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SUBORDINATIONS BY CERTAIN UNIVALENT FUNCTIONS ASSOCIATED WITH A FAMILY OF MULTIPLIER TRANSFORMATIONS*

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Abstract The purpose of the present paper is to obtain some implications of subordinations by univalent functions in the open unit disk associated with a family of multiplier transformations. Moreover, applications for integral operators are also considered.

Keywords Subordination, multivalent function, hadamard product, integral operator, multiplier transformation.

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1. Introduction

Let $\mathcal{H} = \mathcal{H}(\mathbb{U})$ denote the class of analytic functions in the open unit disk

$$\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}.$$

For $a \in \mathbb{C}$ and $n \in \mathbb{N} = \{1, 2, \cdots\}$, let

$$\mathcal{H}[a, n] = \{f \in \mathcal{H} : f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots\}.$$

Let f and F be members of $\mathcal{H}$. The function f is said to be subordinate to F, or F is said to be superordinate to f, if there exists a function w analytic in $\mathbb{U}$, with $w(0) = 0$ and $|w(z)| < 1$ for $z \in \mathbb{U}$, such that

$$f(z) = F(w(z)) \quad (z \in \mathbb{U}).$$

In such a case, we write

$$f \prec F \quad \text{or} \quad f(z) \prec F(z) \quad (z \in \mathbb{U}).$$

If the function F is univalent in $\mathbb{U}$, then we have (cf. [23])

$$f \prec F \quad \Longleftrightarrow \quad f(0) = F(0) \quad \text{and} \quad f(\mathbb{U}) \subset F(\mathbb{U}).$$

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Let $\mathcal{Q}$ be the class of functions f that are analytic and injective on $\overline{\mathbb{U}}\backslash E(f)$, where

$$E(f) = \left\{ \zeta \in \partial\mathbb{U} : \lim_{z\to\zeta} f(z) = \infty \right\},$$

and are such that $f'(\zeta) \neq 0$ for $\zeta \in \partial\mathbb{U}\backslash E(f)$.

We also denote $\mathcal{M}_\beta^*$ by the class of univalent functions $q \in \mathcal{H}$ with $q(0) = 1$ satisfying the following condition:

$$\Re\left[(1-\beta)\frac{zq'(z)}{q(z)} + \beta\left(1 + \frac{zq''(z)}{q'(z)}\right)\right] > 0 \quad (\beta \in \mathbb{R};\ z \in \mathbb{U}).$$

Then we also note that $\mathcal{M}_1^*$ is the class of convex (not necessarily normalized) functions in $\mathbb{U}$.

Let $\mathcal{A}_p$ denote the class of all analytic functions of the form

$$f(z) = z^p + \sum_{k=1}^{\infty} a_{k+p} z^{k+p} \quad (n, p \in \mathbb{N}; z \in \mathbb{U}). \tag{1.1}$$

For any complex number κ, we define the multiplier transformations I_λ^κ of functions $f \in \mathcal{A}_p$ by

$$I_{\lambda,p}^{\kappa} f(z) = z^p + \sum_{k=1}^{\infty} \left(\frac{k+p+\lambda}{p+\lambda}\right)^{\kappa} a_{k+p} z^{k+p},$$
$$(\lambda \in \mathbb{C}\backslash\mathbb{Z}_0^-;\ \mathbb{Z}_0^- := \{-p, -p+1, \cdots\}). \tag{1.2}$$

The operator $I_{\lambda,1}^{\kappa}$ was introduced and studied by Srivastava and Attiya [36], which was called as the Srivastava-Attiya operator [34]. Several interesting operators as special cases of the Srivastava-Attiya operator have been widely studied by (for examples) Attiya and Yassen [4], Cho and Srivastava [4], Deniz *et al.* [13], Jung *et al.* [14], Mostafa *et al.* [25], Owa and Srivastava [30], Sălăgean [35], Uralegaddi and Somanatha [37].

Let

$$f_{\lambda,p}^{\kappa}(z) = z^p + \sum_{k=1}^{\infty} \left(\frac{k+p+\lambda}{p+\lambda}\right)^{\kappa} z^{k+p},$$
$$(\kappa \in \mathbb{C};\ \lambda \in \mathbb{C}\backslash\mathbb{Z}_0^-;\ \mathbb{Z}_0^- := \{-p, -p+1, \cdots\})$$

and let $f_{\lambda,p}^{\kappa,\mu}$ be defined such that

$$f_{\lambda,p}^{\kappa}(z) * f_{\lambda,p}^{\kappa,\mu}(z) = \frac{z^p}{(1-z)^{\mu+p}} \ \ (\mu > -p;\ z \in \mathbb{U}), \tag{1.3}$$

where the symbol $*$ stands for the Hadamard product(or convolution). Then, motivated essentially by the Choi-Sagio-Srivastava operator [9] (see, also [20], [21], [27] and [28]), we now introduce the operator $I_{\lambda,p}^{\kappa,\mu} : \mathcal{A}_p \to \mathcal{A}_p$, which are defined here by

$$I_{\lambda,p}^{\kappa,\mu} f(z) = \left(f_{\lambda,p}^{\kappa,\mu} * f\right)(z), \tag{1.4}$$
$$(f \in \mathcal{A}_p;\ \kappa \in \mathbb{C};\ \mu > -p).$$

In view of (1.3) and (1.4), we obtain the following relations:

$$z\left(I_{\lambda,p}^{\kappa+1,\mu}f(z)\right)' = (\lambda+p)I_{\lambda,p}^{\kappa,\mu}f(z) - \lambda I_{\lambda,p}^{\kappa+1,p}f(z), \tag{1.5}$$

and

$$z\left(I_{\lambda,p}^{\kappa,\mu}f(z)\right)' = (\mu+p)I_{\lambda,p}^{\kappa,\mu+1}f(z) - \mu I_{\lambda,p}^{\kappa,\mu}f(z). \tag{1.6}$$

By using the principle of subordination, various subordination theorems involving certain integral operators for analytic functions in $\mathbb{D}$ were investigated Bulboacă [6]- [8], Miller *et al.* [24] and Owa and Srivastava [31]. Also Kumar *et al.* [18] gave an unified approach to study the properties of all these linear operators by considering the aspect that these operators satisfy recurrence relation of some common forms. They studied properties of integral transforms in a similar way. Furthermore, the study of the subordinaton properties for various operators is a important role in pure and applied mathematics. For some recent developments one may refer to [3], [11] and [12] (see, also [1], [2], [14], [26], [29] and [34]).

The aim of the present paper, motivated by the works mentioned above, is to investigate some subordination properties for multivalent functions associated with the multiplier transformation $I_{\lambda,p}^{\kappa,\mu}$ defined by (1.1). Also we consider some applications to the integral operator.

The following lemmas will be required in our present investigation.

Lemma 1.1. [22] *Let $p \in \mathcal{Q}$ with $p(0) = a$ and let*

$$q(z) = a + a_n z^n + \cdots$$

be analytic in $\mathbb{U}$ with

$$q(z) \not\equiv a \quad \text{and} \quad n \in \mathbb{N}.$$

If q is not subordinate to p, then there exist points

$$z_0 = r_0 e^{i\theta} \in \mathbb{U} \quad \text{and} \quad \zeta_0 \in \partial\mathbb{U}\backslash E(f),$$

for which

$$q(\mathbb{U}_{r_0}) \subset p(\mathbb{U}),\ q(z_0) = p(\zeta_0) \quad \textit{and} \quad z_0 q'(z_0) = m\zeta_0 p'(\zeta_0) \quad (m \geq n).$$

Lemma 1.2. [23] *Let k be convex (univalent) and let $A \geq 0$. Suppose that $M > 4/k'(0)$ and that $B(z)$ and $D(z)$ are analytic with $D(0) = 0$ and satisfy*

$$\Re\{B(z)\} \geq A + M|D(z)| \quad (z \in \mathbb{U}).$$

If $p \in \mathcal{H}$, with $p(0) = k(0)$ satisfies

$$Az^2p''(z) + B(z)zp'(z) + p(z) + D(z) \prec k(z) \quad (z \in \mathbb{U}),$$

then

$$p(z) \prec k(z) \quad (z \in \mathbb{U}).$$

A function $L(z,t)$ defined on $\mathbb{U} \times [0,\infty)$ is said to be the subordination chain (or Löwner chain) if $L(\cdot,t)$ is analytic and univalent in $\mathbb{U}$ for all $t \in [0,\infty)$, $L(z,\cdot)$ is continuously differentiable on $[0,\infty)$ for all $z \in \mathbb{U}$ and

$$L(z,s) \prec L(z,t) \quad (z \in \mathbb{U};\ 0 \leq s < t).$$

Lemma 1.3. [32] *The function*

$$L(z,t) = a_1(t)z + \cdots$$

with

$$a_1(t) \neq 0 \quad \text{and} \quad \lim_{t\to\infty} |a_1(t)| = \infty.$$

Suppose that $L(\cdot,t)$ ia analytic in $\mathbb{U}$ for all $t \geq 0$, $L(z,\cdot)$ is continuously differentiable on $[0,\infty)$ for all $z \in \mathbb{U}$. If $L(z,t)$ satisfies

$$\Re\left\{\frac{\frac{z\partial L(z,t)}{\partial z}}{\frac{\partial L(z,t)}{\partial t}}\right\} > 0 \quad (z \in \mathbb{U};\ 0 \leq t < \infty)$$

and

$$|L(z,t)| \leq K_0|a_1(t)| \quad (|z| < r_0 < 1;\ 0 \geq t < \infty))$$

for some positive constants K_0 and r_0, then $L(z,t)$ is a subordination chain.

2. Main results

Firstly, we begin by proving the following subordination theorem involving the multiplier transformation $I_{\lambda,p}^{\kappa,\mu}$ defined by (1.1).

Theorem 2.1. *Let $f,g \in \mathcal{A}_p$ with*

$$\Re\left\{(\lambda+p)\gamma\frac{I_{\lambda,p}^{\kappa,\mu}g(z)}{I_{\lambda,p}^{\kappa+1,\mu}g(z)}\right\} > 0 \quad (\gamma \in \mathbb{C};\ z \in \mathbb{U}), \tag{2.1}$$

and suppose also that $k \in \mathcal{M}_\beta^*$. Then the following subordination relation:

$$\left[\left(\frac{I_{\lambda,p}^{\kappa+1,\mu}f(z)}{I_{\lambda,p}^{\kappa+1,\mu}g(z)}\right)^{\gamma}\right]^{1-\beta}\left[\frac{I_{\lambda,p}^{\kappa,\mu}f(z)}{I_{\lambda,p}^{\kappa,\mu}g(z)}\left(\frac{I_{\lambda,p}^{\kappa+1,\mu}f(z)}{I_{\lambda,p}^{\kappa+1,\mu}g(z)}\right)^{\gamma-1}\right]^{\beta} \prec k(z), \tag{2.2}$$

$$(\gamma \in \mathbb{C};\ 0 \leq \beta \leq 1;\ z \in \mathbb{U})$$

implies that

$$\left(\frac{I_{\lambda,p}^{\kappa+1,\mu}f(z)}{I_{\lambda,p}^{\kappa+1,\mu}g(z)}\right)^{\gamma} \prec k(z) \quad (z \in \mathbb{U}).$$

Proof. Let us define the function q by

$$q(z) := \left(\frac{I_{\lambda,p}^{\kappa+1,\mu}f(z)}{I_{\lambda,p}^{\kappa+1,\mu}g(z)}\right)^{\gamma}, \tag{2.3}$$

$$(f,g \in \mathcal{A}_p;\ \gamma \in \mathbb{C};\ z \in \mathbb{U}).$$

By using the equation (1.5) to (2.3) and also, by a simple calculation, we have

$$\frac{I_{\lambda,p}^{\kappa,\mu}f(z)}{I_{\lambda,p}^{\kappa,\mu}g(z)}\left(\frac{I_{\lambda,p}^{\kappa+1,\mu}f(z)}{I_{\lambda,p}^{\kappa+1,\mu}g(z)}\right)^{\gamma-1} = q(z) + \frac{zq'(z)}{(\lambda+p)\gamma H(z)}, \tag{2.4}$$

where
$$H(z)=\frac{I_{\lambda,p}^{\kappa,\mu}g(z)}{I_{\lambda,p}^{\kappa+1,\mu}g(z)}\quad(z\in\mathbb{U}).$$
We note that the assumption (2.1) implies that
$$H(z)\neq 0\quad(z\in\mathbb{U}).$$
Hence, combining (2.3) and (2.4), we obtain
$$\begin{aligned}&\left[\left(\frac{I_{\lambda,p}^{\kappa+1,\mu}f(z)}{I_{\lambda,p}^{\kappa+1,\mu}g(z)}\right)^{\gamma}\right]^{1-\beta}\left[\frac{I_{\lambda,p}^{\kappa,\mu}f(z)}{I_{\lambda,p}^{\kappa,\mu}g(z)}\left(\frac{I_{\lambda,p}^{\kappa+1,\mu}f(z)}{I_{\lambda,p}^{\kappa+1,\mu}g(z)}\right)^{\gamma-1}\right]^{\beta}\\&=q(z)\left(1+\frac{zq'(z)}{q(z)}\frac{1}{(\lambda+p)\gamma H(z)}\right)^{\beta}.\end{aligned}\tag{2.5}$$
Thus, from (2.5), we need to prove the following subordination implication:
$$q(z)\left(1+\frac{zq'(z)}{q(z)}\frac{1}{(\lambda+p)\gamma H(z)}\right)^{\beta}\prec k(z)\ (z\in\mathbb{U})\quad\Longrightarrow\quad q(z)\prec k(z)\ (z\in\mathbb{U}).\tag{2.6}$$

For the particular case $\beta=1$, the implication (2.6) becomes
$$q(z)+\frac{1}{(\lambda+p)\gamma H(z)}zq'(z)\prec k(z)\ (z\in\mathbb{U})\quad\Longrightarrow\quad q(z)\prec k(z)\ (z\in\mathbb{U}).\tag{2.7}$$
According to Lemma 1,2 for $A=0$ and $D=0$ and by using the inequality (2.1), we deduce that the above implication (2.7) holds true.

Now we will prove that our result for the case $\beta\neq 1$. Without loss of generality, we can assume that k satisfies the conditions of Theorem 2.1 on the closed disk $\overline{\mathbb{U}}$ and
$$k'(\zeta)\neq 0\quad(\zeta\in\partial\mathbb{U}).$$
If it does not hold the generality stated above, then we replace f,g,k and H by
$$f_r(z)=f(rz),\ g_r(z)=g(rz),\ k_r(z)=k(rz)\text{ and }H_r(z)=H(rz),$$
respectively, where $0<r<1$ and then k_r is univalent on $\overline{\mathbb{U}}$. Since
$$q_r(z)\left(1+\frac{zq_r'(z)}{q_r(z)}\frac{1}{(\lambda+p)\gamma H_r(z)}\right)^{\beta}\prec k_r(z)\ (z\in\mathbb{U}),$$
where
$$q_r(z)=q(rz)\quad(0<r<1;\ z\in\mathbb{U}),$$
we would then prove that
$$q_r(z)\prec k_r(z)\quad(0<r<1;\ z\in\mathbb{U}),$$
and by letting $r\to 1^-$, we obtain
$$q(z)\prec k(z)\quad(z\in\mathbb{U}).$$

If we suppose that the implication (2.6) is not true, that is,

$$q(z) \not\prec k(z) \quad (z \in \mathbb{U}),$$

then, from Lemma 1.1, there exist points

$$z_0 \in \mathbb{U} \quad \text{and} \quad \zeta_0 \in \partial\mathbb{U}$$

such that

$$q(z_0) = k(\zeta_0) \quad \text{and} \quad z_0 q'(z_0) = m\zeta_0 k'(\zeta_0) \;\; (m \geq 1). \tag{2.8}$$

To prove the implication (2.6), we define the function

$$L : \mathbb{U} \times [0, \infty) \longrightarrow \mathbb{C}$$

by

$$\begin{aligned} L(z,t) &= k(z)\left[1 + t\frac{zk'(z)}{k(z)}\frac{1}{(\lambda+p)\gamma H(z_0)}\right]^{\beta} \\ &= a_1(t)z + \cdots , \end{aligned}$$

and we will show that $L(z,t)$ is a subordination chain. At first, we note that $L(z,t)$ is analytic in $|z| < r < 1$, for sufficient small $r > 0$ and for all $t \geq 0$. We also have that $L(z,t)$ is continuously differentiable on $[0,\infty)$ for each $|z| < r < 1$. A simple calculation shows that

$$a_1(t) = \frac{\partial L(0,t)}{\partial z} = k'(0)\left[1 + \frac{t\beta}{(\lambda+p)\gamma H(z_0)}\right].$$

From the assumptions $k'(0) \neq 0$ and (2.1) with $0 < \beta \leq 1$, we deduce

$$\Re\left\{1 + \frac{t\beta}{(\lambda+p)\gamma H(z_0)}\right\} \geq 1 > 0 \quad (t \geq 0). \tag{2.9}$$

Hence we obtain

$$a_1(t) \neq 0 \quad (t \geq 0)$$

and also we can see that

$$\lim_{t\to\infty} |a_1(t)| = \infty.$$

While, by a direct computation of $L(z,t)$, we have

$$\begin{aligned} \Re\left\{\frac{\frac{z\partial L(z,t)}{\partial z}}{\frac{\partial L(z,t)}{\partial t}}\right\} =& \frac{t}{\beta}\Re\left[(1-\beta)\frac{zk'(z)}{k(z)} + \beta\left(1 + \frac{zk''(z)}{k'(z)}\right)\right] \\ &+ \frac{1}{\beta}\Re\{(\lambda+p)\gamma H(z_0)\}. \end{aligned} \tag{2.10}$$

By using the fact that $k \in \mathcal{M}^*_\beta$ and the assumption (2.1) to (2.10), we obtain

$$\Re\left\{\frac{\frac{z\partial L(z,t)}{\partial z}}{\frac{\partial L(z,t)}{\partial t}}\right\} > 0 \quad (z \in \mathbb{U};\; 0 \leq t < \infty),$$

which completes the proof of the first condition of Lemma 1.2. Moreover, we have

$$\begin{aligned}
&\left|\frac{L(z,t)}{a_1(t)}\right|^{1/\beta}\\
=&\left|\frac{k(z)}{k'(0)}\right|^{1/\beta}\frac{\left|1+t\frac{zk'(z)}{k(z)}\frac{1}{(\lambda+p)\gamma H(z_0)}\right|}{\left|1+\frac{t\beta}{(\lambda+p)\gamma H(z_0)}\right|^{1/\beta}}\\
\leq&\frac{1}{\beta}\left|\frac{k(z)}{k'(0)}\right|^{1/\beta}\left[\left|\frac{zk'(z)}{k(z)}\right|+\frac{\left|\beta-\frac{zk'(z)}{k(z)}\right|}{\left|1+\frac{\beta t}{(\lambda+p)\gamma H(z_0)}\right|}\right]\frac{1}{\left|1+\frac{\beta t}{(\lambda+p)\gamma H(z_0)}\right|^{1/\beta-1}}\\
\leq&\frac{1}{\beta k'(0)}\left|\frac{k(z)}{k'(0)}\right|^{1/\beta-1}\left[|zk'(z)|+\frac{\beta|k(z)|+|zk'(z)|}{\left|1+\frac{\beta t}{(\lambda+p)\gamma H(z_0)}\right|}\right]\frac{1}{\left|1+\frac{\beta t}{(\lambda+p)\gamma H(z_0)}\right|^{1/\beta-1}}.
\end{aligned} \tag{2.11}$$

Since $k\in\mathcal{M}^*_\beta$, the function k may be written by

$$k(z)=k(0)+k'(0)K(z)\quad(z\in\mathbb{U}), \tag{2.12}$$

where K is a normalized univalent function in $\mathbb{U}$. We also note that for function K, we have the following sharp growth and distortion results (cf. [16] and [32])

$$\frac{r}{(1+r)^2}\leq|K(z)|\leq\frac{r}{(1-r)^2}\quad(|z|=r<1) \tag{2.13}$$

and

$$\frac{1-r}{(1+r)^3}\leq|K'(z)|\leq\frac{1+r}{(1-r)^3}\quad(|z|=r<1). \tag{2.14}$$

Hence, by applying the equations (2.9), (2.12), (2.13) and (2.14) to (2.11), we can find easily an upper bound for the right-hand side of (2.11). Thus the function $L(z,t)$ satisfies the second condition of Lemma 1.2, which proves that $L(z,t)$ is a subordination chain. In particular, we note from the definition of subordination chain that

$$k(z)=L(z,0)\prec L(z,t)\quad(z\in\mathbb{U};\ t\geq 0). \tag{2.15}$$

Now, by using the equality (2.5) and the relation (2.8), we obtain

$$\begin{aligned}
&\left[\left(\frac{I^{\kappa+1,\mu}_{\lambda,p}f(z_0)}{I^{\kappa+1,\mu}_{\lambda,p}g(z_0)}\right)^{\mu}\right]^{1-\beta}\left[\frac{I^{\kappa,\mu}_{\lambda,p}f(z_0)}{I^{\kappa,\mu}_{\lambda,p}g(z_0)}\left(\frac{I^{\kappa+1,\mu}_{\lambda,p}f(z_0)}{I^{\kappa+1,\mu}_{\lambda,p}g(z_0)}\right)^{\mu-1}\right]^{\beta}\\
=&q(z_0)\left(1+\frac{zq'(z_0)}{q(z_0)}\frac{1}{(\lambda+p)\gamma H(z_0)}\right)^{\beta}\\
=&k(\zeta_0)\left(1+m\frac{\zeta_0k'(\zeta_0)}{q(\zeta_0)}\frac{1}{(\lambda+p)\gamma H(z_0)}\right)^{\beta}\\
=&L(\zeta_0,m)\quad(m\geq 1).
\end{aligned}$$

Then, according to (2.15), we deduce that

$$\left[\left(\frac{I_{\lambda,p}^{\kappa+1,\mu}f(z_0)}{I_{\lambda,p}^{\kappa+1,\mu}g(z_0)}\right)^{\gamma}\right]^{1-\beta}\left[\frac{I_{\lambda,p}^{\kappa,\mu}f(z_0)}{I_{\lambda,p}^{\kappa,\mu}g(z_0)}\left(\frac{I_{\lambda,p}^{\kappa+1,\mu}f(z_0)}{I_{\lambda,p}^{\kappa+1,\mu}g(z_0)}\right)^{\gamma-1}\right]^{\beta} = L(\zeta_0, m) \notin k(\mathbb{U}). \tag{2.16}$$

But, the last relation (2.16) contradicts the assumption (2.2), and hence we finally conclude that

$$q(z) \prec k(z) \quad (z \in \mathbb{U}).$$

Therefore we complete the proof of Theorem 2.1. □

If we take $g(z) = z^p$ in Theorem 2.1, we have the following result.

Corollary 2.1. *Let $f \in \mathcal{A}_p$ and $k \in \mathcal{M}_\beta^*$. Then the following subordination relation:*

$$\left[\left(\frac{I_{\lambda,p}^{\kappa+1,\mu}f(z)}{z^p}\right)^{\gamma}\right]^{1-\beta}\left[\frac{I_{\lambda,p}^{\kappa,\mu}f(z)}{z^p}\left(\frac{I_{\lambda,p}^{\kappa+1,\mu}f(z)}{z^p}\right)^{\gamma-1}\right]^{\beta} \prec k(z),$$

$$(\gamma \in \mathbb{C};\ \Re\{(\lambda+p)\gamma\} > 0;\ 0 \le \beta \le 1;\ z \in \mathbb{U})$$

implies that

$$\left(\frac{I_{\lambda,p}^{\kappa+1,\mu}f(z)}{z^p}\right)^{\gamma} \prec k(z) \quad (z \in \mathbb{U}).$$

If we let $\gamma = 1$ and $\beta = 1$ in Theorem 2.1, we have the following result.

Corollary 2.2. *Let $f, g \in \mathcal{A}_p$*

$$\Re\left\{(\lambda+p)\frac{I_{\lambda,p}^{\kappa,\mu}g(z)}{I_{\lambda,p}^{\kappa+1,\mu}g(z)}\right\} > 0 \quad (z \in \mathbb{U}).$$

Suppose also that $k \in \mathcal{M}_1^*$. Then the following subordination relation:

$$\frac{I_{\lambda,p}^{\kappa,\mu}f(z)}{I_{\lambda,p}^{\kappa,\mu}g(z)} \prec k(z) \quad (z \in \mathbb{U})$$

implies that

$$\frac{I_{\lambda,p}^{\kappa+1,\mu}f(z)}{I_{\lambda,p}^{\kappa+1,\mu}g(z)} \prec k(z) \quad (z \in \mathbb{U}).$$

By using (1.6) and a similar method given in the proof of Theorem 2.1, we have the following theorem below.

Theorem 2.2. *Let $f, g \in \mathcal{A}_p$ with*

$$\Re\left\{(\mu+p)\gamma\frac{I_{\lambda,p}^{\kappa,\mu+1}g(z)}{I_{\lambda,p}^{\kappa,\mu}g(z)}\right\} > 0 \quad (\gamma \in \mathbb{C};\ z \in \mathbb{U}). \tag{2.17}$$

Suppose also that $k \in \mathcal{M}_\beta^*$. Then the following subordination relation:

$$\left[\left(\frac{I_{\lambda,p}^{\kappa,\mu}f(z)}{I_{\lambda,p}^{\kappa,\mu}g(z)}\right)^{\gamma}\right]^{1-\beta}\left[\frac{I_{\lambda,p}^{\kappa,\mu+1}f(z)}{I_{\lambda,p}^{\kappa,\mu+1}g(z)}\left(\frac{I_{\lambda,p}^{\kappa,\mu}f(z)}{I_{\lambda,p}^{\kappa,\mu}g(z)}\right)^{\gamma-1}\right]^{\beta} \prec k(z), \tag{2.18}$$

$(\gamma \in \mathbb{C};\ 0 \leq \beta \leq 1;\ z \in \mathbb{U})$

implies that

$$\left(\frac{I_{\lambda,p}^{\kappa,\mu} f(z)}{I_{\lambda,p}^{\kappa,\mu} g(z)}\right)^{\gamma} \prec k(z) \quad (z \in \mathbb{U}).$$

Theorem 2.3. *Let* $f, g \in \mathcal{A}_p$

$$\Re\left\{(\lambda + p)\gamma \frac{I_{\lambda,p}^{\kappa,\mu} g(z)}{I_{\lambda,p}^{\kappa+1,\mu} g(z)}\right\} > 0 \quad (\gamma \in \mathbb{C};\ z \in \mathbb{U}).$$

Suppose also that $k \in \mathcal{M}_1^*$. *Then the following subordination relation:*

$$(1-\beta)\left(\frac{I_{\lambda,p}^{\kappa+1,\mu} f(z)}{I_{\lambda,p}^{\kappa+1,\mu} g(z)}\right)^{\gamma} + \beta \frac{I_{\lambda,p}^{\kappa,\mu} f(z)}{I_{\lambda,p}^{\kappa,\mu} g(z)} \left(\frac{I_{\lambda,p}^{\kappa+1,\mu} f(z)}{I_{\lambda,p}^{\kappa+1,\mu} g(z)}\right)^{\gamma-1} \prec k(z),$$

$(\gamma \in \mathbb{C};\ \beta \geq 0;\ z \in \mathbb{U})$

implies that

$$\left(\frac{I_{\lambda,p}^{\kappa+1,\mu} f(z)}{I_{\lambda,p}^{\kappa+1,\mu} g(z)}\right)^{\gamma} \prec k(z) \quad (z \in \mathbb{U}).$$

Proof. Let us define the function q as in the proof of Theorem 2.1 by

$$q(z) := \left(\frac{I_{\lambda,p}^{\kappa+1,\mu} f(z)}{I_{\lambda,p}^{\kappa+1,\mu} g(z)}\right)^{\gamma} \quad (f, g \in \mathcal{A}_p;\ \gamma \in \mathbb{C};\ z \in \mathbb{U}).$$

Then, by using the equations (2.3) and (2.4), we obtain

$$\begin{aligned}&(1-\beta)\left(\frac{I_{\lambda,p}^{\kappa+1,\mu} f(z)}{I_{\lambda,p}^{\kappa+1,\mu} g(z)}\right)^{\gamma} + \beta \frac{I_{\lambda,p}^{\kappa,\mu} f(z)}{I_{\lambda,p}^{\kappa,\mu} g(z)} \left(\frac{I_{\lambda,p}^{\kappa+1,\mu} f(z)}{I_{\lambda,p}^{\kappa+1,\mu} g(z)}\right)^{\gamma-1}\\ =&q(z)\left(1 + \frac{zq'(z)}{q(z)} \frac{\beta}{(\lambda+p)\gamma H_0(z)}\right).\end{aligned}$$

The remaining part of the proof in Theorem 2.3 is similar to that of Theorem 2.1 and so we omit the detailed proof. □

If we take $\gamma = 1$ in Theorem 2.3, we have the following result.

Corollary 2.3. *Let* $f, g \in \mathcal{A}_p$

$$\Re\left\{(\lambda + p) \frac{I_{\lambda,p}^{\kappa,\mu} g(z)}{I_{\lambda,p}^{\kappa+1,\mu} g(z)}\right\} > 0 \quad (z \in \mathbb{U}).$$

Suppose also that $k \in \mathcal{M}_1^*$. *Then the following subordination relation:*

$$(1-\beta)\left(\frac{I_{\lambda,p}^{\kappa+1,\mu} f(z)}{I_{\lambda,p}^{\kappa+1,\mu} g(z)}\right) + \beta \frac{I_{\lambda,p}^{\kappa,\mu} f(z)}{I_{\lambda,p}^{\kappa+1,\mu} g(z)} \prec k(z) \quad (\beta \geq 0;\ z \in \mathbb{U})$$

implies that

$$\frac{I_{\lambda,p}^{\kappa+1,\mu} f(z)}{I_{\lambda,p}^{\kappa+1,\mu} g(z)} \prec k(z) \quad (z \in \mathbb{U}).$$

Also, by using (1.3) and a similar method given in the proof of Theorem 2.3, we have the following Theorem below.

Theorem 2.4. *Let* $f, g \in \mathcal{A}_p$

$$\Re\left\{(\mu+p)\gamma\frac{I_{\lambda,p}^{\kappa,\mu+1}g(z)}{I_{\lambda,p}^{\kappa,\mu}g(z)}\right\}>0 \quad (\gamma\in\mathbb{C};\ z\in\mathbb{U}).$$

Suppose also that $k \in \mathcal{M}_1^*$. *Then the following subordination relation:*

$$(1-\beta)\left(\frac{I_{\lambda,p}^{\kappa,\mu}f(z)}{I_{\lambda,p}^{\kappa,\mu}g(z)}\right)^{\gamma}+\beta\frac{I_{\lambda,p}^{\kappa,\mu+1}f(z)}{I_{\lambda,p}^{\kappa,\mu+1}g(z)}\left(\frac{I_{\lambda,p}^{\kappa,\mu}f(z)}{I_{\lambda,p}^{\kappa,\mu}g(z)}\right)^{\gamma-1}\prec k(z),$$

$$(\gamma\in\mathbb{C};\ \beta\geq 0;\ z\in\mathbb{U})$$

implies that

$$\left(\frac{I_{\lambda,p}^{\kappa,\mu}f(z)}{I_{\lambda,p}^{\kappa,\mu+1}g(z)}\right)^{\gamma}\prec k(z)\quad (z\in\mathbb{U}).$$

Next, we consider the integral operator F_ν ($\Re\{\nu\}>-p$) defined by (cf. [5], [15] and [19])

$$F_\nu(f)(z):=\frac{\nu+p}{z^\nu}\int_0^z t^{\nu-1}f(t)dt \quad (f\in\mathcal{A}_p;\ \Re\{\nu\}>-p). \tag{2.19}$$

Now, we obtain the following subordination property involving the integral operator defined by (2.19).

Theorem 2.5. *Let* $f, g \in \mathcal{A}_p$

$$\Re\left\{(\nu+p)\gamma\frac{I_{\lambda,p}^{\kappa,\mu}g(z)}{I_{\lambda,p}^{\kappa,\mu}F_\nu(g)(z)}\right\}>0 \quad (\Re\{\nu\}>-p;\ \gamma\in\mathbb{C};\ z\in\mathbb{U}),$$

where F_ν *is the integral operator defined by (2.16). Suppose also that* $k \in \mathcal{M}_\beta^*$. *Then the following subordination relation:*

$$\left[\left(\frac{I_{\lambda,p}^{\kappa,\mu}F_\nu(f)(z)}{I_{\lambda,p}^{\kappa,\mu}F_\nu(g)(z)}\right)^{\gamma}\right]^{1-\beta}\left[\frac{I_{\lambda,p}^{\kappa,\mu}f(z)}{I_{\lambda,p}^{\kappa,\mu}g(z)}\left(\frac{I_{\lambda,p}^{\kappa,\mu}F_\nu(f)(z)}{I_{\lambda,p}^{\kappa,\mu}F_\nu(g)(z)}\right)^{\gamma-1}\right]^{\beta}\prec k(z),$$

$$(\gamma\in\mathbb{C};\ 0\leq\beta\leq 1;\ z\in\mathbb{U})$$

implies that

$$\left(\frac{I_{\lambda,p}^{\kappa,\mu}F_\nu(f)(z)}{I_{\lambda,p}^{\kappa,\mu}F_\nu(g)(z)}\right)^{\gamma}\prec k(z)\quad (z\in\mathbb{U}).$$

Proof. Let us define the function q by

$$q(z):=\left(\frac{I_{\lambda,p}^{\kappa,\mu}F_\nu(f)(z)}{I_{\lambda,p}^{\kappa,\mu}F_\nu(g)(z)}\right)^{\mu}\quad (f,g\in\mathcal{A}_p;\ \mu\in\mathbb{C};\ z\in\mathbb{U}). \tag{2.20}$$

From the definition of the integral operator F_ν defined by (2.19), we obtain

$$z(I_{\lambda,p}^{\kappa,\mu}F_\nu(f)(z))' = (\nu+p)I_{\lambda,p}^{\kappa,\mu}f(z) - \nu I_{\lambda,p}^{\kappa,\mu}F_\nu(f)(z). \tag{2.21}$$

By using the equation (2.21) and also, by a simple calculation, we have

$$\frac{I_{\lambda,p}^{\kappa,\mu}f(z)}{I_{\lambda,p}^{\kappa,\mu}g(z)}\left(\frac{I_{\lambda,p}^{\kappa,\mu}F_\nu(f)(z)}{I_{\lambda,p}^{\kappa,\mu}F_\nu(g)(z)}\right)^{\gamma-1} = q(z) + \frac{zq'(z)}{(\nu+p)\gamma H(z)}, \tag{2.22}$$

where

$$H(z) = \frac{I_{\lambda,p}^{\kappa,\mu}g(z)}{I_{\lambda,p}^{\kappa,\mu}F_\nu(g)(z)} \quad (z \in \mathbb{U}).$$

We also note that from the assumption,

$$H(z) \neq 0 \quad (z \in \mathbb{U}).$$

Hence, combining (2.20) and (2.22), we obtain

$$\begin{aligned}&\left[\left(\frac{I_{\lambda,p}^{\kappa,\mu}F_\nu(f)(z)}{I_{\lambda,p}^{\kappa,\mu}F_\nu(g)(z)}\right)^{\gamma}\right]^{1-\beta}\left[\frac{I_{\lambda,p}^{\kappa,\mu}f(z)}{I_{\lambda,p}^{\kappa,\mu}g(z)}\left(\frac{I_{\lambda,p}^{\kappa,\mu}F_\nu(f)(z)}{I_{\lambda,p}^{\kappa,\mu}F_\nu(g)(z)}\right)^{\gamma-1}\right]^{\beta}\\ =&q(z)\left(1+\frac{zq'(z)}{q(z)}\frac{1}{(\nu+p)\gamma H(z)}\right)^{\beta}.\end{aligned}$$

The remaining part of the proof is similar to that of Theorem 2.1 and so we may omit for the proof involved. □

If we let $\gamma = 1$ and $\beta = 1$ in Theorem 2.5, we have the following result.

Corollary 2.4. *Let $f, g \in \mathcal{A}_p$*

$$\Re\left\{(\nu+p)\frac{I_{\lambda,p}^{\kappa,\mu}g(z)}{I_{\lambda,p}^{\kappa,\mu}F_\nu(g)(z)}\right\} > 0 \quad (\Re\{\nu\} > -p;\ z \in \mathbb{U}),$$

where F_ν is the integral operator defined by (2.19). Suppose also that $k \in \mathcal{M}_1^*$. Then the following subordination relation:

$$\frac{I_{\lambda,p}^{\kappa,\mu}f(z)}{I_{\lambda,p}^{\kappa,\mu}g(z)} \prec k(z) \quad (z \in \mathbb{U})$$

implies that

$$\frac{I_{\lambda,p}^{\kappa,\mu}F_\nu(f)(z)}{I_{\lambda,p}^{\kappa,\mu}F_\nu(g)(z)} \prec k(z) \quad (z \in \mathbb{U}).$$

The proof of Theorem 2.6 below is much akin to that of Theorem 2.3 and so the details may be omitted.

Theorem 2.6. *Let $f, g \in \mathcal{A}_p$*

$$\Re\left\{(\nu+p)\gamma\frac{I_{\lambda,p}^{\kappa,\mu}g(z)}{I_{\lambda,p}^{\kappa,\mu}F_\nu(g)(z)}\right\} > 0 \quad (\Re\{\nu\} > -p;\ \gamma \in \mathbb{C};\ z \in \mathbb{U}).$$

Suppose also that $k \in \mathcal{M}_1^*$. Then the following subordination relation:

$$(1-\beta)\left(\frac{I_{\lambda,p}^{\kappa,\mu}F_\nu(f)(z)}{I_{\lambda,p}^{\kappa,\mu}F_\nu(g)(z)}\right)^\gamma + \beta\frac{I_{\lambda,p}^{\kappa,\mu}f(z)}{I_{\lambda,p}^{\kappa,\mu}g(z)}\left(\frac{I_{\lambda,p}^{\kappa,\mu}F_\nu(f)(z)}{I_{\lambda,p}^{\kappa,\mu}F_\nu(g)(z)}\right)^{\gamma-1} \prec k(z),$$

$$(\Re\{\nu\} > -p;\ \gamma \in \mathbb{C};\ \beta \geq 0;\ z \in \mathbb{U})$$

implies that

$$\left(\frac{I_{\lambda,p}^{\kappa,\mu}F_\nu(f)(z)}{I_{\lambda,p}^{\kappa,\mu}F_\nu(g)(z)}\right)^\gamma \prec k(z) \quad (z \in \mathbb{U}).$$

If we take $\gamma = 1$ in Theorem 2.6, we have the following result.

Corollary 2.5. *Let $f, g \in \mathcal{A}_p$*

$$\Re\left\{(\nu+p)\frac{I_{\lambda,p}^{\kappa,\mu}g(z)}{I_{\lambda,p}^{\kappa,\mu}F_\nu g(z)}\right\} > 0 \quad (\Re\{\nu\} > -p;\ z \in \mathbb{U}).$$

Suppose also that $k \in \mathcal{M}_1^*$. Then the following subordination relation:

$$(1-\beta)\left(\frac{I_{\lambda,p}^{\kappa,\mu}F_\nu(f)(z)}{I_{\lambda,p}^{\kappa,\mu}F_\nu(g)(z)}\right) + \beta\frac{I_{\lambda,p}^{\kappa,\mu}f(z)}{I_{\lambda,p}^{\kappa,\mu}g(z)} \prec k(z) \quad (\beta \geq 0;\ z \in \mathbb{U})$$

implies that

$$\frac{I_{\lambda,p}^{\kappa,\mu}F_\nu(f)(z)}{I_{\lambda,p}^{\kappa,\mu}F_\nu(g)(z)} \prec k(z) \quad (z \in \mathbb{U}).$$

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