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THE SECOND-ORDER DIFFERENTIAL EQUATION METHOD FOR SOLVING THE VARIATIONAL INEQUALITY PROBLEM*

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Abstract In this paper, we smooth the Karush-Kuhn-Tucker (KKT) functions for the classical variational inequality problem, then it is transformed to an unconstrained optimization problem. We firstly establish the second-order differential equation system involving two time-dependent parameters for solving the unconstrained optimization problem. The global convergence theorem for the second-order differential equation system is proved. At last, four numerical experiments are reported to verify the effectiveness of the second-order differential equation method for solving the classical variational inequality problem.

Keywords Second-order differential equation, variational inequality, KKT condition.

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1. Introduction

As one of the important branches of optimization theory and algorithm, the variational inequality problems are widely used in many fields such as physics, engineering, economics, transportation, complementarity problems and control problems. Many numerical methods have emerged in large numbers, such as projection method, interior point method, nonsmooth equation method, smoothing method, merit function method and so on.

In this paper, we will consider the following classical variational inequality problem: finding x^* such that

$$\langle F(x^*), y - x^* \rangle \geq 0, \ y \in C \tag{1.1}$$

where

$$C = \{x \in \Re^n | h(x) = 0, -g(x) \geq 0\}, \tag{1.2}$$

$\langle \cdot, \cdot \rangle$ is the Euclidean inner product, $h : \Re^n \to \Re^l$, $g : \Re^n \to \Re^m$ are continuously differentiable, and $\Re^n$ is n dimensional real column vector space.

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We can obtain the KKT conditions for the variational inequality problem (1.1) as:

$$\begin{aligned} &L(x,\mu,\lambda) = F(x) + Jh(x)^{\mathrm{T}}\mu + Jg(x)^{\mathrm{T}}\lambda = 0,\\ &h(x) = 0, \\ &g(x) \perp \lambda. \end{aligned} \tag{1.3}$$

In recent years, the differential equation method for solving the variational inequality problems has attracted much attention among many scholars. Since 1980, a series of artificial network methods based on differential equation systems have been proposed by Hopfield and Tank [10], and used to solve complementarity problems and variational inequality problems. He and Yang [9] proposed a differential equation system for nonsymmetric linear variational inequality problems based on projection operators and contraction methods. Gao, Liao and Qi [8] proposed a differential equation model for solving variational inequalities with linear and nonlinear constraints based on projection operator theory. Without using the Lyapunov function for the stability or convergence, Antipin [1] established systems of differential equations with the controlled process for solving variational inequality problems with coupled constraints. And the global convergence for the differential equation system was proved based on the properties of the projection operator and the symmetric function. The second-order differential equation system with the controlled process was established for solving the variational inequality with constraints in Wang et al [19]. Most of all, we focus on Attouch et al [2,4,5], in which the second-order differential equation systems were established to solve convex optimization. In this general framework, the fast convergence property was studied under the certain conditions and the convergence rate thus was discussed.

Inspired by the above results, we construct the second-order differential equation system involving two time-dependent parameters to solve classical variational inequality problem (1.1). Firstly, we use the smoothed complementary function to transform the KKT conditions (1.3) of the variational inequality problem (1.1) into a smoothing equation system problem, and introduce the merit function to transform it into an unconstrained optimization problem. Secondly, the second-order differential equation system is constructed to solve the transformed unconstrained optimization problem. And the differential equation system involves two time-dependent parameters which are a positive viscous damping coefficient $\gamma(t)$ and a time scale coefficient $\beta(t)$. The fast convergence property is closely related to the asymptotic vanishing property $\gamma(t) \to 0$, and to the temporal scaling $\beta(t) \to \infty$. At last, two numerical experiments are reported to verify the effectiveness of the second-order differential equation method for solving the classical variational inequality problem.

2. Preliminaries

In order to smooth the KKT conditions of the variational inequalities problem (1.1), we apply the complementary function $\varphi : \Re^m \times \Re^m \to \Re^m$ to satisfy $\varphi(x,y) = 0$ if and only if $x \perp y$. Consider the following smooth NR function:

$$\varphi_{NR}^{\varepsilon}(x,y) = x - \phi(\varepsilon, x-y), \tag{2.1}$$

for any $\varepsilon > 0$, $x, y \in \Re^m$, and $\phi : \Re_+ \times \Re^m \to \Re^m$ satisfying

$$\phi\left(\varepsilon, x-y\right) = \frac{1}{2}\left(x - y + \sqrt{\varepsilon^2 e + \left(x-y\right)^2}\right).$$

The KKT conditions (1.3) for the variational inequality problem (1.1) is equivalent to the following

$$S\left(\varepsilon, x, \mu, \lambda\right) = \begin{pmatrix} \varepsilon \\ L\left(x, \mu, \lambda\right) \\ h\left(x\right) \\ \varphi_{NR}^{\varepsilon}\left(-g\left(x\right), \lambda\right) \end{pmatrix} = 0, \tag{2.2}$$

where

$$\varphi_{NR}^{\varepsilon}\left(-g\left(x\right), \lambda\right) = \begin{pmatrix} \varphi_{NR}^{\varepsilon}\left(-g_1\left(x\right), \lambda_1\right) \\ \varphi_{NR}^{\varepsilon}\left(-g_2\left(x\right), \lambda_2\right) \\ \vdots \\ \varphi_{NR}^{\varepsilon}\left(-g_n\left(x\right), \lambda_n\right) \end{pmatrix}.$$

Note that the unconstrained optimization problem as follows.

$$\min \Phi\left(\varepsilon, x, \mu, \lambda\right) := \frac{1}{2}\left\|S\left(\varepsilon, x, \mu, \lambda\right)\right\|^2. \tag{2.3}$$

In fact, Φ is called merit function of the problem (1.1). It is easy to see that x^* is the solution to the unconstrained optimization problem (2.3) means that x^* is the solution to the variational inequality problem (1.1).

Let $z = (\varepsilon, x, \mu, \lambda) \in \Re \times \Re^n \times \Re^l \times \Re^m$, then the unconstrained optimization problem (2.3) can be simplified as

$$\min \Phi\left(z\right) := \frac{1}{2}\left\|S\left(z\right)\right\|^2. \tag{2.4}$$

3. Second-order differential equation system

In this section, we will discuss the second-order differential equation method for solving the variational inequalities problem (1.1).

In order to solve the variational inequalities problem (1.1), inspired by Attouch [5], the second-order differential equation system involves two time-dependent parameters which are a positive viscous damping coefficient $\gamma(t)$ and a time scale coefficient $\beta(t)$ is established in the following.

$$\begin{pmatrix} \ddot{\varepsilon}\left(t\right) \\ \ddot{x}\left(t\right) \\ \ddot{\mu}\left(t\right) \\ \ddot{\lambda}\left(t\right) \end{pmatrix} + \gamma\left(t\right) \begin{pmatrix} \dot{\varepsilon}\left(t\right) \\ \dot{x}\left(t\right) \\ \dot{\mu}\left(t\right) \\ \dot{\lambda}\left(t\right) \end{pmatrix} + \beta\left(t\right) \begin{pmatrix} \nabla_{\varepsilon} S(z)^{\mathrm{T}} S\left(z\right) \\ \nabla_{x} S(z)^{\mathrm{T}} S\left(z\right) \\ \nabla_{\mu} S(z)^{\mathrm{T}} S\left(z\right) \\ \nabla_{\lambda} S(z)^{\mathrm{T}} S\left(z\right) \end{pmatrix} = 0, \tag{3.1}$$

where $\gamma, \beta \in [t_0, +\infty) \to \Re_+$ are nonnegative continuous functions.

We take for granted the existence of a solution to this system, and the corresponding equilibrium point of the system is the solution of the variational inequality problem (1.1). Based on the parameter tuning of the positive viscous damping coefficient $\gamma(t)$ and the time scaling parameter $\beta(t)$, we will analyze the global convergence of the solution trajectory of the second-order differential equation system (3.1).

In the elementary case $\Phi(\varepsilon, x, \mu, \lambda) \equiv 0$, by direct integration of a system of second order differential equations, we obtain

$$p(t) = e^{\int_{t_0}^{t} \gamma(u) du},$$

and it is assumed that the following conditions are satisfied,

$$H_0 : \int_{t_0}^{+\infty} \frac{du}{p(u)} < +\infty.$$

Under this assumption, we can define the function $\Gamma : [t_0, +\infty) \to \Re_+$ and

$$\Gamma(t) = p(t) \int_{t}^{+\infty} \frac{du}{p(u)}. \tag{3.2}$$

By differentiating (3.2), we obtain the relation

$$\dot{\Gamma}(t) = \gamma(t) \Gamma(t) - 1. \tag{3.3}$$

Finally, let us define the global energy function $W(t)$

$$W(t) = \frac{1}{2} \left\| \begin{pmatrix} \ddot{\varepsilon}(t) \\ \ddot{x}(t) \\ \ddot{\mu}(t) \\ \ddot{\lambda}(t) \end{pmatrix} \right\|^2 + \beta(t) \left(\Phi(\varepsilon(t), x(t), \mu(t), \lambda(t) - \min \Phi \right),$$

and the anchor function $h(t)$

$$h(t) = \frac{1}{2} \left\| \begin{pmatrix} \varepsilon(t) \\ x(t) \\ \mu(t) \\ \lambda(t) \end{pmatrix} - \begin{pmatrix} \varepsilon^* \\ x^* \\ \mu^* \\ \lambda^* \end{pmatrix} \right\|^2,$$

where $\begin{pmatrix} \varepsilon^* \\ x^* \\ \mu^* \\ \lambda^* \end{pmatrix} \in \arg\min \Phi \neq \emptyset$ is given. They are the basic constitutive blocks of the function $\xi : [t_0, +\infty) \to \Re_+$. And $\xi(t)$ is a nonnegative function, we define

$$\xi(t) = \Gamma(t)^2 W(t) + h(t) + \Gamma(t) \dot{h}(t), \tag{3.4}$$

which is equivalent to that

$$\xi(t) = \Gamma(t)^2\beta(t)\left(\Phi(\varepsilon(t), x(t), \mu(t), \lambda(t) - \min\Phi\right) + \frac{1}{2}\left\| \begin{pmatrix} \varepsilon(t) \\ x(t) \\ \mu(t) \\ \lambda(t) \end{pmatrix} - \begin{pmatrix} \varepsilon^* \\ x^* \\ \mu^* \\ \lambda^* \end{pmatrix} + \Gamma(t) \begin{pmatrix} \dot{\varepsilon}(t) \\ \dot{x}(t) \\ \dot{\mu}(t) \\ \dot{\lambda}(t) \end{pmatrix} \right\|^2.$$

Theorem 3.1. *Let $\beta : [t_0, +\infty) \to \Re_+$ be a continuous positive function, $\gamma : [t_0, +\infty) \to \Re_+$ is a continuous function that satisfies the assumptions H_0. We assume that the following growth conditions $H_{\gamma,\beta}$ is satisfied linking $\gamma(t)$ and $\beta(t)$:*

$$H_{\gamma,\beta}: \qquad \Gamma(t)\dot{\beta}(t) \le \beta(t)\left(3 - 2\gamma(t)\Gamma(t)\right).$$

Then, for every solution trajectory $(\varepsilon, x, \mu, \lambda) : [t_0 \to +\infty) \to \Re \times \Re^n \times \Re^l \times \Re^m$ of the second-order differential equation system (3.1), the convergence rate of the values is satisfied

$$\Phi(\varepsilon(t), x(t), \mu(t), \lambda(t)) - \min\Phi = O\left(\frac{1}{\beta(t)\Gamma(t)^2}\right), \tag{3.5}$$

as $t \to +\infty$. In addition, the trajectory of the solution is bounded on $[t_0, +\infty)$.

Proof. For simplicity, let $m = \min\Phi(\varepsilon, x, \mu, \lambda)$. Computing the derivative of $\xi(t)$ as follows.

$$\dot{\xi}(t) = 2\Gamma(t)\dot{\Gamma}(t)W(t) + \Gamma(t)^2\dot{W}(t) + \dot{h}(t) + \dot{\Gamma}(t)\dot{h}(t) + \Gamma(t)\ddot{h}(t).$$

We firstly calculate the derivatives of the principal components, including the derivatives of the global energy function $W(t)$ and the anchor function $h(t)$. By the classical chain rule, we can obtain that

$$\begin{aligned} \dot{W}(t) &= \left\langle \begin{pmatrix} \dot{\varepsilon}(t) \\ \dot{x}(t) \\ \dot{\mu}(t) \\ \dot{\lambda}(t) \end{pmatrix}, \begin{pmatrix} \ddot{\varepsilon}(t) \\ \ddot{x}(t) \\ \ddot{\mu}(t) \\ \ddot{\lambda}(t) \end{pmatrix} \right\rangle + \beta(t) \left\langle \begin{pmatrix} \dot{\varepsilon}(t) \\ \dot{x}(t) \\ \dot{\mu}(t) \\ \dot{\lambda}(t) \end{pmatrix}, \begin{pmatrix} \nabla_\varepsilon S(z)^{\mathrm{T}} S(z) \\ \nabla_x S(z)^{\mathrm{T}} S(z) \\ \nabla_\mu S(z)^{\mathrm{T}} S(z) \\ \nabla_\lambda S(z)^{\mathrm{T}} S(z) \end{pmatrix} \right\rangle \\ &\quad + \dot{\beta}(t)\left(\Phi(\varepsilon(t), x(t), \mu(t), \lambda(t) - m\right) \\ &= \left\langle \begin{pmatrix} \dot{\varepsilon}(t) \\ \dot{x}(t) \\ \dot{\mu}(t) \\ \dot{\lambda}(t) \end{pmatrix}, \begin{pmatrix} \ddot{\varepsilon}(t) \\ \ddot{x}(t) \\ \ddot{\mu}(t) \\ \ddot{\lambda}(t) \end{pmatrix} + \beta(t) \begin{pmatrix} \nabla_\varepsilon S(z)^{\mathrm{T}} S(z) \\ \nabla_x S(z)^{\mathrm{T}} S(z) \\ \nabla_\mu S(z)^{\mathrm{T}} S(z) \\ \nabla_\lambda S(z)^{\mathrm{T}} S(z) \end{pmatrix} \right\rangle \\ &\quad + \dot{\beta}(t)\left(\Phi(\varepsilon(t), x(t), \mu(t), \lambda(t) - m\right) \end{aligned}$$

$$= \left\langle \begin{pmatrix} \dot{\varepsilon}(t) \\ \dot{x}(t) \\ \dot{\mu}(t) \\ \dot{\lambda}(t) \end{pmatrix}, -\gamma(t) \begin{pmatrix} \dot{\varepsilon}(t) \\ \dot{x}(t) \\ \dot{\mu}(t) \\ \dot{\lambda}(t) \end{pmatrix} \right\rangle + \dot{\beta}(t)\left(\Phi(\varepsilon(t), x(t), \mu(t), \lambda(t) - m\right)$$

$$= -\gamma(t) \left\| \begin{pmatrix} \dot{\varepsilon}(t) \\ \dot{x}(t) \\ \dot{\mu}(t) \\ \dot{\lambda}(t) \end{pmatrix} \right\|^2 + \dot{\beta}(t)\left(\Phi(\varepsilon(t), x(t), \mu(t), \lambda(t) - m\right).$$

On the other hand, the derivative for the anchor function $h(t)$ yields that

$$\dot{h}(t) = \left\langle \begin{pmatrix} \dot{\varepsilon}(t) \\ \dot{x}(t) \\ \dot{\mu}(t) \\ \dot{\lambda}(t) \end{pmatrix}, \begin{pmatrix} \varepsilon(t) - \varepsilon^* \\ x(t) - x^* \\ \mu(t) - \mu^* \\ \lambda(t) - \lambda^* \end{pmatrix} \right\rangle,$$

$$\ddot{h}(t) = \left\| \begin{pmatrix} \dot{\varepsilon}(t) \\ \dot{x}(t) \\ \dot{\mu}(t) \\ \dot{\lambda}(t) \end{pmatrix} \right\|^2 + \left\langle \begin{pmatrix} \dot{\varepsilon}(t) \\ \dot{x}(t) \\ \dot{\mu}(t) \\ \dot{\lambda}(t) \end{pmatrix}, \begin{pmatrix} \varepsilon(t) - \varepsilon^* \\ x(t) - x^* \\ \mu(t) - \mu^* \\ \lambda(t) - \lambda^* \end{pmatrix} \right\rangle.$$

Thus we can get that

$$\gamma(t)\dot{h}(t) + \ddot{h}(t)$$

$$= \left\langle \gamma(t) \begin{pmatrix} \dot{\varepsilon}(t) \\ \dot{x}(t) \\ \dot{\mu}(t) \\ \dot{\lambda}(t) \end{pmatrix}, \begin{pmatrix} \varepsilon(t) - \varepsilon^* \\ x(t) - x^* \\ \mu(t) - \mu^* \\ \lambda(t) - \lambda^* \end{pmatrix} \right\rangle + \left\langle \begin{pmatrix} \ddot{\varepsilon}(t) \\ \ddot{x}(t) \\ \ddot{\mu}(t) \\ \ddot{\lambda}(t) \end{pmatrix}, \begin{pmatrix} \varepsilon(t) - \varepsilon^* \\ x(t) - x^* \\ \mu(t) - \mu^* \\ \lambda(t) - \lambda^* \end{pmatrix} \right\rangle + \left\| \begin{pmatrix} \dot{\varepsilon}(t) \\ \dot{x}(t) \\ \dot{\mu}(t) \\ \dot{\lambda}(t) \end{pmatrix} \right\|^2$$

$$= \left\langle \gamma(t) \begin{pmatrix} \dot{\varepsilon}(t) \\ \dot{x}(t) \\ \dot{\mu}(t) \\ \dot{\lambda}(t) \end{pmatrix} + \begin{pmatrix} \ddot{\varepsilon}(t) \\ \ddot{x}(t) \\ \ddot{\mu}(t) \\ \ddot{\lambda}(t) \end{pmatrix}, \begin{pmatrix} \varepsilon(t) - \varepsilon^* \\ x(t) - x^* \\ \mu(t) - \mu^* \\ \lambda(t) - \lambda^* \end{pmatrix} \right\rangle + \left\| \begin{pmatrix} \dot{\varepsilon}(t) \\ \dot{x}(t) \\ \dot{\mu}(t) \\ \dot{\lambda}(t) \end{pmatrix} \right\|^2$$

$$= -\beta(t)\left\langle \begin{pmatrix} \nabla_\varepsilon S(z)^{\mathrm{T}} S(z) \\ \nabla_x S(z)^{\mathrm{T}} S(z) \\ \nabla_\mu S(z)^{\mathrm{T}} S(z) \\ \nabla_\lambda S(z)^{\mathrm{T}} S(z) \end{pmatrix}, \begin{pmatrix} \varepsilon(t)-\varepsilon^* \\ x(t)-x^* \\ \mu(t)-\mu^* \\ \lambda(t)-\lambda^* \end{pmatrix} \right\rangle + \left\| \begin{pmatrix} \dot\varepsilon(t) \\ \dot x(t) \\ \dot\mu(t) \\ \dot\lambda(t) \end{pmatrix} \right\|^2$$

$$\le -\beta(t)\left(\Phi\begin{pmatrix} \varepsilon(t) \\ x(t) \\ \mu(t) \\ \lambda(t) \end{pmatrix}^{\mathrm{T}} - m \right) + \left\| \begin{pmatrix} \dot\varepsilon(t) \\ \dot x(t) \\ \dot\mu(t) \\ \dot\lambda(t) \end{pmatrix} \right\|^2$$

$$\le \left\| \begin{pmatrix} \dot\varepsilon(t) \\ \dot x(t) \\ \dot\mu(t) \\ \dot\lambda(t) \end{pmatrix} \right\|^2.$$

The above inequality holds from the convex function property of $\Phi(\varepsilon, x, \mu, \lambda)$. So we can get the derivative of $\xi(t)$ as follows.

$$\dot\xi(t) = 2\Gamma(t)\dot\Gamma(t)W(t) + \Gamma(t)^2\dot W(t) + \dot h(t) + \dot\Gamma(t)\dot h(t) + \Gamma(t)\ddot h(t)$$

$$= 2\Gamma(t)\dot\Gamma(t)\left(\frac{1}{2}\left\| \begin{pmatrix} \dot\varepsilon(t) \\ \dot x(t) \\ \dot\mu(t) \\ \dot\lambda(t) \end{pmatrix} \right\|^2 + \beta(t)\left(\Phi\begin{pmatrix} \varepsilon(t) \\ x(t) \\ \mu(t) \\ \lambda(t) \end{pmatrix}^{\mathrm{T}} - m \right)\right)$$

$$+\Gamma(t)^2\left(-\gamma(t)\left\| \begin{pmatrix} \dot\varepsilon(t) \\ \dot x(t) \\ \dot\mu(t) \\ \dot\lambda(t) \end{pmatrix} \right\|^2 + \dot\beta(t)\left(\Phi\begin{pmatrix} \varepsilon(t) \\ x(t) \\ \mu(t) \\ \lambda(t) \end{pmatrix}^{\mathrm{T}} - m \right)\right)$$

$$+\Gamma(t)\left(\gamma(t)\dot h(t) + \ddot h(t)\right)$$

$$\le 2\Gamma(t)\dot\Gamma(t)\left(\frac{1}{2}\left\| \begin{pmatrix} \dot\varepsilon(t) \\ \dot x(t) \\ \dot\mu(t) \\ \dot\lambda(t) \end{pmatrix} \right\|^2 + \beta(t)\left(\Phi\begin{pmatrix} \varepsilon(t) \\ x(t) \\ \mu(t) \\ \lambda(t) \end{pmatrix}^{\mathrm{T}} - m \right)\right)$$

$$
+\Gamma(t)^2\left(-\gamma(t)\left\|\begin{pmatrix}\dot{\varepsilon}(t)\\ \dot{x}(t)\\ \dot{\mu}(t)\\ \dot{\lambda}(t)\end{pmatrix}\right\|^2+\dot{\beta}(t)\left(\Phi\begin{pmatrix}\varepsilon(t)\\ x(t)\\ \mu(t)\\ \lambda(t)\end{pmatrix}^{\mathrm{T}}-m\right)\right)
$$

$$
+\Gamma(t)\left(-\beta(t)\left(\Phi\begin{pmatrix}\varepsilon(t)\\ x(t)\\ \mu(t)\\ \lambda(t)\end{pmatrix}^{\mathrm{T}}-m\right)+\left\|\begin{pmatrix}\dot{\varepsilon}(t)\\ \dot{x}(t)\\ \dot{\mu}(t)\\ \dot{\lambda}(t)\end{pmatrix}\right\|^2\right)
$$

$$
\le\Gamma(t)\left\|\begin{pmatrix}\dot{\varepsilon}(t)\\ \dot{x}(t)\\ \dot{\mu}(t)\\ \dot{\lambda}(t)\end{pmatrix}\right\|^2\left(1+\dot{\Gamma}(t)-\gamma(t)\Gamma(t)\right)
$$

$$
+\left(\Phi\begin{pmatrix}\varepsilon(t)\\ x(t)\\ \mu(t)\\ \lambda(t)\end{pmatrix}^{\mathrm{T}}-m\right)\Gamma(t)\left(\Gamma(t)\dot{\beta}(t)+2\dot{\Gamma}(t)\beta(t)-\beta(t)\right).
$$

We finally obtain by equation (3.3)

$$
\dot{\xi}(t)\le(\Phi(\varepsilon(t),x(t),\mu(t),\lambda(t))-m)\Gamma(t)\left(\Gamma(t)\dot{\beta}(t)+2\dot{\Gamma}(t)\beta(t)-\beta(t)\right),
$$

which means that

$$
\dot{\xi}(t)\le(\Phi(\varepsilon(t),x(t),\mu(t),\lambda(t))-m)\Gamma(t)\left(\Gamma(t)\dot{\beta}(t)+\beta(t)(2\gamma(t)\Gamma(t)-3)\right).
$$

Therefore, from the assumed growth conditions $H_{\gamma,\beta}$, it can be inferred that $\dot{\xi}(t)\le 0$. That is, the function $\xi(t)$ is decreasing on $[t_0,+\infty)$. And then we get $\xi(t)\le\xi(t_0)$. According to the formulation (3.4) of $\xi(t)$, we deduce that, for all $t\ge t_0$,

$$
\Phi(\varepsilon(t),x(t),\mu(t),\lambda(t))-\min\Phi\le\frac{\xi(t_0)}{\beta(t)\Gamma(t)^2},
$$

that is,

$$
\Phi(\varepsilon(t),x(t),\mu(t),\lambda(t))-\min\Phi=\mathrm{O}\left(\frac{1}{\beta(t)\Gamma(t)^2}\right).
$$

In addition, let us show that the trajectory remains bounded. According to the formulation and the decreasing of $\xi(t)$, the following inequality holds.

$$\left\| \begin{pmatrix} \varepsilon(t) \\ x(t) \\ \mu(t) \\ \lambda(t) \end{pmatrix} - \begin{pmatrix} \varepsilon^* \\ x^* \\ \mu^* \\ \lambda^* \end{pmatrix} + \Gamma(t) \begin{pmatrix} \dot{\varepsilon}(t) \\ \dot{x}(t) \\ \dot{\mu}(t) \\ \dot{\lambda}(t) \end{pmatrix} \right\|^2 \leq 2\xi(t) \leq 2\xi(t_0).$$

After developing the above inequality, we obtain

$$\left\| \begin{pmatrix} \varepsilon(t) \\ x(t) \\ \mu(t) \\ \lambda(t) \end{pmatrix} - \begin{pmatrix} \varepsilon^* \\ x^* \\ \mu^* \\ \lambda^* \end{pmatrix} \right\|^2 + 2\Gamma(t) \left\langle \begin{pmatrix} \varepsilon(t) \\ x(t) \\ \mu(t) \\ \lambda(t) \end{pmatrix} - \begin{pmatrix} \varepsilon^* \\ x^* \\ \mu^* \\ \lambda^* \end{pmatrix}, \begin{pmatrix} \dot{\varepsilon}(t) \\ \dot{x}(t) \\ \dot{\mu}(t) \\ \dot{\lambda}(t) \end{pmatrix} \right\rangle \leq 2\xi(t_0). \quad (3.6)$$

Set $q(t) := \int_{t_0}^{+\infty} \frac{ds}{p(s)}$, considering the assumptions $H_0 : \int_{t_0}^{+\infty} \frac{du}{p(u)} < +\infty$, we can obtain that the function $q(t) := \int_{t_0}^{+\infty} \frac{ds}{p(s)}$ is bounded in $[t_0, +\infty)$. Moreover, we can obtain the following equivalence relation.

$$q(t) = \frac{\Gamma(t)}{p(t)},$$
$$\dot{q}(t) = -\frac{1}{p(t)}.$$

After dividing (3.6) by $p(t)$, associative that above equivalence relation, we obtain, with $C = \xi(t_0)$,

$$\frac{1}{p(t)} h(t) + q(t)\dot{h}(t) \leq \frac{C}{p(t)}, \forall t \in [t_0, +\infty),$$

equivalently,

$$q(t)\dot{h}(t) - \dot{q}(t)(h(t) - C) \leq 0, \forall t \in [t_0, +\infty),$$

where $h(t)$ is the anchor function.

After dividing by $q(t)^2$, we finally obtain

$$\frac{1}{q(t)^2}\left[q(t)\dot{h}(t) - \dot{q}(t)(h(t) - C_1)\right] = \frac{d}{dt}\left(\frac{h(t) - C_1}{q(t)}\right) \leq 0, \forall t \in [t_0, +\infty).$$

Integration of this inequality gives $h(t) \leq \xi(t_0)(1 + q(t))$. Therefore, the solution trajectory is bounded, and this completes the proof. □

4. Theoretical comparison of the first-order system and the second-order system

In past research, the differential equation approach to solving the variational inequality problem has often been the first-order differential equation, i.e. Sun [16] et al. On account of the second-order cone constrained variational inequality problem as a specific case of the variational inequality problem, here we introduce the approach of the neural network based on the metric projector for solving the second-order cone constrained variational inequality problem from Sun [16] et al. The first-order system is as follows.

$$\begin{cases} \dfrac{\mathrm{d}z\left(t\right)}{\mathrm{d}t} = -\rho\nabla\Phi\left(z\left(t\right)\right), \\ z\left(t_0\right) = z_0, \end{cases} \tag{4.1}$$

where

$$\nabla\Phi\left(z\left(t\right)\right) = \begin{pmatrix} \nabla_\varepsilon S(z)^{\mathrm{T}} S\left(z\right) \\ \nabla_x S(z)^{\mathrm{T}} S\left(z\right) \\ \nabla_\mu S(z)^{\mathrm{T}} S\left(z\right) \\ \nabla_\lambda S(z)^{\mathrm{T}} S\left(z\right) \end{pmatrix},$$

and $z = (\varepsilon, x, \mu, \lambda) \in \Re \times \Re^n \times \Re^l \times \Re^m$, $\rho > 0$ is a scaling factor.

Comparing with Sun [16] et al, we can get the following results. And the comparison of theories with the first-order system (4.1) and the second-order system (3.1) is shown in Table 4.1.

Table 1. The comparison of theories conditions with the first-order system (4.1) and the second-order system (3.1) for solving the variational inequality problem.

Condition	The First-Order System	The Second-Order System
the convexity of Φ		✓
the nonsingularity of $\nabla\Phi$	✓	
$\arg\min \Phi \neq \emptyset$	✓	✓
the compact of $\arg\min \Phi$	✓	
the positive semidefinite of $JF\left(x^*\right)$	✓	

5. Numerical experiments

In order to demonstrate the effectiveness of the second-order differential equation system (3.1) involving two time-dependent parameters for solving the variational inequality problem (1.1), four examples are tested. The numerical implementation is coded by Matlab 9.0 and the ordinary differential equation solver adopted is ode45. In the following tests, the parameter $\gamma\left(t\right) = \frac{\alpha}{t}$ with $\alpha = 25$ and $\beta\left(t\right) \equiv 1$.

Example 5.1. Let

$$\left\langle \frac{1}{2}Dx, y - x \right\rangle \geq 0, \quad \forall y \in C,$$

where

$$C = \left\{x \in \Re^n \left| Ax - a = 0, Bx - b \leq 0 \right.\right\}.$$

$A \in \Re^{l\times n}$, $B \in \Re^{m\times n}$ and $D \in \Re^{n\times n}$ is a symmetric matrix. And $a \in \Re^{l}$, $b \in \Re^{m}$ with $l+m \le n$. Similarly to [15], we let

$$D = (D_{ij})_{n\times n},$$

where

$$D_{ij} = \begin{cases} 2, & i = j, \\ 1, & |i-j| = 1, \\ 0, & \text{otherwise.} \end{cases}$$

$A = \left[I_{l\times l} 0_{l\times(n-l)}\right]_{l\times n}$, $B = \left[0_{m\times(n-m)} I_{m\times m}\right]_{m\times n}$, $a = 0_{l\times 1}$ and $b = \left(e_{m_1}, \cdots, e_{m_p}\right)$, where $e_{m_i} = (1, 0, \cdots, 0)^T \in \Re^{m_i}$. In our experiment, we set $l = m = 3$ and $n = 6$ for the simulations, and the variational inequality problem has $x^* = (0,0,0,0,0,0)$ as its solution.

The CPU-time to reach termination condition (seconds) and x^* of Example 5.1 with the first-order differential equation system (4.1) and the second-order differential equation system (3.1) for Example 5.1 are summarized in Table 5.1. The solution trajectories $x(t)$ of the second-order differential equation system (3.1) from the given initial random points for Example 5.1 are showed in Figure 1. The comparison of error rates of $\|x(t) - x^*\|_2$ for the first-order differential equation system (4.1) and the second-order differential equation system (3.1) for Example 5.1 are showed in Figure 2.

Table 2. The CPU-time to reach termination condition (seconds) and numerical results of Example 5.1 with the first-order system (4.1) and the second-order system (3.1).

	The First-Order System (4.1)	The Second-Order System (3.1)
CPU-time(s)	0.052114	0.358340
x^* of Example 5.1	$\begin{pmatrix} 3.1614e-07 \\ 5.3598e-07 \\ 6.0267e-07 \\ 4.8481e-07 \\ 2.0909e-07 \\ 1.2831e-07 \end{pmatrix}$	$\begin{pmatrix} 5.4031e-05 \\ 4.6095e-05 \\ 1.6225e-05 \\ 2.1543e-05 \\ 5.4784e-05 \\ 4.7972e-05 \end{pmatrix}$

Example 5.2. Consider the following variational inequality

$$\langle F(x^*), y - x^*\rangle \ge 0, \ \ y \in C,$$

where

$$C = \left\{x \in \Re^5 \,|\, -g(x) = x \ge 0\right\},$$

and

$$F(x) = [2x_1 - 4, e^{x_2} - 1, 3x_3 - 4, -\sin(x_4), x_5]^{\mathrm{T}}.$$

Here $x^* = (2, 0, 1.333, 0, 0)$.

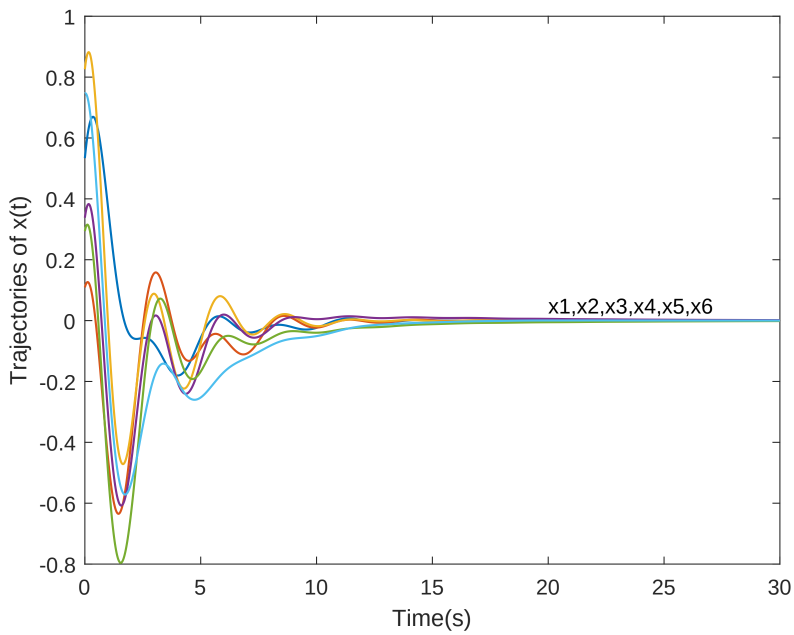
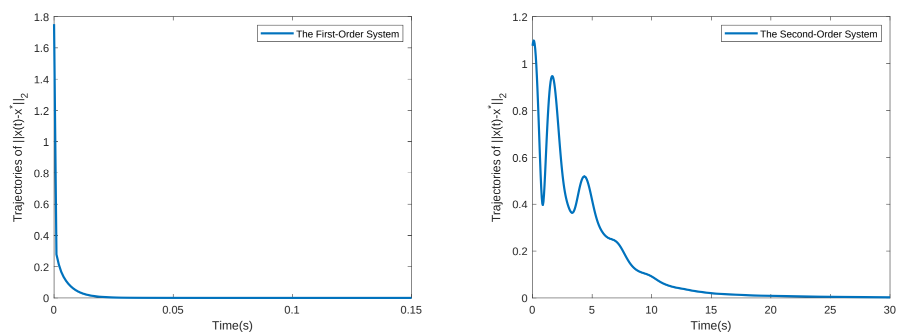


Figure 1. Trajectories of $x(t)$ of the second-order differential equation system (3.1) for Example 5.1 from six random initial points about x.



(a) The First-Order System

(b) The Second-Order System

Figure 2. Comparison of error rates of $\|x(t) - x^*\|_2$ for the first-order system (4.1) and the second-order system (3.1) for Example 5.1.

The CPU-time to reach termination condition (seconds) and x^* of Example 5.2 with the first-order differential equation system (4.1) and the second-order differential equation system (3.1) for Example 5.2 are summarized in Table 5.2. The solution trajectories $x(t)$ of the second-order differential equation system (3.1) from the given initial random points for Example 5.2 are showed in Figure 3. The comparison of error rates of $\|x(t) - x^*\|_2$ for the first-order differential equation system (4.1) and the second-order differential equation system (3.1) for Example 5.2 are showed in Figure 4.

Table 3. The CPU-time to reach termination condition (seconds) and numerical results of Example 5.2 with the first-order system (4.1) and the second-order system (3.1).

	The First-Order System (4.1)	The Second-Order System (3.1)
CPU-time(s)	0.046629	2.508700
x^* of Example 5.2	$\begin{pmatrix} 2 \\ 1.1276e-16 \\ 1.333 \\ 1.3591e-17 \\ 1.1789e-16 \end{pmatrix}$	$\begin{pmatrix} 2 \\ 3.5662e-05 \\ 1.333 \\ 4.0498e-06 \\ 3.5662e-05 \end{pmatrix}$

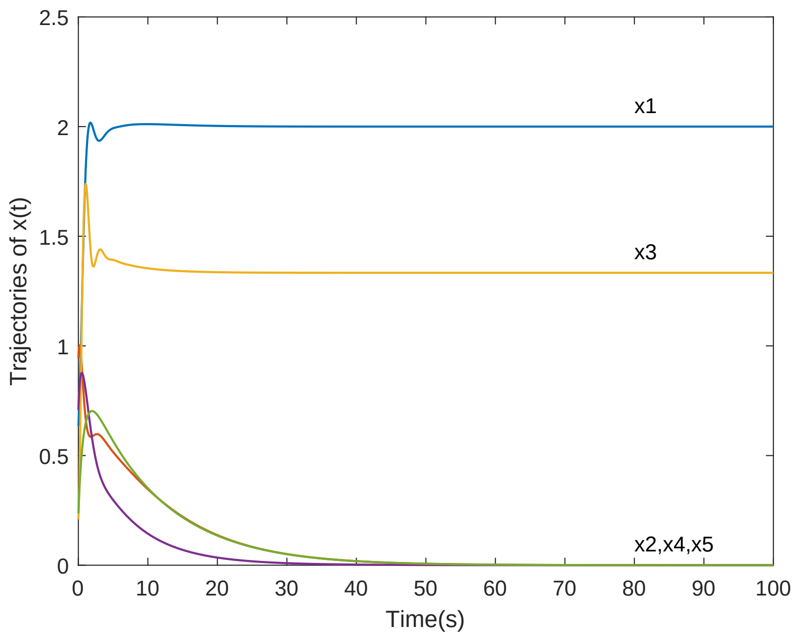


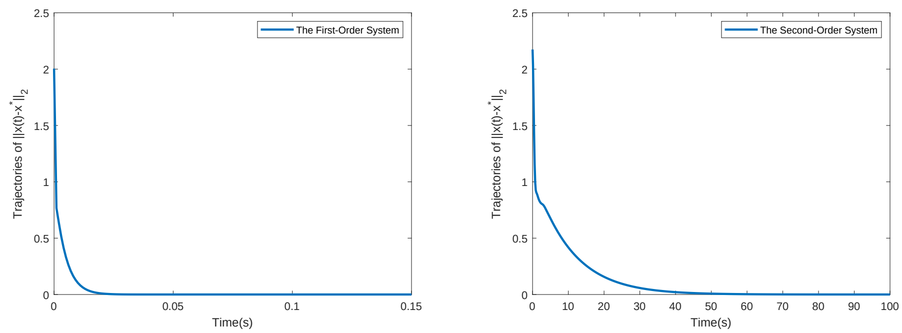
Figure 3. Trajectories of $x(t)$ of the second-order differential equation system (3.1) for Example 5.2 from five random initial points about x.

Example 5.3. Let

$$\langle F(x^*), y - x^* \rangle \geq 0, \ \forall y \in C$$

where

$$F(x) = \begin{pmatrix} x_1^3 - 8 \\ x_1 + x_2^3 - 3 \\ x_3^3 + 8 \\ x_4^2 + x_2 - 10 \\ x_5 + 1 \end{pmatrix}.$$



(a) The First-Order System

(b) The Second-Order System

Figure 4. Comparison of error rates of $\|x(t) - x^*\|_2$ for the first-order system (4.1) and the second-order system (3.1) for Example 5.2.

$C = \{x \in \Re^5 : a \le x \le b\}$, $a = (-1, 0, -5, 2, -3)^{\mathrm{T}}$ and $b = (4, 6, -1, 7, 0)^{\mathrm{T}}$. In our experiment, the variational inequality problem has $x^* = (2, 1, -2, 3, -1)^{\mathrm{T}}$ as its solution.

The CPU-time to reach termination condition (seconds) and x^* of Example 5.1 with the first-order differential equation system (4.1) and the second-order differential equation system (3.1) for Example 5.1 are summarized in Table 5.3. The solution trajectories $x(t)$ of the second-order differential equation system (3.1) from the given initial random points for Example 5.3 are showed in Figure 5. The comparison of error rates of $\|x(t) - x^*\|_2$ for the first-order differential equation system (4.1) and the second-order differential equation system (3.1) for Example 5.3 are showed in Figure 6.

Table 4. The CPU-time to reach termination condition (seconds) and numerical results of Example 5.3 with the first-order system (4.1) and the second-order system (3.1).

	The First-Order System (4.1)	The Second-Order System (3.1)
CPU-time(s)	0.29512	5.50720
x^* of Example 5.3	$(2, 1, -2, 3, -1)^{\mathrm{T}}$	$(2, 1, -2, 3, -1)^{\mathrm{T}}$

Example 5.4. Consider the following nonlinear variational inequality problem

$$F(x) = \begin{pmatrix} 2x_1 + 0.2x_1^3 - 0.5x_2 + 0.1x_3 - 4 \\ -0.5x_1 + x_2 + 0.1x_2^3 + 0.5 \\ 0.5x_1 - 0.2x_2 + 2x_3 - 0.5 \end{pmatrix},$$

where

$$C = \left\{ x \in \Re^3 \,\middle|\, x_1^2 + 0.4x_2^2 + 0.6x_3^2 - 1 \le 0 \right\}.$$

The problem has a unique optimal solution $x^* = (1, 0, 0)^{\mathrm{T}}$.

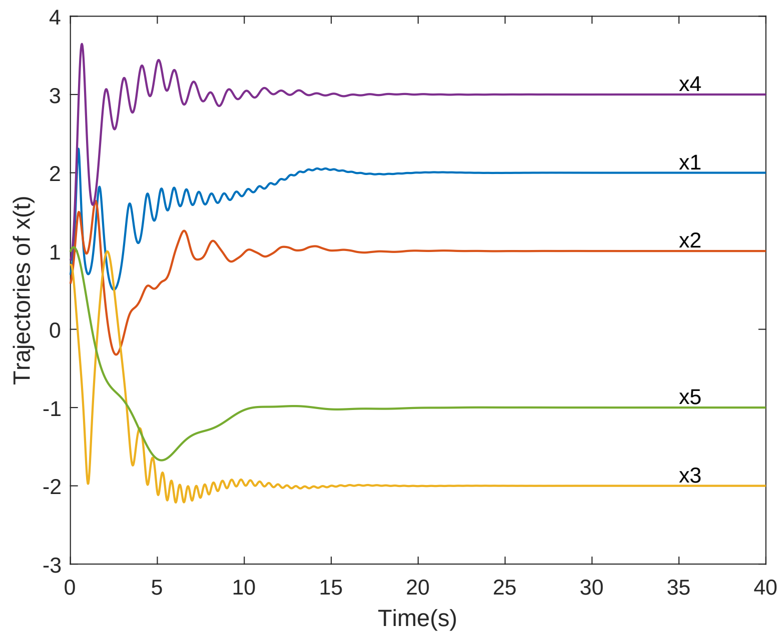
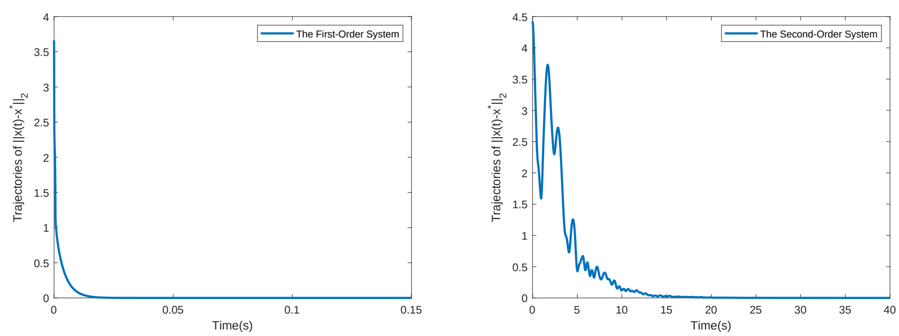


Figure 5. Trajectories of $x(t)$ of the second-order differential equation system (3.1) for Example 5.3 from five random initial points about x.



(a) The First-Order System

(b) The Second-Order System

Figure 6. Comparison of error rates of $\|x(t) - x^*\|_2$ for the first-order system (4.1) and the second-order system (3.1) for Example 5.3.

The CPU-time to reach termination condition (seconds) and x^* of Example 5.4 with the first-order differential equation system (4.1) and the second-order differential equation system (3.1) for Example 5.4 are summarized in Table 5.4. The solution trajectories $x(t)$ of the second-order differential equation system (3.1) from the given initial random points for Example 5.4 are showed in Figure 7. The comparison of error rates of $\|x(t) - x^*\|_2$ for the first-order differential equation system (4.1) and the second-order differential equation system (3.1) for Example 5.4 are showed in Figure 8.

Table 5. The CPU-time to reach termination condition (seconds) and numerical results of Example 5.4 with the first-order system (4.1) and the second-order system (3.1).

	The First-Order System (4.1)	The Second-Order System (3.1)
CPU-time(s)	0.063617	0.565150
x^* of Example 5.4	$(1,0,0)^{\mathrm{T}}$	$(1,0,0)^{\mathrm{T}}$

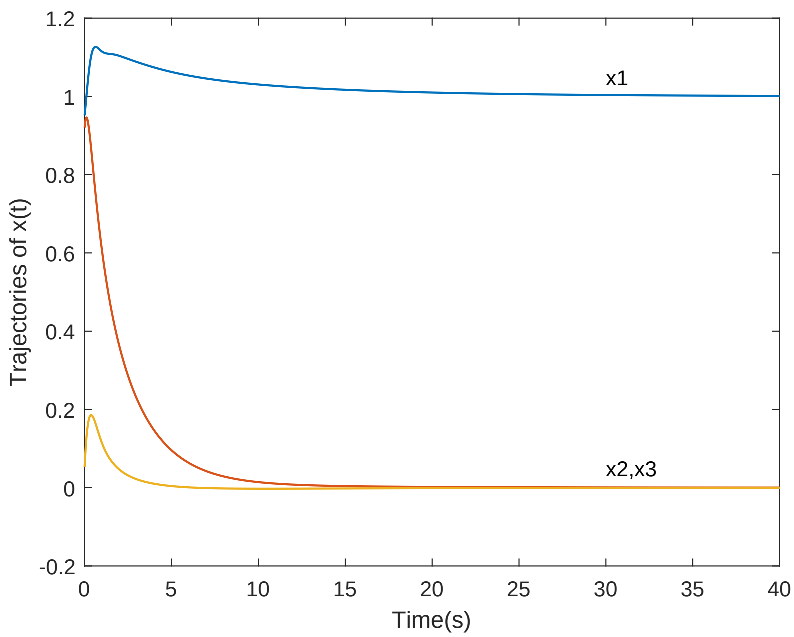
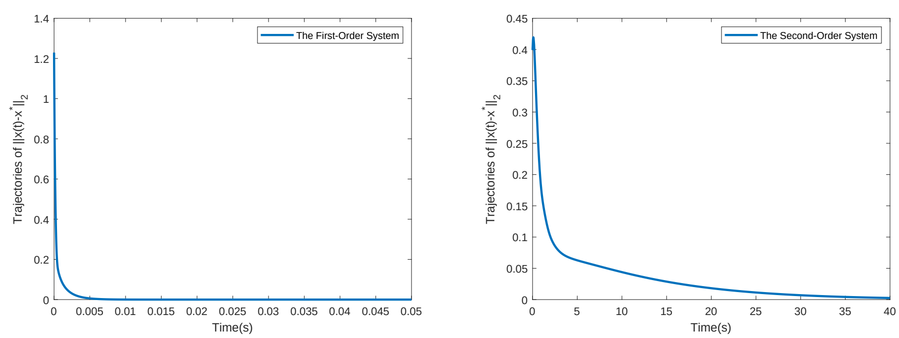


Figure 7. Trajectories of $x\,(t)$ of the second-order differential equation system (3.1) for Example 5.4 from three random initial points about x.



(a) The First-Order System (b) The Second-Order System

Figure 8. Comparison of error rates of $\|x\,(t)-x^*\|_2$ for the first-order system (4.1) and the second-order system (3.1) for Example 5.4.

6. Conclusions

In this paper, for the first time we establish the second-order differential equation system (3.1) involving a positive viscous damping coefficient $\gamma(t)$ and a time scale coefficient $\beta(t)$ inspired by Attouch et al [5] to solve the classical variational inequalities problem (1.1). We study the asymptotic behavior and convergence rate of the trajectory of the second-order differential equation system (3.1). According to the comparison of the theoretical conditions, numerical results and calculation time with the first-order differential equation system (4.1), we can get that the theoretical constraints of the second-order system are looser than that of the first-order system, but the accuracy of the numerical results and the speed of the calculation time are less than that of the first-order system. At present, the differential equation method for solving variational inequality problems is still very preliminary, it is need to be further improved.

Acknowledgments

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References

[1] A. S. Antipin, *Solving variational inequalities with coupling constraints with the use of different equations*, Differential Equations, 2000, 36(11), 1587–1596.

[2] H. Attouch and A. Cabot, *Asymptotic stabilization of inertial gradient dynamics with time-dependent viscosity*, J. Differential Equations, 2017, 263, 5412–5458.

[3] H. Attouch and A. Cabot, *Convergence rates of inertial forward-backward algorithms*, SIAM J. Optim., 2018, 28(1), 849–874.

[4] H. Attouch, A. Cabot, Z. Chbani and H. Riahi, *Rate of convergence of inertial gradient dynamics with time-dependent viscous damping coefficient*, Evolution Equations and Control Theory, 2018, 7(3), 353–371.

[5] H. Attouch, Z. Chbani and H. Riahi, *Fast Convex Optimization via Time Scaling of Damped Inertial Gradient Dynamics*, 2019. HAL Id: hal-02138954. https://hal.archives-ouvertes.fr/hal-02138954.

[6] H. Attouch, Z. Chbani and H. Riahi, *Rate of convergence of the Nesterov accelerated gradient method in the subcritical case* $\alpha \leq 3$, ESAIM: COCV, 2019, 25, No. 2.

[7] J.-S. Chen, C.-H. Ko and S.-H. Pan, *A neural network based on the generalized Fischer-Burmeister function for nonlinear complementarity problems*, Inf. Sci., 2010, 180, 697–711.

[8] X. B. Gao, L. Z. Liao and L. Qi, *A novel neural network for variational inequalities with linear and nonlinear constraints*, IEEE Trans Neural Netw, 2005, 16(6), 1305–1317. DOI: 10.1109/TNN.2005.852974. PMID: 16342476.

[9] B. He and H. Yang, *A neural network model for monotone linear asymmetric variational inequalities*, IEEE Trans Neural Netw, 2000, 11(1), 3–16. DOI: 10.1109/72.822505. PMID: 18249734.

[10] J. J. Hopfield, *Neurons with graded response have collective computational properties like those of two-state neurons*, Proc Natl Acad Sci U S A, 1984, 81(10), 3088–3092. DOI: 10.1073/pnas.81.10.3088. PMID: 6587342; PMCID: PMC345226.

[11] L. Jin, H. Y. Huang and H. Huang, *Differential equation method based on approximate augmented Lagrangian for nonlinear programming*, Journal of Industrial & Management Optimization, 2020, 16(5), 2267–2281.

[12] A. Nazemi and A. Sabeghi, *A novel gradient-based neural network for solving convex second-order cone constrained variational inequality problems*, Journal of Computational and Applied Mathematics, 2019, 347, 343–356. DOI: 10.1016/j.cam.2018.08.030.

[13] A. Nazemi and A. Sabeghi, *A new noural network framework for solving convex second-order cone constrained variational inequality problems with an application in multi-finger robot hands*, Journal of Experimental and Theoretical Artificial Intelligence, 2020, 20, 181–203.

[14] R. T. Rockafellar and R. J.-B. Wets, *Variational Analysis*, New York, NY, USA: Springer-Verlag, 1998.

[15] J. H. Sun, J.-S. Chen and C.-H. Ko, *Neural networks for solving second-order cone constrained variational inequality problem*, Comput. Optim. Appl., 2012, 51(2), 623–648.

[16] J. H. Sun, W. C. Fu, J. H. Alcantara and J.-S. Chen, *A neural network based on the metric projector for solving SOCCVI problem*, IEEE Transactions on Neural Networks and Learning Systems, 2021, 32(7), 2886–2900.

[17] J. H. Sun and L. W. Zhang, *A globally convergent method based on Fischer-Burmeister operators for solving second-order-cone constrained variational inequality problems*, Comput. Math. Appl., 2009, 58, 1936–1946.

[18] D. V. Thong and D. V. Hieu, *Modified subgradient extragradient method for variational inequality problems. numerical algorithms*, 2018, 79, 597–610. DOI: 10.1007/s11075-017-0452-4.

[19] L. Wang, X. Chen and J. Sun, *The second-order differential equation system with contrlled process for variational inequality with constraints*, Complexity, 2021, 2021, 1–8.

[20] L. Wang, Y. Li and L. W. Zhang, *A Differential equation method for solving box constrained variational inequality problems*, Journal of Industrial and Management Optimization, 7(1), 2011, 183–198.

[21] L. Wang and S. Wang, *A second-order differential equation method for equilibrium programming with constraints*, Proceeding of the 11th World Congress on Intelligent Control and Automation, 2014, 1279–1284.