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PAINLEVÉ INTEGRABLE PROPERTY, BILINEAR FORM, BÄCKLUND TRANSFORMATION, KINK AND SOLITON SOLUTIONS OF A (2+1)-DIMENSIONAL VARIABLE-COEFFICIENT GENERAL COMBINED FOURTH-ORDER SOLITON EQUATION IN A FLUID OR PLASMA

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Abstract In this paper, we focus our attention on a (2+1)-dimensional variable-coefficient general combined fourth-order soliton equation in a fluid or plasma. Under certain coefficient constraints, we get the Painlevé integrable property. We obtain the bilinear form and bilinear auto-Bäcklund transformation. By virtue of the truncated Painlevé expansion, we derive an auto-Bäcklund transformation. Under certain coefficient constraints, we graphically analyse the one-kink waves, one soliton, two-kink waves and two solitons. We get the expressions of the amplitude and velocity of the one soliton and analyse the types of the two solitons and two-kink waves before and after the interactions.

Keywords Fluid or plasma, (2+1)-dimensional variable-coefficient general combined fourth-order soliton equation, Painlevé integrable property, bilinear form, soliton and kink solutions.

MSC(2010) 35C08, 35G20, 47A07.

1. Introduction

Fluid dynamics has been considered as the basic mechanism of studying the liquids, gases and plasmas and the forces they are subjected to [7, 17, 24]. Plasma physics has been used to study the interactions of charged particles and fluids with the self-consistent electromagnetic fields [2]. Researchers have investigated the nonlinear

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evolution equations (NLEEs) to describe some phenomena in fiber optics, plasma physics and fluid mechanics [1, 5, 7, 8, 19, 23, 24, 26–29, 31–34, 38, 39]. Methods have been found to get some analytic solutions for the NLEEs, including the Bäcklund transformations and Hirota method [6, 18, 21, 41, 43]. Soliton and kink solutions have been associated with the wave phenomena in the plasmas and fluids [4, 9, 12].

Since the variable-coefficient NLEEs have shown us more information than their constant-coefficient counterparts in, e.g., the optical fibers [35], fluids [13] and so on, we will consider a (2+1)-dimensional variable-coefficient general combined fourth-order soliton equation in a fluid or plasma,

$$\begin{aligned}&\alpha\left(6u_xu_{xx}+u_{xxxx}\right)+\beta\left[3\left(u_xu_t\right)_x+u_{xxxt}\right]+\gamma\left[3\left(u_xu_y\right)_x+u_{xxxy}\right]\\&+\delta_1(t)u_{yt}+\delta_2(t)u_{xx}+\delta_3(t)u_{xt}+\delta_4(t)u_{xy}+\delta_5(t)u_{yy}+\delta_6(t)u_{tt}=0,\end{aligned}\tag{1.1}$$

where u is a differentiable function of the variables x, y and t, the real constants α, β and γ cannot be zero simultaneously, $\delta_i(t)$'s $(1\le i\le 6)$ are the real differentiable functions of t, and the subscripts represent the partial derivatives.

Some special cases of Eq. (1.1) have been discussed in types of the fluids or plasmas:

- When $\alpha=\beta=\delta_2(t)=\delta_6(t)=0$ and $\gamma=1$, Eq. (1.1) has been reduced to the (2+1)-dimensional variable-coefficient Boiti-Leon-Manna-Pempinelli equation,

$$u_{xxxy}+3(u_xu_y)_x+\delta_1(t)u_{yt}+\delta_3(t)u_{xt}+\delta_4(t)u_{xy}+\delta_5(t)u_{yy}=0,\tag{1.2}$$

  to describe an incompressible fluid [14, 25, 30].

- When $\delta_1(t)=\zeta_1$, $\delta_2(t)=\zeta_2$, $\delta_3(t)=\zeta_3$, $\delta_4(t)=\zeta_4$, $\delta_5(t)=\zeta_5$ and $\delta_6(t)=\zeta_6$, Eq. (1.1) has become a general combined fourth-order soliton equation,

$$\begin{aligned}&\alpha\left(6u_xu_{xx}+u_{xxxx}\right)+\beta\left[3\left(u_xu_t\right)_x+u_{xxxt}\right]+\gamma\left[3\left(u_xu_y\right)_x+u_{xxxy}\right]\\&+\zeta_1u_{yt}+\zeta_2u_{xx}+\zeta_3u_{xt}+\zeta_4u_{xy}+\zeta_5u_{yy}+\zeta_6u_{tt}=0,\end{aligned}\tag{1.3}$$

  for the shallow-water waves, where ζ_i's are the constants [40, 42].

- When $\beta=\delta_1(t)=\delta_6(t)=0$, $\delta_2(t)=\zeta_2$, $\delta_3(t)=1$, $\delta_4(t)=\zeta_4$ and $\delta_5(t)=\zeta_5$, Eq. (1.1) has become a (2+1)-dimensional generalized Bogoyavlensky-Konopelchenko equation,

$$\begin{aligned}&\alpha\left(6u_xu_{xx}+u_{xxxx}\right)+\gamma\left[3\left(u_xu_y\right)_x+u_{xxxy}\right]\\&+\zeta_2u_{xx}+u_{xt}+\zeta_4u_{xy}+\zeta_5u_{yy}=0,\end{aligned}\tag{1.4}$$

  for the shallow-water waves, stratified internal waves in a fluid or ion-acoustic waves in a plasma [3, 20, 22].

However, to our knowledge, Painlevé integrable property, bilinear form, auto-Bäcklund transformation, kink and soliton solutions of Eq. (1.1) have not been studied. In Section 2, we will get the Painlevé integrable property of Eq. (1.1). In Section 3, we will derive the bilinear form of Eq. (1.1). In Section 4, bilinear Bäcklund transformation and one-kink solutions of Eq. (1.1) will be given. In Section 5, one- and two-soliton solutions of Eq. (1.1) will be constructed. In Section 6, two-kink solutions of Eq. (1.1) will be obtained. Section 7 will be our conclusions.

2. Painlevé integrable property of Eq. (1.1)

According to Refs. [7,15,16], we suppose that

$$u = u_0 \varphi^p, \tag{2.1}$$

where u_0 and φ are the real differentiable functions of x, y and t, p is a non-positive integer. Substituting Expression (2.1) into Eq. (1.1), we find that the leading-order analysis yields $p = -1$, let the coefficient of φ^{-5} equal to zero and get $u_0 = 2\varphi_x$. Next, we rewrite u as

$$u = u_j \varphi^{-1+j} + u_0 \varphi^{-1}, \tag{2.2}$$

where u_j is a real differentiable function of x, y and t and j is an integer. Substituting Expression (2.2) into Eq. (1.1) and making the coefficient of φ^{-5+j} equal to zero, we find that $j = -1, 1, 4, 6$. We set

$$u = \varphi^{-1} \sum_{j=0}^{6} u_j \varphi^j, \tag{2.3}$$

take Expression (2.3) into Eq. (1.1) and sort out the coefficients of φ^{-j}'s $(-1 \leq j \leq 5)$. Afterwards, we find the coefficient of φ^{-4} is zero, which means that u_1 is an arbitrary function. Next, we let the coefficient of φ^{-3} equal to zero and obtain the expression of u_2. Taking u_2 into Eq. (1.1) and letting the coefficient of φ^{-2} equal to zero, we get the the expression of u_3. We substitute u_3 into Eq. (1.1) and obtain the coefficient of φ^{-1} is zero, which means that u_4 is an arbitrary function. Then, in order to simplify the verification of the remaining resonant condition $j = 6$, we adopt Kruskal's simplified expressions as [37]

$$\varphi = \eta + \phi(t), \qquad u_j = u_j(\eta, t), \qquad \eta = x + y. \tag{2.4}$$

Based on Expression (2.4), we let the coefficient of φ^0 equal to zero and obtain the expression of u_5. Taking u_5 into Eq. (1.1), we find that u_6 is an arbitrary function when

$$\delta_3(t) = \zeta_1 - \delta_1(t), \qquad \delta_2(t) = \zeta_2 - \delta_4(t) - \delta_5(t), \qquad \delta_6(t) = \zeta_6. \tag{2.5}$$

Thus, we derive the Painlevé integrable property of Eq. (1.1).

3. Bilinear form of Eq. (1.1)

Motivated by Ref. [40], we assume that

$$u = 2(\ln f)_x, \tag{3.1}$$

where f is the real function of x, y and t. We take Expression (3.1) into Eq. (1.1) and get the bilinear form of Eq. (1.1)

$$\begin{aligned}[\alpha D_x^4 &+ \beta D_x^3 D_t + \gamma D_x^3 D_y + \delta_1(t) D_y D_t + \delta_2(t) D_x^2 \\ &+ \delta_3(t) D_x D_t + \delta_4(t) D_x D_y + \delta_5(t) D_y^2 + \delta_6(t) D_t^2] F \cdot F = 0,\end{aligned} \tag{3.2}$$

where D_x, D_y and D_t are the Hirota bilinear derivative operators [10], defined as

$$D_x^m D_y^n D_t^k(\eta \cdot \sigma) \equiv \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right)^m \left(\frac{\partial}{\partial y} - \frac{\partial}{\partial y'}\right)^n \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\right)^k \eta(x, y, t)\sigma(x', y', t')\Big|_{x'=x, y'=y, t'=t},$$

with x', y' and t' being the formal variables, m, n and k being the non-negative integers, η being the differentiable function of x, y, z and t, while σ being the differentiable function of x', y', z' and t'.

4. Bilinear Bäcklund transformation and one-kink solutions of Eq. (1.1)

Based on Bilinear Form (3.2), we consider the following equation:

$$\begin{aligned} 0 =& g^2[\alpha D_x^4 + \beta D_x^3 D_t + \gamma D_x^3 D_y + \delta_1(t) D_y D_t + \delta_2(t) D_x^2 \\ &+ \delta_3(t) D_x D_t + \delta_4(t) D_x D_y + \delta_5(t) D_y^2 + \delta_6(t) D_y^2] f \cdot f \\ &- f^2[\alpha D_x^4 + \beta D_x^3 D_t + \gamma D_x^3 D_y + \delta_1(t) D_y D_t + \delta_2(t) D_x^2 \\ &+ \delta_3(t) D_x D_t + \delta_4(t) D_x D_y + \delta_5(t) D_y^2 + \delta_6(t) D_y^2] g \cdot g, \end{aligned} \tag{4.1}$$

where g is another solution of Bilinear Form (3.2). According to the exchange identities [10]:

$$\begin{aligned} & g^2\left(D_x^3 D_t f \cdot f\right) - f^2\left(D_x^3 D_t g \cdot g\right) = 2\left[D_t\left(D_x^3 f \cdot g\right) \cdot (fg)\right] \\ & \qquad + 6\left[D_x\left(D_x D_t f \cdot g\right) \cdot \left(D_x g \cdot f\right)\right], \\ & g^2\left(D_x^3 D_y f \cdot f\right) - f^2\left(D_x^3 D_y g \cdot g\right) = 2\left[D_y\left(D_x^3 f \cdot g\right) \cdot (fg)\right] \\ & \qquad + 6\left[D_x\left(D_x D_y f \cdot g\right) \cdot \left(D_x g \cdot f\right)\right], \\ & g^2\left(D_x^4 f \cdot f\right) - f^2\left(D_x^4 g \cdot g\right) = 2\left[D_x\left(D_x^3 f \cdot g\right) \cdot (fg)\right] \\ & \qquad + 6\left[D_x\left(D_x^2 f \cdot g\right) \cdot \left(D_x g \cdot f\right)\right], \\ & g^2\left(D_y D_t f \cdot f\right) - f^2\left(D_y D_t g \cdot g\right) = 2\left[D_t\left(D_y f \cdot g\right) \cdot gf\right], \\ & g^2\left(D_y D_y f \cdot f\right) - f^2\left(D_y D_y g \cdot g\right) = 2\left[D_y\left(D_y f \cdot g\right) \cdot gf\right], \\ & g^2\left(D_x D_t f \cdot f\right) - f^2\left(D_x D_t g \cdot g\right) = 2\left[D_t\left(D_x f \cdot g\right) \cdot gf\right], \\ & g^2\left(D_x D_y f \cdot f\right) - f^2\left(D_x D_y g \cdot g\right) = 2\left[D_y\left(D_x f \cdot g\right) \cdot gf\right], \\ & g^2\left(D_t D_t f \cdot f\right) - f^2\left(D_t D_t g \cdot g\right) = 2\left[D_t\left(D_t f \cdot g\right) \cdot gf\right], \\ & g^2\left(D_x^2 f \cdot f\right) - f^2\left(D_x^2 g \cdot g\right) = 2\left[D_x\left(D_x f \cdot g\right) \cdot gf\right], \end{aligned} \tag{4.2}$$

we obtain the bilinear Bäcklund transformation of Eq. (1.1) as follows:

$$\begin{aligned}
&2\alpha D_x^3 f\cdot g + 2\delta_2(t) D_x f\cdot g + 2\delta_3(t) D_t f\cdot g = \lambda_1 g f,\\
&6\alpha D_x^2 f\cdot g + 6\beta D_x D_t f\cdot g + 6\gamma D_x D_y f\cdot g + \lambda_2 D_x f\cdot g = 0,\\
&2\beta D_x^3 f\cdot g + \delta_6(t) D_t f\cdot g = \lambda_3 g f,\\
&\delta_6(t) g f = \lambda_4 D_t g\cdot f,\\
&2\gamma D_x^3 f\cdot g + 2\delta_4(t) D_x f\cdot g + 2\delta_5(t) D_y f\cdot g + 2\delta_1(t) D_t f\cdot g = \lambda_5 g f,
\end{aligned} \tag{4.3}$$

where λ_κ's, $\kappa = 1, 2, 3, 4, 5$, are the constants.

We take $g = 1$ as a solution of Bilinear Form (3.2) and $f = \exp[b(t)y + c(t)t + d(t)x] + 1$, and substitute them into Bilinear Bäcklund Transformation (4.3), where $b(t)$, $c(t)$ and $d(t)$ are the functions about t. We choose the coefficient constraints as follows:

$$\begin{aligned}
&\delta_6(t) = \beta = 0,\\
&\gamma = \alpha,\\
&\delta_1(t) = \delta_3(t),\\
&\delta_2(t) = \delta_4(t) + \delta_5(t).
\end{aligned} \tag{4.4}$$

Via Coefficient Constraints (4.4), we take $\lambda_1 = \lambda_3 = \lambda_4 = \lambda_5 = 0$, $b(t) = d(t) = 1$ and $\lambda_2 = -12\alpha$, then obtain the one-kink solutions of Eq. (1.1) as

$$u = \frac{2\exp[c(t)t + x + y]}{\exp[c(t)t + x + y] + 1}, \tag{4.5}$$

where $c(t) = \frac{\int \frac{\delta_2(t)+\alpha}{-\delta_3(t)} dt}{t}$.

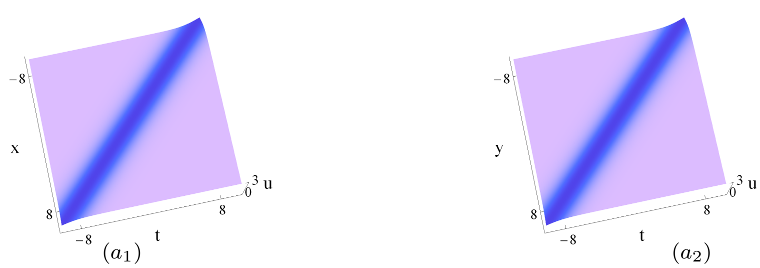


Figure 1. One kink wave via Solutions (4.5) under Coefficient Constraints (4.4), $(a_1 - c_1)$ y=0; $(a_2 - c_2)$ x=0. Parameters are (a_1) and (a_2): $\delta_2(t)$=$t - 1$, $\delta_3(t)$=$-t$, $\delta_4(t)$=-1 and α=1.

Figs. 1 and 2 present the propagation of the one kink waves via Solutions (4.5) on the x-t and y-t planes when the coefficient $\delta_2(t)$ varies but $\delta_3(t)$ and $\delta_4(t)$ keep unchanged. According to Figs. 1 and 2, we find that the amplitudes of the one kink waves keep unchanged during the propagation. This behavior shows that the amplitudes of one kink waves are not related to the coefficient $\delta_2(t)$.

Figs. 1 and 3 present the propagation of the one kink waves via Solutions (4.5) on the x-t and y-t planes when the coefficient $\delta_3(t)$ varies but $\delta_2(t)$ and $\delta_4(t)$ keep

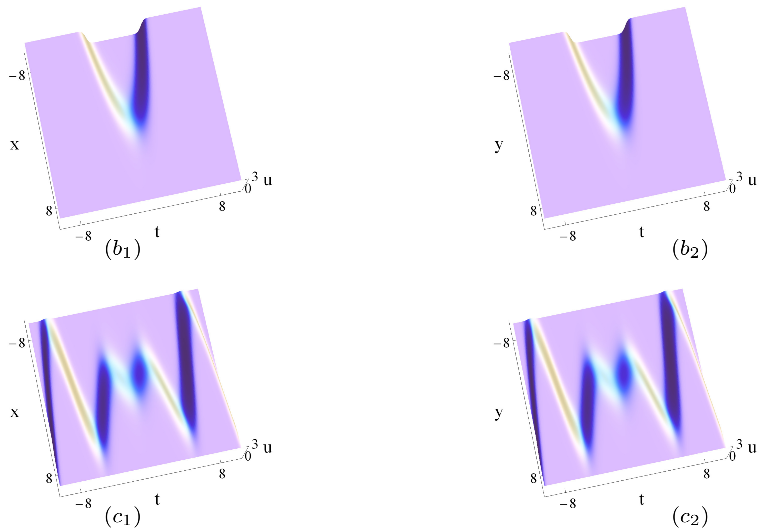


Figure 2. The same as Figs. 1 (a_1) and (a_2) except that (b_1) and (b_2): $\delta_2(t)=2t^2-1$; (c_1) and (c_2): $\delta_2(t)=t(\sin t+t\cos t)-1$.

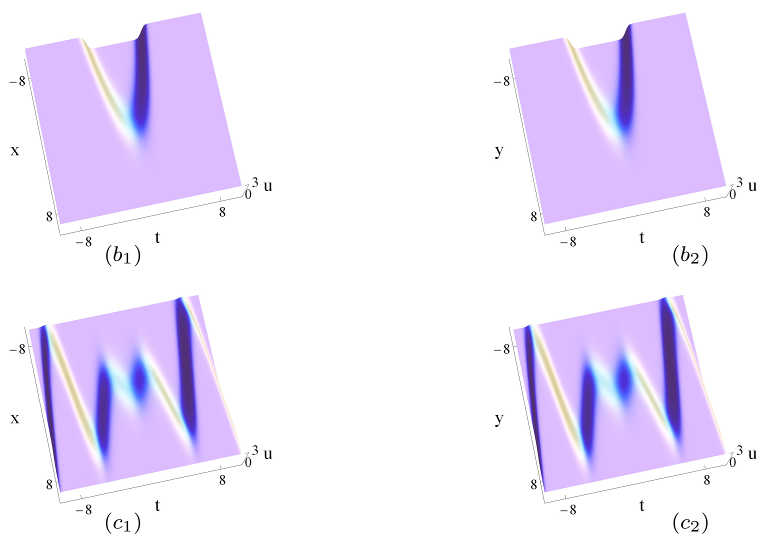


Figure 3. The same as Figs. 1 (a_1) and (a_2) except that (b_1) and (b_2): $\delta_3(t)=-0.5$; (c_1) and (c_2): $\delta_3(t)=-\frac{t}{\sin t+t\cos t}$.

unchanged. According to Figs. 1 and 3, we find that the amplitudes of the one kink waves keep unchanged during the propagation. This behavior shows that the amplitudes of one kink waves are not related to the coefficient $\delta_3(t)$.

Figs. 1 and 4 present the propagation of the one kink waves via Solutions (4.5) on the x-t and y-t planes when the coefficient $\delta_4(t)$ varies but $\delta_2(t)$ and $\delta_3(t)$ keep unchanged. According to Figs. 1 and 4, we find that the amplitudes of the one kink waves keep unchanged during the propagation. This behavior shows that the amplitudes of one kink waves are not related to the coefficient $\delta_4(t)$.

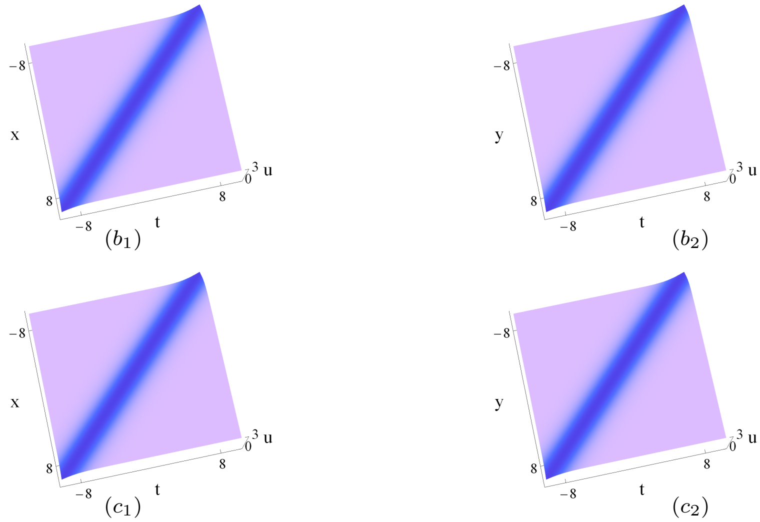


Figure 4. The same as Figs. 1 (a_1) and (a_2) except that (b_1) and (b_2): $\delta_4(t)$=$2t^2-1$; (c_1) and (c_2): $\delta_4(t)$=$t(\sin t + t\cos t) - 1$.

5. One- and two-soliton solutions of Eq. (1.1)

Motivated by Ref. [11], we assume that

$$f = \exp\left\{\frac{d_1(t)\exp[b_1(t)y + c_1(t)t + d_1(t)x]}{\exp[b_1(t)y + c_1(t)t + d_1(t)x] + 1}\right\}, \tag{5.1}$$

where $b_1(t)$, $c_1(t)$ and $d_1(t)$ are the real functions of t. Then we take Expression (5.1) into Bilinear Form (3.2) and get the coefficient constraints as

$$\begin{aligned} &\delta_6(t) = \beta = 0,\\ &\alpha + \gamma = 0,\\ &\delta_1(t) + \delta_3(t) = 0,\\ &\delta_4(t) + \delta_2(t) + \delta_5(t) = 0. \end{aligned} \tag{5.2}$$

Based on Coefficient Constraints (5.2), we take $b_1(t) = d_1(t) = 1$ and $c_1(t) = \delta_1(t)$, then give the one-soliton solutions of Eq. (1.1) as

$$u = 2\left\{\frac{\exp[\delta_1(t)t + x + y]}{\exp[\delta_1(t)t + x + y] + 1} - \frac{\exp[2\delta_1(t)t + 2x + 2y]}{\{\exp[\delta_1(t)t + x + y] + 1\}^2}\right\}. \tag{5.3}$$

Meanwhile, the amplitude B and velocity $V = (v_x, v_y)^T$ of One-Soliton Solutions (5.3) can be given, respectively

$$B = \frac{1}{2}, \tag{5.4a}$$

$$v_x = -\left\{t[\delta_1(t)]_t + \delta_1(t)\right\}, \tag{5.4b}$$

$$v_y = -\left\{t[\delta_1(t)]_t + \delta_1(t)\right\}, \tag{5.4c}$$

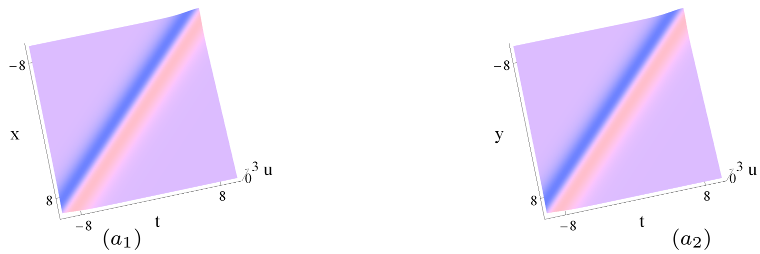


Figure 5. One soliton via Solutions (5.3) under Coefficient Constraints (5.2), $(a_1 - c_1)$ $y = 0$; $(a_2 - c_2)$ $x = 0$. Parameters are (a_1) and (a_2): $\delta_2(t)$=t, $\delta_1(t)$=1, $\delta_4(t)$=1 and α=1.

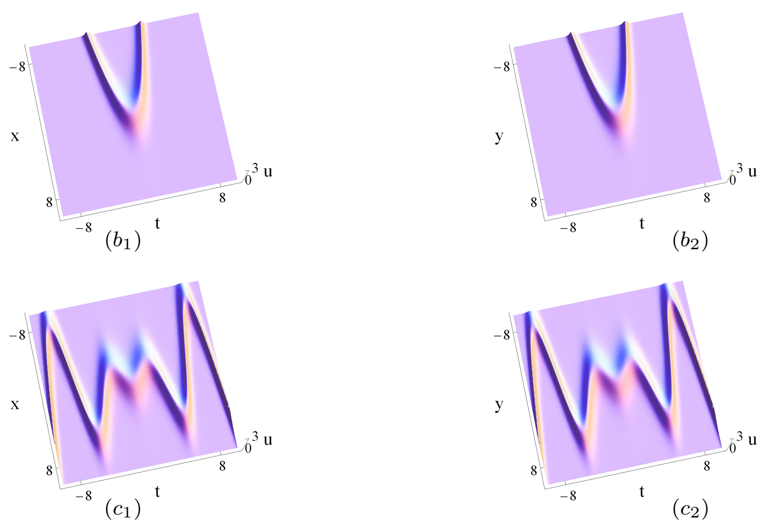


Figure 6. The same as Figs. 5 (a_1) and (a_2) except that (b_1) and (b_2): $\delta_1(t)$=t; (c_1) and (c_2): $\delta_1(t)$=$\sin t$.

where T represents the transpose of a vector.

Figs. 5 and 6 present the propagation of the one soliton via Solutions (5.3) on the x-t and y-t planes when the coefficient $\delta_1(t)$ is a constant or a function of t. According to Eq. (5.4a), the amplitudes of one soliton are not related to the coefficient $\delta_1(t)$, which is consistent with the amplitudes of one soliton keep unchanged in Figs. 5 and 6. Based on Eqs. (5.4b) and (5.4c), the velocities of one soliton are related to the coefficient $\delta_1(t)$.

At the same time, we take $c_1(t) = \delta_2(t)$.

We take $c_1(t) = \delta_4(t)$.

Through Figs. 5, 6, 7 and 8, we keep any two of the $\delta_1(t)$, $\delta_2(t)$ and $\delta_4(t)$ unchanged, the propagation of one soliton is similar. It can be found that the coefficients $\delta_2(t)$ and $\delta_4(t)$ have the similar effects on the one soliton. With the coefficients $\delta_1(t)$, $\delta_2(t)$ and $\delta_4(t)$ taking the constants, we observe that the one soliton presents the linear type. With the coefficients $\delta_1(t)$, $\delta_2(t)$ and $\delta_4(t)$ taking the linear functions, we observe that the one soliton presents the parabolic type. With the coefficients $\delta_1(t)$, $\delta_2(t)$ and $\delta_4(t)$ taking the periodic functions, we observe that the one soliton presents the periodic type.

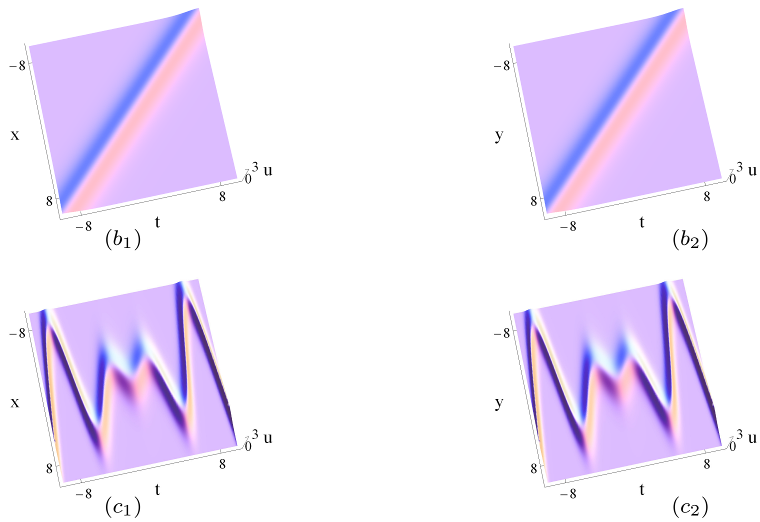


Figure 7. The same as Figs. 5 (a_1) and (a_2) except that (b_1) and (b_2): $\delta_2(t)=1$; (c_1) and (c_2): $\delta_2(t)=\sin t$.

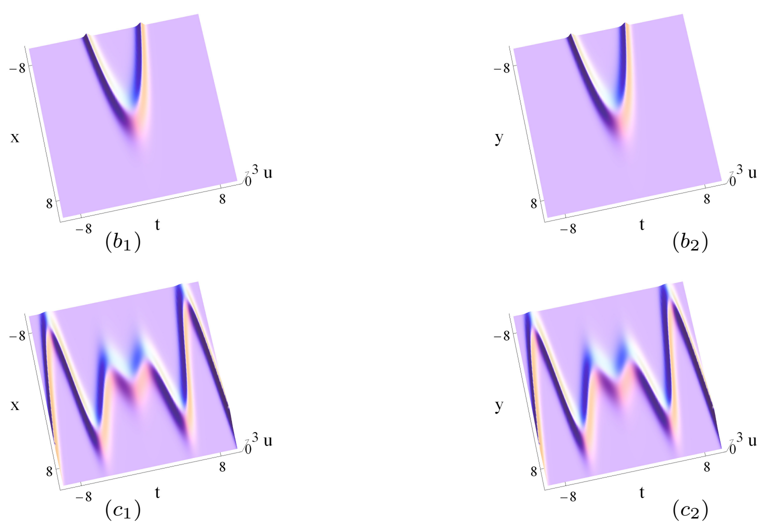


Figure 8. The same as Figs. 5 (a_1) and (a_2) except that (b_1) and (b_2): $\delta_4(t)=t$; (c_1) and (c_2): $\delta_4(t)=\sin t$.

We suppose that

$$f = \exp\left\{ \frac{k[d_2(t)+e(t)]\exp[b_2(t)y + c_2(t)t + d_2(t)x + e(t)x + f(t)y + g(t)t]}{A} \right. \\ \left. + \frac{d_2(t)\exp[b_2(t)y + c_2(t)t + d_2(t)x] + e(t)\exp[e(t)x + f(t)y + g(t)t]}{A} \right\}, \quad (5.5)$$

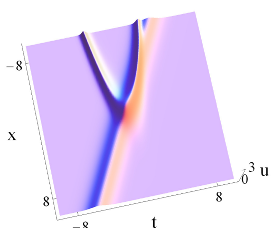
where $e(t)$, $f(t)$, $g(t)$, $b_2(t)$, $c_2(t)$ and $d_2(t)$ are the real functions of t, k is a real constant and $A = \exp[b_2(t)y + c_2(t)t + d_2(t)x + e(t)x + f(t)y + g(t)t]k + \exp[b_2(t)y +$

$c_2(t)t+d_2(t)x]+\exp[e(t)x+f(t)y+g(t)t]+1$. Taking Expression (5.5) into Bilinear Form (3.2), we get the coefficient constraints as

$$\begin{aligned}&\delta_6(t)=\beta=0,\\&\alpha+\gamma=0,\\&\delta_1(t)+\delta_3(t)=0,\\&\delta_4(t)+\delta_2(t)+\delta_5(t)=0.\end{aligned}\tag{5.6}$$

According to Coefficient Constraints (5.6), we take $k=3$, $e(t)=f(t)=b_2(t)=d_2(t)=1$ and $c_2(t)=\delta_1(t), g(t)=\delta_2(t)$, then give the two-soliton solutions of Eq. (1.1) as

$$\begin{aligned}u=&2\left\{\frac{12\exp[\delta_1(t)t+\delta_2(t)t+2x+2y]+\exp[\delta_1(t)t+x+y]+\exp[\delta_2(t)t+x+y]}{3\exp[\delta_1(t)t+\delta_2(t)t+2x+2y]+\exp[\delta_1(t)t+x+y]+\exp[\delta_2(t)t+x+y]+1}\right.\\&\left.-\frac{\{6\exp[\delta_1(t)t+\delta_2(t)t+2x+2y]+\exp[\delta_1(t)t+x+y]+\exp[\delta_2(t)t+x+y]\}^2}{\{3\exp[\delta_1(t)t+\delta_2(t)t+2x+2y]+\exp[\delta_1(t)t+x+y]+\exp[\delta_2(t)t+x+y]+1\}^2}\right\}.\end{aligned}\tag{5.7}$$



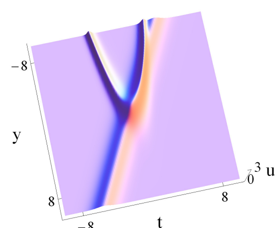


Figure 9. Two solitons via Solutions (5.7) under Coefficient Constraints (5.6), (a_1-c_1) $y=0$; (a_2-c_2) $x=0$. Parameters are (a_1) and (a_2): $\delta_4(t)$=1, α=1, $\delta_1(t)$=2 and $\delta_2(t)$=t.

We assume that $c_2(t)=\delta_1(t)$ and $g(t)=\delta_4(t)$.

We assume that $c_2(t)=\delta_2(t)$ and $g(t)=\delta_4(t)$.

Figs. 9 and 10 present the propagation of the two solitons via Solutions (5.7) on the x-t and y-t planes. According to Figs. 9 and 10, we find that the two solitons are composed of two one solitons which are similar to those in Figs. 5-8. Figs. 9 and 11 present the coefficient $\delta_4(t)$ has the similar effects on the two solitons with $\delta_2(t)$. Figs. 9 and 12 present the coefficient $\delta_2(t)$ has the similar effects on the two solitons with $\delta_1(t)$.

From Figs. 9, 10, 11 and 12, we observe three types of the two solitons, i.e., two solitons composed of linear-type one soliton and parabolic-type one soliton, two solitons composed of two parabolic-type one solitons and two solitons composed of periodic-type one soliton and parabolic-type one soliton. For the two solitons composed of linear-type one soliton and parabolic-type one soliton, we find that the linear-type one soliton keeps linear type and parabolic-type one soliton keeps parabolic type after the interaction. For the two solitons composed of two parabolic-type one solitons, we find that the two parabolic-type one solitons keep parabolic type after the interaction. For the two solitons composed of periodic-type one soliton and parabolic-type one soliton, we find that the periodic-type one soliton

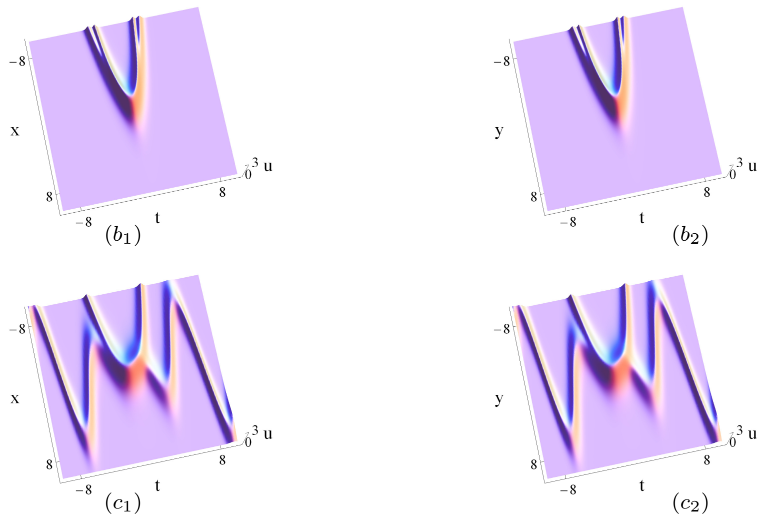


Figure 10. The same as Figs. 9 (a_1) and (a_2) except that (b_1) and (b_2): $\delta_1(t)$=t and $\delta_2(t)$=$2t$; (c_1) and (c_2): $\delta_1(t)$=t and $\delta_2(t)$=$\cos t$.

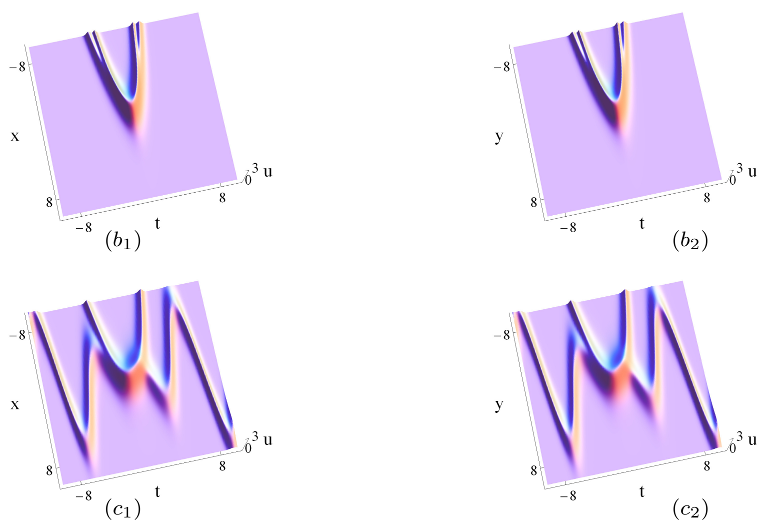


Figure 11. The same as Figs. 9 (a_1) and (a_2) except that (b_1) and (b_2): $\delta_1(t)$=t and $\delta_4(t)$=$2t$; (c_1) and (c_2): $\delta_1(t)$=t and $\delta_4(t)$=$\cos t$.

keeps periodic type and parabolic-type one soliton keeps parabolic type after the interaction.

6. Two-kink waves of Eq. (1.1)

Motivated by Ref. [36], we take

$$u = \varphi^{-1} \left(2\varphi_x + u_1 \varphi \right), \tag{6.1}$$

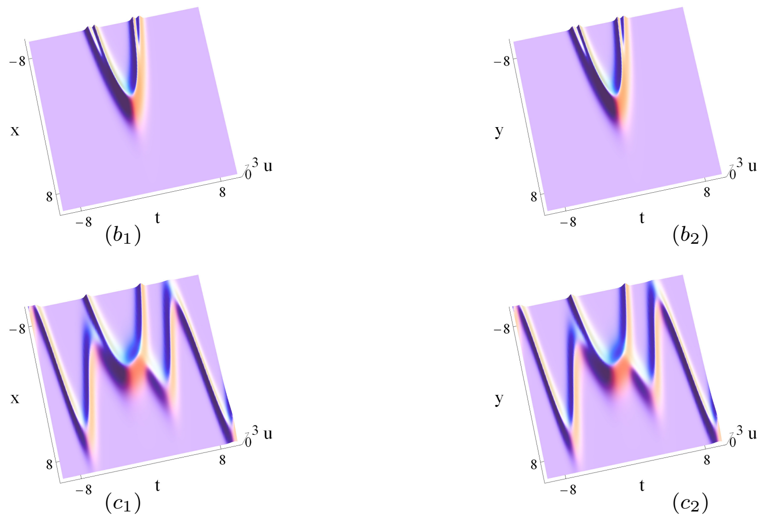


Figure 12. The same as Figs. 9 (a_1) and (a_2) except that (b_1) and (b_2): $\delta_4(t)$=$2t$; (c_1) and (c_2): $\delta_4(t)$=$\cos t$.

where φ and u_1 are the functions about x, y and t. Substituting Expression (6.1) into Eq. (1.1), we equate φ^{-i+1}'s to zero and sort out the Painlevé-Bäcklund equations

$$\varphi^{-5} : u_0 = 2\varphi_x, \tag{6.2a}$$

$$\varphi^{-4} : 0, \tag{6.2b}$$

$$\begin{aligned}\varphi^{-3} : \ & \delta_6(t)\varphi_t^2 + \delta_1(t)\varphi_t\varphi_y + \delta_5(t)\varphi_y^2 + \delta_3(t)\varphi_t\varphi_x + \delta_4(t)\varphi_y\varphi_x + \delta_2(t)\varphi_x^2 \\ & + 3\beta u_{1t}\varphi_x^2 + 3\gamma u_{1y}\varphi_x^2 + 3\beta u_{1x}\varphi_t\varphi_x + 3\gamma u_{1x}\varphi_y\varphi_x + 6\alpha u_{1x}\varphi_x^2 \\ & - 3\beta\varphi_{xt}\varphi_{xx} - 3\gamma\varphi_{xy}\varphi_{xx} - 3\alpha\varphi_x^2 + 3\beta\varphi_x\varphi_{xxt} + 3\gamma\varphi_x\varphi_{xxy} \\ & + \beta\varphi_t\varphi_{xxx} + \gamma\varphi_y\varphi_{xxx} + 4\alpha\varphi_x\varphi_{xxx} = 0,\end{aligned} \tag{6.2c}$$

$$\begin{aligned}\varphi^{-2} : \ & \delta_5(t)\varphi_{yy}\varphi_x + 2\delta_3(t)\varphi_x\varphi_{xt} + 6\beta u_{1x}\varphi_x\varphi_{xt} + \delta_6(t)\left(\varphi_{tt}\varphi_x + 2\varphi_t\varphi_{xt}\right) \\ & + 3\beta u_{1xt}\varphi_x^2 + 2\delta_5(t)\varphi_y\varphi_{xy} + 2\delta_4(t)\varphi_x\varphi_{xy} + 6\gamma u_{1x}\varphi_x\varphi_{xy} \\ & + \delta_1(t)\left(\varphi_{yt}\varphi_x + \varphi_y\varphi_{xt} + \varphi_t\varphi_{xy}\right) + 3\gamma u_{1xy}\varphi_x^2 + \delta_3(t)\varphi_t\varphi_{xx} + \delta_4(t)\varphi_y\varphi_{xx} \\ & + 3\delta_2(t)\varphi_x\varphi_{xx} + 9\beta u_{1t}\varphi_x\varphi_{xx} + 9\gamma u_{1y}\varphi_x\varphi_{xx} + 3\beta u_{1x}\varphi_t\varphi_{xx} \\ & + 3\gamma u_{1x}\varphi_y\varphi_{xx} + 18\alpha u_{1x}\varphi_x\varphi_{xx} + 3\beta u_{1xx}\varphi_t\varphi_x + 3\gamma u_{1xx}\varphi_y\varphi_x + 6\alpha u_{1xx}\varphi_x^2 \\ & - 2\beta\varphi_{xt}\varphi_{xxx} - 2\gamma\varphi_{xy}\varphi_{xxx} - 2\alpha\varphi_{xx}\varphi_{xxx} + 4\beta\varphi_x\varphi_{xxxt} + 4\gamma\varphi_x\varphi_{xxxy} \\ & + \beta\varphi_t\varphi_{xxxx} + \gamma\varphi_y\varphi_{xxxx} + 5\alpha\varphi_t\varphi_{xxxx} = 0,\end{aligned} \tag{6.2d}$$

$$\begin{aligned}\varphi^{-1} : \ & \delta_6(t)\varphi_{xtt} + \delta_1(t)\varphi_{xyt} + \delta_5(t)\varphi_{xyy} + 3\beta u_{1xt}\varphi_{xx} + 3\gamma u_{1xy}\varphi_{xx} + 3\beta u_{1xx}\varphi_{xt} \\ & + 3\gamma u_{1xx}\varphi_{xy} + 6\alpha u_{1xx}\varphi_{xx} + 3\beta u_{1x}\varphi_{xxt} + \delta_3(t)\varphi_{xxt} + 3\gamma u_{1x}\varphi_{xxy} \\ & + \delta_4(t)\varphi_{xxy} + \delta_2(t)\varphi_{xxx} + 3\beta u_{1t}\varphi_{xxx} + 3\gamma u_{1y}\varphi_{xxx} + 6\alpha u_{1x}\varphi_{xxx} \\ & + \alpha\varphi_{xxxxx} + \beta\varphi_{xxxxt} + \gamma\varphi_{xxxxy} = 0,\end{aligned} \tag{6.2e}$$

$$\begin{aligned}\varphi^0\ :\ &\delta_6(t)u_{1tt}+\delta_1(t)u_{1yt}+\delta_5(t)u_{1yy}+\delta_3(t)u_{1xt}+3\beta u_{1x}u_{1xt}+\delta_4(t)u_{1xy}\\ &+3\gamma u_{1x}u_{1xy}+\delta_2(t)u_{1xx}+3\beta u_{1t}u_{1xx}+3\gamma u_{1y}u_{1xx}+6\alpha u_{1x}u_{1xx}\\ &+\alpha u_{1xxxx}+\beta u_{1xxxt}+\gamma u_{1xxxy}=0.\end{aligned}\tag{6.2f}$$

Eqs. (6.1) and (6.2) are the auto-Bäcklund transformations of Eq. (1.1). For Eq. (1.1), we select a seed solution and obtain another solution via Auto-Bäcklund Transformations (6.1) and (6.2).

We suppose that $u_1 = 0$ and

$$\begin{aligned}\varphi(x,y,t)=&k_1\exp[d_3(t)x+b_3(t)y+c_3(t)t+e_1(t)x+f_1(t)y+g_1(t)t]\\ &+\exp[d_3(t)x+b_3(t)y+c_3(t)t]+\exp[e_1(t)x+f_1(t)y+g_1(t)t]+1,\end{aligned}\tag{6.3}$$

where $e_1(t)$, $f_1(t)$, $g_1(t)$, $b_3(t)$, $c_3(t)$ and $d_3(t)$ are all the real functions of t, and k_1 is a real constant. Substituting Expression (6.3) into Painlevé-Bäcklund Eqs. (6.2), we obtain that

$$\begin{aligned}&\delta_6(t)=\beta=0,\\ &\alpha+\gamma=0,\\ &\delta_1(t)+\delta_3(t)=0,\\ &\delta_4(t)+\delta_2(t)+\delta_5(t)=0.\end{aligned}\tag{6.4}$$

Via Coefficient Constraints (6.4), we take $k_1 = 3$, $e_1(t) = f_1(t) = b_3(t) = d_3(t) = 1$ and $c_3(t) = \delta_1(t), g_1(t) = \delta_2(t)$, then get the two-kink solutions of Eq. (1.1) as

$$u=\frac{2\left\{2\exp[\delta_1(t)t+\delta_2(t)t+2x+2y]+\exp[\delta_1(t)t+x+y]+\exp[\delta_2(t)t+x+y]\right\}}{3\exp[\delta_1(t)t+\delta_2(t)t+2x+2y]+\exp[\delta_1(t)t+x+y]+\exp[\delta_2(t)t+x+y]+1}.\tag{6.5}$$

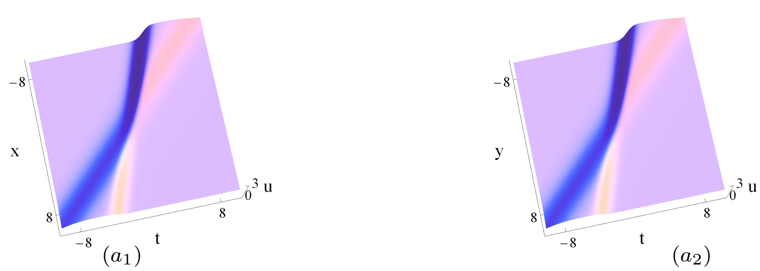


Figure 13. Two kink waves via Solutions (6.5) under Coefficient Constraints (6.4), $(a_1 - c_1)$ $y = 0$; $(a_2 - c_2)$ $x = 0$. Parameters are (a_1) and (a_2): $\delta_4(t)$=1, α=1, $\delta_1(t)$=1 and $\delta_2(t)$=3.

Figs. 13 and 14 present the propagation of the two kink waves via Solutions (6.5) on the x-t and y-t planes. According to Figs. 13 and 14, we find that the two kink waves are composed of two one kink waves which are similar to those in Figs. 1-4.

We take $c_3(t) = \delta_1(t)$ and $g_1(t) = \delta_4(t)$.

We take $c_3(t) = \delta_2(t)$ and $g_1(t) = \delta_4(t)$.

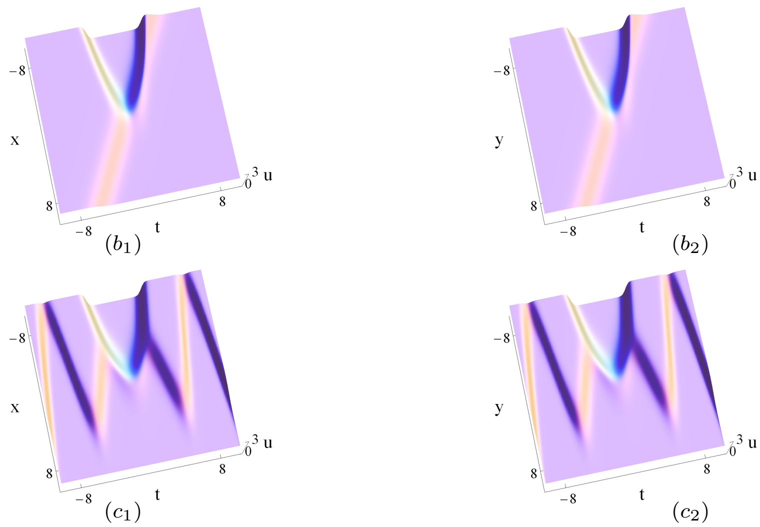


Figure 14. The same as Figs. 13 (a_1) and (a_2) except that (b_1) and (b_2): $\delta_1(t){=}t$ and $\delta_2(t){=}2$; (c_1) and (c_2): $\delta_1(t){=}t$ and $\delta_2(t){=}\sin t$.

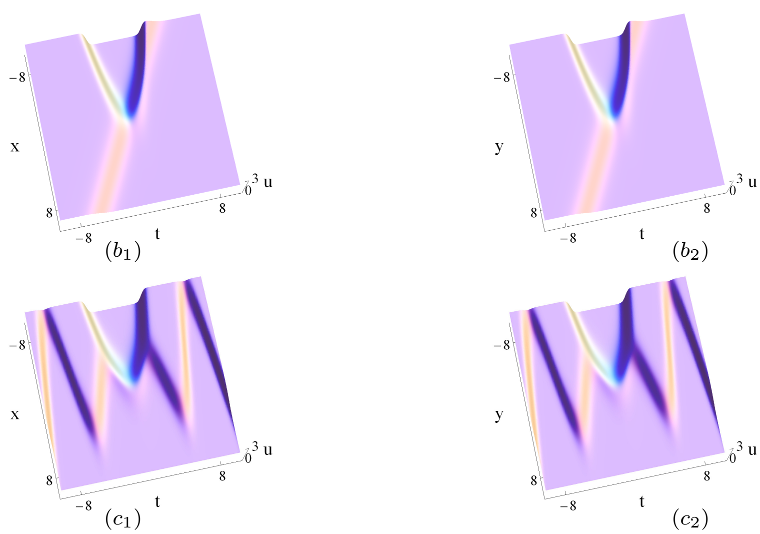


Figure 15. The same as Figs. 13 (a_1) and (a_2) except that (b_1) and (b_2): $\delta_1(t){=}t$ and $\delta_4(t){=}2$; (c_1) and (c_2): $\delta_1(t){=}t$ and $\delta_4(t){=}\sin t$.

Figs. 13 and 15 present the coefficient $\delta_4(t)$ has the similar effects on the two kink waves with $\delta_2(t)$. Figs. 13 and 16 present the coefficient $\delta_2(t)$ has the similar effects on the two kink waves with $\delta_1(t)$.

Though the analysis of one- and two-kink waves, we find one- and two-kink waves present linear, parabolic and periodic types when the coefficients $\delta_1(t)$, $\delta_2(t)$ and $\delta_4(t)$ take constants, linear functions and periodic functions. For the two-kink waves composed of linear-type one-kink wave and parabolic-type one-kink wave, we find that the linear-type one-kink wave keeps linear type and parabolic-type one-kink wave keeps parabolic type after the interaction. For the two-kink waves composed

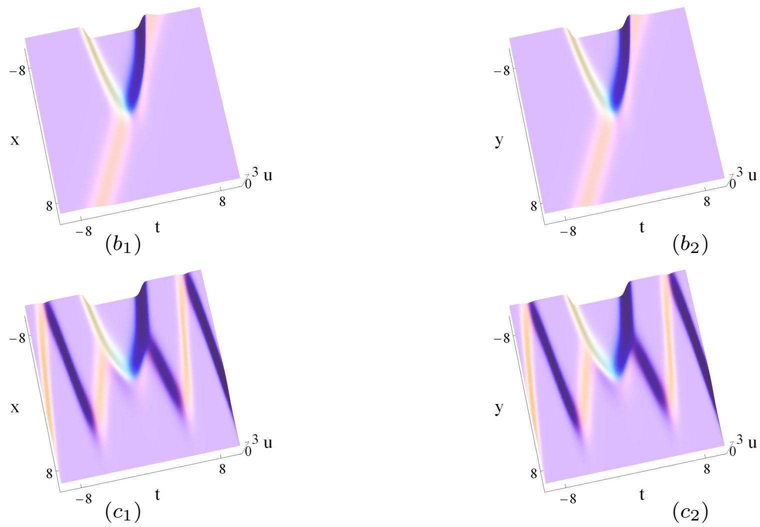


Figure 16. The same as Figs. 13 (a_1) and (a_2) except that (b_1) and (b_2): $\delta_2(t)=t$ and $\delta_4(t)=2$; (c_1) and (c_2): $\delta_2(t)=t$ and $\delta_4(t)=\sin t$.

of two parabolic-type one-kink waves, we find that the two parabolic-type one-kink waves keep parabolic type after the interaction. For the two-kink waves composed of periodic-type one-kink wave and parabolic-type one-kink wave, we find that the periodic-type one-kink wave keeps periodic type and parabolic-type one-kink wave keeps parabolic type after the interaction.

7. Conclusions

In this paper, we have investigated a (2+1)-dimensional variable-coefficient general combined fourth-order soliton equation in a fluid or plasma. The Painlevé integrable property has been derived under Coefficient Constraints (2.5). We have obtained Bilinear Form (3.2) to get Bilinear Bäcklund Transformation (4.3). By virtue of Truncated Painlevé Expansion (6.1), we have gotten Auto-Bäcklund Transformations (6.1) and (6.2).

Though the analysis of one- and two-kink waves, we have found that one- and two-kink waves present linear, parabolic and periodic types when the coefficients $\delta_1(t)$, $\delta_2(t)$ and $\delta_4(t)$ take constants, linear functions and periodic functions. For the two-kink waves composed of linear-type one-kink wave and parabolic-type one-kink wave, we have found that the linear-type one-kink wave keeps linear type and parabolic-type one-kink wave keeps parabolic type after the interaction. For the two-kink waves composed of two parabolic-type one-kink waves, we have found that the two parabolic-type one-kink waves keep parabolic type after the interaction. For the two-kink waves composed of periodic-type one-kink wave and parabolic-type one-kink wave, we have found that the periodic-type one-kink wave keeps periodic type and parabolic-type one-kink wave keeps parabolic type after the interaction. Based on Eqs. (5.4a) and (5.4c), we have gotten that the amplitudes of one soliton keep unchanged but the velocities of one soliton change with the coefficient $\delta_1(t)$. We have also observed that the behavior of two solitons is similar to the two-kink waves.

Data availability

No data was created during this research.

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