n \tau_i x_i\Big)} \leq \max_{i=1,\ldots,n}\left\{\frac{\mu_i}{\tau_i}\right\}. \tag{1.10}$$

On the other hand, raising (1.9) to the power $\lambda \geq 1$, implies

$$\min_{i=1,\ldots,n}\left\{\frac{\mu_i}{\tau_i}\right\}^\lambda \leq \frac{\Big(\sum_{i=1}^n \mu_i f(x_i) - f\Big(\sum_{i=1}^n \mu_i x_i\Big)\Big)^\lambda}{\Big(\sum_{i=1}^n \tau_i f(x_i) - f\Big(\sum_{i=1}^n \tau_i x_i\Big)\Big)^\lambda} \leq \max_{i=1,\ldots,n}\left\{\frac{\mu_i}{\tau_i}\right\}^\lambda. \tag{1.11}$$

However, neither (1.10) nor (1.11) can be used to find upper or lower bounds for the quotient

$$\frac{\Big(\sum_{i=1}^n \mu_i f(x_i)\Big)^\lambda - f^\lambda\Big(\sum_{i=1}^n \mu_i x_i\Big)}{\Big(\sum_{i=1}^n \tau_i f(x_i)\Big)^\lambda - f^\lambda\Big(\sum_{i=1}^n \tau_i x_i\Big)}.$$

When $n = 2$, Sababheh [26] showed that

$$\min_{i=1,\ldots,n}\left\{\frac{\mu_i}{\tau_i}\right\}^\lambda \leq \frac{\Big(\sum_{i=1}^n \mu_i f(x_i)\Big)^\lambda - f^\lambda\Big(\sum_{i=1}^n \mu_i x_i\Big)}{\Big(\sum_{i=1}^n \tau_i f(x_i)\Big)^\lambda - f^\lambda\Big(\sum_{i=1}^n \tau_i x_i\Big)} \leq \max_{i=1,\ldots,n}\left\{\frac{\mu_i}{\tau_i}\right\}^\lambda. \tag{1.12}$$

The first goal of this paper is to prove that (1.12) is valid for any $n \in \mathbb{N}$. The method used in [26] to prove (1.12) for $n = 2$ was a differential Calculus approach, that we cannot use for the general case. To prove the general case we will need the following lemma, which will enable us to prove a more general result that implies (1.12) upon selecting certain parameters.

Lemma 1.1 ([3]). *Let ϕ be a strictly increasing convex function defined on an interval I. If x, y, z and w are points in I such that*

$$z - w \leq x - y$$

where $w \leq z \leq x$ and $y \leq x$, then

$$(0 \leq) \quad \phi(z) - \phi(w) \leq \phi(x) - \phi(y).$$

This lemma will be simply used to prove

$$\begin{aligned}
&\phi\Big(\min_{i=1,\ldots,n}\left\{\frac{\mu_i}{\tau_i}\right\}\sum_{i=1}^n \tau_i f(x_i)\Big) - \phi\Big(\min_{i=1,\ldots,n}\left\{\frac{\mu_i}{\tau_i}\right\} f\Big(\sum_{i=1}^n \tau_i x_i\Big)\Big) \\
\leq&\phi\Big(\sum_{i=1}^n \mu_i f(x_i)\Big) - \phi \circ f\Big(\sum_{i=1}^n \mu_i x_i\Big) \\
\leq&\phi\Big(\max_{i=1,\ldots,n}\left\{\frac{\mu_i}{\tau_i}\right\}\sum_{i=1}^n \tau_i f(x_i)\Big) - \phi\Big(\max_{i=1,\ldots,n}\left\{\frac{\mu_i}{\tau_i}\right\} f\Big(\sum_{i=1}^n \tau_i x_i\Big)\Big),
\end{aligned} \tag{1.13}$$

for the pre-stated parameters. Then by selecting the proper function ϕ, we will be able to prove (1.12) for any n.

Another strongly related goal of this paper is to prove

$$\min_{i=1,\dots,n}\left\{\frac{\mu_i^m}{\tau_i}\right\} \leq \frac{\left(\sum_{i=1}^n \mu_i f(x_i)\right)^m - f^m\left(\sum_{i=1}^n \mu_i x_i\right)}{\sum_{i=1}^n \tau_i f^m(x_i) - f^m\Big(\sum_{i=1}^n \tau_i x_i\Big)}, \tag{1.14}$$

as a mixture of (1.10) and (1.12), for a positive integer m and the log-convex function f.

Once we show our main inequalities, we will present some applications that include new comparisons between means, and new matrix inequalities that involve the Heinz means and related inequalities for unitarily invariant norms.

To prove (1.14), we will need the following lemma from [13].

Lemma 1.2. *Let n and m be two integers and let $x_i \in \mathbb{R}$. Set $i_0 := m$, $i_n := 0$ and*

$$A := \{(i_1, \dots, i_{n-1}) : 0 \leq i_k \leq i_{k-1},\ 1 \leq k \leq n-1\}.$$

Then

$$\left(\sum_{i=1}^n x_i\right)^m = \sum_{(i_1,\dots,i_{n-1})\in A} \binom{i_0}{i_1}\binom{i_1}{i_2}\dots\binom{i_{n-2}}{i_{n-1}} x_1^{i_0-i_1} x_2^{i_1-i_2}\dots x_n^{i_{n-1}-i_n}.$$

We will present our results in the next two sections, where inequalities for convex functions with emphasize on (1.12) will be presented in Section 2, while (1.14) will be presented in Section 3.

Remark 1.1. In the sequel, our functions are assumed not to have the form $f(x) = ax+b$. This is a trivial convex function in the sense that the inequality (1.4) becomes an identity. The reason for this assumption is stated in Remark 2.1 below.

2. Convexity results

2.1. Inequalities for convex functions

The main result of this section is to prove (1.13).

Theorem 2.1. *Let $f : I \longrightarrow [0,\infty)$ be convex and ϕ be a strictly increasing convex function defined on $[0,\infty)$, $\{x_1,\dots,x_n\} \subset I$, $\{\mu_1,\dots,\mu_n\} \subset (0,1)$ and $\{\tau_1,\dots,\tau_n\} \subset (0,1)$ be such that $\sum_{i=1}^n \mu_i = \sum_{i=1}^n \tau_i = 1$. Then*

$$\begin{aligned}
&\phi\Big(\min_{i=1,\dots,n}\left\{\frac{\mu_i}{\tau_i}\right\}\sum_{i=1}^n \tau_i f(x_i)\Big) - \phi\Big(\min_{i=1,\dots,n}\left\{\frac{\mu_i}{\tau_i}\right\} f\Big(\sum_{i=1}^n \tau_i x_i\Big)\Big)\\
\leq&\phi\Big(\sum_{i=1}^n \mu_i f(x_i)\Big) - \phi\circ f\Big(\sum_{i=1}^n \mu_i x_i\Big) \qquad (2.1)\\
\leq&\phi\Big(\max_{i=1,\dots,n}\left\{\frac{\mu_i}{\tau_i}\right\}\sum_{i=1}^n \tau_i f(x_i)\Big) - \phi\Big(\max_{i=1,\dots,n}\left\{\frac{\mu_i}{\tau_i}\right\} f\Big(\sum_{i=1}^n \tau_i x_i\Big)\Big).
\end{aligned}$$

Proof. Let $x=\sum_{i=1}^n \mu_i f(x_i)$, $y=f\Big(\sum_{i=1}^n \mu_i x_i\Big)$,
$z=\min_{i=1,\dots,n}\left\{\frac{\mu_i}{\tau_i}\right\}\sum_{i=1}^n \tau_i f(x_i)$, $w=\min_{i=1,\dots,n}\left\{\frac{\mu_i}{\tau_i}\right\} f\Big(\sum_{i=1}^n \tau_i x_i\Big)$,
$z'=\max_{i=1,\dots,n}\left\{\frac{\mu_i}{\tau_i}\right\}\Big(\sum_{i=1}^n \tau_i f(x_i)\Big)$ and $w'=\max_{i=1,\dots,n}\left\{\frac{\mu_i}{\tau_i}\right\} f\Big(\sum_{i=1}^n \tau_i x_i\Big)$. Then based on Theorem 1.1, we have

$$z-w\le x-y\le z'-w'.$$

The first and the second inequalities in (2.1) follow directly by applying Lemma 1.1 to the inequalities $z-w\le x-y$, with $w\le z\le x$, $y\le x$ and $x-y\le z'-w'$ with $y\le x\le z'$, $w'\le z'$, respectively. This completes the proof. $\square$

The significance of Theorem 2.1 is how certain selections of the function ϕ imply some interesting inequalities. For example, letting $\phi(x)=x^\lambda$ with $\lambda\ge 1$ in Theorem 2.1 implies (1.12) for any positive integer n, as follows.

Corollary 2.1. *Let $f:I\longrightarrow[0,\infty)$ be convex, $\{x_1,\dots,x_n\}\subset I$, $\{\mu_1,\dots,\mu_n\}\subset(0,1)$ and $\{\tau_1,\dots,\tau_n\}\subset(0,1)$ be such that $\sum_{i=1}^n\mu_i=\sum_{i=1}^n\tau_i=1$. Then for all real number $\lambda\ge1$,*

$$\min_{i=1,\dots,n}\left\{\frac{\mu_i}{\tau_i}\right\}^\lambda\le\frac{\Big(\sum_{i=1}^n\mu_i f(x_i)\Big)^\lambda-f^\lambda\Big(\sum_{i=1}^n\mu_i x_i\Big)}{\Big(\sum_{i=1}^n\tau_i f(x_i)\Big)^\lambda-f^\lambda\Big(\sum_{i=1}^n\tau_i x_i\Big)}\le\max_{i=1,\dots,n}\left\{\frac{\mu_i}{\tau_i}\right\}^\lambda. \tag{2.2}$$

The limiting case $\lambda\to\infty$ implies the following interesting inequality for convex functions.

Corollary 2.2. *Let $f:I\longrightarrow[0,\infty)$ be convex, $\{x_1,\dots,x_n\}\subset I$, $\{\mu_1,\dots,\mu_n\}\subset(0,1)$ and $\{\tau_1,\dots,\tau_n\}\subset(0,1)$ be such that $\sum_{i=1}^n\mu_i=\sum_{i=1}^n\tau_i=1$. Then*

$$\min\left\{\frac{\mu_i}{\tau_i}\right\}\le\frac{\sum_{i=1}^n\mu_i f(x_i)}{\sum_{i=1}^n\tau_i f(x_i)}\le\max\left\{\frac{\mu_i}{\tau_i}\right\}.$$

Proof. This follows from Corollary 2.1 by letting $\lambda\to\infty$, and noting that

$$f\left(\sum_{i=1}^n\mu_i x_i\right)\le\sum_{i=1}^n\mu_i f(x_i), f\left(\sum_{i=1}^n\tau_i x_i\right)\le\sum_{i=1}^n\tau_i f(x_i),$$

and that

$$\lim_{\lambda\to\infty}\left(a^\lambda-b^\lambda\right)^{\frac{1}{\lambda}}=a,$$

when $a>b$. $\square$

We should remark that immediate computations show that, for any positive function f,

$$\frac{\sum_{i=1}^n\mu_i f(x_i)}{\sum_{i=1}^n\tau_i f(x_i)}\le\frac{\max\{\mu_i\}}{\min\{\tau_i\}}.$$

However, Corollary 2.2 provides a better estimate for convex functions because

$$\max\left\{\frac{\mu_i}{\tau_i}\right\}\le\frac{\max\{\mu_i\}}{\min\{\tau_i\}}.$$

A similar argument applies for the first inequality in Corollary 2.2.

Remark 2.1. Tracking the proof of Corollary 2.1, we have

$$\begin{aligned}
&\min_{i=1,\dots,n}\left\{\frac{\mu_i}{\tau_i}\right\}^{\lambda}\left(\Big(\sum_{i=1}^{n}\tau_i f(x_i)\Big)^{\lambda}-f^{\lambda}\Big(\sum_{i=1}^{n}\tau_i x_i\Big)\right)\\
\le&\Big(\sum_{i=1}^{n}\mu_i f(x_i)\Big)^{\lambda}-f^{\lambda}\Big(\sum_{i=1}^{n}\mu_i x_i\Big)\\
\le&\max_{i=1,\dots,n}\left\{\frac{\mu_i}{\tau_i}\right\}^{\lambda}\left(\Big(\sum_{i=1}^{n}\tau_i f(x_i)\Big)^{\lambda}-f^{\lambda}\Big(\sum_{i=1}^{n}\tau_i x_i\Big)\right).
\end{aligned}$$

If it happens that $\sum_{i=1}^{n}\tau_i f(x_i)=f\Big(\sum_{i=1}^{n}\tau_i x_i\Big)$, then

$$\Big(\sum_{i=1}^{n}\tau_i f(x_i)\Big)^{\lambda}-f^{\lambda}\Big(\sum_{i=1}^{n}\tau_i x_i\Big)=0,$$

and hence $\sum_{i=1}^{n}\mu_i f(x_i)=f\Big(\sum_{i=1}^{n}\mu_i x_i\Big)$ for any choice of the μ_i. The only function that satisfies this identity is the linear function $f(x)=ax+b$. However, our default assumption is that f is not linear, see Remark 1.1 above.

On the other hand, applying Theorem 2.1 with $\phi(x)=\exp(\lambda x), \lambda>0$ implies the following new refinement and reverse of the corresponding inequalities of log-convex functions.

Corollary 2.3. *Let $f: I \longrightarrow (0,\infty)$ be log-convex, $\{x_1,\dots,x_n\}\subset I$, $\{\mu_1,\dots,\mu_n\}\subset(0,1)$ and $\{\tau_1,\dots,\tau_n\}\subset(0,1)$ be such that $\sum_{i=1}^{n}\mu_i=\sum_{i=1}^{n}\tau_i=1$. Then for all real number $\lambda>0$,*

$$\begin{aligned}
&\Big(\prod_{i=1}^{n}f^{\tau_i}(x_i)\Big)^{\lambda m}-f^{\lambda m}\Big(\sum_{i=1}^{n}\tau_i x_i\Big)\\
\le&\Big(\prod_{i=1}^{n}f^{\mu_i}(x_i)\Big)^{\lambda}-f^{\lambda}\Big(\sum_{i=1}^{n}\mu_i x_i\Big)\\
\le&\Big(\prod_{i=1}^{n}f^{\tau_i}(x_i)\Big)^{\lambda M}-f^{\lambda M}\Big(\sum_{i=1}^{n}\tau_i x_i\Big),
\end{aligned}\tag{2.3}$$

where $m=\min_{i=1,\dots,n}\{\frac{\mu_i}{\tau_i}\}$ and $M=\max_{i=1,\dots,n}\{\frac{\mu_i}{\tau_i}\}$.

2.2. Scalar inequalities

In this subsection, we present concrete applications of the inequalities obtained earlier. When $x>0$ and $0\neq p<1$ the function $f(x)=x^{\frac{1}{p}}$ is convex. Applying Corollary 2.1, we obtain the following new bounds for the difference between the arithmetic and power means. Here, we recall that given positive numbers $x_1,\cdots,x_n$ and $\mu_1,\cdots,\mu_n$ such that $\sum_{i=1}^{n}\mu_i=1$, the quantity $A:=\sum_{i=1}^{n}\mu_i x_i$ is called the arithmetic mean of the $\{x_i\}$. On the other hand, if $p\in\mathbb{R}$, the power mean of $\{x_i\}$ is defined by $M_p:=(\sum_{i=1}^{n}\mu_i x_i^p)^{\frac{1}{p}}$. When $p=0$, the power mean is calculated via a limit to obtain the geometric mean, namely $\prod_{i=1}^{n}x_i^{\mu_i}$. It is well known that, as a function of p, $(\sum_{i=1}^{n}\mu_i x_i^p)^{\frac{1}{p}}$ is an increasing function. Thus, when $p\le 1$, we have

$(\sum_{i=1}^n \mu_i x_i^p)^{\frac{1}{p}} \le \sum_{i=1}^n \mu_i x_i$. The following is a refinement and a reverse for this celebrated result.

Theorem 2.2. *Let n be a positive integer and $0 \ne p < 1$. For $i = 1, 2, \dots, n$, let $x_i > 0$ and $\{\mu_1, \dots, \mu_n\} \subset (0,1)$ and $\{\tau_1, \dots, \tau_n\} \subset (0,1)$ be such that $\sum_{i=1}^n \mu_i = \sum_{i=1}^n \tau_i = 1$. Then for $\lambda \ge 1$,*

$$\min_{i=1,\dots,n}\left\{\frac{\mu_i}{\tau_i}\right\}^\lambda \le \frac{(\sum_{i=1}^n \mu_i x_i)^\lambda - (\sum_{i=1}^n \mu_i x_i^p)^{\frac{\lambda}{p}}}{(\sum_{i=1}^n \tau_i x_i)^\lambda - (\sum_{i=1}^n \tau_i x_i^p)^{\frac{\lambda}{p}}} \le \max_{i=1,\dots,n}\left\{\frac{\mu_i}{\tau_i}\right\}^\lambda. \tag{2.4}$$

On the other hand, letting $p = -1$ in Theorem 2.2, we have the following bounds for the difference between the arithmetic and harmonic means. We recall here that the harmonic, denoted H, mean of the above parameters corresponds to the power mean when $p = -1$. Since the power means increase in p, it is clear that $H \le A$. The following is a reverse and a refinement of this inequality.

Corollary 2.4. *Let n be a positive integer. For $i = 1, 2, \dots, n$, let $x_i > 0$ and $\{\mu_1, \dots, \mu_n\} \subset (0,1)$ and $\{\tau_1, \dots, \tau_n\} \subset (0,1)$ be such that $\sum_{i=1}^n \mu_i = \sum_{i=1}^n \tau_i = 1$. Then for all real number $\lambda \ge 1$,*

$$\min_{i=1,\dots,n}\left\{\frac{\mu_i}{\tau_i}\right\}^\lambda \le \frac{(\sum_{i=1}^n \mu_i x_i)^\lambda - \left(\sum_{i=1}^n \mu_i x_i^{-1}\right)^{-\lambda}}{(\sum_{i=1}^n \tau_i x_i)^\lambda - \left(\sum_{i=1}^n \tau_i x_i^{-1}\right)^{-\lambda}} \le \max_{i=1,\dots,n}\left\{\frac{\mu_i}{\tau_i}\right\}^\lambda. \tag{2.5}$$

If we let $p \longrightarrow 0$ in Theorem 2.2, we get the following bounds for the difference between the arithmetic and geometric means.

Corollary 2.5. *Let n be a positive integer. For $i = 1, 2, \dots, n$, let $x_i > 0$ and $\{\mu_1, \dots, \mu_n\} \subset (0,1)$ and $\{\tau_1, \dots, \tau_n\} \subset (0,1)$ be such that $\sum_{i=1}^n \mu_i = \sum_{i=1}^n \tau_i = 1$. Then for all real number $\lambda \ge 1$,*

$$\min_{i=1,\dots,n}\left\{\frac{\mu_i}{\tau_i}\right\}^\lambda \le \frac{(\sum_{i=1}^n \mu_i x_i)^\lambda - (\prod_{i=1}^n x_i^{\mu_i})^\lambda}{(\sum_{i=1}^n \tau_i x_i)^\lambda - (\prod_{i=1}^n x_i^{\tau_i})^\lambda} \le \max_{i=1,\dots,n}\left\{\frac{\mu_i}{\tau_i}\right\}^\lambda. \tag{2.6}$$

When $x > 0$ and $p \in (-\infty, 0)$, the function $f(x) = x^{\frac{1}{p}}$ is log-convex. Applying Corollary 2.3, we obtain the following new bounds for the difference between the arithmetic and power means.

Theorem 2.3. *Let n be a positive integer and $p \in (-\infty, 0)$. For $i = 1, 2, \dots, n$, let $x_i > 0$ and $\{\mu_1, \dots, \mu_n\} \subset (0,1)$ and $\{\tau_1, \dots, \tau_n\} \subset (0,1)$ be such that $\sum_{i=1}^n \mu_i = \sum_{i=1}^n \tau_i = 1$. Then for all real number $\lambda > 0$,*

$$\begin{aligned}
&\left(\prod_{i=1}^n x_i^{\tau_i}\right)^{\lambda m} - \left(\sum_{i=1}^n \tau_i x_i^p\right)^{\frac{\lambda m}{p}} \\
&\le \left(\prod_{i=1}^n x_i^{\mu_i}\right)^{\lambda} - \left(\sum_{i=1}^n \mu_i x_i^p\right)^{\frac{\lambda}{p}} \\
&\le \left(\prod_{i=1}^n x_i^{\tau_i}\right)^{\lambda M} - \left(\sum_{i=1}^n \tau_i x_i^p\right)^{\frac{\lambda M}{p}},
\end{aligned}$$

where $m = \min_{i=1,\dots,n}\{\frac{\mu_i}{\tau_i}\}$ and $M = \max_{i=1,\dots,n}\{\frac{\mu_i}{\tau_i}\}$.

On the other hand, letting $p=-1$ in Theorem 2.3, we have the following bounds for the difference between the geometric and harmonic means.

Corollary 2.6. *For $i=1,2,\ldots,n$, let $x_i>0$ and $\{\mu_1,\ldots,\mu_n\}\subset(0,1)$ and $\{\tau_1,\ldots,\tau_n\}\subset(0,1)$ be such that $\sum_{i=1}^n\mu_i=\sum_{i=1}^n\tau_i=1$. Then for $\lambda>0$,*

$$\begin{aligned}&\left(\prod_{i=1}^n x_i^{\tau_i}\right)^{\lambda m}-\left(\sum_{i=1}^n\tau_i x_i^{-1}\right)^{-\lambda m}\\ \leq&\left(\prod_{i=1}^n x_i^{\mu_i}\right)^{\lambda}-\left(\sum_{i=1}^n\mu_i x_i^{-1}\right)^{-\lambda}\\ \leq&\left(\prod_{i=1}^n x_i^{\tau_i}\right)^{\lambda M}-\left(\sum_{i=1}^n\tau_i x_i^{-1}\right)^{-\lambda M},\end{aligned}$$

where $m=\min_{i=1,\ldots,n}\{\frac{\mu_i}{\tau_i}\}$ and $M=\max_{i=1,\ldots,n}\{\frac{\mu_i}{\tau_i}\}$.

2.3. The special case $n=2$

In this part of the paper, we present the special case $n=2$. This was treated in [26]. Here, we have the more general result than (1.12), as follows.

Theorem 2.4. *Let $f:[0,1]\longrightarrow[0,\infty)$ be convex and ϕ be a strictly increasing convex function defined on $\mathbb{R}$ and let τ,μ be real numbers with $0<\mu<\tau<1$. Then*

$$\begin{aligned}&\phi\Big(\frac{\mu}{\tau}(\tau f(1)+(1-\tau)f(0))\Big)-\phi\Big(\frac{\mu}{\tau}f(\tau)\Big)\\ \leq&\phi(\mu f(1)+(1-\mu)f(0))-\phi\circ f(\mu)\\ \leq&\phi\Big(\frac{1-\mu}{1-\tau}(\tau f(1)+(1-\tau)f(0))\Big)-\phi\Big(\frac{1-\mu}{1-\tau}f(\tau)\Big).\end{aligned}$$

This entails the following result for log-convex functions.

Corollary 2.7. *Let $f:[0,1]\longrightarrow[0,\infty)$ be log-convex and let τ,μ and λ be real numbers with $\lambda>0$ and $0<\mu<\tau<1$. Then*

$$\begin{aligned}&\Big(f^{\tau}(1)f^{1-\tau}(0)\Big)^{\frac{\lambda\mu}{\tau}}-f^{\frac{\lambda\mu}{\tau}}(\tau)\\ \leq&\Big(f^{\mu}(1)f^{1-\mu}(0)\Big)^{\lambda}-f^{\lambda}(\mu)\\ \leq&\Big(f^{\tau}(1)f^{1-\tau}(0)\Big)^{\frac{\lambda(1-\mu)}{1-\tau}}-f^{\frac{\lambda(1-\mu)}{1-\tau}}(\tau).\end{aligned}$$

We refer the reader to [26] to see the significance of the case $n=2$, where then the reader can apply Theorem 2.4 to obtain some applications. It is important to notice that the motivation of [26] is to complement the study of [4, 19]. In [28] it was noted that the result of [4] is better than the main result in [2]. Therefore, the results in this paper extend in a more general setting the results in these references.

2.4. Matrix norm inequalities

Let $\mathbb{M}_n$ be the algebra of all complex matrices of order $n\times n$. The positive semi-definite matrix $A\in\mathbb{M}_n$ written as $A\geq0$, is a Hermitian matrix with $\langle Ax,x\rangle\geq0$

for all $x \in \mathbb{C}^n$. If $A \in \mathbb{M}_n$ is a Hermitian matrix with $\langle Ax, x\rangle > 0$ for all nonzero $x \in \mathbb{C}^n$, then A is called a positive definite matrix, written as $A > 0$. The set of all positive semi-definite matrices is denoted by $\mathbb{M}_n^+$ and the set of all positive definite matrices in $\mathbb{M}_n$ is denoted by $\mathbb{M}_n^{++}$. The singular values of a matrix $A \in \mathbb{M}_n$ are the eigenvalues of the positive semi-definite matrix $|A| = (A^*A)^{1/2}$, denoted by $s_i(A)$ for $i = 1, 2, 3, \dots, n$. A matrix norm $|||\cdot|||$ on $\mathbb{M}_n$ is called unitarily invariant if $|||UAV||| = |||A|||$ for all $A \in \mathbb{M}_n$ and all unitary matrices $U, V \in \mathbb{M}_n$. The trace norm is given by $||A||_1 = tr|A| = \sum_{k=1}^n s_k(A)$, where tr is the usual trace. This norm is unitarily invariant. An important example of unitarily invariant norms is the Hilbert-Schmidt norm $||\cdot||_2$ defined by

$$||A||_2 = (tr(A^*A))^{\frac{1}{2}} = \Big(\sum_{i,j} |a_{i,j}|^2\Big)^{\frac{1}{2}}, \quad (A = (a_{i,j})).$$

Let $A, B \in \mathbb{M}_n^+, X \in \mathbb{M}_n$, and $\tau \in [0, 1]$. The celebrated Hölder's inequality asserts that [15]

$$|||A^\tau X B^{1-\tau}||| \le |||AX|||^\tau |||XB|||^{1-\tau}. \tag{2.7}$$

In particular

$$tr|A^\tau B^{1-\tau}| \le (trA)^\tau (trB)^{1-\tau}. \tag{2.8}$$

It is known that when $A, B \in \mathbb{M}_n^+$ and $X \in \mathbb{M}_n$, the function $f(\mu) = |||A^\mu X B^{1-\mu}|||$ is log-convex on $[0, 1]$, (see [27]) for any unitarily invariant norm $|||\cdot|||$ on $\mathbb{M}_n$. Applying Corollary 2.7, we obtain the following new refinement and reverse of the Hölder's inequality.

Theorem 2.5. *Let $A, B \in \mathbb{M}_n^+$, $X \in \mathbb{M}_n$ and let τ, μ and λ be real numbers with $\lambda > 0$ and $0 < \mu < \tau < 1$. Then*

$$\begin{aligned}&\Big(|||AX|||^\tau |||XB|||^{1-\tau}\Big)^{\frac{\lambda\mu}{\tau}} - |||A^\tau X B^{1-\tau}|||^{\frac{\lambda\mu}{\tau}}\\ \le& |||AX|||^\mu |||XB|||^{1-\mu} - |||A^\mu X B^{1-\mu}|||\\ \le&\Big(|||AX|||^\tau |||XB|||^{1-\tau}\Big)^{\frac{\lambda(1-\mu)}{1-\tau}} - |||A^\tau X B^{1-\tau}|||^{\frac{\lambda(1-\mu)}{1-\tau}}.\end{aligned}$$

In particular,

$$\begin{aligned}&\Big(tr(A)^\tau tr(B)^{1-\tau}\Big)^{\frac{\lambda\mu}{\tau}} - tr(|A^\tau B^{1-\tau}|)^{\frac{\lambda\mu}{\tau}}\\ \le& tr(A)^\mu tr(B)^{1-\mu} - tr(|A^\mu B^{1-\mu}|)\\ \le&\Big(tr(A)^\tau tr(B)^{1-\tau}\Big)^{\frac{\lambda(1-\mu)}{1-\tau}} - tr(|A^\tau B^{1-\tau}|)^{\frac{\lambda(1-\mu)}{1-\tau}}.\end{aligned}$$

It is known that when $A, B \in \mathbb{M}_n^+$ and $X \in \mathbb{M}_n$, the function $f(\mu) = |||A^\mu X B^{1-\mu} + A^{1-\mu} X B^\mu|||$ is convex on $[0, 1]$, (see [5, Theorem IX.4.8]) for any unitarily invariant norm $|||\cdot|||$ on $\mathbb{M}_n$. Then by using the Theorem 2.4 we have the following result.

Theorem 2.6. *Let ϕ be a strictly increasing convex function defined on $\mathbb{R}$ and let τ, μ be real numbers with $0 < \mu < \tau < 1$. Then*

$$\phi\Big(\frac{\mu}{\tau}|||AX + XB|||\Big) - \phi\Big(\frac{\mu}{\tau}|||A^\tau X B^{1-\tau} + A^{1-\tau} X B^\tau|||\Big)$$

$$
\begin{aligned}
&\leq \phi(|||AX+XB|||) - \phi(|||A^{\mu}XB^{1-\mu}+A^{1-\mu}XB^{\mu}|||)\\
&\leq \phi\Big(\frac{1-\mu}{1-\tau}|||AX+XB|||\Big) - \phi\Big(\frac{1-\mu}{1-\tau}|||A^{\tau}XB^{1-\tau}+A^{1-\tau}XB^{\tau}|||\Big).
\end{aligned}
$$

The next lemma provides a technical result which we will need in the next result.

Lemma 2.1 ([26]). *Let $A, B \in \mathbb{M}_n^+$ and $X \in \mathbb{M}_n$ and let $f(x) = ||A^x XB^{1-x} + A^{1-x}XB^x||_2$. Then f is log-convex on $[0, 1]$.*

Using this lemma, together with Theorem 2.4, we have the following Theorem.

Theorem 2.7. *Let $A, B \in \mathbb{M}_n^+$ and $X \in \mathbb{M}_n$ and τ, μ and λ be real numbers with $\lambda > 0$ and $0 < \mu < \tau < 1$. Then*

$$
\begin{aligned}
&||AX+XB||_2^{\frac{\lambda\mu}{\tau}} - ||A^{\tau}XB^{1-\tau}+A^{1-\tau}XB^{\tau}||_2^{\frac{\lambda\mu}{\tau}}\\
\leq &||AX+XB||_2^{\lambda} - ||A^{\mu}XB^{1-\mu}+A^{1-\mu}XB^{\mu}||_2^{\lambda}\\
\leq &||AX+XB||_2^{\frac{\lambda(1-\mu)}{1-\tau}} - ||A^{\tau}XB^{1-\tau}+A^{1-\tau}XB^{\tau}||_2^{\frac{\lambda(1-\mu)}{1-\tau}}.
\end{aligned}
$$

These inequalities are interesting refinement and reversal of the well known Heinz inequality (see [5, Theorem IX.4.8])

$$||A^{\tau}XB^{1-\tau}+A^{1-\tau}XB^{\tau}||_2 \leq ||AX+XB||_2.$$

3. Log-convexity results

3.1. Inequalities for log-convex functions

In this part of the paper, we prove (1.14) to complement our analysis for log-convex functions. The main result of this section reads as follows.

Theorem 3.1. *Let $f : I \longrightarrow (0, \infty)$ be log-convex, $\{x_1, \ldots, x_n\} \subset I$, $\{\mu_1, \ldots, \mu_n\} \subset (0, 1)$ and $\{\tau_1, \ldots, \tau_n\} \subset (0, 1)$ be such that $\sum_{i=1}^n \mu_i = \sum_{i=1}^n \tau_i = 1$. Then for all positive integers m,*

$$
\begin{aligned}
&f^m\left(\sum_{i=1}^n \mu_i x_i\right) + \min_{i=1,\ldots,n}\left\{\frac{\mu_i^m}{\tau_i}\right\}\left(\sum_{i=1}^n \tau_i f^m(x_i) - f^m\Big(\sum_{i=1}^n \tau_i x_i\Big)\right)\\
\leq &\left(\sum_{i=1}^n \mu_i f(x_i)\right)^m.
\end{aligned}
\tag{3.1}
$$

Proof. We claim that

$$
\begin{aligned}
&\left(\sum_{i=1}^n \mu_i f(x_i)\right)^m - \min_{i=1,\ldots,n}\left\{\frac{\mu_i^m}{\tau_i}\right\}\left(\sum_{i=1}^n \tau_i f^m(x_i) - f^m\Big(\sum_{i=1}^n \tau_i x_i\Big)\right)\\
\geq &f^m\left(\sum_{i=1}^n \mu_i x_i\right).
\end{aligned}
$$

Indeed, by Lemma 1.2, we have the following equality

$$\left(\sum_{i=1}^n \mu_i f(x_i)\right)^m - \min_{i=1,\ldots,n}\left\{\frac{\mu_i^m}{\tau_i}\right\}\left(\sum_{i=1}^n \tau_i f^m(x_i) - f^m\Big(\sum_{i=1}^n \tau_i x_i\Big)\right)$$

$$= \sum_{(i_1,\dots,i_{n-1})\in A} \binom{i_0}{i_1}\binom{i_1}{i_2}\dots\binom{i_{n-2}}{i_{n-1}} \mu_1^{i_0-i_1}\mu_2^{i_1-i_2}\dots\mu_n^{i_{n-1}-i_n} f^{i_0-i_1}(x_1)$$
$$\times f^{i_1-i_2}(x_2)\dots f^{i_{n-1}-i_n}(x_n)$$
$$-\min_{i=1,\dots,n}\left\{\frac{\mu_i^m}{\tau_i}\right\}\left(\sum_{i=1}^n \tau_i f^m(x_i) - f^m\Big(\sum_{i=1}^n \tau_i x_i\Big)\right).$$

Let B be a subset of A such that

$$\sum_{(i_1,\dots,i_{n-1})\in A} \binom{i_0}{i_1}\binom{i_1}{i_2}\dots\binom{i_{n-2}}{i_{n-1}} \mu_1^{i_0-i_1}\mu_2^{i_1-i_2}\dots\mu_n^{i_{n-1}-i_n}$$
$$\times f^{i_0-i_1}(x_1) f^{i_1-i_2}(x_2)\dots f^{i_{n-1}-i_n}(x_n)$$
$$= \sum_{(i_1,\dots,i_{n-1})\in B} \binom{i_0}{i_1}\binom{i_1}{i_2}\dots\binom{i_{n-2}}{i_{n-1}} \mu_1^{i_0-i_1}\mu_2^{i_1-i_2}\dots\mu_n^{i_{n-1}-i_n}$$
$$\times f^{i_0-i_1}(x_1) f^{i_1-i_2}(x_2)\dots f^{i_{n-1}-i_n}(x_n)$$
$$+\sum_{i=1}^n \mu_i^m f^m(x_i).$$

Hence, by the log-convexity of the function f^m it follows that.

$$\left(\sum_{i=1}^n \mu_i f(x_i)\right)^m - \min_{i=1,\dots,n}\left\{\frac{\mu_i^m}{\tau_i}\right\}\left(\sum_{i=1}^n \tau_i f^m(x_i) - f^m\Big(\sum_{i=1}^n \tau_i x_i\Big)\right)$$
$$= \sum_{(i_1,\dots,i_{n-1})\in B} \binom{i_0}{i_1}\binom{i_1}{i_2}\dots\binom{i_{n-2}}{i_{n-1}}$$
$$\mu_1^{i_0-i_1}\mu_2^{i_1-i_2}\dots\mu_n^{i_{n-1}-i_n} f^{i_0-i_1}(x_1) f^{i_1-i_2}(x_2)\dots f^{i_{n-1}-i_n}(x_n)$$
$$+\sum_{i=1}^n \left(\mu_i^m - \min_{i=1,\dots,n}\left\{\frac{\mu_i^m}{\tau_i}\right\}\tau_i\right) f^m(x_i) + \min_{i=1,\dots,n}\left\{\frac{\mu_i^m}{\tau_i}\right\} f^m\Big(\sum_{i=1}^n \tau_i x_i\Big)$$
$$\geq \sum_{(i_1,\dots,i_{n-1})\in B} \binom{i_0}{i_1}\binom{i_1}{i_2}\dots\binom{i_{n-2}}{i_{n-1}} \mu_1^{i_0-i_1}\mu_2^{i_1-i_2}\dots\mu_n^{i_{n-1}-i_n} f^m\Big(\sum_{k=1}^n \frac{i_{k-1}-i_k}{m} x_k\Big)$$
$$+\sum_{i=1}^n (\mu_i^m - \min_{i=1,\dots,n}\left\{\frac{\mu_i^m}{\tau_i}\right\})\tau_i f^m(x_i) + \min_{i=1,\dots,n}\left\{\frac{\mu_i^m}{\tau_i}\right\} f^m\Big(\sum_{i=1}^n \tau_i x_i\Big). \tag{3.2}$$

We have

$$\sum_{(i_1,\dots,i_{n-1})\in B} \binom{i_0}{i_1}\binom{i_1}{i_2}\dots\binom{i_{n-2}}{i_{n-1}} \mu_1^{i_0-i_1}\mu_2^{i_1-i_2}\dots\mu_n^{i_{n-1}-i_n}$$
$$+\sum_{i=1}^n (\mu_i^m - \min_{i=1,\dots,n}\left\{\frac{\mu_i^m}{\tau_i}\right\})\tau_i + \min_{i=1,\dots,n}\left\{\frac{\mu_i^m}{\tau_i}\right\}$$
$$=1.$$

Thus (3.2) is a convex combination of positive numbers. Therefore, by the Jensen's inequality and Lemma 1.2, we get

$$\left(\sum_{i=1}^n \mu_i f(x_i)\right)^m - \min_{i=1,\dots,n}\left\{\frac{\mu_i^m}{\tau_i}\right\}\left(\sum_{i=1}^n \tau_i f^m(x_i) - f^m\Big(\sum_{i=1}^n \tau_i x_i\Big)\right)$$

$$
\begin{aligned}
&\geq f^m\Big(\sum_{(i_1,\ldots,i_{n-1})\in B}\binom{i_0}{i_1}\binom{i_1}{i_2}\cdots\binom{i_{n-2}}{i_{n-1}}\mu_1^{i_0-i_1}\mu_2^{i_1-i_2}\cdots\mu_n^{i_{n-1}-i_n}\\
&\quad\times\Big(\sum_{k=1}^{n}\frac{i_{k-1}-i_k}{m}x_k\Big)+\sum_{i=1}^{n}(\mu_i^m-\min_{i=1,\ldots,n}\Big\{\frac{\mu_i^m}{\tau_i}\Big\})\tau_i x_i+\min_{i=1,\ldots,n}\Big\{\frac{\mu_i^m}{\tau_i}\Big\}\sum_{i=1}^{n}\tau_i x_i\Big)\\
&=f^m\left(\sum_{i=1}^{n}\mu_i x_i\right).
\end{aligned}
$$

This completes the proof. □

3.2. Scalar inequalities

When $x>0$ and $p\in(-\infty,0)$ the function $f(x)=x^{\frac{1}{p}}$ is log-convex. Applying Theorems 3.1, we obtain the following new bounds for the difference between the arithmetic and power means.

Corollary 3.1. *Let n be a positive integer and $p\in(-\infty,0)$. For $i=1,2,\ldots,n$, let $x_i>0$, $\{\mu_1,\ldots,\mu_n\}\subset(0,1)$ and $\{\tau_1,\ldots,\tau_n\}\subset(0,1)$ be such that $\sum_{i=1}^n\mu_i=\sum_{i=1}^n\tau_i=1$. Then for all positive integers m,*

$$
\left(\sum_{i=1}^{n}\mu_i x_i^p\right)^{\frac{m}{p}}+\min_{i=1,\ldots,n}\Big\{\frac{\mu_i^m}{\tau_i}\Big\}\left(\sum_{i=1}^{n}\tau_i x_i^m-\Big(\sum_{i=1}^{n}\tau_i x_i^p\Big)^{\frac{m}{p}}\right)\leq\left(\sum_{i=1}^{n}\mu_i x_i\right)^m. \tag{3.3}
$$

On the other hand, letting $p=-1$ in Corollary 3.1, we have the following bounds for the difference between the arithmetic and harmonic means.

Corollary 3.2. *Let n be a positive integer and $p\in(-\infty,0)$. For $i=1,2,\ldots,n$, let $x_i>0$, $\{\mu_1,\ldots,\mu_n\}\subset(0,1)$ and $\{\tau_1,\ldots,\tau_n\}\subset(0,1)$ be such that $\sum_{i=1}^n\mu_i=\sum_{i=1}^n\tau_i=1$. Then for all positive integers m,*

$$
\left(\sum_{i=1}^{n}\mu_i x_i^{-1}\right)^{-m}+\min_{i=1,\ldots,n}\Big\{\frac{\mu_i^m}{\tau_i}\Big\}\left(\sum_{i=1}^{n}\tau_i x_i^m-\Big(\sum_{i=1}^{n}\tau_i x_i^{-1}\Big)^{-m}\right)\leq\left(\sum_{i=1}^{n}\mu_i x_i\right)^m. \tag{3.4}
$$

If we let $p\longrightarrow 0$ in Corollary 3.1, we get the following bounds for the difference between the arithmetic and geometric means.

Corollary 3.3. *Let n be a positive integer and $p\in(-\infty,0)$. For $i=1,2,\ldots,n$, let $x_i>0$, $\{\mu_1,\ldots,\mu_n\}\subset(0,1)$ and $\{\tau_1,\ldots,\tau_n\}\subset(0,1)$ be such that $\sum_{i=1}^n\mu_i=\sum_{i=1}^n\tau_i=1$. Then for all positive integers m,*

$$
\left(\prod_{i=1}^{n}x_i^{\mu_i}\right)^m+\min_{i=1,\ldots,n}\Big\{\frac{\mu_i^m}{\tau_i}\Big\}\left(\sum_{i=1}^{n}\tau_i x_i^m-\Big(\prod_{i=1}^{n}x_i^{\tau_i}\Big)^m\right)\leq\left(\sum_{i=1}^{n}\mu_i x_i\right)^m. \tag{3.5}
$$

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