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DYNAMICAL BEHAVIOR OF THE FECAL-ORAL TRANSMISSION DISEASES MODEL ON A T-PERIODIC EVOLUTION DOMAIN*

You Zhou[1], Beibei Zhang[1] and Zhi Ling[1,†]

Abstract We study the transmission dynamics of a fecal-oral diseases model on a T-periodic evolution domain. We introduce the basic reproduction number $R_0(\rho)$ as a threshold by some operator semigroup theory and give the relationship between it and that of the fixed domain, where $\rho(t)$ is the domain evolution rate. By means of upper and lower solutions method, we investigate the existence, uniqueness and attractivity of endemic and disease-free equilibria respectively. Under certain conditions, there exists a unique global asymptotically stable positive periodic solution if $R_0(\rho) > 1$. When $R_0(\rho) \leq 1$, the model possesses only zero solutions and is globally asymptotically stable. The final numerical simulations further verify our conclusions and illustrate the effect of the evolution rate. Based on the index $\overline{\rho^{-2}} := \frac{1}{T}\int_0^T \frac{1}{\rho(t)^2}\mathrm{d}t$, compared with the model on a fixed domain, we show that the transmission risk of the diseases increases if the index is lower than 1 and the risk decreases if the index is equal or greater than 1.

Keywords Periodic evolution domain, fecal-oral transmission diseases, basic reproduction number, dynamical behavior.

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1. Introduction

Emerging infectious diseases such as SARS, dengue fever, malaria, COVID-19, etc. are major public health problems worldwide. Some characteristic features of these emerging infectious diseases are sudden and widespread outbreaks. Thousands of people suffer from these emerging infectious diseases every year, meanwhile, many easy-to-treat diseases have not been paid enough attention to, for example cholera, schistosomiasis, rabies and so on, resulting in the endemic diseases. As an endemic disease, the fecal-oral transmission diseases (FOTD) are caused by the bacteria and viruses from feces of infected persons entering the respiratory and alimentary tracts of the susceptible. Take hand-foot-and-mouth diseases (HFMD) as an example, millions of children who are mainly under five-years old have been affected in the

[†]The corresponding author.
[1]School of Mathematical Science, Yangzhou University, 225002 Yangzhou, China
[*]The authors were supported by National Natural Science Foundation of China (12171418 and 12371505).
Email: DX120220050@stu.yzu.edu.cn(Y. Zhou), beibzhang96@163.com(B. Zhang), zhling@yzu.edu.cn(Z. Ling)

Western Pacific [25, 33]. Since the viruses can persist in feces for several weeks, HFMD is prone to outbreaks in nurseries, summer camps or within households etc., putting family members and close contacts at risk. It follows then that it is very important and urgent to study the transmission law and diffusion mechanism of viruses so as to effectively prevent and cure endemic infectious diseases similar to the FOTD.

In view of the non-experimental characteristic of infectious diseases, mathematical modeling [23] is one of the important tools to investigate the outbreak and transmission laws of contagious diseases and then to prevent and control contagion [15]. As early as 1760, Bernoulli [6] applied the mathematical model to investigate the transmission of smallpox. In 1911, Ross [32] established the compartment model to study the process of malaria transmission, which further advanced the process of researching the dynamics of infectious diseases. By 1927, Kermack and McKendrick [21] established the SIR epidemic model to explore the transmission law of Black Death and provided the threshold theory to predict the risk of its outbreaks, which are still being perfected and widely applied. In 1981, Capasso and Maddalena [8] proposed the following reaction-diffusion system to understand the dynamics of the FOTD in the Mediterranean region of Europe:

$$\begin{cases} u_{1t}(x,t) - d_1\Delta u_1(x,t) = -a_{11}u_1(x,t) + a_{12}u_2(x,t), & x \in \Omega,\ t > 0, \\ u_{2t}(x,t) - d_2\Delta u_2(x,t) = -a_{22}u_2(x,t) + g(u_1(x,t)), & x \in \Omega,\ t > 0, \\ \dfrac{\partial u_1}{\partial \eta} + \alpha_1 u_1 = \dfrac{\partial u_2}{\partial \eta} + \alpha_2 u_2 = 0, & x \in \partial\Omega,\ t > 0, \\ u_1(x,0) = u_{10}(x) \geq 0,\ u_2(x,0) = u_{20}(x) \geq 0, & x \in \Omega. \end{cases} \tag{1.1}$$

Here $u_1(x,t)$ and $u_2(x,t)$ represent the densities of bacteria and the infective at time t and location $x \in \Omega \subseteq \mathbb{R}^n$ $(n = 1, 2, 3)$, respectively. The bounded habitat Ω is an open domain with a sufficiently smooth boundary $\partial\Omega$. Parameters $d_1,\ d_2 \geq 0$ are the diffusion coefficients of u_1 and u_2. The natural mortality of bacteria is a_{11}. The ratio a_{12} denotes the infective contribution to the growth of bacteria. While a_{22} represents the natural damping rate of the infected human population due to the finite average duration of humans' infectiousness. The normal derivative along η on $\partial\Omega$ is denoted by $\frac{\partial}{\partial \eta}$, where η is the outward unit normal vector. Under the assumption that the total susceptible is constant during the evolution of the epidemic, the last term $g(u_1)$ describes the infection rate of humans which satisfies the following conditions:

(C1) $g \in C^2([0,\infty)),\ g(0) = 0$;

(C2) $g^{'}(z) > 0$ and $g^{''}(z) < 0$ for all $z \geq 0$;

(C3) $\dfrac{g(z)}{z}$ is decreasing and $\lim\limits_{z\to\infty} \dfrac{g(z)}{z} < \dfrac{a_{11}a_{22}}{a_{12}}$.

They considered the Robin boundary conditions $\dfrac{\partial u_i}{\partial \eta} + \alpha_i u_i = 0$ for $x \in \partial\Omega$ and $i = 1, 2$, where more detailed biological interpretations on its boundary conditions can see for example [7]. The initial conditions satisfied $u_{i0} \in C^2(\overline{\Omega})$ and were both not identically zero for all $x \in \partial\Omega$. This model can be applied to explain many epidemics through fecal-oral transmission such as typhoid, infectious hepatitis, colitis and so on.

In recent years, multiple issues based on the FOTD model (1.1) have been researched by scholars from various countries. S.L. Wu, G.S. Chen, C.H. Hsu [37] and W.B. Xu, W.T. Li, S.G. Ruan [39] considered the nonlocal diffusion epidemic model arising from the spread of fecally–orally transmitted diseases, respectively. Specifically, Wu et al. were concerned with the well-posedness of entire solutions to the model while Xu et al. focused on the effects of initial values and nonlocal dispersal on its spatial propagation. Tang and Ouyang [35] introduced the chemotaxis into model (1.1) with Neumann boundary conditions generating Turing bifurcations and also proved the stability of positive equilibrium. In addition, based on model (1.1), Huang and Tan [17] introduced the pathogens' incubation period to investigate the global dynamical behaviors of the model under homogeneous Dirichlet boundary conditions by utilizing operator semigroup theory and dynamical systems approach. More work about the variants of model (1.1) can refer to [16, 24, 28, 41] and the references therein.

Compared with the work on changing underlying areas, most of the foregoing work is confined to fixed domains, which exists certain biological limitations. In real life, the species' habitat may expand or shrink in response to the changing ecological environment. For instance, influenced by geography, the boundaries of species' habitats expand or contract over time as populations invade into the new environments, which is mathematically reduced to a free boundary problem [11, 12, 18, 28]. In particular, some bounded habitats may present certain regular changes in reality under the influence of natural environment. For example, some insects inhabit on growing leaves [34]; the habitats of some fish expand with the rise of river temperatures [9,31]; the area of lakes and vegetation are periodically changeful on account of the seasonal replacement [22]. In fact, we often attribute these types of the biological habitat's evolution to the problems of growing domain [13,26,40,43] and periodic evolution [19,20,27,36,38,44], describing the process of species dispersal and its pattern formation and long term behavior. Madzvamuse [26] discovered the conditions under which standard linear stability theory holds for reaction-diffusion systems with constant coefficients on growing domains. Garduño et al. [13] focused on the spatial and spatio-temporal patterns in the reaction-diffusion FitzHugh-Nagumo model on growing curved domains. Zhang et al. [40] established a FOTD model with homogeneous Robin boundary conditions on a growing domain to explore the effect of growth function $\rho(t)$ on its asymptotic behavior. Zhu et al. were concerned with the dengue fever model on a growing domain [43] and a T-periodic evolution domain [44], respectively, to investigate the long term dynamic behaviors of dengue virus in terms of the corresponding basic reproduction numbers and compare with results obtained in the case of fixed domains. Moreover, for spatially isotropic and temporally periodic evolution domains, Montano and Lisena [27] dealt with the dynamics of a diffusive Lotka-Volterra model with periodic coefficients and zero-flux boundary conditions. Kavallaris et al. [20] were devoted to the dynamics of the shadow system of a singular Gierer–Meinhardt model and gave blow-up results for the nonlocal equation to interpret instability patterns. Jiang and Wang [19] derived a n-dimensional diffusive logistic equation and analyzed the effect of evolution on the persistence of single species. Tong and Lin [36] considered an SIS reaction-diffusion model with logistic term. They explored the effect of periodically evolving domains on the spread of diseases and discussed the stability of the disease-free equilibrium by the basic reproduction number. Xu et al. [38] proposed a generalized logistic model with impulses in an evolving domain to research the impacts of regional evolution and impulses on the persistence or extinction of species.

Inspired by the work mentioned above, we introduce a periodic evolution domain into the FOTD model (1.1). Compared with the FOTD model on a growing domain which can see for example [40], our aim here is to discuss the effect of periodic evolution on the dynamics of bacteria and reveal some new phenomena caused by the domain evolution.

The rest of this paper is organized as follows. We first derive the FOTD model and its basic reproduction number R_0 followed by some preliminaries in section 2. Existence, uniqueness and attractivity of endemic and disease-free equilibria on the evolution and fixed domains are analysed in section 3. In section 4, we give some numerical simulations and epidemiological explanations. We end with a brief summary in section 5.

2. Preparatory work

In this section, we first deduce the FOTD model on the evolution domain and then transform it into a model with time-period coefficients on the fixed domain followed by derivation of basic reproduction number R_0 and its some properties which can be useful for later sections.

Suppose that the evolution domain $\Omega(t) \subset \mathbb{R}^n$ $(n \geq 1)$ is simply connected and bounded with evolution boundary $\partial\Omega(t)$ for time $t \geq 0$. Considering any point $x = (x_1(t), x_2(t), \ldots, x_n(t))$ in $\Omega(t)$, we denote the spatial densities of bacteria and the infective at location x and time t by $u_1(x,t)$ and $u_2(x,t)$, respectively. Based on Law of conservation of mass and Reynold transport theorem [1], we can establish the following evolution FOTD model:

$$\begin{cases} \frac{\partial u_1}{\partial t} + \nabla u_1 \cdot \boldsymbol{a} + u_1(\nabla \cdot \boldsymbol{a}) = d_1 \nabla^2 u_1 - a_{11}(t) u_1(x,t) + a_{12}(t) u_2(x,t), \\ \frac{\partial u_2}{\partial t} + \nabla u_2 \cdot \boldsymbol{a} + u_2(\nabla \cdot \boldsymbol{a}) = d_2 \nabla^2 u_2 - a_{22}(t) u_2(x,t) + g(u_1(x,t)), \end{cases} \tag{2.1}$$

where $(x,t) \in \Omega(t) \times (0, +\infty)$, the domain evolution produces a flow velocity field $\boldsymbol{a} = (\dot{x}_1(t), \dot{x}_2(t), \ldots, \dot{x}_n(t))$, terms $\nabla u_i \cdot \boldsymbol{a}$ $(i = 1,2)$ are the advection terms which represent the transport of species around $\Omega(t)$ at a rate determined by the flow $\boldsymbol{a}$, and $u_i(\nabla \cdot \boldsymbol{a})$ denote dilution terms due to local volume expansion (contraction) [10], parameters d_i represent the diffusion coefficients of u_i which are assumed to be constants. We typically consider homogeneous Neumann boundary conditions

$$\frac{\partial u_i}{\partial \eta} = 0, \ \text{for } i = 1,2, \ x \in \partial\Omega(t), \tag{2.2}$$

which mean that there is no population flux across the boundary $\partial\Omega(t)$ and both bacteria and the infective live in a self-contained environment. These equations are supplemented with initial conditions

$$u_i(x,0) = u_{i0}(x), \ \text{for } i = 1,2, \ x \in \overline{\Omega}(0), \tag{2.3}$$

where u_{i0} are nonnegative bounded functions and $\Omega(0)$ is the initial domain.

However, due to problem (2.1)–(2.3) containing advection and dilution terms, studying the properties of its solutions directly is difficult in most instances. Fortunately, we can consider this problem on a continuously deforming domain $\Omega(t)$

from a transformation of Lagrangian coordinates [5] point of view. Let the fixed cartesian coordinate $y = (y_1, y_2, \dots, y_n) \in \Omega(0)$ satisfy

$$x_i(t) = \hat{x}_i(y_1, y_2, \dots, y_n, t), \text{ for } i = 1, 2, \dots, n.$$

These positions $x_i(t)$ are then mapped to fixed positions determined by the coordinate y. Thus, we assume u_1 and u_2 are mapped into the new functions defined by, respectively,

$$\begin{aligned} u_1(x(t), t) &= u_1(x_1(t), x_2(t), \dots, x_n(t), t) := u(y_1, y_2, \dots, y_n, t) = u(y, t), \\ u_2(x(t), t) &= u_2(x_1(t), x_2(t), \dots, x_n(t), t) := v(y_1, y_2, \dots, y_n, t) = v(y, t). \end{aligned} \tag{2.4}$$

Then problem (2.1)–(2.3) can be converted into the model on a fixed domain $\Omega(0)$ that remains complicated. In order to further simplify the model, we assume that domain evolution is bounded, temporally periodic and spatially isotropic. Then $x \in \Omega(t)$ can be denoted by

$$x = \rho(t) y, \ y \in \Omega(0), \tag{2.5}$$

where the domain evolution rate $\rho(t)$ is subject to $\rho(t) \in C^1 : [0, T] \to (0, \infty)$, $\rho(0) = 1$ and $\rho(t) = \rho(t + T)$ for a given positive period T, the boundary of $\Omega(0)$ satisfies $\partial\Omega(0) \in C^2$. Combining (2.4) with (2.5), we obtain that

$$\begin{aligned} &\boldsymbol{a} = \dot{x} = \dot{\rho}(t) y = \frac{\dot{\rho}}{\rho} x, \quad \nabla \cdot \boldsymbol{a} = \frac{n\dot{\rho}}{\rho}, \\ &u_t = u_{1t} + \nabla u_1 \cdot \boldsymbol{a}, \quad v_t = u_{2t} + \nabla u_2 \cdot \boldsymbol{a}, \\ &\Delta u_1 = \frac{1}{\rho^2(t)} \Delta u, \quad \Delta u_2 = \frac{1}{\rho^2(t)} \Delta v. \end{aligned}$$

Similarly, the boundary condition can be transmuted into

$$\frac{\partial u}{\partial \eta} = \frac{\partial v}{\partial \eta}, \ y \in \partial\Omega(0).$$

Then, we can transform the FOTD model with time-periodic coefficients on a T-period evolution domain into the model on a fixed domain as follows:

$$\begin{cases} u_t = \dfrac{d_1}{\rho^2(t)} \Delta u - \dfrac{n\dot{\rho}(t)}{\rho(t)} u - a_{11}(t) u + a_{12}(t) v, & y \in \Omega(0), \ t > 0, \\ v_t = \dfrac{d_2}{\rho^2(t)} \Delta v - \dfrac{n\dot{\rho}(t)}{\rho(t)} v + g(u) - a_{22}(t) v, & y \in \Omega(0), \ t > 0, \\ \dfrac{\partial u}{\partial \eta} = \dfrac{\partial v}{\partial \eta} = 0, & y \in \partial\Omega(0), \ t > 0, \end{cases} \tag{2.6}$$

with the initial condition

$$u(y, 0) = u_0(y) := u_{10}(x(0)) \geq 0, \ v(y, 0) = v_0(y) := u_{20}(x(0)) \geq 0, \quad y \in \overline{\Omega}(0). \tag{2.7}$$

We also consider the problem under the periodic condition

$$u(y, 0) = u(y, T) \geq 0, \ v(y, 0) = u(y, T) \geq 0, \quad y \in \overline{\Omega}(0). \tag{2.8}$$

Here, the corresponding coefficients of reaction terms $a_{11}(t)$, $a_{12}(t)$ and $a_{22}(t)$ are all assumed to be nonnegative, T-periodic and sufficiently smooth. The Laplacian Δ is with respect to y and g satisfies (C1), (C2) and (C3*) $\frac{g(z)}{z}$ is decreasing and $\lim\limits_{z\to\infty}\frac{g(z)}{z} < \frac{(a_{11}^m - n\max_{t\in[0,T]}|\frac{\dot\rho}{\rho}|)(a_{22}^m - n\max_{t\in[0,T]}|\frac{\dot\rho}{\rho}|)}{a_{12}^M}$, where $a_{11}^m = \min\limits_{t\in[0,T]} a_{11}(t)$, $a_{22}^m = \min\limits_{t\in[0,T]} a_{22}(t)$, $a_{12}^M = \max\limits_{t\in[0,T]} a_{12}(t)$ and $\max_{t\in[0,T]}|\frac{\dot\rho}{\rho}| < \frac{\min\{a_{11}^m, a_{22}^m\}}{n}$.

Particularly, domain $\Omega(t)$ can be regarded as a fixed one $\Omega(0)$ if $\rho(t)\equiv 1$, that is, problem (2.6), (2.7) is a reaction-diffusion FOTD model on the fixed domain.

We will reveal the long term behavior of this problem and draw the effect of evolution on the spread of FOTD by means of the basic reproduction number R_0. Next, we first give the definition of R_0 according to operator semigroup theory and study some of its properties by spectral analysis method [42] and eigenvalue problem theory [2, 7].

Consider the following linearized system of (2.6) at the disease-free equilibrium $(0,0)$

$$\begin{cases} \boldsymbol{U}_t - \boldsymbol{D}(t)\Delta\boldsymbol{U} = \boldsymbol{F}(t)\boldsymbol{U} - \boldsymbol{V}(t)\boldsymbol{U}, & y\in\Omega(0),\ t>0,\\ \dfrac{\partial\boldsymbol{U}}{\partial\eta} = 0, & y\in\partial\Omega(0),\ t>0,\end{cases}$$

where the initial conditions here are the same as in problem (2.6) and

$$\boldsymbol{U} = \begin{pmatrix} u\\ v\end{pmatrix},\ \boldsymbol{F}(t) = \begin{pmatrix} 0 & a_{12}(t)\\ g'(0) & 0\end{pmatrix},$$

$$\boldsymbol{D}(t) = \begin{pmatrix} \frac{d_1}{\rho^2(t)} & 0\\ 0 & \frac{d_2}{\rho^2(t)}\end{pmatrix},\ \boldsymbol{V}(t) = \begin{pmatrix} a_{11}(t) + \frac{n\dot\rho(t)}{\rho(t)} & 0\\ 0 & a_{22}(t) + \frac{n\dot\rho(t)}{\rho(t)}\end{pmatrix}.$$

Let $\boldsymbol{E}(t,s)$ be the evolution operator of the following system

$$\begin{cases} \boldsymbol{U}_t - \boldsymbol{D}(t)\Delta\boldsymbol{U} = -\boldsymbol{V}(t)\boldsymbol{U}, & y\in\Omega(0),\ t>0,\\ \dfrac{\partial\boldsymbol{U}}{\partial\eta} = 0, & y\in\partial\Omega(0),\ t>0.\end{cases}$$

According to the standard operator semigroup theory, there exist positive constants K and c_0 such that

$$\|\boldsymbol{E}(t,s)\| \le Ke^{-c_0(t-s)},\ \forall t>s,\ t,s\in\mathbb{R}.$$

Moreover, suppose that

$$C_T = \{f\in C(\mathbb{R}, C(\overline{\Omega}(0),\mathbb{R})) : \forall t\in\mathbb{R},\ f(t) = f(t+T) \text{ for given positive period } T\},$$

then we define a positive cone C_T^+ with the maximum norm $\|\cdot\|$ by

$$C_T^+ := \{\varphi\in C_T : \varphi(t)y\ge 0,\ (y,t)\in\overline{\Omega}(0)\times\mathbb{R}\},$$

which is an ordered Banach space and denote $\varphi(t)y$ by $\varphi(y,t)$ for any $\varphi\in C_T$. Assume that $\zeta = (\phi,\psi)\in C_T\times C_T$ is the density distribution of bacteria and the infective at location $y \in \Omega(0)$ and time s. It follows from the statements in [42] that the next generation infection operator $\mathfrak{L} : C_T \times C_T \to C_T \times C_T$ can be defined by

$$\mathfrak{L}(\zeta)(t) := \int_{-\infty}^{t} \boldsymbol{E}(t,s)\boldsymbol{F}(s)\zeta(\cdot,s)\mathrm{d}s = \int_{0}^{\infty} \boldsymbol{E}(t,t-a)\boldsymbol{F}(t-a)\zeta(\cdot,t-a)\mathrm{d}a.$$

Here positive operator $\mathfrak{L}$ is continuous and compact on $C_T \times C_T$, that is, $\mathfrak{L}(C_T^+ \times C_T^+) \subset C_T^+ \times C_T^+$. Therefore, consulting [42], we formulate R_0, the basic reproduction number of model (2.6), by the spectral radius of operator $\mathfrak{L}$, denoted by

$$R_0 = \boldsymbol{\rho}(\mathfrak{L}).$$

In addition, in order to study some properties of R_0, we need the following Lemmas which can refer to Proposition 3.9, Theorem 3.7 and Theorem 3.8 in [42].

Lemma 2.1. $R_0 = \mu_0$, *where* $\mu_0 > 0$ *is the unique principal eigenvalue of the following periodic parabolic eigenvalue problem*

$$\begin{cases} \dfrac{\partial \Phi}{\partial t} - \dfrac{d_1}{\rho^2(t)}\Delta\Phi = \dfrac{a_{12}(t)\Psi}{\mu} - \left(a_{11}(t) + \dfrac{n\dot{\rho}(t)}{\rho(t)}\right)\Phi, & y \in \Omega(0),\ t > 0, \\ \dfrac{\partial \Psi}{\partial t} - \dfrac{d_2}{\rho^2(t)}\Delta\Psi = \dfrac{g^{'}(0)\Phi}{\mu} - \left(a_{22}(t) + \dfrac{n\dot{\rho}(t)}{\rho(t)}\right)\Psi, & y \in \Omega(0),\ t > 0, \\ \dfrac{\partial \Phi}{\partial \eta} = \dfrac{\partial \Psi}{\partial \eta} = 0, & y \in \partial\Omega(0),\ t > 0, \\ \Phi(y,0) = \Phi(y,T),\ \Psi(y,0) = \Psi(y,T), & y \in \overline{\Omega}(0), \end{cases} \tag{2.9}$$

and corresponds to an eigenfunction pair $(\Phi_0, \Psi_0) \in C_T \times C_T$ *satisfying* $\Phi_0, \Psi_0 > 0$ *in* $\overline{\Omega}(0) \times (0, \infty)$.

Lemma 2.2. $\operatorname{sign}(1 - R_0) = \operatorname{sign}(\lambda_0)$, *where* λ_0 *is the unique principal eigenvalue of the following periodic parabolic eigenvalue problem*

$$\begin{cases} \dfrac{\partial \phi}{\partial t} - \dfrac{d_1}{\rho^2(t)}\Delta\phi = a_{12}(t)\psi - \left(a_{11}(t) + \dfrac{n\dot{\rho}(t)}{\rho(t)}\right)\phi + \lambda\phi, & y \in \Omega(0),\ t > 0, \\ \dfrac{\partial \psi}{\partial t} - \dfrac{d_2}{\rho^2(t)}\Delta\psi = g^{'}(0)\phi - \left(a_{22}(t) + \dfrac{n\dot{\rho}(t)}{\rho(t)}\right)\psi + \lambda\psi, & y \in \Omega(0),\ t > 0, \\ \dfrac{\partial \phi}{\partial \eta} = \dfrac{\partial \psi}{\partial \eta} = 0, & y \in \partial\Omega(0),\ t > 0, \\ \phi(y,0) = \phi(y,T),\ \psi(y,0) = \psi(y,T), & y \in \overline{\Omega}(0), \end{cases} \tag{2.10}$$

and corresponds to an eigenfunction pair $(\phi_0, \psi_0) \in C_T \times C_T$ *satisfying* $\phi_0, \psi_0 > 0$ *in* $\overline{\Omega}(0) \times (0, \infty)$.

Remark 2.1. It follows from Lemma 2.1 and [3,4] that R_0 monotonically increases with respect to $a_{12}(t)$ and $g^{'}(0)$. Hence, if $a_{12}(t)$ and $g^{'}(0)$ are sufficiently big, then $R_0 > 1$ holds.

Meanwhile, in order to describe the dependence of R_0 with respect to periodic evolution rate $\rho(t)$, we denote $R_0(\rho) = R_0$ and the integral average of any continuous function $f(t)$ over $(0, T)$ by

$$\overline{f} = \frac{1}{T}\int_0^T f(t)\mathrm{d}t.$$

In what follows, we give the estimation of lower bound with respect to $R_0(\rho)$.

Theorem 2.1. *Fix variable coefficients $a_{11}(t)$, $a_{12}(t)$ and $a_{22}(t)$ in eigenvalue problem* (2.9)*. Then we can deduce that*

$$R_0(\rho) \geq \frac{\sqrt{\overline{a_{12}(t)g'(0)}}}{\sqrt{\overline{a_{11}(t)} \cdot \overline{a_{22}(t)}}}. \tag{2.11}$$

In particular, if there exist constants a_{11}^*, a_{12}^* and a_{22}^* such that $a_{11}(t)$, $a_{12}(t)$ and $a_{22}(t)$ are respectively identically equal to them, then

$$R_0(\rho) \geq \sqrt{\frac{a_{12}^* g'(0)}{a_{11}^* a_{22}^*}} := \mathfrak{R}_o, \tag{2.12}$$

where $\mathfrak{R}_o$ is the basic reproduction number of the FOTD model (1.1) *with homogeneous Neumann boundary conditions.*

Proof. First, due to the homogeneous Neumann boundary conditions, we assume that one eigenfunction pair of problem (2.9) corresponding to principal eigenvalue μ_0 are $\Phi_0(y,t) = z_1(t)$ and $\Psi_0(y,t) = z_2(t)$, where

$$z_i(t) = z_i(t+T) > 0, \text{ for } t \in [0,\infty) \text{ and } i = 1,2,$$

are to be determined later. By simplifying, we can obtain

$$\begin{cases} \dot{z_1}(t) = \dfrac{a_{12}(t)z_2(t)}{R_0(\rho)} - \left(a_{11}(t) + \dfrac{n\dot{\rho}(t)}{\rho(t)}\right) z_1(t), \\ \dot{z_2}(t) = \dfrac{g'(0)z_1(t)}{R_0(\rho)} - \left(a_{22}(t) + \dfrac{n\dot{\rho}(t)}{\rho(t)}\right) z_2(t). \end{cases} \tag{2.13}$$

In fact, according to the assumptions of $\rho(t)$ and $\dot{\rho}(t)$ and some ordinary differential equations arguments [14], there exist a differentiable T-periodic solution $(z_1(t), z_2(t))$ satisfying system (2.13). With the appropriate transformation, system (2.13) can be rewritten as

$$\begin{cases} \dfrac{\dot{z_1}(t)}{z_1(t)} = \dfrac{a_{12}(t)}{R_0(\rho)} \dfrac{z_2(t)}{z_1(t)} - \left(a_{11}(t) + \dfrac{n\dot{\rho}(t)}{\rho(t)}\right), \\ \dfrac{\dot{z_2}(t)}{z_2(t)} = \dfrac{g'(0)}{R_0(\rho)} \dfrac{z_1(t)}{z_2(t)} - \left(a_{22}(t) + \dfrac{n\dot{\rho}(t)}{\rho(t)}\right). \end{cases}$$

Thanks to the periodicity of $z_i(t)(i = 1,2)$ and the assumption of $\rho(t)$, we integrate and average the above equations over $[0,T]$ and achieve

$$\begin{cases} \dfrac{1}{R_0(\rho)} \cdot \dfrac{1}{T} \displaystyle\int_0^T \dfrac{a_{12} z_2(t)}{z_1(t)} \mathrm{d}t \quad = \overline{a_{11}(t)}, \\ \dfrac{1}{R_0(\rho)} \cdot \dfrac{1}{T} \displaystyle\int_0^T \dfrac{g'(0) z_1(t)}{z_2(t)} \mathrm{d}t = \overline{a_{22}(t)}. \end{cases}$$

Therefore,

$$\overline{a_{11}(t)} \cdot \overline{a_{22}(t)} = \frac{1}{T} \int_0^T a_{11}(t)\mathrm{d}t \cdot \frac{1}{T} \int_0^T a_{22}(t)\mathrm{d}t$$

$$=\frac{1}{R_0^2(\rho)}\cdot\frac{1}{T}\int_0^T\frac{a_{12}z_2(t)}{z_1(t)}\mathrm{d}t\cdot\frac{1}{T}\int_0^T\frac{g^{'}(0)z_1(t)}{z_2(t)}\mathrm{d}t$$
$$\geq\frac{1}{R_0^2(\rho)}\cdot\left(\frac{1}{T}\int_0^T\sqrt{a_{12}(t)g^{'}(0)}\mathrm{d}t\right)^2,$$

where the inequality relation is obtained by the Hölder inequality. It implies that

$$R_0^2(\rho)\geq\frac{\left(\overline{\sqrt{a_{12}(t)g^{'}(0)}}\right)^2}{\overline{a_{11}(t)}\cdot\overline{a_{22}(t)}}.$$

The proof of inequality (2.11) is completed.

Obviously, when $a_{11}(t)\equiv a_{11}^*$, $a_{12}(t)\equiv a_{12}^*$ and $a_{22}(t)\equiv a_{22}^*$, we can directly deduce inequality (2.12) from the above inequality. Furthermore, if $\rho(t)\equiv 1$, then the corresponding problem can be regarded as the problem on a fixed domain, i.e., problem (1.1) with homogeneous Neumann boundary conditions. Similarly, we can obtain

$$\begin{cases}\dot{z_1}(t)=-a_{11}^*z_1(t)+\dfrac{a_{12}^*z_2(t)}{\mathfrak{R}_\mathfrak{o}},\\[2ex] \dot{z_2}(t)=\dfrac{g^{'}(0)z_1(t)}{\mathfrak{R}_\mathfrak{o}}-a_{22}^*z_2(t),\end{cases}\tag{2.14}$$

we can easily calculate that

$$z_1(t)=e^{(-a_{11}^*+\frac{Ka_{12}^*}{\mathfrak{R}_\mathfrak{o}})t},\quad z_2(t)=Ke^{(-a_{11}^*+\frac{Ka_{12}^*}{\mathfrak{R}_\mathfrak{o}})t}$$

are a set of solutions to system (2.14), where

$$K=\frac{a_{11}^*-a_{22}^*+\sqrt{(a_{11}^*-a_{22}^*)^2+\frac{4a_{12}^*g^{'}(0)}{{\mathfrak{R}_\mathfrak{o}}^2}}}{2a_{12}^*/\mathfrak{R}_\mathfrak{o}}.$$

By periodicity hypothesis of $z_i(t)(i=1,2)$, we further calculate to get

$$\mathfrak{R}_\mathfrak{o}=\sqrt{\frac{a_{12}^*g^{'}(0)}{a_{11}^*a_{22}^*}}$$

which is consistent with the second equality relation of Formula (2.12).

Thus we can summarize what we have proved as Theorem 2.1. $\square$

Remark 2.2. $R_0(1)\geq\mathfrak{R}_\mathfrak{o}$ and the equality holds if and only if $a_{11}(t)$, $a_{12}(t)$ and $a_{22}(t)$ are all constants.

3. Main results

In this section, we first give the definition of upper and lower solutions to our problem and with that show the existence, uniqueness and attractivity of disease-free and endemic equilibria on the periodic evolution and fixed domains, respectively.

Notice that problem (2.6) with the periodic condition (2.8) is equivalent to

$$\begin{cases} u_t - \dfrac{d_1}{\rho^2(t)}\Delta u = f_1(t,u,v), & y \in \Omega(0),\ t>0, \\ v_t - \dfrac{d_2}{\rho^2(t)}\Delta v = f_2(t,u,v), & y \in \Omega(0),\ t>0, \\ \dfrac{\partial u}{\partial \eta} = \dfrac{\partial v}{\partial \eta} = 0, & y \in \partial\Omega(0),\ t>0, \\ u(y,0) = u(y,T),\ v(y,0) = v(y,T), & y \in \overline{\Omega}(0), \end{cases} \tag{3.1}$$

where

$$f_1(t,u,v) = -\left(a_{11}(t) + \frac{n\dot{\rho}(t)}{\rho(t)}\right)u + a_{12}(t)v, \quad y \in \Omega(0),\ t>0,$$
$$f_2(t,u,v) = g(u) - \left(a_{22}(t) + \frac{n\dot{\rho}(t)}{\rho(t)}\right)v, \qquad y \in \Omega(0),\ t>0.$$

Owing to the nonnegativity of $a_{12}(t)$ and hypotheses in $g(u)$, the reaction terms f_1 and f_2 are all quasimonotone nondecreasing. In what follows, we define the ordered upper and lower solutions to problem (3.1).

Definition 3.1. Suppose that $\tilde{u}(y,t)$, $\hat{u}(y,t)$, $\tilde{v}(y,t)$ and $\hat{v}(y,t)$ are nonnegative functions in $C^{2,1}(\Omega(0)\times(0,\infty))\cap C(\overline{\Omega}(0)\times[0,\infty))$. Then $\tilde{\mathbf{u}} = (\tilde{u},\tilde{v})$ and $\hat{\mathbf{u}} = (\hat{u},\hat{v})$ are called ordered upper and lower solutions to problem (3.1) if $\tilde{\mathbf{u}} \ge \hat{\mathbf{u}} \ge 0$ and if

$$\begin{cases} \tilde{u}_t - \dfrac{d_1}{\rho^2(t)}\Delta\tilde{u} \ge f_1(t,\tilde{u},\tilde{v}),\ \tilde{v}_t - \dfrac{d_2}{\rho^2(t)}\Delta\tilde{v} \ge f_2(t,\tilde{u},\tilde{v}), & y \in \Omega(0),\ t>0, \\ \hat{u}_t - \dfrac{d_1}{\rho^2(t)}\Delta\hat{u} \le f_1(t,\hat{u},\hat{v}),\ \hat{v}_t - \dfrac{d_2}{\rho^2(t)}\Delta\hat{v} \le f_2(t,\hat{u},\hat{v}), & y \in \Omega(0),\ t>0, \\ \dfrac{\partial\tilde{u}}{\partial\eta} \ge 0 \ge \dfrac{\partial\hat{u}}{\partial\eta},\ \dfrac{\partial\tilde{v}}{\partial\eta} \ge 0 \ge \dfrac{\partial\hat{v}}{\partial\eta}, & y \in \partial\Omega(0),\ t>0, \\ \tilde{u}(y,0) \ge \tilde{u}(y,T),\ \tilde{v}(y,0) \ge \tilde{v}(y,T), & y \in \overline{\Omega}(0), \\ \hat{u}(y,0) \le \hat{u}(y,T),\ \hat{v}(y,0) \le \hat{v}(y,T), & y \in \overline{\Omega}(0). \end{cases} \tag{3.2}$$

Remark 3.1. We call $\tilde{\mathbf{u}} \ge \hat{\mathbf{u}}$ if $\tilde{u} \ge \hat{u}$ and $\tilde{v} \ge \hat{v}$. Moreover, we denote any function pair $\hat{\mathbf{u}} \le \mathbf{u} := (u,v) \le \tilde{\mathbf{u}}$ by $\mathbf{u} \in \langle\hat{\mathbf{u}},\tilde{\mathbf{u}}\rangle$.

Indeed, these foregoing ordered upper and lower solutions do not have to be T-periodic in time.

Next, employing some standard upper and lower solutions arguments, we will describe the well-posedness of solutions to problem (3.1).

3.1. Existence and uniqueness of periodic solutions

Theorem 3.1. *If $R_0(\rho) > 1$, then there exist two sets of positive periodic solutions $\underline{\mathbf{u}} := (\underline{u},\underline{v})$ and $\overline{\mathbf{u}} := (\overline{u},\overline{v})$ satisfying* (3.1) *such that any solution to problem* (3.1) $\mathbf{u} \in \langle\underline{\mathbf{u}},\overline{\mathbf{u}}\rangle$. *Moreover, if $\overline{\mathbf{u}}(y,0) = \underline{\mathbf{u}}(y,0)$, then $\overline{\mathbf{u}} = \underline{\mathbf{u}} = \mathbf{u}^* := (u^*,v^*)$ is the unique solution to problem* (3.1).

Proof. Let

$$\tilde{\mathbf{u}}(y,t) = (M_1, M_2),\ \ \hat{\mathbf{u}}(y,t) = (\delta\phi_0, \delta\psi_0) \in \overline{\Omega}(0) \times [0, \infty), \tag{3.3}$$

where $(\phi_0, \psi_0) > 0$ is an eigenfunction pair of problem (2.10) corresponding to principal eigenvalue λ_0 and $\delta > 0$ is to be selected later. Constants M_1, M_2 are positive and should satisfy the following relationships

$$\begin{cases} -a_{11}(t)M_1 - \dfrac{n\dot{\rho(t)}}{\rho(t)}M_1 + a_{12}(t)M_2 \le 0, & t > 0, \\ g(M_1) - a_{22}(t)M_2 - \dfrac{n\dot{\rho(t)}}{\rho(t)}M_2 \le 0, & t > 0. \end{cases} \tag{3.4}$$

It is sufficient to show that

$$\begin{cases} -a_{11}^m M_1 + n\max_{t\in[0,T]}|\frac{\dot{\rho}}{\rho}|M_1 + a_{12}^M M_2 \le 0, \\ g(M_1) - a_{22}^m M_2 + n\max_{t\in[0,T]}|\frac{\dot{\rho}}{\rho}|M_2 \le 0, \end{cases} \tag{3.5}$$

which is equivalent to

$$\begin{cases} \dfrac{M_2}{M_1} \le \dfrac{a_{11}^m - n\max_{t\in[0,T]}|\frac{\dot{\rho}}{\rho}|}{a_{12}^M}, \\ \dfrac{g(M_1)}{M_1} \le (a_{22}^m - n\max\limits_{t\in[0,T]}|\dfrac{\dot{\rho}}{\rho}|) \cdot \dfrac{M_2}{M_1}. \end{cases} \tag{3.6}$$

In fact, substituting (3.3) back into (3.2), it follows from hypothesis (C3*) that there exists $M^* > 0$ such that

$$\frac{g(M)}{M} \le \frac{(a_{11}^m - n\max_{t\in[0,T]}|\frac{\dot{\rho}}{\rho}|)(a_{22}^m - n\max_{t\in[0,T]}|\frac{\dot{\rho}}{\rho}|)}{a_{12}^M}$$

for any $M \ge M^*$, which implies that (3.6) holds. Hence, we can take $M_1 = M^*$ and $M_2 = \frac{(a_{11}^m - n\max_{t\in[0,T]}|\frac{\dot{\rho}}{\rho}|)M^*}{a_{12}^M}$. Meanwhile, the rest inequalities of (3.2) are transformed into

$$\begin{aligned} (\phi_0)_t - \frac{d_1}{\rho^2(t)}\Delta\phi_0 &\le -\left(a_{11}(t) + \frac{n\dot{\rho}(t)}{\rho(t)}\right)\phi_0 + a_{12}(t)\psi_0, \\ (\psi_0)_t - \frac{d_2}{\rho^2(t)}\Delta\psi_0 &\le -\left(a_{22}(t) + \frac{n\dot{\rho}(t)}{\rho(t)}\right)\psi_0 + \frac{g(\delta\phi_0)}{\delta}. \end{aligned} \tag{3.7}$$

Combining with (2.10), we observe that

$$\lambda_0 \le 0 \text{ and } \lambda_0 \le \left(\frac{g(\delta\phi_0)}{\delta} - g^{'}(0)\phi_0\right)/\psi_0.$$

Owing to hypotheses (C1) and (C2), we know $\frac{g(\delta\phi_0)}{\delta} - g^{'}(0)\phi_0 < 0$ if $\delta > 0$ is sufficiently small. By now we deduce that $\lambda_0 < 0$ which is consistent with $R_0(\rho) > 1$ according to Lemma 2.2. In other words, we construct a pair of ordered upper and lower solutions (3.3) satisfying condition (3.4). Thanks to Theorem 2.1 in [30], there exists at least one periodic solution to problem (3.1).

On the other hand, problem (3.1) is further equivalent to the following system

$$\begin{cases} u_t - \dfrac{d_1}{\rho^2(t)}\Delta u + K_1 u = F_1(t,u,v), & y \in \Omega(0),\ t>0, \\ v_t - \dfrac{d_2}{\rho^2(t)}\Delta v + K_2 v = F_2(t,u,v), & y \in \Omega(0),\ t>0, \\ \dfrac{\partial u}{\partial \eta} = \dfrac{\partial v}{\partial \eta} = 0, & y \in \partial\Omega(0),\ t>0, \\ u(y,0) = u(y,T),\ v(y,0) = v(y,T), & y \in \overline{\Omega}(0), \end{cases}$$

where K_1, K_2 are constants and

$$K_1 \geq \max_{t\in[0,T]}\left\{a_{11}(t) + \frac{n\dot{\rho}(t)}{\rho(t)}\right\}, \quad K_2 \geq \max_{t\in[0,T]}\left\{a_{22}(t) + \frac{n\dot{\rho}(t)}{\rho(t)}\right\},$$
$$F_1(t,u,v) = K_1 u + f_1(t,u,v), \qquad F_2(t,u,v) = K_2 v + f_2(t,u,v).$$

Notice that F_1 and F_2 are all monotone nondecreasing with respect to u and v. Let $\underline{\mathbf{u}}^{(0)} = \hat{\mathbf{u}}$ and $\overline{\mathbf{u}}^{(0)} = \tilde{\mathbf{u}}$ as the initial iterations. We can construct sequences $\{\overline{\mathbf{u}}^{(k)}\}_{k=1}^{\infty}$ and $\{\underline{\mathbf{u}}^{(k)}\}_{k=1}^{\infty}$ from the following iteration process:

$$\begin{cases} \overline{u}_t^{(k)} - \dfrac{d_1}{\rho^2(t)}\Delta\overline{u}^{(k)} + K_1\overline{u}^{(k)} = F_1(t,\overline{u}^{(k-1)},\overline{v}^{(k-1)}), & y\in\Omega(0), t>0, \\ \overline{v}_t^{(k)} - \dfrac{d_2}{\rho^2(t)}\Delta\overline{v}^{(k)} + K_1\overline{v}^{(k)} = F_2(t,\overline{u}^{(k-1)},\overline{v}^{(k-1)}), & y\in\Omega(0), t>0, \\ \underline{u}_t^{(k)} - \dfrac{d_1}{\rho^2(t)}\Delta\underline{u}^{(k)} + K_1\underline{u}^{(k)} = F_1(t,\underline{u}^{(k-1)},\underline{v}^{(k-1)}), & y\in\Omega(0), t>0, \\ \underline{v}_t^{(k)} - \dfrac{d_2}{\rho^2(t)}\Delta\underline{v}^{(k)} + K_1\underline{v}^{(k)} = F_2(t,\underline{u}^{(k-1)},\underline{v}^{(k-1)}), & y\in\Omega(0), t>0, \\ \dfrac{\partial\overline{u}^{(k)}}{\partial\eta} = \dfrac{\partial\overline{v}^{(k)}}{\partial\eta} = \dfrac{\partial\underline{u}^{(k)}}{\partial\eta} = \dfrac{\partial\underline{v}^{(k)}}{\partial\eta} = 0, & y\in\partial\Omega(0), t>0, \end{cases} \tag{3.8}$$

with the periodic condition

$$\begin{cases} \overline{u}^{(k)}(y,0) = \overline{u}^{(k-1)}(y,T),\ \overline{v}^{(k)}(y,0) = \overline{v}^{(k-1)}(y,T), & y\in\overline{\Omega}(0), \\ \underline{u}^{(k)}(y,0) = \underline{u}^{(k-1)}(y,T),\ \underline{v}^{(k)}(y,0) = \underline{v}^{(k-1)}(y,T), & y\in\overline{\Omega}(0). \end{cases} \tag{3.9}$$

According to [30, Lemma 3.1] and [29, Lemma 2.1], the sequences generated by process (3.8) with (3.9) possess the monotone property as follows:

$$\hat{\mathbf{u}} \leq \underline{\mathbf{u}}^{(k)} \leq \underline{\mathbf{u}}^{(k+1)} \leq \overline{\mathbf{u}}^{(k+1)} \leq \overline{\mathbf{u}}^{(k)} \leq \tilde{\mathbf{u}}, \tag{3.10}$$

where $\overline{\mathbf{u}}^{(k)}$ and $\underline{\mathbf{u}}^{(k)}$ satisfy (3.2) for $k = 1, 2, \ldots$. It follows from some classical partial differential equations arguments that $\{\overline{\mathbf{u}}^{(k)}\}_{k=1}^{\infty}$ and $\{\underline{\mathbf{u}}^{(k)}\}_{k=1}^{\infty}$ belong to $C^{2,1}(\Omega(0)\times(0,\infty)) \cap C(\overline{\Omega}(0)\times[0,\infty)$. Owing to the bounded monotonic principle, the limit of sequences $\{\overline{\mathbf{u}}^{(k)}\}_{k=1}^{\infty}$ and $\{\underline{\mathbf{u}}^{(k)}\}_{k=1}^{\infty}$ exist as $k\to\infty$, which are denoted by

$$\lim_{k\to\infty}\overline{\mathbf{u}}^{(k)} = \overline{\mathbf{u}}, \quad \lim_{k\to\infty}\underline{\mathbf{u}}^{(k)} = \underline{\mathbf{u}}.$$

Combining with (3.10), the following monotonicity property holds:

$$\hat{\mathbf{u}} \leq \underline{\mathbf{u}}^{(k)} \leq \underline{\mathbf{u}}^{(k+1)} \leq \underline{\mathbf{u}} \leq \overline{\mathbf{u}} \leq \overline{\mathbf{u}}^{(k+1)} \leq \overline{\mathbf{u}}^{(k)} \leq \tilde{\mathbf{u}} \tag{3.11}$$

and $\underline{\mathbf{u}}, \overline{\mathbf{u}}$ satisfy model (3.8) with (3.9) which implies that $\underline{\mathbf{u}}, \overline{\mathbf{u}}$ are both solutions to problem (3.1). In fact, $\underline{\mathbf{u}}, \overline{\mathbf{u}}$ are also T-periodic referring to the argument of Theorem 2.1 in [29].

Moreover, we claim that for any periodic solution $\mathbf{u}$ of problem (3.1), the function pair $\mathbf{u}$ belongs to the sector $\langle \underline{\mathbf{u}}, \overline{\mathbf{u}} \rangle$. Set

$$S = \left\{ \mathbf{u} \in C(\overline{\Omega}(0) \times [0, \infty)) : \mathbf{u} \in \langle \hat{\mathbf{u}}, \tilde{\mathbf{u}} \rangle \right\}.$$

For any periodic solution $\mathbf{u}$ in S, let $\mathbf{u}$ be a solution to problem (3.1) and consider $\tilde{\mathbf{u}}$ and $\mathbf{u}$ as a pair of ordered upper and lower solutions to (3.1). Choosing $\overline{\mathbf{u}}^{(0)} = \tilde{\mathbf{u}}$ and $\underline{\mathbf{u}}^{(0)} = \mathbf{u}$ as the initial iterations and repeating the above argument, we obtain

$$\mathbf{u} = \underline{\mathbf{u}}^{(0)} = \underline{\mathbf{u}}^{(k)} \leq \overline{\mathbf{u}} \leq \overline{\mathbf{u}}^{(k)} \leq \tilde{\mathbf{u}},$$

which implies that

$$\mathbf{u} \leq \overline{\mathbf{u}} \text{ for } (y, t) \in \overline{\Omega}(0) \times [0, \infty).$$

By the similar argument for $\mathbf{u}$ and $\hat{\mathbf{u}}$, it is sufficient to prove that

$$\mathbf{u} \geq \underline{\mathbf{u}} \text{ for } (y, t) \in \overline{\Omega}(0) \times [0, \infty).$$

The above two inequalities indicate that $\mathbf{u} \in \langle \underline{\mathbf{u}}, \overline{\mathbf{u}} \rangle$.

Finally, we assert that $\overline{\mathbf{u}} = \underline{\mathbf{u}}$ if $\overline{\mathbf{u}}(y, 0) = \underline{\mathbf{u}}(y, 0)$, which one can see for example [30, Theorem B]. Actually, choosing the initial condition (2.7), that is, $\mathbf{u}(y, 0) = (u_0(y), v_0(y))$, problem (3.1) can be regarded as an initial-boundary value parabolic problem. For this type of initial-boundary value (IBV) problems, the uniqueness of solutions is classical. We will not repeat it here. The proof is completed. $\square$

It follows from the foregoing theorem that periodic solutions to problem (3.1) exist in the case of $R_0(\rho) > 1$. Naturally, we also want to know what would happen when $R_0(\rho) \leq 1$. The following theorem answers this question.

Theorem 3.2. *If $R_0(\rho) \leq 1$, then there is none positive periodic solution satisfying* (3.1).

Proof. Proof by contradiction. Suppose that there exists a positive periodic solution $\mathbf{u}^+ := (u^+, v^+)(y, t)$ such that

$$\begin{cases} u_t^+ - \dfrac{d_1}{\rho^2(t)} \Delta u^+ = -\left(a_{11}(t) + \dfrac{n\dot{\rho}(t)}{\rho(t)} \right) u^+ + a_{12}(t) v^+, & y \in \Omega(0),\ t > 0, \\ v_t^+ - \dfrac{d_2}{\rho^2(t)} \Delta v^+ = -\left(a_{22}(t) + \dfrac{n\dot{\rho}(t)}{\rho(t)} \right) v^+ + g(u^+), & y \in \Omega(0),\ t > 0, \\ \dfrac{\partial u^+}{\partial \eta} = \dfrac{\partial v^+}{\partial \eta} = 0, & y \in \partial\Omega(0),\ t > 0, \\ u^+(y, 0) = u^+(y, T),\ v^+(y, 0) = v^+(y, T), & y \in \overline{\Omega}(0). \end{cases} \tag{3.12}$$

Due to hypotheses (C1) and (C2), we have

$$v_t^+ - \frac{d_2}{\rho^2(t)} \Delta v^+ < -\left(a_{22}(t) + \frac{n\dot{\rho}(t)}{\rho(t)} \right) v^+ + g^{'}(0) u^+, \quad y \in \Omega(0),\ t > 0. \tag{3.13}$$

Without loss of generality, we can define $\mathbf{u}^+ := (C_1\phi_0, C_2\psi_0)$, where $(\phi_0, \psi_0) > 0$ is the eigenfunctions pair of problem (2.10) corresponding to principal eigenvalue λ_0 and C_1, C_2 are given positive constants.

Notice that the principal eigenvalue λ_0 satisfies

$$\begin{cases} \dfrac{\partial \phi_0}{\partial t} - \dfrac{d_1}{\rho^2(t)}\Delta\phi_0 = a_{12}(t)\psi_0 - \left(a_{11}(t) + \dfrac{n\dot{\rho}(t)}{\rho(t)}\right)\phi_0 + \lambda\phi_0, & y \in \Omega(0),\ t > 0, \\ \dfrac{\partial \psi_0}{\partial t} - \dfrac{d_2}{\rho^2(t)}\Delta\psi_0 = g^{'}(0)\phi_0 - \left(a_{22}(t) + \dfrac{n\dot{\rho}(t)}{\rho(t)}\right)\psi_0 + \lambda_0\psi_0, & y \in \Omega(0),\ t > 0, \\ \dfrac{\partial \phi_0}{\partial \eta} = \dfrac{\partial \psi_0}{\partial \eta} = 0, & y \in \partial\Omega(0),\ t > 0, \\ \phi_0(y,0) = \phi_0(y,T),\ \psi_0(y,0) = \psi_0(y,T), & y \in \overline{\Omega}(0). \end{cases} \tag{3.14}$$

Consulting [3, 4], we know that λ_0 monotonically decreases with respect to $a_{12}(t)$ and $g^{'}(0)$. Thus we deduce that $\lambda_0 < 0$ by comparing (3.13) with (3.14). However, Lemma 2.2 indicates $\mathrm{sign}(1 - R_0(\rho)) = \mathrm{sign}(\lambda_0)$ which implies that $R_0(\rho) > 1$ contrary to condition $R_0(\rho) \le 1$. The proof is completed. □

3.2. Attractivity of disease-free and endemic equilibria

Our aim in this subsection is to present the attractivity of periodic solutions to problem (3.1). For this purpose, we need some lemmas which one can refer to [30, Theorem B, Lemma 3.1, Lemma 3.2]. We omit the details of their proofs here.

Lemma 3.1. *Suppose that* $\{\overline{\mathbf{u}}^{(k)}\}_{k=1}^{\infty}$ *and* $\{\underline{\mathbf{u}}^{(k)}\}_{k=1}^{\infty}$ *are the sequences constructed by* (3.8) *and the initial condition changed by* (2.7), *that is,*

$$\{\overline{\mathbf{u}}^{(k)}\}_{k=1}^{\infty}(y,0) = \{\underline{\mathbf{u}}^{(k)}\}_{k=1}^{\infty}(y,0) = \mathbf{u}_0 := (u_0(y), v_0(y)),$$

where $\mathbf{u}_0 \in \langle \hat{\mathbf{u}}, \tilde{\mathbf{u}} \rangle$ *in* $\overline{\Omega}(0)$. *Then* $\{\overline{\mathbf{u}}^{(k)}\}_{k=1}^{\infty}$ *and* $\{\underline{\mathbf{u}}^{(k)}\}_{k=1}^{\infty}$ *converge monotonically to the unique solution* $\mathbf{u}$ *of problem* (2.6) *equipped with the initial condition* (2.7) *and satisfy the relation*

$$\hat{\mathbf{u}} = \underline{\mathbf{u}}^{(0)} \le \underline{\mathbf{u}}^{(k-1)} \le \underline{\mathbf{u}}^{(k)} \le \mathbf{u} \le \overline{\mathbf{u}}^{(k)} \le \overline{\mathbf{u}}^{(k-1)} \le \overline{\mathbf{u}}^{(0)} = \tilde{\mathbf{u}}$$

for $k = 1, 2, \ldots$.

For IBV problem (2.6),(2.7), we exhibit the existence-comparison theorem (Lemma 3.1). Besides, the above sequences possess some additional properties.

Lemma 3.2. *For any two positive integers* k *and* $k^{'}$, *the functions pair* $\overline{\mathbf{u}}^{(k)}$, $\underline{\mathbf{u}}^{(k^{'})}$ *are a pair of ordered upper and lower solutions to problem* (3.1) *and problem* (2.6),(2.7), *respectively, if* $\mathbf{u}_0 \in \langle \underline{\mathbf{u}}^{(k^{'})}, \overline{\mathbf{u}}^{(k)} \rangle$ *in* $\overline{\Omega}(0)$.

Lemma 3.3. *Let* $\mathbf{u}(y,t;\mathbf{u}_0) := (u(y,t;u_0), v(y,t;v_0))$ *be the solution to problem* (2.6),(2.7) *with any* $\mathbf{u}_0 \in S_0$, *where*

$$S_0 = \left\{ \mathbf{u}_0 \in C(\overline{\Omega}(0)) : \mathbf{u}_0 \in \langle \hat{\mathbf{u}}, \tilde{\mathbf{u}} \rangle (y,0) \right\}.$$

Then for every k, *the following relationship holds:*

$$\underline{\mathbf{u}}^{(k)}(\cdot,t) \le \mathbf{u}(\cdot, t + kT; \mathbf{u}_0) \le \overline{\mathbf{u}}^{(k)}(\cdot,t) \ \text{in}\ \overline{\Omega}(0) \times [0,\infty). \tag{3.15}$$

Next, we show the attractivity of disease-free and endemic equilibria.

Theorem 3.3. *Let* $\underline{\mathbf{u}}(y,t), \overline{\mathbf{u}}(y,t)$ *be the* T*-periodic solutions determined in Theorem 3.1. Denote by* $\mathbf{u}(y,t;\mathbf{u}_0)$ *the solution to problem* (2.6),(2.7). *If* $R_0(\rho) > 1$, *then*

$$\lim_{k\to\infty} \mathbf{u}(y,t+kT;\mathbf{u}_0) = \begin{cases} \underline{\mathbf{u}}(y,t), & \text{if } \mathbf{u}_0 \in \langle \hat{\mathbf{u}}, \underline{\mathbf{u}} \rangle \text{ in } \overline{\Omega}(0), \\ \overline{\mathbf{u}}(y,t), & \text{if } \mathbf{u}_0 \in \langle \overline{\mathbf{u}}, \tilde{\mathbf{u}} \rangle \text{ in } \overline{\Omega}(0), \end{cases}$$

and the following relationship holds for any $\mathbf{u}_0 \in S_0$*:*

$$\underline{\mathbf{u}}(y,t) \le \mathbf{u}(y,t+kT;\mathbf{u}_0) \le \overline{\mathbf{u}}(y,t) \tag{3.16}$$

in $\overline{\Omega}(0)\times[0,\infty)$ *as* $k\to\infty$. *Moreover, if* $\underline{\mathbf{u}}(y,0) = \overline{\mathbf{u}}(y,0) = \mathbf{u}^*(y,0)$, *then* $\underline{\mathbf{u}}(y,t) = \overline{\mathbf{u}}(y,t) = \mathbf{u}^*(y,t)$ *and*

$$\lim_{k\to\infty} \mathbf{u}(y,t+kT;\mathbf{u}_0) = \mathbf{u}^*(y,t). \tag{3.17}$$

Proof. For the case $\mathbf{u}_0 \in \langle \hat{\mathbf{u}}, \underline{\mathbf{u}} \rangle$, regarding $\underline{\mathbf{u}}$ as an upper solution to problem (2.6),(2.7), then it follows from Lemma 3.1 that

$$\hat{\mathbf{u}} = \underline{\mathbf{u}}^{(0)} \le \mathbf{u} \le \overline{\mathbf{u}}^{(0)} = \underline{\mathbf{u}} \text{ in } \overline{\Omega}(0)\times[0,\infty), \tag{3.18}$$

and particularly, for every k,

$$\mathbf{u}(y,t+kT;\mathbf{u}_0) \le \underline{\mathbf{u}}(y,t+kT) = \underline{\mathbf{u}}(y,t). \tag{3.19}$$

On the other hand, according to Lemma 3.3, we obtain

$$\mathbf{u}(y,t+kT;\mathbf{u}_0) \ge \underline{\mathbf{u}}^{(k)}(y,t) \text{ in } \overline{\Omega}(0)\times[0,\infty)$$

for every k. Letting $k\to\infty$ and noticing that $\lim\limits_{k\to\infty} \underline{\mathbf{u}}^{(k)} = \underline{\mathbf{u}}$ yield

$$\lim_{k\to\infty} \mathbf{u}(y,t+kT;\mathbf{u}_0) = \underline{\mathbf{u}}(y,t).$$

For the other case $\mathbf{u}_0 \in \langle \overline{\mathbf{u}}, \tilde{\mathbf{u}} \rangle$, it is similar to obtain

$$\lim_{k\to\infty} \mathbf{u}(y,t+kT;\mathbf{u}_0) = \overline{\mathbf{u}}(y,t).$$

And we can easily check that relationship (3.16) holds due to (3.11) and Lemma 3.3.

In addition, the claim that $\underline{\mathbf{u}} = \overline{\mathbf{u}} = \mathbf{u}^*$ if $\underline{\mathbf{u}}(y,0) = \overline{\mathbf{u}}(y,0) = \mathbf{u}^*(y,0)$ has been verified in the proof of Theorem 3.1. Meanwhile, relationship (3.17) is apparent under the result $\underline{\mathbf{u}} = \overline{\mathbf{u}}$. The proof is completed. □

Theorem 3.4. *If* $R_0(\rho) \le 1$ *and* $\mathbf{u}(y,t;\mathbf{u}_0)$ *is the solution to problem* (2.6),(2.7), *then the following relationship holds for any* $\mathbf{u}_0 \ge 0$*:*

$$\lim_{t\to\infty} \mathbf{u}(y,t;\mathbf{u}_0) = (0,0) \text{ uniformly for } y \in \overline{\Omega}(0). \tag{3.20}$$

Proof. Let positive constant-pair (M_1, M_2) satisfy (3.6). Then for any constant-pair $(M_1^*, M_2^*) \ge (M_1, M_2)$, it is easy to check that $\tilde{\mathbf{u}} \equiv (M_1^*, M_2^*)$ and $\hat{\mathbf{u}} \equiv (0,0)$ satisfy Definition 3.1. By the iteration process established in the proof of Theorem 3.1, we can similarly obtain two T-periodic solutions $\underline{\mathbf{u}}$ and $\overline{\mathbf{u}}$ which satisfy (3.1) and

$$(0,0) \le \underline{\mathbf{u}} \le \overline{\mathbf{u}} \le (M_1^*, M_2^*).$$

Besides, it follows from Theorem 3.2 that there is none positive periodic solution satisfying (3.1), which implies that $\underline{\mathbf{u}} = \overline{\mathbf{u}} \equiv 0$. Analogous to the method used in Theorem 3.3 and with the help of the arbitrariness of (M_1^*, M_2^*), the uniform convergence property of $\mathbf{u}(y,t;\mathbf{u}_0)$ is valid:

$$\mathbf{u}(y, t+kT; \mathbf{u}_0) \to (0,0) \text{ as } k \to \infty \text{ in } \overline{\Omega}(0) \times [0, \infty).$$

The proof is completed. □

3.3. Existence and attractivity of periodic solutions on the fixed domain

Assume that the domain $\Omega(t)$ is fixed, then problem (2.6)–(2.8) is rewritten by the following FOTD model

$$\begin{cases} u_t - d_1 \Delta u = -a_{11}(t)u + a_{12}(t)v, & y \in \Omega(0),\ t > 0, \\ v_t - d_2 \Delta v = g(u) - a_{22}(t)v, & y \in \Omega(0),\ t > 0, \\ \dfrac{\partial u}{\partial \eta} = \dfrac{\partial v}{\partial \eta} = 0, & y \in \partial\Omega(0),\ t > 0, \end{cases} \tag{3.21}$$

with two different conditions, respectively, which one is the initial condition (2.7) and the other is periodic condition (2.8).

According to the next generation infection operator, we can similarly define the corresponding basic reproduction number of (3.21),(2.8) by $R_0(1)$ which also represents the principal eigenvalue of system (3.22) as follows:

$$\begin{cases} \dfrac{\partial \hat{\Phi}_0}{\partial t} - d_1 \Delta \hat{\Phi}_0 = \dfrac{a_{12}(t)\hat{\Psi}_0}{\hat{\mu}} - a_{11}(t)\hat{\Phi}_0, & y \in \Omega(0),\ t > 0, \\ \dfrac{\partial \hat{\Psi}_0}{\partial t} - d_2 \Delta \hat{\Psi}_0 = \dfrac{g'(0)\hat{\Phi}_0}{\hat{\mu}} - a_{22}(t)\hat{\Psi}_0, & y \in \Omega(0),\ t > 0, \\ \dfrac{\partial \hat{\Phi}_0}{\partial \eta} = \dfrac{\partial \hat{\Psi}_0}{\partial \eta} = 0, & y \in \partial\Omega(0),\ t > 0, \\ \hat{\Phi}_0(y,0) = \hat{\Phi}_0(y,T),\ \hat{\Psi}_0(y,0) = \hat{\Psi}_0(y,T), & y \in \overline{\Omega}(0). \end{cases} \tag{3.22}$$

Notice that $\text{sign}(1 - R_0(1)) = \text{sign}(\hat{\lambda}_0)$, where $\hat{\lambda}_0$ denotes the principal eigenvalue of problem (3.23) as follows:

$$\begin{cases} \dfrac{\partial \hat{\phi}_0}{\partial t} - d_1 \Delta \hat{\phi}_0 = a_{12}(t)\hat{\psi}_0 - a_{11}(t)\hat{\phi}_0 + \hat{\lambda}\hat{\phi}_0, & y \in \Omega(0),\ t > 0, \\ \dfrac{\partial \hat{\psi}_0}{\partial t} - d_2 \Delta \hat{\psi}_0 = g'(0)\hat{\phi}_0 - a_{22}(t)\hat{\psi}_0 + \hat{\lambda}\hat{\psi}_0, & y \in \Omega(0),\ t > 0, \\ \dfrac{\partial \hat{\phi}_0}{\partial \eta} = \dfrac{\partial \hat{\psi}_0}{\partial \eta} = 0, & y \in \partial\Omega(0),\ t > 0, \\ \hat{\phi}_0(y,0) = \hat{\phi}_0(y,T),\ \hat{\psi}_0(y,0) = \hat{\psi}_0(y,T), & y \in \overline{\Omega}(0). \end{cases} \tag{3.23}$$

Here, $\hat{\Phi}_0, \hat{\Psi}_0, \hat{\phi}_0$ and $\hat{\psi}_0$ belonging to C_T are strictly positive principal eigenfunctions in $\overline{\Omega}(0) \times (0, \infty)$. The first two correspond to principal eigenvalue $R_0(1)$ while the rest correspond to $\hat{\lambda}_0$.

Similar to Definition 3.1, we can define the ordered upper and lower solutions to (3.21),(2.8) by $\tilde{\mathbf{u}} = (\tilde{u}, \tilde{v})$ and $\hat{\mathbf{u}} = (\hat{u}, \hat{v})$ as follows:

Definition 3.2. Suppose that $\tilde{u}(y,t)$, $\tilde{v}(y,t)$, $\hat{u}(y,t)$ and $\hat{v}(y,t)$ are nonnegative functions in $C^{2,1}(\Omega(0)\times(0,\infty))\cap C(\overline{\Omega}(0)\times[0,\infty))$. Then $\tilde{\mathbf{u}}$ and $\hat{\mathbf{u}}$ are called ordered upper and lower solutions to problem (3.21),(2.8) if $\tilde{\mathbf{u}}\geq\hat{\mathbf{u}}\geq 0$ and if

$$\begin{cases}\tilde{u}_t-d_1\Delta\tilde{u}\geq -a_{11}(t)\tilde{u}+a_{12}(t)\tilde{v},\ \tilde{v}_t-d_2\Delta\tilde{v}\geq g(\tilde{u})-a_{22}(t)\tilde{v}, & y\in\Omega(0),\ t>0,\\ \hat{u}_t-d_1\Delta\hat{u}\leq -a_{11}(t)\hat{u}+a_{12}(t)\hat{v},\ \hat{v}_t-d_2\Delta\hat{v}\leq g(\hat{u})-a_{22}(t)\hat{v}, & y\in\Omega(0),\ t>0,\\ \dfrac{\partial\tilde{u}}{\partial\eta}\geq 0\geq\dfrac{\partial\hat{u}}{\partial\eta},\ \dfrac{\partial\tilde{v}}{\partial\eta}\geq 0\geq\dfrac{\partial\hat{v}}{\partial\eta}, & y\in\partial\Omega(0),\ t>0,\\ \tilde{u}(y,0)\geq\tilde{u}(y,T),\ \tilde{v}(y,0)\geq\tilde{v}(y,T), & y\in\overline{\Omega}(0),\\ \hat{u}(y,0)\leq\hat{u}(y,T),\ \hat{v}(y,0)\leq\hat{v}(y,T), & y\in\overline{\Omega}(0).\end{cases}$$

Subsequently, we present some results on problem (3.21) analogous to Theorem 3.1–Theorem 3.4.

Theorem 3.5. *The following statements hold.*

(i) If $R_0(1)>1$, then there exist two positive periodic solutions $\underline{\mathbf{u}}$ and $\overline{\mathbf{u}}$ satisfying (3.21),(2.8) *such that any solution to problem* (3.21),(2.8) $\mathbf{u}\in\langle\underline{\mathbf{u}},\overline{\mathbf{u}}\rangle$*. Moreover, if $\underline{\mathbf{u}}(y,0)=\overline{\mathbf{u}}(y,0)$, then $\underline{\mathbf{u}}=\overline{\mathbf{u}}=\mathbf{u}^*$ is the unique solution to this problem;*

(ii) If $R_0(1)\leq 1$, then problem (3.21),(2.8) *possesses no positive T-periodic solution.*

Theorem 3.6. *Let $\mathbf{u}(y,t;\mathbf{u}_0)$ be the solution to problem* (3.21),(2.7) *and $\tilde{\mathbf{u}},\hat{\mathbf{u}}\geq 0$ be a pair of ordered upper and lower solutions to it.*

(i) If $R_0(1)>1$, then

$$\lim_{k\to\infty}\mathbf{u}(y,t+kT;\mathbf{u}_0)=\begin{cases}\underline{\mathbf{u}}(y,t), & \textit{if}\ \ \mathbf{u}_0\in\langle\hat{\mathbf{u}},\underline{\mathbf{u}}\rangle\ \textit{in}\ \overline{\Omega}(0),\\ \overline{\mathbf{u}}(y,t), & \textit{if}\ \ \mathbf{u}_0\in\langle\overline{\mathbf{u}},\tilde{\mathbf{u}}\rangle\ \textit{in}\ \overline{\Omega}(0).\end{cases}$$

Moreover, for any $\mathbf{u}_0\in\hat{S}_0$,

$$\underline{\mathbf{u}}(y,t)\leq\mathbf{u}(y,t+kT;\mathbf{u}_0)\leq\overline{\mathbf{u}}(y,t)$$

holds as $k\to\infty$ in $\overline{\Omega}(0)\times[0,\infty)$, where

$$\hat{S}_0:=\Big\{\mathbf{u}_0\in C(\overline{\Omega}(0)):\mathbf{u}_0\in\langle\hat{\mathbf{u}},\tilde{\mathbf{u}}\rangle(y,0)\Big\}.$$

Particularly, if $\underline{\mathbf{u}}(y,0)=\overline{\mathbf{u}}(y,0)=\mathbf{u}^*(y,0)$, then

$$\lim_{k\to\infty}\mathbf{u}(y,t+kT;\mathbf{u}_0)=\mathbf{u}^*(y,t);$$

(ii) If $R_0(1)\leq 1$, then for any $\mathbf{u}_0\in\hat{S}_0$,

$$\lim_{t\to\infty}\mathbf{u}(y,t;\mathbf{u}_0)=(0,0)\ \textit{uniformly for}\ y\in\overline{\Omega}(0).$$

4. Numerical simulations

The aim of this section is to present the effect of evolution on the spread of fecal-oral diseases. We will provide a numerical illustration on $\mathbb{R}^1$ to support theoretical results established above.

Assume first that the reaction and diffusion coefficients are all constants, where $d_1 = 0.2$ and $d_2 = 0.4$. We choose $g(u) = (u+1)^{2/3} - 1$ and then $g'(0) = \frac{2}{3}$. Let the initial domain $\Omega(0) = (0,1)$. Then $\Omega(t) = (0, x(t)) = (0, \rho(t)y)$, where $y \in (0,1)$ and $\rho(t) \in C^1([0,T],(0,\infty))$ satisfying $\rho(0) = 1$ is periodic. We take $u_0(y) = 5+0.1\sin(\pi y), v_0(y) = 3+0.1\sin(\pi y)+0.05\sin(3\pi y)$ as the initial condition and set the periodic evolution rates as follows:

$$\rho_1(t) \equiv 1,\ \rho_2(t) = e^{0.3(1-\cos 2t)} \text{ and } \rho_3(t) = e^{-0.25(1-\cos 2t)}.$$

Example 4.1. Let $a_{11}(t) \equiv 0.75$, $a_{12}(t) \equiv 2/3$ and $a_{22}(t) \equiv 0.3$. Then

$$\mathfrak{R}_o = \sqrt{\frac{\frac{2}{3} \times \frac{2}{3}}{0.75 \times 0.3}} \approx 1.4055.$$

It follows from Theorem 2.1 that $R_0(\rho) \geq \mathfrak{R}_o > 1$.

First, we consider the fixed domain case $\rho(t) = \rho_1(t)$. Then $R_0(1) = \mathfrak{R}_o$. Due to Theorem 3.5(i) and Theorem 3.6(i), the endemic equilibrium exists periodically and is globally asymptotically stable if $R_0(1) > 1$. As shown in Figures (a) and (b), the infective tends to a positive stable solution which implies that fecal-oral transmission diseases will persist and become endemic.

Then take $\rho(t) = \rho_2(t)$ and $\rho(t) = \rho_3(t)$, respectively. We compute $\max_{t\in[0,T]}|\frac{\dot{\rho_2}}{\rho_2}| \approx 0.2338$ and $\max_{t\in[0,T]}|\frac{\dot{\rho_3}}{\rho_3}| \approx 0.1624$ satisfying (3.4). It is also easy to get $\overline{\rho_2^{-2}} \approx 0.5993 < 1$ and $\overline{\rho_3^{-2}} \approx 1.7534 > 1$ which imply that the periodic evolution rate ρ_2 is larger than ρ_3. It follows from Theorem 3.1 and Theorem 3.3 that the endemic disease persists periodically and is globally asymptotically stable if $R_0(\rho) > 1$. From figures (c) and (e), we present that the infective all stabilizes to a positive periodic solution. It means that fecal-oral transmission diseases will persist and become endemic periodically in the both cases of $\rho(t) = \rho_2(t)$ and $\rho(t) = \rho_3(t)$. Figures (d) and (f) show the periodic evolution of the domain.

Example 4.2. Choose $a_{11}(t) \equiv 0.3$, $a_{12}(t) \equiv 1/3$ and $a_{22} \equiv 0.95$. Then

$$\mathfrak{R}_o = \sqrt{\frac{\frac{1}{3} \times \frac{2}{3}}{0.3 \times 0.95}} \approx 0.8830.$$

Considering the fixed domain case, it follows from Theorem 3.5(ii) and Theorem 3.6(ii) that the disease-free equilibrium exists and is globally asymptotically stable if $R_0(1) < 1$. Here $R_0(1) = \mathfrak{R}_o < 1$. As shown in Figure 2(a) and Figure 2(b), the infective converges to zero which implies that fecal-oral transmission diseases will become extinct.

Then take $\rho(t) = \rho_2(t)$ and $\rho(t) = \rho_3(t)$, respectively. In these cases, we can not estimate the relationship of size between $R_0(\rho)$ and 1. From Figure 2(c), we present that the infective v will converge to a positive periodic solution. It means that fecal-oral transmission diseases will persist and become endemic periodically in the case of $\rho(t) = \rho_2(t)$. Figure 2(d) shows the periodic evolution of the domain. Meanwhile, it is easy to see from Figure 2(e) and Figure 2(f) that the infective v will decay to zero as the domain periodically evolves, that is, FOTD will vanish on the evolution domain with the evolution rate $\rho(t) = \rho_3(t)$. Moreover, we notice that the periodic evolution rate ρ_2 is larger than ρ_3. Hence, we can conjecture that the risk of spreading would become smaller as the evolution rate $\rho(t)$ decreases.

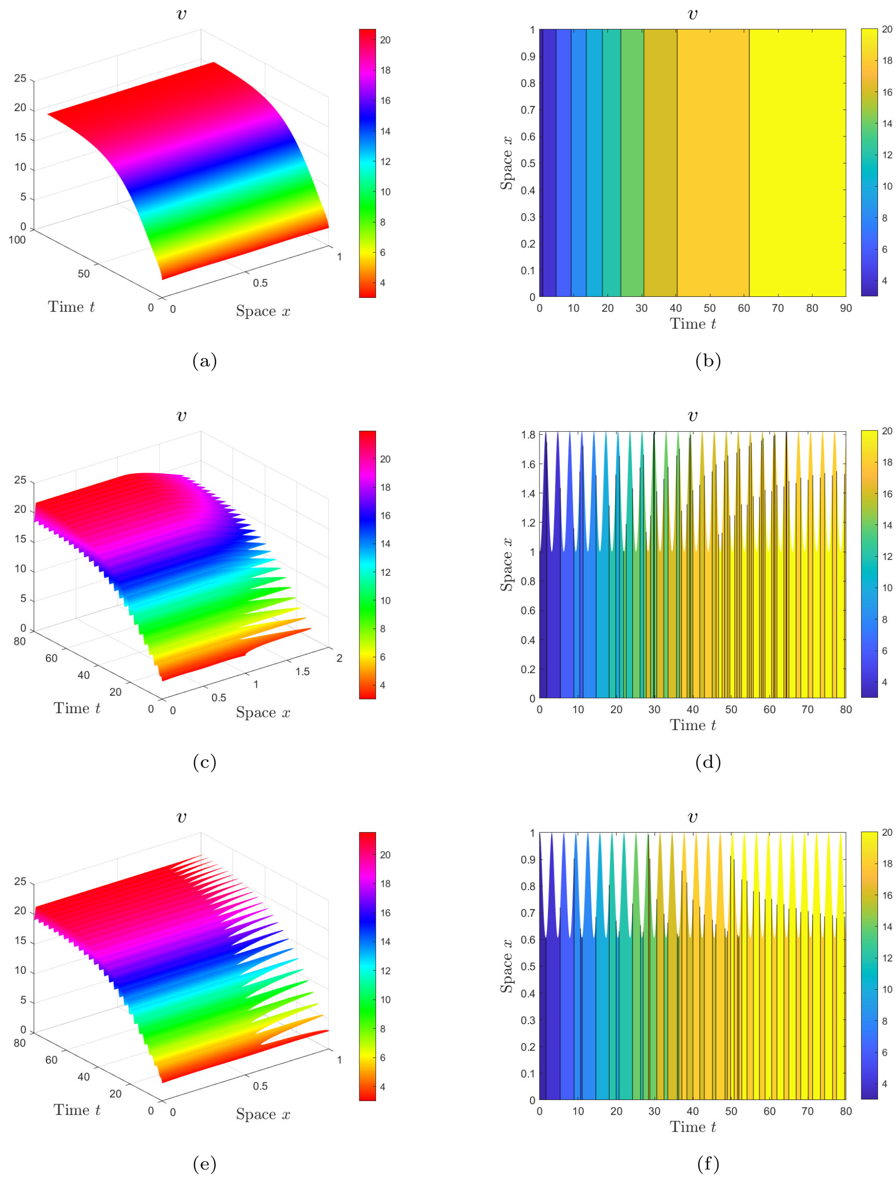

Figure 1. $a_{11} = 0.75$, $a_{12} = 2/3$, $a_{22} = 0.3$ and $R_0(\rho) \geq \mathfrak{R}_o > 1$. Figures (a), (c) and (e) all show that infected individuals stabilize to a positive periodic solution when $R_0(\rho) > 1$ while pictures (b), (d) and (f) are the corresponding contour maps of the infective, respectively. Figures (a)(b), (c)(d) and (e)(f) correspond to the case of $\rho(t) = \rho_1(t)$, $\rho_2(t)$ and $\rho_3(t)$, respectively.

5. Conclusions

On account of the foregoing analysis, we realize that evolution plays significant effect on the transmission of fecal-oral diseases. Compared with existing studies by reaction-diffusion model (1.1), in order to explore the effect of evolution on the prevention and control of FOTD, we first transform model (2.1)–(2.3) into that

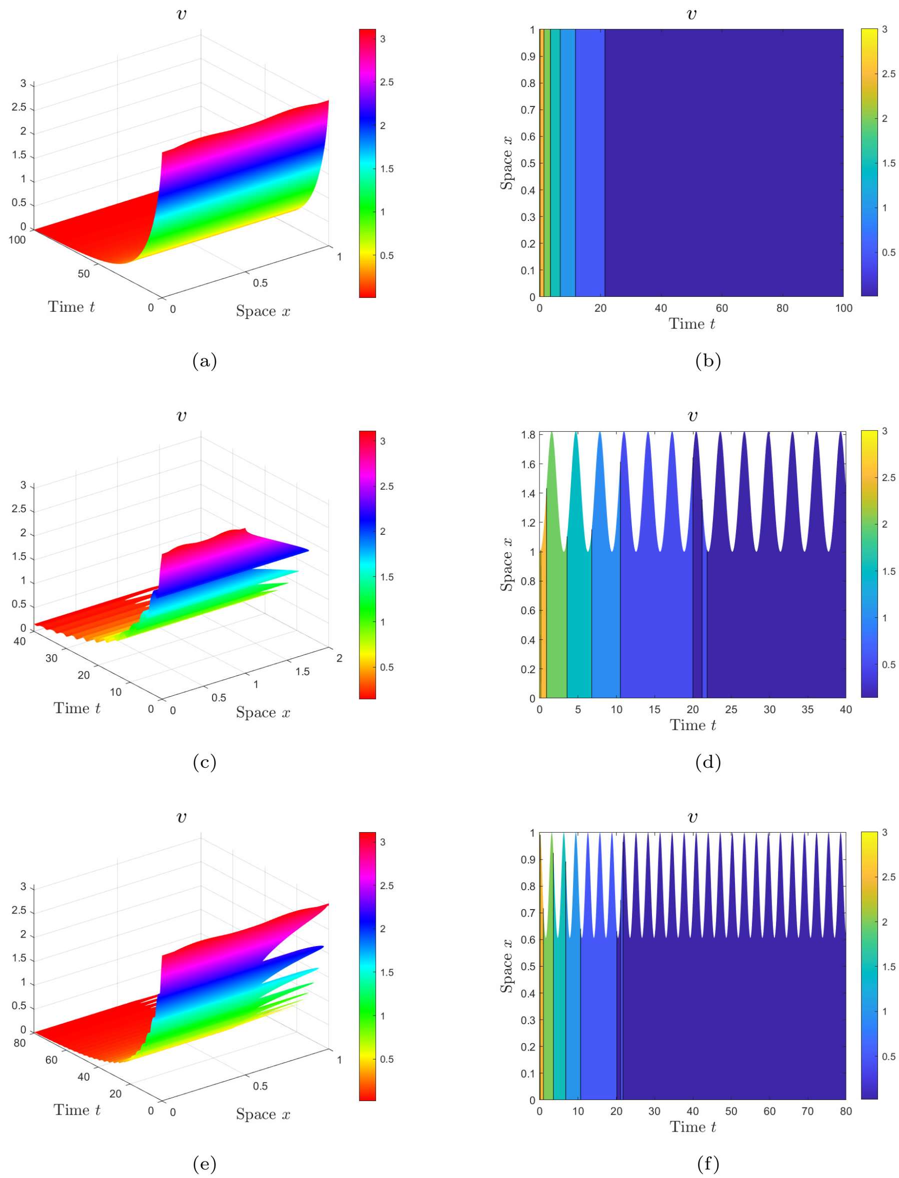

Figure 2. $a_{11} = 0.3$, $a_{12} = 1/3$, $a_{22} = 0.95$ and $\mathfrak{R}_o < 1$. Figures (a) and (e) all show that infected individuals will decrease to zero while picture (c) stabilizes to a positive periodic solution. The corresponding contour maps of (a), (c) and (e) are figures (b), (d) and (f), respectively. Figures (a)(b), (c)(d) and (e)(f) correspond to the case of $\rho(t) = \rho_1(t)$, $\rho_2(t)$ and $\rho_3(t)$, respectively.

on a fixed domain at the expense of turning the constant diffusion coefficients to time-period ones. Such transformation in coefficients is a key difficulty to be solved in investigating the asymptotic properties. To see this, we give the hypothesis of spatial isotropy. We then define the basic reproduction number $R_0(\rho)$ as the corresponding threshold parameter. Due to the complexity of FOTD model, it is difficult for $R_0(\rho)$ to find the clear analytic expression. However, we can estimate its lower bound by one important tool, the integral average, which can be used to predict the development trend of fecal-oral bacteria to some extent. Based on it, the effect of evolution on the spreading of FOTD is analyzed.

According to Theorem 2.1, we deduce the relationship $R_0(\rho) \geq \mathfrak{R}_\mathfrak{o}$ which implies that the periodic evolution can increase the risk of transmissions. Here, $R_0(\rho)$ and $\mathfrak{R}_\mathfrak{o}$ are the basic reproduction numbers of FOTD model on a periodic evolution and fixed domain with constant coefficients, respectively.

When $\mathfrak{R}_\mathfrak{o} > 1$, we have $R_0(\rho) \geq \mathfrak{R}_\mathfrak{o} > 1$. It follows from Theorem 3.3 and Theorem 3.6(i) that the solution of problem (2.6),(2.7) will tend to a positive steady-state periodic solution, which means that the diseases will spread and eventually become endemic on the both fixed and periodic evolution domain. Moreover, combining with Example 4.1, we show that the diseases will spread regardless of whether $\overline{\rho^{-2}} < 1$ or $\overline{\rho^{-2}} > 1$ when $\mathfrak{R}_\mathfrak{o} > 1$.

When $\mathfrak{R}_\mathfrak{o} \leq 1$, accurately estimating the relationship of size between $R_0(\rho)$ and 1 comes to nothing. According to Theorem 3.3 and Theorem 3.6(ii), if $R_0(\rho) > 1 > \mathfrak{R}_\mathfrak{o}$, then the development tendency of bacteria and the infective will happen a fundamental change, that is, the bacteria and infected individuals will vanish on the initial fixed domain $\Omega(0)$ but persist on the evolution domain $\Omega(t)$. Combining with the numerical simulation Example 4.2 and related conclusions in [19, 40], we can conjecture that the transmission risk of bacteria and the infective may become larger in the case of $\overline{\rho^{-2}} < 1$ and oppose in the case of $\overline{\rho^{-2}} > 1$. If $1 \geq R_0(\rho) \geq \mathfrak{R}_\mathfrak{o}$, then it follows from Theorem 3.4 and Theorem 3.6(ii) that the bacteria and infected individuals will die out whether on the evolution domain or fixed one.

Through the research and analysis in this paper, we clearly recognize the basic fact that the virus living environment presents seasonal changes. Theses results indicate that evolution will increase the transmission risk of FOTD, where the risk of the large evolution rate may be larger than that of small evolution rate. It will be detrimental to the prevention and control of FOTD. Therefore, only by improving the living environment of human beings, increasing the mortality rate of bacteria, and at the same time reducing the mutual transmission rate between bacteria and people can the spread of fecal-oral transmission diseases be effectively prevented and controlled.

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Authors' contributions

Y. Zhou, B. B. Zhang and Z. Ling wrote the main manuscript text and Y. Zhou prepared the full numerical simulations. All authors reviewed the manuscript.

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