Journal of Applied Analysis and Computation
Volume 14, Number 2, April 2024, 816–846
Website:http://www.jaac-online.com
DOI:10.11948/20230177

UPPER SEMI-CONTINUITY AND REGULARITY OF RANDOM ATTRACTORS FOR STOCHASTIC FRACTIONAL POWER DISSIPATIVE EQUATIONS

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Abstract This paper is devoted to the study of the asymptotic dynamics of stochastic fractional power dissipative equations with additive noise. A priori estimates for solutions are derived under certain growth conditions for the nonlinearity. We then prove that the random dynamical system has a unique (L^2, L^p)-random attractor with $p > 2$, and furthermore, the family of random attractors is upper semi-continuous and regular at any point in $[0, \infty)$ under the topology of p-norms.

Keywords Stochastic fractional power dissipative equation, (L^2, L^p)-random attractor, upper semi-continuity, regularity.

MSC(2010) 37L55, 35B40, 60H15.

1. Introduction

Fractional differential equations have a wide range of applications in physics, biology, chemistry and other fields of science (see [5, 24, 27, 28, 30, 35–37] for example). Recently, some of the classical equations from physics have been postulated with fractional derivatives to better describe complex phenomena (e.g., [11, 14, 29, 32]).

Our main interest in this work is to investigate the limiting behavior of random attractors of the following stochastic fractional power dissipative equation with additive noise defined in the entire space $\mathbb{R}^n$:

$$du + \Big((-\triangle)^\alpha u + \lambda u\Big)dt = (f(x, u) + g(x))dt + \varepsilon h dW(t), \tag{1.1}$$

where $\varepsilon > 0$, $\alpha \in (1/2, 1)$ and λ are positive constants, $g \in L^p(\mathbb{R}^n) \cap H^\alpha(\mathbb{R}^n)$, $h \in H^2(\mathbb{R}^n) \cap W^{2,p}(\mathbb{R}^n)$ for some $p > 2$, and W is a two-sided real-valued Wiener process on a complete probability space that results from the fact that small irregularity has to be taken into account in some circumstances. In the various lemmas that follow we assume the nonlinear function $f(\cdot, \cdot)$ satisfies some of the following conditions:

$$f(x, s)s \le -\beta_1 |s|^p + \gamma_1(x), \tag{1.2}$$

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$$|f(x,s)| \le \beta_2 |s|^{p-1} + \gamma_2(x), \tag{1.3}$$

$$\left|\frac{\partial}{\partial s} f(x,s)\right| \le \beta_3, \tag{1.4}$$

$$\left|\frac{\partial}{\partial x} f(x,s)\right| \le \gamma_3(x), \tag{1.5}$$

where $p > 1, \beta_i\,(i = 1, 2, 3)$ are positive constants, $\gamma_1(x) \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, and $\gamma_2(x) \in L^2(\mathbb{R}^n) \cap L^q(\mathbb{R}^n)$ with $1/p + 1/q = 1$, and $\gamma_3(x) \in L^2(\mathbb{R}^n)$.

Stochastic differential equations of this type arise naturally from physical systems when random forcing is considered. These random perturbations, such as molecular collisions in gases and liquids and electric fluctuations in resistors [13], very often are neglected during the derivation of these ideal models. On the other hand, to build a more realistic model, one should include the perturbations, and this will further help better understand the dynamical behavior of the model. One way to characterize the micro effects by random perturbations in the dynamics of the macro observable is through additive or multiplicative noise in the governing equation. A crucial step to study stochastic partial differential equation is to examine the asymptotic behavior of the random dynamical systems generated by its solutions. Some nice works along these lines have been done (see [2,4,6,8–10,12,16] for example). In particular, in [18–23], the authors studied stochastic fractional PDEs, and proved that the corresponding random dynamical system has a random attractor in L^2. Furthermore, when the state space is a Hilbert space, the upper semi-continuity of attractors for PDEs or stochastic PDEs is discussed, such as [15] for deterministic case and [26] for stochastic case.

However, to the authors' best knowledge, there are no such results for *stochastic fractional* PDEs while the state space is a *Banach space*. The purpose of the present paper is to close this gap and study the upper semi-continuity of random attractors in the Banach space L^p for stochastic fractional power dissipative equation.

The paper is organized as follows. In Section 2, some preliminaries, notations and random attractor theories for random dynamical systems are introduced. A continuous random dynamical system for the stochastic fractional power dissipative equation is defined in Section 3. In Section 4, we derive uniform estimates for solutions in L^2. Uniform estimates for solutions in L^p are established in Section 5, while the upper semi-continuity and regularity of random attractors are investigated in Section 6.

2. Preliminaries and notations

For convenience, we first recall some concepts related to random attractors for stochastic dynamical systems (see [2,3,9] for more details).

Given two state spaces $L^2(\mathbb{R}^n)$ and $L^p(\mathbb{R}^n)$ $(p > 2)$ equipped with norms $\|\cdot\|$ and $\|\cdot\|_p$ respectively. Let $\Phi : \mathbb{R}^+ \times \Omega \times L^2 \to L^2$, $(t, \omega, u) \mapsto \Phi(t, \omega, u)$ be a random dynamical system (RDS) over a measurable dynamical system $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t\in\mathbb{R}})$ on L^2. The RDS Φ is always assumed to satisfy

$$\Phi(t, \omega, L^2) \subset L^p \quad \text{for every } t > 0,\ \mathbb{P}\text{-}a.s.\ \omega \in \Omega. \tag{2.1}$$

Definition 2.1. $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t\in\mathbb{R}})$ is called a measurable dynamical system, if $\theta : \mathbb{R} \times \Omega \to \Omega$ is $(\mathcal{B}(\mathbb{R}) \times \mathcal{F}, \mathcal{F})$-measurable, $\theta_0 = \mathbb{I}$, $\theta_{t+s} = \theta_t \circ \theta_s$ for all $t, s \in \mathbb{R}$, and $\theta_t A = A$ for all $t \in \mathbb{R}$ and $A \in \mathcal{F}$.

Definition 2.2. A stochastic process $\Phi(t,\omega)$ is called a continuous random dynamical system (RDS) over $(\Omega,\mathcal{F},\mathbb{P},(\theta_t)_{t\in\mathbb{R}})$ if Φ is $(\mathcal{B}(\mathbb{R}^+)\times\mathcal{F}\times\mathcal{B}(X),\mathcal{B}(X))$-measurable, and for all $\omega\in\Omega$,

(i) the mapping $\Phi:\mathbb{R}^+\times\Omega\times X\to X$ is continuous;

(ii) $\Phi(0,\omega)=\mathbb{I}$ on X;

(iii) $\Phi(t+s,\omega,\chi)=\Phi(t,\theta_s\omega,\Phi(s,\omega,\chi))$ for all $t,s\geq 0$ and $\chi\in X$ (cocycle property).

Definition 2.3. A set-valued mapping $B:\Omega\to\mathbb{S}(L^2)$ is said to be a random set if the mapping $\omega\mapsto d(u,B(\omega))$ is $\mathcal{F}$-measurable for each $u\in L^2$, where $\mathbb{S}(L^2)$ denotes the class of all subsets in L^2. A set-valued mapping $B:\Omega\to\mathbb{S}(L^2)$ is called tempered with respect to $(\theta_t)_{t\in\mathbb{R}}$ if for P-a.s. $\omega\in\Omega$ and all $\epsilon>0$,

$$\lim_{t\to\infty} e^{-\epsilon t}d(B(\theta_{-t}\omega))=0,$$

where $d(B)=\sup_{u\in B}\|u\|$.

Let $\Phi(t,w)$ be a continuous random dynamical system over $(\Omega,\mathcal{F},\mathbb{P},(\theta_t)_{t\in\mathbb{R}})$, and let $\mathcal{D}$ be the collection of all tempered random set in L^2.

Definition 2.4. Let $\mathcal{D}$ be a collection of random subsets of X and $\{K(\omega)\}_{\omega\in\Omega}\in\mathcal{D}$. Then $\{K(\omega)\}_{\omega\in\Omega}$ is called an absorbing set of Φ in $\mathcal{D}$ if for all $B\in\mathcal{D}$ and P-a.s. $\omega\in\Omega$, there exists $t_B(\omega)>0$ such that

$$\Phi(t,\theta_{-t}\omega,B(\theta_{-t}\omega))\subseteq K(\omega),\qquad t\geq t_B(\omega).$$

Definition 2.5. A set-valued mapping $\mathcal{A}:\Omega\to\mathbb{S}(L^2\cap L^p)$ is called a (L^2,L^p)-random attractor for a RDS Φ if the following conditions are satisfied,

(i) $\mathcal{A}$ is tempered and random set in L^2, and $\mathcal{A}(\omega)$ is compact in L^p for P-a.s. $\omega\in\Omega$;

(ii) $\mathcal{A}$ is invariant under Φ, that is,

$$\Phi(t,\omega,\mathcal{A}(\omega))=\mathcal{A}(\theta_t\omega),\ \text{for all } t\geq 0 \text{ and for a.s. } \omega\in\Omega;$$

(iii) $\mathcal{A}$ attracts every element $B\in\mathcal{D}$ under the topology of L^p, that is,

$$\lim_{t\to\infty} d_p(\Phi(t,\theta_{-t}\omega,B(\theta_{-t}\omega)),\mathcal{A}(\omega))=0,$$

where d_p is the Hausdorff semi-metric given by $d_p(Y,Z)=\sup_{u\in Y}\inf_{v\in Z}\|u-v\|_p$.

We show some concepts of convergence and uniform asymptotic compactness for a family of RDS Φ as follows.

Definition 2.6. A family Φ_ε with $\varepsilon\in(0,a]$ of RDS is said to be convergent on $(0,a]$ under the topology of L^2 if $\|\Phi_\varepsilon(t,\omega,u)-\Phi_{\varepsilon_0}(t,\omega,u_0)\|\to 0$ when $\varepsilon\to\varepsilon_0$ and $\|u-u_0\|\to 0$ for all $\varepsilon_0\in(0,a]$, $t>0$ and $\omega\in\Omega$. A family Φ_ε with $\varepsilon\in(0,a]$ of RDS is said to converges to a deterministic DS Φ_0 in L^2 if $\|\Phi_\varepsilon(t,\omega,u)-\Phi_{\varepsilon_0}(t,\omega,u_0)\|\to 0$ when $\varepsilon\downarrow\varepsilon_0$ and $\|u-u_0\|\to 0$ for all $\varepsilon_0\in(0,a]$, $t>0$ and $\omega\in\Omega$.

Definition 2.7. A family $\Phi_\varepsilon(\varepsilon\in I, \textit{an indexed set})$ of RDS is said to be uniformly asymptotically compact in L^2 (resp. L^p) if the sequence $\{\Phi_{\varepsilon_n}(t_n,\theta_{-t_n}\omega,u_n)\}$ is precompact in L^2 (resp. L^p) whenever $\varepsilon_n\in I$, $t_n\to+\infty$ and $u_n\to+\infty$ and $u_n\in B(\theta_{-t_n}\omega)$ with $B\in\mathcal{D}$.

From [38], the following statement holds:

$$u_n \in L^2 \cap L^p,\ \|u_n - u_0\| \to 0 \text{ and } \|u_n - u_0'\|_p \to 0, \Rightarrow u_0 = u_0' \in L^2 \cap L^p.$$

By [16], we have the following result.

Theorem 2.1. *Let $\Phi_\varepsilon(\varepsilon \in (0,a])$ be a family of continuous RDS on L^2 over $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t\in\mathbb{R}})$ and Φ_0 a DS on L^2. Both Φ_ε and Φ_0 take their values in L^p in the sense of 2.5. Assume the following four statements hold.*

(i) *Φ_ε is convergent on $(0,a]$ and Φ_ε converges to Φ_0 as $\varepsilon \downarrow 0$ in L^2;*

(ii) *every Φ_ε has a closed and random absorbing set $E_\varepsilon(\omega)$ in L^2 such that $E := \{\cup_{\varepsilon\in(0,a]} E_\varepsilon(\omega)\}$ is a tempered set and, for a deterministic positive constant c,*

$$\limsup_{\varepsilon\downarrow 0} \|E_\varepsilon(\omega)\| \le c;$$

(iii) *the family Φ_ε is uniformly asymptotically compact in L^2;*

(iv) *the family Φ_ε is uniformly asymptotically compact in L^p.*

Then every Φ_ε has a unique (L^2, L^p)-random attractor $\mathcal{A}_\varepsilon$ such that $\mathcal{A}(\omega) := \cup_{\varepsilon\in(0,a]} E_\varepsilon(\omega)$ is precompact in $L^2 \cap L^p$ and the family $\mathcal{A}_\varepsilon$ is upper semi-continuous at any point $\varepsilon_0 \in (0,a]$, i.e.,

$$\lim_{\varepsilon\to\varepsilon_0} d(\mathcal{A}_\varepsilon(\omega), \mathcal{A}_{\varepsilon_0}(\omega)) = \lim_{\varepsilon\to\varepsilon_0} d_p(\mathcal{A}_\varepsilon(\omega), \mathcal{A}_{\varepsilon_0}(\omega)) = 0.$$

Moreover, if the DS Φ_0 has a (L^2, L^p)-attracting set $\mathcal{A}_0$, then $\mathcal{A}_\varepsilon$ converges to $\mathcal{A}_0$ under the p-norm

$$\lim_{\varepsilon\downarrow 0} d_p(\mathcal{A}_\varepsilon(\omega), \mathcal{A}_0) = 0.$$

In [16], a condition (iv)' is given to replace the condition (iv), which is stated as

(iv)' For any $\eta > 0$ and $B \in \mathcal{D}$, there exist two positive random variables T and M (independent of ε) such that P-a.s.

$$\sup_{0<\varepsilon\le a} \sup_{u_0\in B(\theta_{-t}\omega)} \int_{D(|\Phi_\varepsilon|\ge M)} |\Phi_\varepsilon(t, \theta_{-t}\omega, u_0)|^p dx < \eta \quad \text{for all } t \ge T,$$

where $D(|\Phi_\varepsilon| \ge M) := \{x \in D | (\Phi_\varepsilon(t, \theta_{-t}\omega, u_0))(x) \ge M\}$.

We next briefly recall some notations that are related to the fractional derivative and fractional Sobolev space. To get started, we present the definition and the properties of $(-\triangle)^\alpha$ by Fourier series ([29]). The negative powers $(-\triangle)^{\frac{\beta}{2}}$ (that is $(-\triangle)^{-\frac{\beta}{2}}$, Re$\beta > 0$), can be represented by Riesz potentials

$$(\mathcal{I}^\beta \varphi)(x) = \frac{1}{\gamma(\beta)} \int_{\mathbb{R}^n} |x-y|^{-n+\beta} \varphi(y) dy,$$

where $\gamma(\beta) = \pi^{n/2} 2^\beta \Gamma(\frac{\beta}{2}) / \Gamma(\frac{n}{2} - \frac{\beta}{2})$. Using the Fourier transform

$$\Phi(\xi) = \int_{\mathbb{R}^n} \varphi(x,t) e^{-\mathrm{i}x\cdot\xi} dx,$$

one can define $(-\triangle)^{\frac{\beta}{2}}$ as

$$\mathcal{F}\{(-\triangle)^{\frac{\beta}{2}}\varphi\} = |k|^\beta \Phi,$$
$$(-\triangle)^{\frac{\beta}{2}}\varphi = \mathcal{F}^{-1}\{|k|^\beta \Phi\} = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} |k|^\beta \Phi e^{\mathrm{i}k\cdot x} dk.$$

Let $H^{2\alpha}(\mathbb{R}^n)$ denote the complete Sobolev space of order α under the norm:

$$\|u\|^2_{H^{2\alpha}(\mathbb{R}^n)} = \int_{\mathbb{R}^n} (1+|k|^{4\alpha})|\hat{u}(k)|^2 dk.$$

By the definition of $(-\triangle)^\alpha$, the following formula for integration by parts can be easily established.

Lemma 2.1. *For $f, g \in H^{2\alpha}(\mathbb{R}^n)$, one has*

$$\int_{\mathbb{R}^n} (-\triangle)^\alpha f \cdot g dx = \int_{\mathbb{R}^n} (-\triangle)^{\alpha_1} f \cdot (-\triangle)^{\alpha_2} g dx, \tag{2.2}$$

where α_1, α_2 are nonnegative constants with $\alpha_1 + \alpha_2 = \alpha$.

By [7], we have the following lemmas. Let $C_0^2(\mathbb{R}^n)$ denote the functions in $C^2(\mathbb{R}^n)$ with compact support and $C^2(\mathbb{T}^n) = \{u : \mathbb{T}^n \to R | u$ is twice continuously differentiable$\}$.

Lemma 2.2. *If $0 \le \alpha \le 2$, $x \in \mathbb{R}^n$, $\mathbb{T}^n$ and $\theta \in C_0^2(\mathbb{R}^n)$, $C^2(\mathbb{T}^n)$. Then the following inequality holds:*

$$2\theta(-\triangle)^{\alpha/2}\theta(x) \ge (-\triangle)^{\alpha/2}\theta^2(x). \tag{2.3}$$

Lemma 2.3. *If $0 \le \alpha \le 2$, and $\theta \in C_0^2(\mathbb{R}^n)(C^2(\mathbb{T}^n))$, it follows that*

$$\int |\theta|^{p-2}\theta(-\triangle)^{\alpha/2}\theta(x) \ge \frac{1}{p}\int |(-\triangle)^{\alpha/4}\theta^{p/2}|^2 dx, \tag{2.4}$$

where $p = 2^j$ and j is a positive integer.

We now recall the Gagliardo-Nirenberg's inequality ([25]), which is used very often in our analysis.

Lemma 2.4. *Let u belong to $L^q(\mathbb{R}^n)$ and its derivatives of order m, $D^m u$, belong to $L^r(\mathbb{R}^n)$, $1 \le q, r \le \infty$. For the derivatives $D^j u$, $0 \le j < m$, the following inequality holds*

$$\left\|D^j u\right\|_{L^p} \le c\left\|D^m u\right\|^\theta_{L^r} \|u\|^{1-\theta}_{L^q}, \tag{2.5}$$

where

$$\frac{1}{p} = \frac{j}{n} + \theta\left(\frac{1}{r} - \frac{m}{n}\right) + (1-\theta)\frac{1}{q}, \quad \text{for all } \theta \text{ in the interval } \frac{j}{m} \le \theta \le 1$$

(the constant c depending only on n, m, j, q, r, θ), with the following exceptional cases

(i) If $j = 0, rm < n$ and $q = \infty$, then we make the additional assumption that either u tends to zero at infinite or $u \in L^{\tilde{q}}$ for some finite $\tilde{q} > 0$.

(ii) If $1 < r < \infty$ and $m - j - n/r$ is a nonnegative integer, then (2.5) holds only for θ satisfying $j/m \le \theta < 1$.

To end this section, we comment that, in our following discussions, we denote by $\|\cdot\|$ and $(\cdot,\cdot)$ the norm and the inner product in $L^2(\mathbb{R}^n)$ and use $\|\cdot\|_p$ to denote the norm in $L^p(\mathbb{R}^n)$. The letters $c, c_j (j = 1, 2, \cdots)$ are generic positive constants which may change their values from line to line or even in the same line.

3. Stochastic semilinear fractional power dissipative equation

In the sequel, we consider the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ where

$$\Omega = \{\omega \in C(\mathbb{R}, \mathbb{R}) : \omega(0) = 0\}.$$

$\mathcal{F}$ is the Borel σ-algebra induced by the compact-open topology of Ω, and $\mathbb{P}$ the corresponding Wiener measure on $(\Omega, \mathcal{F})$. Define a shift on ω by

$$\theta_t\omega(\cdot) = \omega(\cdot + t) - \omega(t), \quad \omega \in \Omega, \quad t \in \mathbb{R}.$$

Then $\big(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t\in\mathbb{R}}\big)$ is a metric dynamical system.

In this section, we discuss the existence of a continuous random dynamical system for the stochastic semilinear fractional power equation perturbed by additive noise. Thanks to the special linear additive noise, the stochastic semilinear fractional power equation can be reduced to an equation with random coefficients by a suitable change of variable. Let $z(\omega) = h\phi(\omega)$ where $\phi(\omega) = -\lambda \int_{-\infty}^{0} e^{\lambda\tau}\omega(\tau)d\tau$ such that $t \mapsto \phi(\theta_t\omega)$ is a pathwise continuous solution of the stochastic equation:

$$d\phi + \lambda\phi dt = dW(t). \tag{3.1}$$

Note that the random variable $|\phi(\omega)|$ is tempered and $\phi(\theta_t\omega)$ is $\mathbb{P}$-a.e. continuous. Therefore, from Proposition 4.3.3 in [1], there exists a tempered function $r(\omega) > 0$ such that

$$|\phi(\omega)|^2 + |\phi(\omega)|^p \le r(\omega), \tag{3.2}$$

where $r(\omega)$ satisfies, for $\mathbb{P}$-a.s. $\omega \in \Omega$,

$$r(\theta_t\omega) \le e^{\frac{\lambda}{2}|t|}r(\omega), \quad t \in \mathbb{R}. \tag{3.3}$$

From (3.2) and (3.3), we obtain that, for $\mathbb{P}$-a.s. $\omega \in \Omega$,

$$|\phi(\theta_t\omega)|^2 + |\phi(\theta_t\omega)|^p \le e^{\frac{\lambda}{2}|t|}r(\omega), \quad t \in \mathbb{R}. \tag{3.4}$$

We rewrite the unknown $v(t)$ as $v(t) = u(t) - \varepsilon z(\theta_t\omega)$ to obtain the following random differential equation

$$v_t + (-\triangle)^\alpha v + \lambda v = f\big(x, v + \varepsilon z(\theta_t\omega)\big) + g - \varepsilon(-\triangle)^\alpha z(\theta_t\omega) \tag{3.5}$$

with the initial data

$$v(0, \omega, v_0) = v_0 = u_0 - \varepsilon z(\omega), \qquad x \in \mathbb{R}^n. \tag{3.6}$$

It has been shown in [19] that, for any initial L^2-valued random element $u_0 = u_0(\omega)$ and for P-a.s. $\omega \in \Omega$, there is a unique solution

$$v(\cdot, \omega, v_0) \in C([0,\infty), L^2) \cap L^2((0,T), H^1) \cap L^p((0,T), L^p) \ \text{ for any } T > 0. \tag{3.7}$$

Next, we construct a random dynamical system modeling the stochastic fractional power dissipative equation. The existence and uniqueness of the solution for the problem (3.5)–(3.6) can be obtained (see [17]), which defines a stochastic dynamical system $(\Phi(t))_{t\geq 0}$ over $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t\in\mathbb{R}})$ by

$$\Phi(t,\omega,v_0) = v(t,\omega,v_0), \quad \text{for} \;\; v_0 \in L^2(\mathbb{R}^n), \; t \geq 0 \;\; \text{and for all} \;\; \omega \in \Omega.$$

Therefore the stochastic process $u(t,\omega,u_0) = v(t,\omega,u_0 - \varepsilon z(\omega)) + \varepsilon z(\theta_t\omega)$ is a solution of (1.1) such that $u(0,\omega) = u_0(\omega)$.

The measurable mapping $\Psi_\varepsilon : \mathbb{R}^+ \times \Omega \times L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ for every $\varepsilon > 0$ given by

$$\Psi_\varepsilon(t,\omega,u_0) = u(t,\omega,u_0) = v(t,\omega,u_0 - \varepsilon z(\omega)) + \varepsilon z(\theta_t\omega)$$

forms a continuous RDS over $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t\in\mathbb{R}})$ on the state space L^2. It is obvious from (3.7) that the system Ψ_ε take its values in L^p in the sense of (2.1). When $\varepsilon = 0$, equation (1.1) defines an operator semigroup Ψ_0 in L^2 with a global (L^2, L^p)-attractor $\mathcal{A}_0$.

4. Uniform estimates of solutions in $L^2(\mathbb{R}^n)$

In this section, we deduce uniform estimates on the solutions with respect to the small parameter ε in $L^2(\mathbb{R}^n)$. From now on, we always suppose that $\mathcal{D}$ is the collection of all tempered random subsets of $L^2(\mathbb{R}^n)$. And we assume that the stochastic process u_ε (sometimes we write u if no confusion) and v_ε are the solutions of equations (1.1) and (3.5) respectively.

Lemma 4.1. *For every $0 < \varepsilon \leq 1$ and $B \in \mathcal{D}$, there exist a random time $T = T(B) > 0$ and a positive constant c (independent of ε) such that P-a.s. for all $t \geq T$*

$$\sup_{u_0\in B(\theta_{-t}\omega)} \left\|u_\varepsilon(t,\theta_{-t}\omega,u_0)\right\|^2 \leq c(1+\varepsilon r(\omega)) \tag{4.1}$$

and

$$\sup_{u_0\in B(\theta_{-t}\omega)} \int_0^t e^{\lambda(\tau-t)} \left\|u_\varepsilon(\tau,\theta_{-t}\omega,u_0)\right\|_p^p d\tau \leq c(1+\varepsilon r(\omega)) \tag{4.2}$$

where $r(\omega)$ is a tempered random variable such that $\tau \mapsto r(\theta_\tau\omega)$ is continuous.

Proof. Taking the inner product in L^2 of (3.5) with v, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|v\|^2 + \|(-\triangle)^{\frac{\alpha}{2}}v\|^2 + \lambda\|v\|^2 \\ &= \int_{\mathbb{R}^n} f\Big(x, v + \varepsilon z(\theta_t\omega)\Big) v dx + \big(g - \varepsilon(-\triangle)^\alpha z(\theta_t\omega), v\big).\end{aligned} \tag{4.3}$$

Now we estimate the two terms on the right-hand side of (4.3). For the first term, applying conditions (1.2) and (1.3), together with Young's inequality, we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^n} f\big(x, v+\varepsilon z(\theta_t\omega)\big) v dx \\
=&\int_{\mathbb{R}^n} f\big(x, v+\varepsilon z(\theta_t\omega)\big)\big(v+\varepsilon z(\theta_t\omega)\big) dx-\varepsilon \int_{\mathbb{R}^n} f\big(x, v+\varepsilon z(\theta_t\omega)\big) z(\theta_t\omega) dx \\
\leq&-\beta_1 \int_{\mathbb{R}^n}|u|^p dx+\int_{\mathbb{R}^n} \gamma_1(x) dx-\varepsilon \int_{\mathbb{R}^n} f\big(x, u\big) z(\theta_t\omega) dx \\
\leq&-\beta_1 \int_{\mathbb{R}^n}|u|^p dx+\int_{\mathbb{R}^n} \gamma_1(x) dx+\varepsilon \int_{\mathbb{R}^n}\big(\beta_2|u|^{p-1}+\gamma_2(x)\big)\big|z(\theta_t\omega)\big| dx \\
\leq&-\frac{1}{2} \beta_1\|u\|_p^p+\varepsilon c_2\big(\|z(\theta_t\omega)\|_p^p+\|z(\theta_t\omega)\|^2\big)+c_3,
\end{aligned} \tag{4.4}$$

where c_2 and c_3 do not depend on ε, and we have used the fact $0<\varepsilon \leq 1$. For the second one, integrating by parts and applying Young's inequality, we have

$$\begin{aligned}
&\big(g-\varepsilon(-\triangle)^\alpha z(\theta_t\omega), v\big) \\
=&\big(g, v\big)-\big(\varepsilon(-\triangle)^{\frac{\alpha}{2}} z(\theta_t\omega),(-\triangle)^{\frac{\alpha}{2}} v\big) \\
\leq& \varepsilon\big\|(-\triangle)^{\frac{\alpha}{2}} z(\theta_t\omega)\big\|^2+\frac{1}{2}\big\|(-\triangle)^{\frac{\alpha}{2}} v\big\|^2+\frac{1}{2\lambda}\|g\|^2+\frac{\lambda}{2}\|v\|^2 .
\end{aligned} \tag{4.5}$$

Substituting (4.4) and (4.5) into (4.3), we get

$$\begin{aligned}
&\frac{d}{dt}\|v\|^2+\big\|(-\triangle)^{\frac{\alpha}{2}} v\big\|^2+\lambda\|v\|^2+\beta_1\|u\|_p^p \\
\leq& \varepsilon c_4\left(\|z(\theta_t\omega)\|_p^p+\|z(\theta_t\omega)\|^2+\big\|(-\triangle)^{\frac{\alpha}{2}} z(\theta_t\omega)\big\|^2\right)+c_5 .
\end{aligned} \tag{4.6}$$

With $z(\theta_t\omega)=h\phi(\theta_t\omega)$ and $h \in H^2(\mathbb{R}^n) \cap W^{2,p}(\mathbb{R}^n)$, the right-hand side of (4.6) is bounded by

$$\varepsilon c_4 c_6\big(\big|\phi(\theta_t\omega)\big|^p+\big|\phi(\theta_t\omega)\big|^2\big)+c_7 \triangleq \varepsilon c_4 \eta_1(\theta_t\omega)+c_7 . \tag{4.7}$$

It follows from (4.6) and (4.7) that, for all $t \geq 0$,

$$\frac{d}{dt}\|v\|^2+\big\|(-\triangle)^{\frac{\alpha}{2}} v\big\|^2+\lambda\|v\|^2+\beta_1\|u\|_p^p \leq \varepsilon c_4 \eta_1(\theta_t\omega)+c_7 . \tag{4.8}$$

Multiplying (4.8) by $e^{\lambda t}$ and then integrating the inequality, we deduce that, for all $t \geq 0$,

$$\begin{aligned}
&\|v(t, \omega, v_0(\omega))\|^2+\beta_1 \int_0^t e^{\lambda(s-t)}\big\|u(s, \omega, u_0(\omega))\big\|_p^p ds \\
\leq& e^{-\lambda t}\|v_0(\omega)\|^2+\varepsilon c_4 \int_0^t e^{\lambda(s-t)} \eta_1(\theta_s\omega) ds+\frac{c_7}{\lambda} .
\end{aligned} \tag{4.9}$$

By (3.4), we obtain that for P-a.s. $\omega \in \Omega$,

$$\eta_1(\theta_s\omega) \leq c_6 e^{\frac{1}{2}\lambda|s|} r(\omega), \quad \forall\, s \in \mathbb{R} . \tag{4.10}$$

Replacing ω by $\theta_{-t}\omega$ in (4.9) and using (4.10), we have, for all $t \geq 0$,

$$\big\|v(t, \theta_{-t}\omega, v_0(\theta_{-t}\omega))\big\|^2+\beta_1 \int_0^t e^{\lambda(s-t)}\big\|u(s, \theta_{-t}\omega, u_0(\theta_{-t}\omega))\big\|_p^p ds$$

$$\leq e^{-\lambda t}\big\|v_0(\theta_{-t}\omega)\big\|^2+\varepsilon c_4c_6\int_{-t}^{0}e^{\frac{1}{2}\lambda\tau}r(\omega)d\tau+\frac{c_7}{\lambda}$$
$$\leq e^{-\lambda t}\big\|v_0(\theta_{-t}\omega)\big\|^2+\varepsilon c_8 r(\omega)+\frac{c_7}{\lambda}. \tag{4.11}$$

Note that

$$u(t,\omega,u_0(\omega))=v\big(t,\omega,v_0(\omega)\big)+\varepsilon z(\theta_t\omega).$$

It follows from (4.11) that, for all $t\geq 0$,

$$\begin{aligned}
&\big\|u(t,\theta_{-t}\omega,u_0(\theta_{-t}\omega))\big\|^2\\
=&\big\|v(t,\theta_{-t}\omega,v_0(\theta_{-t}\omega))+\varepsilon z(\omega)\big\|^2\\
\leq&2\big\|v(t,\theta_{-t}\omega,v_0(\theta_{-t}\omega))\big\|^2+2\varepsilon\|z(\omega)\|^2\\
\leq&2e^{-\lambda t}\big\|u_0(\theta_{-t}\omega)-\varepsilon z(\theta_{-t}\omega)\big\|^2+\varepsilon c_8 r(\omega)+\frac{c_7}{\lambda}+2\varepsilon\|z(\omega)\|^2\\
\leq&4e^{-\lambda t}\left(\big\|u_0(\theta_{-t}\omega)\big\|^2+\big\|z(\theta_{-t}\omega)\big\|^2\right)+\varepsilon c_8 r(\omega)+\frac{c_7}{\lambda}+2\varepsilon\|z(\omega)\|^2.
\end{aligned} \tag{4.12}$$

By the assumption, $\{B(\omega)\}_{\omega\in\Omega}\in\mathcal{D}$ is tempered. On the other hand, by Definition 2.3, $\|z(\omega)\|^2$ is also tempered. Therefore, if $u_0(\theta_{-t}\omega)\in B(\theta_{-t}\omega)$ and $v_0(\theta_{-t}\omega)\in B(\theta_{-t}\omega)$, then there exists $T=T(B,\omega)>0$ such that for all $t\geq T$,

$$e^{-\lambda t}\big\|v_0(\theta_{-t}\omega)\big\|^2\leq 4e^{-\lambda t}\left(\big\|u_0(\theta_{-t}\omega)\big\|^2+\big\|z(\theta_{-t}\omega)\big\|^2\right)\leq 1.$$

It follows from (4.11) and (4.12) that, for all $t\geq T$,

$$\int_0^t e^{\lambda(s-t)}\big\|u(s,\theta_{-t}\omega,u_0(\theta_{-t}\omega))\big\|_p^p ds\leq c\big(1+\varepsilon r(\omega)\big),$$
$$\big\|u(t,\theta_{-t}\omega,u_0(\theta_{-t}\omega))\big\|^2\leq c\big(1+\varepsilon r(\omega)\big).$$

This completes the proof. □

Lemma 4.2. *For every $0<\varepsilon\leq 1$ and $B\in\mathcal{D}$. Then, for every $T_1\geq 0$ and $\mathbb{P}$-a.s. $\omega\in\Omega$, one has, for all $t\geq T_1$,*

$$\int_{T_1}^t e^{\lambda(s-t)}\big\|(-\triangle)^{\frac{\alpha}{2}}v(s,\theta_{-t}\omega,v_0(\theta_{-t}\omega))\big\|^2 ds\leq e^{-\lambda t}\big\|v_0(\theta_{-t}\omega)\big\|^2+c\big(\varepsilon r(\omega)+1\big) \tag{4.13}$$

and

$$\beta_1\int_{T_1}^t e^{\lambda(s-t)}\big\|u(s,\theta_{-t}\omega,u_0(\theta_{-t}\omega))\big\|_p^p ds\leq e^{-\lambda t}\big\|v_0(\theta_{-t}\omega)\big\|^2+c\big(\varepsilon r(\omega)+1\big). \tag{4.14}$$

Proof. Replacing t by T_1 and then replacing ω by $\theta_{-t}\omega$ in (4.9), we have

$$\big\|v(T_1,\theta_{-t}\omega,v_0(\theta_{-t}\omega))\big\|^2\leq e^{-\lambda T_1}\big\|v_0(\theta_{-t}\omega)\big\|^2+\varepsilon c_4\int_0^{T_1}e^{\lambda(s-T_1)}\eta_1(\theta_{s-t}\omega)ds+c.$$

Multiplying the above inequality by $e^{\lambda(T_1-t)}$ and using (4.10), we obtain

$$e^{\lambda(T_1-t)}\big\|v(T_1,\theta_{-t}\omega,v_0(\theta_{-t}\omega))\big\|^2$$

$$\le e^{-\lambda t}\big\|v_0(\theta_{-t}\omega)\big\|^2+\varepsilon c_4\int_0^{T_1}e^{\lambda(s-t)}\eta_1(\theta_{s-t}\omega)ds+ce^{\lambda(T_1-t)}$$
$$\le e^{-\lambda t}\big\|v_0(\theta_{-t}\omega)\big\|^2+\varepsilon c_4c_6r(\omega)\int_{-t}^{T_1-t}e^{\frac{1}{2}\lambda s}ds+ce^{\lambda(T_1-t)}$$
$$\le e^{-\lambda t}\big\|v_0(\theta_{-t}\omega)\big\|^2+\frac{2}{\lambda}\varepsilon c_4c_6r(\omega)e^{\frac{1}{2}\lambda(T_1-t)}+ce^{\lambda(T_1-t)}. \tag{4.15}$$

By (4.8), one has, for all $t\ge T_1$,

$$\begin{aligned}&\|v(t,\omega,v_0(\omega))\|^2+\int_{T_1}^t e^{\lambda(s-t)}\big\|(-\triangle)^{\frac{\alpha}{2}}v(s,\omega,v_0(\omega))\big\|^2ds\\&+\beta_1\int_{T_1}^t e^{\lambda(s-t)}\|u(s,\omega,u_0(\omega))\|_p^pds\\&\le e^{\lambda(T_1-t)}\|v(T_1,\omega,v_0(\omega))\|^2+\varepsilon c_4\int_{T_1}^t e^{\lambda(s-t)}\eta_1(\theta_s\omega)ds+c\int_{T_1}^t e^{\lambda(s-t)}ds.\end{aligned}\tag{4.16}$$

Dropping the first term on the left-hand side of (4.16), replacing ω by $\theta_{-t}\omega$ in (4.16) and using (4.10), we obtain, for all $t\ge T_1$,

$$\begin{aligned}&\int_{T_1}^t e^{\lambda(s-t)}\big\|(-\triangle)^{\frac{\alpha}{2}}v(s,\theta_{-t}\omega,v_0(\theta_{-t}\omega))\big\|^2ds\\&+\beta_1\int_{T_1}^t e^{\lambda(s-t)}\big\|u(s,\theta_{-t}\omega,u_0(\theta_{-t}\omega))\big\|_p^pds\\&\le e^{\lambda(T_1-t)}\big\|v(T_1,\theta_{-t}\omega,v_0(\theta_{-t}\omega))\big\|^2+\varepsilon c_4\int_{T_1}^t e^{\lambda(s-t)}\eta_1(\theta_{s-t}\omega)ds+c\int_{T_1}^t e^{\lambda(s-t)}ds\\&\le e^{\lambda(T_1-t)}\big\|v(T_1,\theta_{-t}\omega,v_0(\theta_{-t}\omega))\big\|^2+\varepsilon c_4c_6r(\omega)\int_{T_1-t}^0 e^{\frac{1}{2}\lambda\tau}d\tau+\frac{c}{\lambda}\\&\le e^{\lambda(T_1-t)}\big\|v(T_1,\theta_{-t}\omega,v_0(\theta_{-t}\omega))\big\|^2+c\big(\varepsilon r(\omega)+1\big).\end{aligned}\tag{4.17}$$

Combining (4.15) and (4.17), we have

$$\begin{aligned}&\int_{T_1}^t e^{\lambda(s-t)}\big\|(-\triangle)^{\frac{\alpha}{2}}v(s,\theta_{-t}\omega,v_0(\theta_{-t}\omega))\big\|^2ds\\&+\beta_1\int_{T_1}^t e^{\lambda(s-t)}\big\|u(s,\theta_{-t}\omega,u_0(\theta_{-t}\omega))\big\|_p^pds\\&\le e^{-\lambda t}\big\|v_0(\theta_{-t}\omega)\big\|^2+c\big(\varepsilon r(\omega)+1\big).\end{aligned}$$

This completes the proof. □

Lemma 4.3. *For every $0<\varepsilon\le 1$, $B\in\mathcal{D}$ and $u_0(\omega)\in B(\omega)$. Then, for $\omega\in\Omega$, there exists $T=T(B,\omega)>0$ such that for all $t\ge T$,*

(i) *the solution $u(t,\omega,u_0(\omega))$ of problem (1.1) satisfies*

$$\beta_1\int_t^{t+1}\big\|u\big(s,\theta_{-t-1}\omega,v_0(\theta_{-t-1}\omega)\big)\big\|_p^pds\le c\big(\varepsilon r(\omega)+1\big),\tag{4.18}$$
$$\int_t^{t+1}\big\|(-\triangle)^{\frac{\alpha}{2}}u\big(s,\theta_{-t-1}\omega,u_0(\theta_{-t-1}\omega)\big)\big\|^2ds\le c\big(\varepsilon r(\omega)+1\big).\tag{4.19}$$

(ii) *the solution* $v(t,\omega,v_0(\omega))$ *of problem* (3.5)–(3.6) *satisfies*

$$\int_t^{t+1} \big\|(-\triangle)^{\frac{\alpha}{2}}v\big(s,\theta_{-t-1}\omega,v_0(\theta_{-t-1}\omega)\big)\big\|^2 ds \le c\big(\varepsilon r(\omega)+1\big). \tag{4.20}$$

Proof. Replacing t by $t+1$ and then replacing T_1 by t in (4.13), we have

$$\begin{aligned}&\int_t^{t+1} e^{\lambda(s-t-1)}\big\|(-\triangle)^{\frac{\alpha}{2}}v\big(s,\theta_{-t-1}\omega,v_0(\theta_{-t-1}\omega)\big)\big\|^2 ds\\ \le& e^{-\lambda(t+1)}\|v_0(\theta_{-t-1}\omega)\|^2+c\big(\varepsilon r(\omega)+1\big).\end{aligned} \tag{4.21}$$

For all $s\in[t,t+1]$, we know $e^{\lambda(s-t-1)}\ge e^{-\lambda}$. So, (4.21) can be rewritten as

$$\begin{aligned}&e^{-\lambda}\int_t^{t+1} \big\|(-\triangle)^{\frac{\alpha}{2}}v\big(s,\theta_{-t-1}\omega,v_0(\theta_{-t-1}\omega)\big)\big\|^2 ds\\ \le& e^{-\lambda(t+1)}\|v_0(\theta_{-t-1}\omega)\|^2+c\big(\varepsilon r(\omega)+1\big)\\ \le& 2e^{-\lambda(t+1)}\big(\|u_0(\theta_{-t-1}\omega)\|^2+\|z(\theta_{-t-1}\omega)\|^2\big)+c\big(\varepsilon r(\omega)+1\big).\end{aligned} \tag{4.22}$$

Since $\|u_0(\omega)\|^2$ and $\|z(\omega)\|^2$ are tempered, there exists $T=T(B,\omega)>0$ such that for all $t\ge T$,

$$2e^{-\lambda(t+1)}\big(\|u_0(\theta_{-t-1}\omega)\|^2+\|z(\theta_{-t-1}\omega)\|^2\big)\le 1.$$

It follows from (4.22) that, for all $t\ge T$,

$$\int_t^{t+1} \big\|(-\triangle)^{\frac{\alpha}{2}}v\big(s,\theta_{-t-1}\omega,v_0(\theta_{-t-1}\omega)\big)\big\|^2 ds \le c\big(\varepsilon r(\omega)+1\big). \tag{4.23}$$

Using (4.14) and repeating the above process, we have, for all $t\ge T$,

$$\beta_1\int_t^{t+1} \big\|u\big(s,\theta_{-t-1}\omega,v_0(\theta_{-t-1}\omega)\big)\big\|_p^p ds \le c\big(\varepsilon r(\omega)+1\big). \tag{4.24}$$

By (4.23), one has

$$\begin{aligned}&\int_t^{t+1} \big\|(-\triangle)^{\frac{\alpha}{2}}u\big(s,\theta_{-t-1}\omega,u_0(\theta_{-t-1}\omega)\big)\big\|^2 ds\\ \le& 2\int_t^{t+1} \big\|(-\triangle)^{\frac{\alpha}{2}}v\big(s,\theta_{-t-1}\omega,v_0(\theta_{-t-1}\omega)\big)\big\|^2 ds+2\varepsilon\int_t^{t+1}\big\|(-\triangle)^{\frac{\alpha}{2}}z(\theta_{s-t-1}\omega)\big\|^2 ds\\ \le& c\big(\varepsilon r(\omega)+1\big)+c\varepsilon\int_t^{t+1} e^{\frac{\lambda}{2}(t+1-s)}r(\omega)ds\le c\big(\varepsilon r(\omega)+1\big).\end{aligned} \tag{4.25}$$

We complete the proof. □

Lemma 4.4. *For every* $0<\varepsilon\le 1$ *and* $B\in\mathcal{D}$. *Then for* $\mathbb{P}$*-a.e.* $\omega\in\Omega$, *there exist a random time* $T=T(B)>0$ *and a positive constant* c *(independent of* ε*) such that P-a.s. for all* $t\ge T$ *the solution* $u(t,\omega,u_0(\omega))$ *of problem* (1.1) *satisfies,*

$$\|(-\triangle)^{\frac{\alpha}{2}}u(t+1,\theta_{-t-1}\omega,u_0(\theta_{-t-1}\omega))\|^2\le c\big(\varepsilon r(\omega)+1\big).$$

Proof. Taking the inner product in L^2 of (3.5) with $(-\triangle)^\alpha v$, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|(-\triangle)^{\frac{\alpha}{2}}v\|^2+\lambda\|(-\triangle)^{\frac{\alpha}{2}}v\|^2+\|(-\triangle)^\alpha v\|^2\\=&\int_{\mathbb{R}^n}f(x,u)(-\triangle)^\alpha v dx+\big(g-\varepsilon(-\triangle)^\alpha z(\theta_t\omega),(-\triangle)^\alpha v\big).\end{aligned}\tag{4.26}$$

Now, we estimate the first term on the right-hand side of (4.26). Integrating by parts, using (1.4) and (1.5), then using Young's inequality and Gagliardo-Nirenberg inequality, we infer that

$$\begin{aligned}&\int_{\mathbb{R}^n}f(x,u)(-\triangle)^\alpha v dx\\\le&\int_{\mathbb{R}^n}\left|\frac{\partial f}{\partial x}(x,u)\right|\left|(-\triangle)^{\alpha-\frac12}v\right|dx+\int_{\mathbb{R}^n}\left|\frac{\partial f}{\partial u}(x,u)\right||\nabla u|\left|(-\triangle)^{\alpha-\frac12}v\right|dx\\\le&\frac12\left(\|\gamma_3\|^2+\beta_3^2\big(\|\nabla v\|^2+\varepsilon\|\nabla z(\theta_t\omega)\|^2\big)+\|(-\triangle)^{\alpha-\frac12}v\|^2\right)\\\le&\frac12\left(\|\gamma_3\|^2+\beta_3^2\|\nabla z(\theta_t\omega)\|^2+c\beta_3^2\|(-\triangle)^\alpha v\|^{\frac1\alpha}\|v\|^{\frac{2\alpha-1}{\alpha}}+c\|(-\triangle)^\alpha v\|^{\frac{2\alpha-1}{\alpha}}\|v\|^{\frac1\alpha}\right)\\\le&\frac14\|(-\triangle)^\alpha v\|^2+c\big(\|\gamma_3\|^2+\|u\|^2\big)+c\varepsilon\big(\|z(\theta_t\omega)\|^2+\|\nabla z(\theta_t\omega)\|^2\big)\end{aligned}\tag{4.27}$$

and

$$\big(g-\varepsilon(-\triangle)^\alpha z(\theta_t\omega),(-\triangle)^\alpha v\big)\le\frac14\|(-\triangle)^\alpha v\|^2+c\varepsilon\big(\|\triangle z(\theta_t\omega)\|^2+\|z(\theta_t\omega)\|^2\big)+c.\tag{4.28}$$

Substituting (4.27) and (4.28) into (4.26), we infer that

$$\frac{d}{dt}\|(-\triangle)^{\frac{\alpha}{2}}v\|^2+2\lambda\|(-\triangle)^{\frac{\alpha}{2}}v\|^2+\|(-\triangle)^\alpha v\|^2\le c\|u\|^2+c\varepsilon\eta_2(\theta_t\omega)+c,\tag{4.29}$$

where $\eta_2(\theta_t\omega)=\|\triangle z(\theta_t\omega)\|^2+\|\nabla z(\theta_t\omega)\|^2+\|z(\theta_t\omega)\|^2$. Note that $z(\theta_t\omega)=h\phi(\theta_t\omega)$ and $h\in H^2(\mathbb{R}^n)\cap W^{2,p}(\mathbb{R}^n)$, together with (3.3), we obtain

$$\eta_2(\theta_t\omega)\le c|\phi(\theta_t\omega)|^2\le ce^{\frac{\lambda}{2}|t|}r(\omega),\quad\text{for all } t\in\mathbb{R}.\tag{4.30}$$

Let T be the positive constant in Lemma 4.3, take $t\ge T$ and $s\in(t,t+1)$. Integrating (4.29) over $(s,t+1)$, one has

$$\begin{aligned}&\|(-\triangle)^{\frac{\alpha}{2}}v(t+1,\omega,v_0(\omega))\|^2\\\le&\|(-\triangle)^{\frac{\alpha}{2}}v(s,\omega,v_0(\omega))\|^2+c\int_s^{t+1}\|u(\tau,\omega,u_0(\omega)\|^2d\tau+\int_s^{t+1}(c\varepsilon\eta_2(\theta_\tau\omega)+c)d\tau\\\le&\|(-\triangle)^{\frac{\alpha}{2}}v(s,\omega,v_0(\omega))\|^2+c\int_t^{t+1}\|u(\tau,\omega,u_0(\omega)\|^2d\tau+\int_t^{t+1}(c\varepsilon\eta_2(\theta_\tau\omega)+c)d\tau.\end{aligned}$$

Now integrating the above inequality with respect to s over $(t,t+1)$, we infer that

$$\|(-\triangle)^{\frac{\alpha}{2}}v(t+1,\omega,v_0(\omega))\|^2$$

$$\leq \int_t^{t+1} \|(-\triangle)^{\frac{\alpha}{2}} v(s,\omega,v_0(\omega))\|^2 ds + \int_t^{t+1} (c\varepsilon\eta_2(\theta_\tau\omega)+c)d\tau$$
$$+c\int_t^{t+1} \|u(\tau,\omega,u_0(\omega))\|^2 d\tau.$$

Replacing ω by $\theta_{-t-1}\omega$, we deduce that

$$\begin{aligned}
&\|(-\triangle)^{\frac{\alpha}{2}} v(t+1,\theta_{-t-1}\omega,v_0(\theta_{-t-1}\omega))\|^2\\
\leq &\int_t^{t+1} \|(-\triangle)^{\frac{\alpha}{2}} v(s,\theta_{-t-1}\omega,v_0(\theta_{-t-1}\omega))\|^2 ds\\
&+c\int_t^{t+1} \|u(\tau,\theta_{-t-1}\omega,u_0(\theta_{-t-1}\omega)\|^2 d\tau + \int_t^{t+1} (c\varepsilon\eta_2(\theta_{\tau-t-1}\omega)+c)d\tau.
\end{aligned}$$

It follows from (4.2) and (4.20) that, for all $t \geq T$,

$$\begin{aligned}
&\|(-\triangle)^{\frac{\alpha}{2}} v(t+1,\theta_{-t-1}\omega,v_0(\theta_{-t-1}\omega))\|^2\\
\leq &c(\varepsilon r(\omega)+1) + c\varepsilon\int_{-1}^0 \eta_2(\theta_\tau\omega)d\tau\\
\leq &c(\varepsilon r(\omega)+1) + \int_{-1}^0 c\varepsilon e^{-\frac{\lambda}{2}\tau} r(\omega)d\tau \leq c(\varepsilon r(\omega)+1).
\end{aligned} \tag{4.31}$$

By (3.3) and (4.31), we obtain that, for all $t \geq T$,

$$\begin{aligned}
&\|(-\triangle)^{\frac{\alpha}{2}} u(t+1,\theta_{-t-1}\omega,u_0(\theta_{-t-1}\omega))\|^2\\
=&\|(-\triangle)^{\frac{\alpha}{2}} v(t+1,\theta_{-t-1}\omega,v_0(\theta_{-t-1}\omega)) - \varepsilon(-\triangle)^{\frac{\alpha}{2}} z(\omega)\|^2\\
\leq &2\big(\|(-\triangle)^{\frac{\alpha}{2}} v(t+1,\theta_{-t-1}\omega,v_0(\theta_{-t-1}\omega))\|^2 + \varepsilon\|(-\triangle)^{\frac{\alpha}{2}} z(\omega)\|^2\big)\\
\leq &c(\varepsilon r(\omega)+1),
\end{aligned}$$

which completes the proof. □

Following from Lemma 4.4, we obtain

Lemma 4.5. *For every $0 < \varepsilon \leq 1$ and $B \in \mathcal{D}$. Then for $\mathbb{P}$-a.e. $\omega \in \Omega$, there exist a random time $T = T(B) > 0$ and a positive constant c (independent of ε) such that P-a.s. for all $t \geq T+1$,*

$$\int_t^{t+1} \big\|(-\triangle)^{\alpha} u\big(s,\theta_{-t-1}\omega,u_0(\theta_{-t-1}\omega)\big)\big\|^2 ds \leq c\big(\varepsilon r(\omega)+1\big)$$

and

$$\int_t^{t+1} \big\|(-\triangle)^{\alpha} v\big(s,\theta_{-t-1}\omega,v_0(\theta_{-t-1}\omega)\big)\big\|^2 ds \leq c\big(\varepsilon r(\omega)+1\big).$$

Proof. From (4.29), one has

$$\frac{d}{dt}\left(e^{\lambda t}\|(-\triangle)^{\frac{\alpha}{2}} v\|^2\right) + e^{\lambda t}\|(-\triangle)^{\alpha} v\|^2 \leq ce^{\lambda t}\|u\|^2 + c\varepsilon e^{\lambda t}\eta_2(\theta_t\omega) + ce^{\lambda t}, \tag{4.32}$$

Integrating (4.32) over (s,t) $(t > s \geq T+1)$, then multiplying $e^{-\lambda t}$, we obtain

$$\|(-\triangle)^{\frac{\alpha}{2}} v(t,\omega,v_0(\omega))\|^2$$

$$
\begin{aligned}
&\leq e^{\lambda(s-t)}\|(-\triangle)^{\frac{\alpha}{2}}v(s,\omega,v_0(\omega))\|^2+c\int_s^t e^{\lambda(\tau-t)}\|u(\tau,\omega,u_0(\omega)\|^2d\tau\\
&\quad+c\varepsilon\int_s^t e^{\lambda(\tau-t)}\eta_2(\theta_\tau\omega)d\tau+c\int_s^t e^{\lambda(\tau-t)}d\tau.
\end{aligned}
$$

Replacing t by T_2 and ω by $\theta_{-s}\omega$, we deduce that

$$
\begin{aligned}
&\|(-\triangle)^{\frac{\alpha}{2}}v(T_2,\theta_{-s}\omega,v_0(\theta_{-s}\omega))\|^2\\
\leq{}& e^{\lambda(s-T_2)}\|(-\triangle)^{\frac{\alpha}{2}}v(s,\theta_{-s}\omega,v_0(\theta_{-s}\omega))\|^2\\
&+c\int_s^{T_2} e^{\lambda(\tau-T_2)}\|u(\tau,\theta_{-s}\omega,u_0(\theta_{-s}\omega)\|^2d\tau\\
&+c\varepsilon\int_s^{T_2} e^{\lambda(\tau-T_2)}\eta_2(\theta_{\tau-s}\omega)d\tau+c\int_s^{T_2} e^{\lambda(\tau-T_2)}d\tau.
\end{aligned}
\tag{4.33}
$$

From (4.31), we have for $s\geq T+1$,

$$
e^{\lambda(s-T_2)}\|(-\triangle)^{\frac{\alpha}{2}}v(s,\theta_{-s}\omega,v_0(\theta_{-s}\omega))\|^2\leq c(\varepsilon r(\omega)+1).
\tag{4.34}
$$

Now we estimate the second term on the right-hand side of (4.33). From (4.9), we deduce that

$$
\begin{aligned}
&\|v(\tau,\theta_{-s}\omega,v_0(\theta_{-s}\omega)\|^2\\
\leq{}& e^{-\lambda\tau}\|v_0(\theta_{-s}\omega)\|^2+c\varepsilon\int_0^\tau e^{\lambda(m-\tau)}\eta_1(\theta_{m-s}\omega)dm+\frac{c_7}{\lambda}\\
\leq{}& e^{-\lambda\tau}\left(\left\|u_0(\theta_{-s}\omega)\right\|^2+\left\|z(\theta_{-s}\omega)\right\|^2\right)+c\varepsilon\int_0^\tau e^{\lambda(m-\tau)}\eta_1(\theta_{m-s}\omega)dm+\frac{c_7}{\lambda}.
\end{aligned}
\tag{4.35}
$$

Applying (4.10), $u_0(\theta_{-t}\omega)\in B(\theta_{-t}\omega)$ and $\tau\in(s,T_2)$, we infer that there exists $T=T(B,\omega)>0$ such that for all $s\geq T$,

$$
\|v(\tau,\theta_{-s}\omega,v_0(\theta_{-s}\omega)\|^2\leq c\left(\varepsilon r(\omega)e^{\lambda(\tau-\frac{1}{2}s)}+1\right).
$$

Then we obtain

$$
\|u(\tau,\theta_{-s}\omega,v_0(\theta_{-s}\omega)\|^2\leq c\left(\varepsilon r(\omega)e^{\lambda(\tau-\frac{1}{2}s)}+\varepsilon r(\omega)+1\right).
$$

Then, the second term on the right-hand side of (4.33) can be bounded by

$$
\begin{aligned}
&\int_s^{T_2} e^{\lambda(\tau-T_2)}\|u(\tau,\theta_{-s}\omega,v_0(\theta_{-s}\omega)\|^2d\tau\\
\leq{}& c\int_s^{T_2} e^{\lambda(\tau-T_2)}\left(\varepsilon r(\omega)e^{\lambda(\tau-\frac{1}{2}s)}+\varepsilon r(\omega)+1\right)d\tau\\
\leq{}& c(\varepsilon r(\omega)+1)\left(1-e^{\lambda(s-T_2)}\right)+c\varepsilon r(\omega)\left(e^{\lambda T_2-\frac{1}{2}\lambda s}-e^{\frac{3}{2}\lambda s-\lambda T_2}\right)\leq c(\varepsilon r(\omega)+1).
\end{aligned}
\tag{4.36}
$$

Applying (4.30), the third term on the right-hand side of (4.33) can be bounded by

$$
\int_s^{T_2} e^{\lambda(\tau-T_2)}\eta_2(\theta_{\tau-s}\omega)d\tau\leq\int_s^{T_2} e^{\lambda(\tau-T_2)}(ce^{\frac{\lambda}{2}|\tau-s|}+c)d\tau\leq ce^{\lambda T_2-\frac{\lambda}{2}s}+c.
\tag{4.37}
$$

Substituting (4.34), (4.36) and (4.37) into (4.33), we obtain

$$\|(-\triangle)^{\frac{\alpha}{2}}v(T_2,\theta_{-s}\omega,v_0(\theta_{-s}\omega))\|^2\leq c\left(\varepsilon r(\omega)+e^{\lambda T_2-\frac{\lambda}{2}s}+1\right). \tag{4.38}$$

Integrating (4.32) over (T_2,t), one has

$$\begin{aligned}&\|(-\triangle)^{\frac{\alpha}{2}}v(t,\omega,v_0(\omega))\|^2+\int_{T_2}^{t}e^{\lambda(s-t)}\|(-\triangle)^{\alpha}v(s,\omega,v_0(\omega))\|^2ds\\ \leq& e^{\lambda(T_2-t)}\|(-\triangle)^{\frac{\alpha}{2}}v(T_2,\omega,v_0(\omega))\|^2+c\int_{T_2}^{t}e^{\lambda(s-t)}\|u(s,\omega,v_0(\omega))\|^2ds \qquad (4.39)\\ &+c\varepsilon\int_{T_2}^{t}e^{\lambda(s-t)}\eta_2(\theta_s\omega)ds+c\int_{T_2}^{t}e^{\lambda(s-t)}ds.\end{aligned}$$

Dropping the first term on the left-side hand of (4.39), and replacing ω by $\theta_{-t}\omega$, one has

$$\begin{aligned}&\int_{T_2}^{t}e^{\lambda(s-t)}\|(-\triangle)^{\alpha}v(s,\theta_{-t}\omega,v_0(\theta_{-t}\omega))\|^2ds\\ \leq& e^{\lambda(T_2-t)}\|(-\triangle)^{\frac{\alpha}{2}}v(T_2,\theta_{-t}\omega,v_0(\theta_{-t}\omega))\|^2\\ &+c\int_{T_2}^{t}e^{\lambda(s-t)}\|u(s,\theta_{-t}\omega,v_0(\theta_{-t}\omega))\|^2ds+c\int_{T_2}^{t}e^{\lambda(s-t)}(\varepsilon\eta_2(\theta_{s-t}\omega)+1)ds\\ \leq& c\left(\varepsilon r(\omega)+e^{2\lambda T_2}+1\right)+c\big(\varepsilon r(\omega)+1\big)+c\left(1-e^{\frac{\lambda}{2}(T_2-t)}\right)\leq c(\varepsilon r(\omega)+1).\end{aligned} \tag{4.40}$$

Replacing t by $t+1$ and then replacing T_2 by t in (4.40), we have

$$\int_{t}^{t+1}e^{\lambda(s-t-1)}\|(-\triangle)^{\alpha}v(s,\theta_{-t-1}\omega,v_0(\theta_{-t-1}\omega))\|^2ds\leq c(\varepsilon r(\omega)+1). \tag{4.41}$$

For all $s\in[t,t+1]$, note that $e^{\lambda(s-t-1)}\geq e^{-\lambda}$. One can rewrite (4.41) as

$$\int_{t}^{t+1}\|(-\triangle)^{\alpha}v(s,\theta_{-t-1}\omega,v_0(\theta_{-t-1}\omega))\|^2ds\leq ce^{\lambda}(\varepsilon r(\omega)+1)\leq c(r(\varepsilon\omega)+1).$$

Then we infer

$$\begin{aligned}&\int_{t}^{t+1}\left\|(-\triangle)^{\alpha}u\big(s,\theta_{-t-1}\omega,u_0(\theta_{-t-1}\omega)\big)\right\|^2ds\\ \leq&2\int_{t}^{t+1}\left\|(-\triangle)^{\alpha}v\big(s,\theta_{-t-1}\omega,v_0(\theta_{-t-1}\omega)\big)\right\|^2ds+2\varepsilon\int_{t}^{t+1}\left\|(-\triangle)^{\alpha}z(\theta_{s-t-1}\omega)\right\|^2ds\\ \leq&c\big(\varepsilon r(\omega)+1\big)+c\varepsilon\int_{t}^{t+1}e^{\frac{\lambda}{2}(t+1-s)}r(\omega)ds\\ \leq&c\big(\varepsilon r(\omega)+1\big).\end{aligned}$$

□

We will also need the following result.

Lemma 4.6. *For every $0 < \varepsilon \le 1$ and $B \in \mathcal{D}$. Then for $\mathbb{P}$-a.e. $\omega \in \Omega$, there exist a random time $T = T(B) > 0$ and a positive constant c (independent of ε) such that P-a.s. for all $t \ge T + 1$,*

$$\|\nabla u\big(s, \theta_{-t-1}\omega, v_0(\theta_{-t-1}\omega)\big)\|^2 \le c\big(\varepsilon r(\omega) + 1\big).$$

Proof. Taking the inner product in L^2 of (3.5) with $-\triangle v$, one has

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\nabla v\|^2 + \lambda\|\nabla v\|^2 + \|(-\triangle)^{\frac{\alpha+1}{2}} v\|^2\\ =&-\int_{\mathbb{R}^n} f(x,u)\triangle v dx + \big(g - \varepsilon(-\triangle)^\alpha z(\theta_t\omega), -\triangle v\big).\end{aligned} \tag{4.42}$$

Now, we estimate the first term on the right-hand side of (4.42). Integrating by parts and using (1.3)–(1.5), then by Young's inequality, we obtain

$$\begin{aligned}&-\int_{\mathbb{R}^n} f(x,u)\triangle v dx\\ =&-\int_{\mathbb{R}^n} f(x,u)\triangle u dx + \varepsilon\int_{\mathbb{R}^n} f(x,u)\triangle z(\theta_t\omega) dx\\ =&\int_{\mathbb{R}^n}\frac{\partial f}{\partial x}(x,u)\nabla u dx + \int_{\mathbb{R}^n}\frac{\partial f}{\partial u}|\nabla u|^2 dx + \varepsilon\int_{\mathbb{R}^n} f(x,u)\triangle z(\theta_t\omega) dx\\ \le&\|\gamma_3\|\|\nabla u\| + \beta_3\|\nabla u\|^2 + \varepsilon\int_{\mathbb{R}^n}\big(\beta_2|u|^{p-1} + |\gamma_2(x)|\big)\,\big|\triangle z(\theta_t\omega)\big| dx\\ \le& c\left(\|\nabla u\|^2 + \|u\|_p^p\right) + c\varepsilon\big(\|\triangle z(\theta_t\omega)\|^2 + \|\triangle z(\theta_t\omega)\|_p^p\big) + c.\end{aligned} \tag{4.43}$$

For the second term on the right-hand side of (4.42), integrating by parts, then using Gagliardo-Nirenberg's inequality and Young's inequality, one has

$$\begin{aligned}&\big(g - \varepsilon(-\triangle)^\alpha z(\theta_t\omega), -\triangle v\big)\\ =&\,\varepsilon\int_{\mathbb{R}^n}(-\triangle)^{\frac{\alpha+1}{2}} z(\theta_t\omega)(-\triangle)^{\frac{\alpha+1}{2}} v dx + \int_{\mathbb{R}^n}(-\triangle)^{\frac{1-\alpha}{2}} g(-\triangle)^{\frac{\alpha+1}{2}} v dx\\ \le&\,\frac{1}{2}\|(-\triangle)^{\frac{\alpha+1}{2}} v\|^2 + c\varepsilon\big(\|\triangle z(\theta_t\omega)\|^2 + \|z(\theta_t\omega)\|^2\big) + c.\end{aligned} \tag{4.44}$$

Substituting (4.43) and (4.44) into (4.42) gives

$$\frac{d}{dt}\|\nabla v\|^2 + 2\lambda\|\nabla v\|^2 + \|(-\triangle)^{\frac{\alpha+1}{2}} v\|^2 \le c\left(\|\nabla u\|^2 + \|u\|_p^p\right) + \varepsilon\eta_3(\theta_t\omega) + c, \tag{4.45}$$

where $\eta_3(\theta_t\omega) = c\big(\|\triangle z(\theta_t\omega)\|^2 + \|\triangle z(\theta_t\omega)\|_p^p + \|z(\theta_t\omega)\|^2\big)$. It implies that

$$\frac{d}{dt}\|\nabla v\|^2 \le c\left(\|\nabla u\|^2 + \|u\|_p^p\right) + \varepsilon\eta_3(\theta_t\omega) + c. \tag{4.46}$$

Let T be the positive constant in Lemma 4.4, take $t \ge T$ and $s \in (t, t+1)$. Integrating (4.46) over $(s, t+1)$, one has

$$\begin{aligned}&\|\nabla v(t+1, \omega, v_0(\omega))\|^2\\ \le&\|\nabla v(s, \omega, v_0(\omega))\|^2 + \int_t^{t+1}\left(\|\nabla u(\tau, \omega, v_0(\omega)\|^2 + \|u(\tau, \omega, v_0(\omega)\|_p^p\right) d\tau\end{aligned}$$

$$+\int_t^{t+1}(\varepsilon\eta_3(\theta_\tau\omega)+c)\,d\tau.$$

Now integrating the above with respect to s over $(t,t+1)$, we infer that

$$\begin{aligned}\|\nabla v(t+1,\omega,v_0(\omega))\|^2 \le \int_t^{t+1}\|\nabla v(s,\omega,v_0(\omega))\|^2 ds+\int_t^{t+1}(\varepsilon\eta_3(\theta_\tau\omega)+c)\,d\tau\\ +c\int_t^{t+1}\big(\|\nabla u(\tau,\omega,v_0(\omega)\|^2+\|u(\tau,\omega,v_0(\omega)\|_p^p\big)\,d\tau.\end{aligned}$$

Replacing ω by $\theta_{-t-1}\omega$, and applying Gagliardo-Nirenberg inequality and Young's inequality, we obtain, for all $t\ge T+1$,

$$\begin{aligned}&\|\nabla v(t+1,\theta_{-t-1}\omega,v_0(\theta_{-t-1}\omega))\|^2\\ \le&\int_t^{t+1}\|(-\triangle)^\alpha v(s,\theta_{-t-1}\omega,v_0(\theta_{-t-1}\omega))\|^2ds+\int_t^{t+1}(\varepsilon\eta_3(\theta_{\tau-t-1}\omega)+c)\,d\tau\\ &+c\int_t^{t+1}\big(\|(-\triangle)^\alpha u(\tau,\theta_{-t-1}\omega,v_0(\theta_{-t-1}\omega)\|^2+\|u(\tau,\theta_{-t-1}\omega,v_0(\theta_{-t-1}\omega)\|_p^p\big)\,d\tau\\ &+c\int_t^{t+1}\big(\|u(\tau,\theta_{-t-1}\omega,v_0(\theta_{-t-1}\omega)\|^2+\|v(\tau,\theta_{-t-1}\omega,v_0(\theta_{-t-1}\omega)\|^2\big)\,d\tau\\ \le& c\big(\varepsilon r(\omega)+1\big)+\int_{-1}^0(\varepsilon\eta_3(\theta_{\tau-t-1}\omega)+c)\,d\tau\\ \le& c\big(\varepsilon r(\omega)+1\big).\end{aligned}\tag{4.47}$$

By (4.47) and (3.3), one has, for all $t\ge T+1$,

$$\begin{aligned}&\|\nabla u(t+1,\theta_{-t-1}\omega,u_0(\theta_{-t-1}\omega))\|^2\\ \le&2\big(\|\nabla v(t+1,\theta_{-t-1}\omega,v_0(\theta_{-t-1}\omega))\|^2+\varepsilon\|\nabla z(\omega)\|^2\big)\\ \le& c\big(\varepsilon r(\omega)+1\big).\end{aligned}$$

□

Next, we deduce uniform estimates of solutions for large space and time variables. Particularly, we show how these estimates depend on the small parameter ε.

Lemma 4.7. *For every $0<\varepsilon\le 1$ and $B\in\mathcal{D}$, suppose that (1.2)–(1.5) hold. Let $B=\{B(\omega)\}\in\mathcal{D}$ and $v_0(\omega)\in B(\omega)$. Then, for every $\eta>0$ and P-a.s. $\omega\in\Omega$, there exist $T^*=T_B^*(\omega,\eta)>0$ and $R^*=R^*(\omega,\eta)>0$ such that for all $t\ge T_B^*(\omega)$, the solution $u(t,\omega,u_0(\omega))$ of problem (1.1) satisfies*

$$\int_{|x|\ge R^*}|u(t,\theta_{-t}\omega,v_0(\theta_{-t}\omega))|^2dx\le\eta.\tag{4.48}$$

Proof. Take a smooth χ such that $0\le\chi(s)\le 1$ for all $s\ge 0$ and

$$\chi(s)=\begin{cases}0, & \text{if } 0\le s\le 1,\\ 1, & \text{if } s\ge 2.\end{cases}\tag{4.49}$$

Then there exists a positive constant c such that $|\chi'(s)| \le c$ for all $s \ge 0$. Taking the inner product in L^2 of (3.5) with $\chi(\frac{|x|^2}{k^2})v$, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^n}\chi\left(\frac{|x|^2}{k^2}\right)|v|^2dx+\int_{\mathbb{R}^n}(-\triangle)^{\alpha}v\chi\left(\frac{|x|^2}{k^2}\right)vdx+\lambda\int_{\mathbb{R}^n}\chi\left(\frac{|x|^2}{k^2}\right)|v|^2dx\\&=\int_{\mathbb{R}^n}f(x,u)\chi\left(\frac{|x|^2}{k^2}\right)vdx+\int_{\mathbb{R}^n}(g-\varepsilon(-\triangle)^{\alpha}z(\theta_t\omega))\chi\left(\frac{|x|^2}{k^2}\right)vdx.\end{aligned}\tag{4.50}$$

Now, we estimate every term in (4.50) as follows. For the second term on the left-side hand, integrating by parts and applying Hölder's inequality, Gagliardo-Nirenberg's inequality and Young's inequality, we have

$$\begin{aligned}&-\int_{\mathbb{R}^n}(-\triangle)^{\alpha}v\chi\left(\frac{|x|^2}{k^2}\right)vdx\\&\le\int_{\mathbb{R}^n}|(-\triangle)^{\alpha-\frac{1}{2}}v|\left(\chi\left(\frac{|x|^2}{k^2}\right)|\nabla v|+\chi'\left(\frac{x^2}{k^2}\right)\frac{2|x|}{k^2}|v|\right)dx\\&\le\left\|(-\triangle)^{\alpha-\frac{1}{2}}v\right\|\|\nabla v\|+\int_{k\le|x|\le\sqrt{2}k}\left|(-\triangle)^{\alpha-\frac{1}{2}}v\right|\left|\chi'\left(\frac{|x|^2}{k^2}\right)\right|\frac{2|x|}{k^2}|v|dx\\&\le\left\|(-\triangle)^{\alpha-\frac{1}{2}}v\right\|\|\nabla v\|+\frac{2\sqrt{2}}{k}\int_{k\le|x|\le\sqrt{2}k}\left|(-\triangle)^{\alpha-\frac{1}{2}}v\right|\left|\chi'\left(\frac{|x|^2}{k^2}\right)\right||v|dx\\&\le\left\|(-\triangle)^{\alpha-\frac{1}{2}}v\right\|\|\nabla v\|+\frac{c}{k}\int_{k\le|x|\le\sqrt{2}k}\left|(-\triangle)^{\alpha-\frac{1}{2}}v\right||v|dx\\&\le c\left(\|v\|^2+\|\nabla v\|^2\right)+\frac{c}{k}\left(\|\nabla v\|^2+\|v\|^2\right).\end{aligned}\tag{4.51}$$

For the first term on the right-hand side of (4.50), applying (1.2) and (1.3), one has

$$\begin{aligned}&\int_{\mathbb{R}^n}f(x,u)\chi\left(\frac{|x|^2}{k^2}\right)vdx\\&=\int_{\mathbb{R}^n}f(x,u)\chi\left(\frac{|x|^2}{k^2}\right)udx-\varepsilon\int_{\mathbb{R}^n}f(x,u)\chi\left(\frac{|x|^2}{k^2}\right)z(\theta_t\omega)dx\\&\le-\beta_1\int_{\mathbb{R}^n}|u|^p\chi\left(\frac{|x|^2}{k^2}\right)dx+\varepsilon\beta_2\int_{\mathbb{R}^n}|u|^{p-1}\chi\left(\frac{|x|^2}{k^2}\right)|z(\theta_t\omega)|dx\\&\quad+\int_{\mathbb{R}^n}\gamma_1(x)\chi\left(\frac{|x|^2}{k^2}\right)dx+\varepsilon\int_{\mathbb{R}^n}\gamma_2(x)\chi\left(\frac{|x|^2}{k^2}\right)|z(\theta_t\omega)|dx\\&\le-\frac{1}{2}\beta_1\int_{\mathbb{R}^n}|u|^p\chi\left(\frac{|x|^2}{k^2}\right)dx+\int_{\mathbb{R}^n}\gamma_1(x)\chi\left(\frac{|x|^2}{k^2}\right)dx\\&\quad+\frac{1}{2}\int_{\mathbb{R}^n}\gamma_2^2(x)\chi\left(\frac{|x|^2}{k^2}\right)dx+c\varepsilon\int_{\mathbb{R}^n}\left(|z(\theta_t\omega)|^2+|z(\theta_t\omega)|^p\right)\chi\left(\frac{|x|^2}{k^2}\right)dx.\end{aligned}\tag{4.52}$$

For the second term on the right-side hand of (4.50), applying Young's inequality, we have

$$\begin{aligned}&\int_{\mathbb{R}^n}(g-\varepsilon(-\triangle)^{\alpha}z(\theta_t\omega))\chi\left(\frac{|x|^2}{k^2}\right)vdx\\&\le\frac{1}{2}\lambda\int_{\mathbb{R}^n}\chi\left(\frac{|x|^2}{k^2}\right)|v|^2dx+\frac{1}{\lambda}\int_{\mathbb{R}^n}(g^2+\varepsilon|(-\triangle)^{\alpha}z(\theta_t\omega)|^2)\chi\left(\frac{|x|^2}{k^2}\right)dx.\end{aligned}\tag{4.53}$$

By (4.51)-(4.53), one can rewrite (4.50) as

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^n}\chi\left(\frac{|x|^2}{k^2}\right)|v|^2dx+\lambda\int_{\mathbb{R}^n}\chi\left(\frac{|x|^2}{k^2}\right)|v|^2dx+\beta_1\int_{\mathbb{R}^n}|u|^p\chi\left(\frac{|x|^2}{k^2}\right)dx\\
\le& c\left(\|v\|^2+\|\nabla v\|^2\right)+\frac{c}{k}\left(\|\nabla v\|^2+\|v\|^2\right)\\
&+\int_{\mathbb{R}^n}\left(2|\gamma_1(x)|+|\gamma_2(x)|^2+g^2\right)\chi\left(\frac{|x|^2}{k^2}\right)dx\\
&+c\varepsilon\int_{\mathbb{R}^n}\left(\left|(-\triangle)^\alpha z(\theta_t\omega)\right|^2+|z(\theta_t\omega)|^2+|z(\theta_t\omega)|^p\right)\chi\left(\frac{|x|^2}{k^2}\right)dx.
\end{aligned}\tag{4.54}$$

By Lemmas 4.1 and 4.6, there exists $T_1=T_1(B,\omega)>0$ such that for all $t\ge T_1$,

$$\|v\big(t,\theta_{-t}\omega,v_0(\theta_{-t}\omega)\big)\|^2_{H^1(\mathbb{R}^n)}\le c\big(r(\omega)+1\big).\tag{4.55}$$

Integrating (4.54) over (T_1,t), then replacing ω by $\theta_{-t}\omega$, we infer that, for all $t\ge T_1$,

$$\begin{aligned}
&\int_{\mathbb{R}^n}\chi\left(\frac{|x|^2}{k^2}\right)\left|v\big(t,\theta_{-t}\omega,v_0(\theta_{-t}\omega)\big)\right|^2dx\\
\le& e^{\lambda(T_1-t)}\int_{\mathbb{R}^n}\chi\left(\frac{|x|^2}{k^2}\right)\left|v\big(T_1,\theta_{-t}\omega,v_0(\theta_{-t}\omega)\big)\right|^2dx\\
&+\int_{T_1}^t e^{\lambda(s-t)}\int_{\mathbb{R}^n}\left(2|\gamma_1(x)|+|\gamma_2(x)|^2+g^2\right)\chi\left(\frac{|x|^2}{k^2}\right)dxds\\
&+c\varepsilon\int_{T_1}^t e^{\lambda(s-t)}\int_{\mathbb{R}^n}\left(\left|(-\triangle)^\alpha z(\theta_{s-t}\omega)\right|^2+|z(\theta_{s-t}\omega)|^2\right.\\
&\left.+|z(\theta_{s-t}\omega)|^p\right)\chi\left(\frac{|x|^2}{k^2}\right)dxds\\
&+c\int_{T_1}^t e^{\lambda(s-t)}\left(\left\|v\big(s,\theta_{-t}\omega,v_0(\theta_{-t}\omega)\big)\right\|^2+\left\|\nabla v\big(s,\theta_{-t}\omega,v_0(\theta_{-t}\omega)\big)\right\|^2\right)ds\\
&+\frac{c}{k}\int_{T_1}^t e^{\lambda(s-t)}\left\|\nabla v\big(s,\theta_{-t}\omega,v_0(\theta_{-t}\omega)\big)\right\|^2ds\\
&+\frac{c}{k}\int_{T_1}^t e^{\lambda(s-t)}\left\|v\big(s,\theta_{-t}\omega,v_0(\theta_{-t}\omega)\big)\right\|^2ds.
\end{aligned}\tag{4.56}$$

In what follows, we estimate each term on the right-hand side of (4.56). For the first term, replacing t by T_1 and replacing ω by $\theta_{-t}\omega$ in (4.9), we infer that

$$\begin{aligned}
&e^{\lambda(T_1-t)}\int_{\mathbb{R}^n}\chi\left(\frac{|x|^2}{k^2}\right)\left|v\big(T_1,\theta_{-t}\omega,v_0(\theta_{-t}\omega)\big)\right|^2dx\\
\le& e^{\lambda(T_1-t)}\left(e^{-\lambda T_1}\|v_0(\theta_{-t}\omega)\|^2dx+\varepsilon c_4\int_0^{T_1}e^{\lambda(s-T_1)}\eta_1(\theta_{s-t}\omega)ds+\frac{c_7}{\lambda}\right)\\
\le& e^{-\lambda t}\|v_0(\theta_{-t}\omega)\|^2dx+ce^{\lambda(T_1-t)}+\varepsilon c_4c_6r(\omega)\int_{-t}^{T_1-t}e^{\frac{1}{2}\lambda\tau}d\tau\\
\le& e^{-\lambda t}\|v_0(\theta_{-t}\omega)\|^2dx+ce^{\lambda(T_1-t)}+\frac{2}{\lambda}\varepsilon c_4c_6r(\omega)e^{\frac{1}{2}\lambda(T_1-t)}.
\end{aligned}\tag{4.57}$$

Therefore, given $\eta > 0$, there exists $T_2 = T_2(B, \omega, \eta) > T_1$ such that for all $t \geq T_2$,

$$e^{\lambda(T_1 - t)} \int_{\mathbb{R}^n} \chi\left(\frac{|x|^2}{k^2}\right) \left|v\big(T_1, \theta_{-t}\omega, v_0(\theta_{-t}\omega)\big)\right|^2 dx \leq \frac{\eta}{6}. \tag{4.58}$$

For the second term, note that $g \in H^\alpha(\mathbb{R}^n)$, $\gamma_1(x) \in L^1(\mathbb{R}^n)$ and $\gamma_2(x) \in L^2(\mathbb{R}^n)$, there exists $R_1 = R_1(\varepsilon)$ such that for all $k \geq R_1$, we have

$$\int_{|x| \geq k} \left(2|\gamma_1(x)| + |\gamma_2(x)|^2 + g^2\right) \chi\left(\frac{|x|^2}{k^2}\right) dx \leq c\eta. \tag{4.59}$$

For the second term, one has

$$\begin{aligned} &\int_{T_1}^t e^{\lambda(s-t)} \int_{\mathbb{R}^n} (2|\gamma_1(x)| + |\gamma_2(x)|^2) \chi\left(\frac{|x|^2}{k^2}\right) dx ds \\ \leq &\int_{T_1}^t e^{\lambda(s-t)} \int_{|x| \geq k} \left(2|\gamma_1(x)| + |\gamma_2(x)|^2\right) dx ds \leq \frac{\eta}{6}. \end{aligned} \tag{4.60}$$

Note that $z(\theta_t\omega) = h\phi(\theta_t\omega)$ and $h \in H^2(\mathbb{R}^n) \cap W^{2,p}(\mathbb{R}^n)$. There exists $R_2 = R_2(\omega, \eta)$ such that for all $k \geq R_2$,

$$\int_{|x| \geq k} \left(\left|\phi\right|^2 + \left|\phi\right|^p\right) dx \leq \frac{1}{6c} \min\left\{\frac{\lambda\eta}{4r(\omega)}, \frac{\eta}{2r(\omega)}\right\}, \tag{4.61}$$

where $r(\omega)$ is the tempered function in (3.2). By (4.61) and (3.2)–(3.3), for the third term, we have

$$\begin{aligned} &\int_{T_1}^t e^{\lambda(s-t)} \int_{\mathbb{R}^n} \Big(\left|(-\triangle)^\alpha z(\theta_{s-t}\omega)\right|^2 + |z(\theta_{s-t}\omega)|^2 + |z(\theta_{s-t}\omega)|^p \Big) \chi\left(\frac{|x|^2}{k^2}\right) dx ds \\ \leq &\int_{T_1}^t e^{\lambda(s-t)} \int_{|x| \geq k} \Big(\left|(-\triangle)^\alpha z(\theta_{s-t}\omega)\right|^2 + |z(\theta_{s-t}\omega)|^2 + |z(\theta_{s-t}\omega)|^p \Big) dx ds \\ \leq &c \int_{T_1}^t e^{\lambda(s-t)} \int_{|x| \geq k} \Big(\left|\phi(\theta_{s-t}\omega)\right|^2 + |\phi(\theta_{s-t}\omega)|^p \Big) dx ds \\ \leq &\frac{c\lambda\eta}{12r(\omega)} \int_{T_1}^t e^{\lambda(s-t)} \Big(\left|\phi(\theta_{s-t}\omega)\right|^2 + |\phi(\theta_{s-t}\omega)|^p \Big) ds \\ \leq &\frac{c\lambda\eta}{12r(\omega)} \int_{T_1}^t e^{\lambda(s-t)} r(\theta_{s-t}\omega) ds \\ \leq &\frac{c\lambda\eta}{12r(\omega)} \int_{T_1 - t}^0 e^{\lambda s} r(\theta_s\omega) ds \\ \leq &\frac{c\lambda\eta}{12r(\omega)} \int_{T_1 - t}^0 e^{\frac{1}{2}\lambda s} r(\omega) ds \\ \leq &\frac{\eta}{6}. \end{aligned} \tag{4.62}$$

For the fourth term, by Lemmas 4.1 and 4.6, there exists $T_3 = T_3(B, \omega, \eta) > T_1$ such that

$$c \int_{T_1}^t e^{\lambda(s-t)} \left(\left\|v\big(s, \theta_{-t}\omega, v_0(\theta_{-t}\omega)\big)\right\|^2 + \left\|\nabla v\big(s, \theta_{-t}\omega, v_0(\theta_{-t}\omega)\big)\right\|^2\right) ds \leq \frac{\eta}{6} \tag{4.63}$$

for all $t \ge T_3$.

By Lemma 4.6, there exists $T_4 = T_4(B, \omega) > T_1$ such that the fifth term on the right-side hand of (4.56) satisfies

$$\frac{c}{k}\int_{T_1}^{t} e^{\lambda(s-t)} \left\|\nabla v\big(s, \theta_{-t}\omega, v_0(\theta_{-t}\omega)\big)\right\|^2 ds \le \frac{c}{k}\big(\varepsilon r(\omega) + 1\big).$$

Then, there is $R_2 = R_2(\omega, \eta)$ such that for all $t \ge T_4$ and $k \ge R_2$,

$$\frac{c}{k}\int_{T_1}^{t} e^{\lambda(s-t)} \left\|\nabla v\big(s, \theta_{-t}\omega, v_0(\theta_{-t}\omega)\big)\right\|^2 ds \le \frac{\eta}{6}. \tag{4.64}$$

For the last term, first replacing t by τ and then replacing ω by $\theta_{-t}\omega$ in (4.9), we deduce that

$$\begin{aligned}
&\frac{c}{k}\int_{T_1}^{t} e^{\lambda(\tau-t)} \left\|v\big(\tau, \theta_{-t}\omega, v_0(\theta_{-t}\omega)\big)\right\|^2 ds\\
\le&\frac{c}{k}\int_{T_1}^{t} e^{-\lambda t}\left\|v_0(\theta_{-t}\omega)\right\|^2 d\tau + \frac{c\varepsilon}{k}\int_{T_1}^{t} e^{\lambda(\tau-t)}\int_0^{\tau} e^{\lambda(s-\tau)}\eta_1(\theta_{s-t}\omega)dsd\tau\\
&+\frac{c}{k}\int_{T_1}^{t} e^{\lambda(\tau-t)}d\tau\\
\le&\frac{c}{k}e^{-\lambda t}(t-T_1)\left\|v_0(\theta_{-t}\omega)\right\|^2 + \frac{c\varepsilon}{k}\int_{T_1}^{t} e^{\lambda(\tau-t)}\int_0^{\tau} e^{\lambda(s-\tau)}\eta_1(\theta_{s-t}\omega)dsd\tau + \frac{c}{k}\\
\le&\frac{c}{k}e^{-\lambda t}(t-T_1)\left\|v_0(\theta_{-t}\omega)\right\|^2 + \frac{c\varepsilon}{k}c_6 r(\omega)\int_{T_1}^{t} e^{\lambda(\tau-t)}\int_{-t}^{\tau-t} e^{\frac{1}{2}\lambda s}dsd\tau + \frac{c}{k}\\
\le&\frac{c}{k}e^{-\lambda t}(t-T_1)\left\|v_0(\theta_{-t}\omega)\right\|^2 + \frac{4c\varepsilon}{\lambda^2 k}c_6 r(\omega) + \frac{c}{k}.
\end{aligned}$$

This implies that there exist $T_5 = T_5(B, \omega, \eta) > T_1$ and $R_3 = R_3(\omega, \eta)$ such that for all $t \ge T_5$ and $k \ge R_3$,

$$\frac{c}{k}\int_{T_1}^{t} e^{\lambda(s-t)} \left\|v\big(s, \theta_{-t}\omega, v_0(\theta_{-t}\omega)\big)\right\|^2 ds \le \frac{\eta}{6}. \tag{4.65}$$

Let $T^* = T^*(B, \omega, \eta) = \max\{T_1, T_2, T_3, T_4, T_5\}$. By (4.58), (4.60), (4.62), (4.63), (4.64) and (4.65), for all $t \ge T^*$ and $k \ge R^* = \max\{R_1, R_2, R_3\}$, one has

$$\int_{\mathbb{R}^n} \chi\left(\frac{x^2}{k^2}\right)|v(t, \theta_{-t}\omega, v_0(\theta_{-t}\omega))|^2 dx \le \eta.$$

It implies that for all $t \ge T^*$ and $k \ge R^*$, one has

$$\int_{|x|\ge k} |v(t, \theta_{-t}\omega, v_0(\theta_{-t}\omega))|^2 dx \le \int_{\mathbb{R}^n} \chi\left(\frac{x^2}{k^2}\right)|v(t, \theta_{-t}\omega, v_0(\theta_{-t}\omega))|^2 dx \le \eta,$$

which implies (4.48). □

5. Uniform estimates of solutions in $L^p(\mathbb{R}^n)$

In this section, we derive uniform estimates of solutions in $L^p(\mathbb{R}^n)$. In what follows, the letter $C = C(\omega)$ is a positive random variable which may change their values everywhere but keeping continuous of $t \mapsto C(\theta_t\omega)$.

Lemma 5.1. *For any $0 < \varepsilon \leq 1$ and $B \in \mathcal{D}$, there exist two random variables $T = T(B)$ and $C = C(B)$ such that P-a.s. for all $t \geq T$*

$$\sup_{0<\varepsilon\leq 1} \sup_{u_0 \in B(\theta_{-t}\omega)} \|u_\varepsilon(\tau, \theta_{-t}\omega, u_0)\|_p^p \leq C(\omega). \tag{5.1}$$

Proof. Taking the inner product in L^2 of (3.5) with $|v|^{p-2}v$, we obtain

$$\begin{aligned} &\frac{1}{p}\frac{d}{dt}\|v\|_p^p + \lambda\|v\|_p^p + \big((-\triangle)^\alpha v, |v|^{p-2}v\big) \\ =& \int_{\mathbb{R}^n} f\Big(x, v + \varepsilon z(\theta_t\omega)\Big)|v|^{p-2}v dx + \big(g - \varepsilon(-\triangle)^\alpha z(\theta_t\omega), |v|^{p-2}v\big). \end{aligned} \tag{5.2}$$

For the third term on the left-hand side of (5.2), we have

$$\big((-\triangle)^\alpha v, |v|^{p-2}v\big) \leq \frac{\beta_1}{2^{p+2}}\|v\|_{2p-2}^{2p-2} + c\|(-\triangle)^\alpha v\|^2. \tag{5.3}$$

Now we estimate the two terms on the right-hand side of (5.2). For the first term, applying conditions (1.2) and (1.3), together with Young's inequality, we obtain

$$\begin{aligned} &\int_{\mathbb{R}^n} f\left(x, v + \varepsilon z(\theta_t\omega)\right) v|v|^{p-2} dx \\ =& \int_{\mathbb{R}^n} \left(f(x,u)u - \varepsilon f(x,u) z(\theta_t\omega)\right)|v|^{p-2} dx \\ \leq& \int_{\mathbb{R}^n} \left(-\frac{\beta_1}{2^p}|v|^p + c(|z(\theta_t\omega)|^2 + |z(\theta_t\omega)|^p) + |\gamma_1| + \gamma_2^2\right)|v|^{p-2} dx \\ \leq& -\frac{\beta_1}{2^{p+1}}\|v\|_{2p-2}^{2p-2} + \lambda\|v\|_p^p + c\left(\|z(\theta_t\omega)\|_{2p-2}^{2p-2} + \|z(\theta_t\omega)\|_p^p + 1\right). \end{aligned} \tag{5.4}$$

The second term on the right-hand side of (5.2) is controlled by

$$\begin{aligned} &\|g\|\|v\|_{2p-2}^{p-1} + \varepsilon\,\|(-\triangle)^\alpha z(\theta_t\omega)\|\,\|v\|_{2p-2}^{p-1} \\ \leq& \frac{\beta_1}{2^{p+2}}\|v\|_{2p-2}^{2p-2} + c\left(\|(-\triangle)^\alpha z(\theta_t\omega)\|^2 + 1\right). \end{aligned} \tag{5.5}$$

Substituting (5.3)-(5.5) into (5.2), we infer

$$\frac{d}{dt}\|v\|_p^p \leq c\|(-\triangle)^\alpha v\|^2 + C(\theta_t\omega), \tag{5.6}$$

where

$$C(\theta_t\omega) = c\left(|\phi(\theta_t\omega)|^{2p-2} + |\phi(\theta_t\omega)|^p + |\phi(\theta_t\omega)|^2 + 1\right).$$

Let $t(\geq T)$ be fixed, where T is the same random time as given in Lemma 4.1. Integrating (5.6) at the sample $\theta_{-t-1}\omega$ over the interval $(s, t+1)$ with $s \in (t, t+1)$, we deduce

$$\begin{aligned} \|v(t+1, \theta_{-t-1}\omega, v_0)\|_p^p \leq& \|v(s, \theta_{-t-1}\omega, v_0)\|_p^p + \int_s^{t+1} C(\theta_{\tau-t-1}\omega)d\tau \\ &+ c\int_s^{t+1} \|(-\triangle)^\alpha v(\tau, \theta_{-t-1}\omega, v_0)\|^2 d\tau. \end{aligned} \tag{5.7}$$

Note that (4.2) and continuous of $\tau \mapsto \phi(\theta_t\omega)$ imply that

$$\begin{aligned}
&\int_t^{t+1} \|v(\tau, \theta_{-t-1}\omega, v_0)\|_p^p d\tau \\
\leq & 2^p \int_t^{t+1} \|u(\tau, \theta_{-t-1}\omega, u_0)\|_p^p d\tau + 2^p \int_t^{t+1} \|z(\theta_{\tau-t-1}\omega)\|_p^p d\tau \\
\leq & c \int_t^{t+1} e^{\lambda(\tau-t-1)} \|u(\tau, \theta_{-t-1}\omega, u_0)\|_p^p d\tau + c\int_{-1}^0 |\phi(\theta_\tau\omega)|^p d\tau \\
\leq & c(1+\varepsilon r(\omega)) + C(\omega) \leq C(\omega).
\end{aligned} \tag{5.8}$$

Integrating (5.7) with respect to s over $(t, t+1)$, then using (5.8), we deduce that

$$\|v(t+1, \theta_{-t-1}\omega, v_0(\theta_{-t-1}\omega))\|_p^p \leq C(\omega),$$

which implies easily (5.1) holding for $t \geq T+1$. □

To prove the uniformly asymptotical compactness of the system in L^p, we give the following auxiliary results.

Lemma 5.2. *Let $\hat{\eta}$ be a small positive constant. Then for each $0 < \varepsilon \leq 1$ and $B \in \mathcal{D}$ there are two positive random variables $T = T(B)$ and $M = M(\eta, B)$ such that P-a.s. for all $s \in [t, t+1]$ with $t \geq T$*

$$\sup_{\varepsilon\in(0,1]} \sup_{u_0\in B(\theta_{-t-1}\omega)} \mathfrak{M}(\mathbb{R}^n(|v_\varepsilon(s, \theta_{-t-1}\omega, v_0)| \geq M)) \leq \hat{\eta}, \tag{5.9}$$

where $\mathfrak{M}$ is the Lebesgue measure.

Proof. By (4.1) and the continuity of $r(\theta_\tau\omega)$, there exists a random time $T = T(B)$ such that for all $s \in [t, t+1]$ with $t \geq T$,

$$\|v(s, \theta_{-t-1}\omega, v_0)\| = \|v(s, \theta_s\theta_{s-t-1}\omega, v_0)\| \leq c(1 + \varepsilon r(\theta_{s-t-1}\omega)) \leq C(\omega)$$

with $s - t - 1 \in [-1, 0]$. Then we obtain that the following estimates about the Lebesgue measure:

$$\begin{aligned}
&M^2\mathfrak{M}(\mathbb{R}^n(|v_\varepsilon(s, \theta_{-t-1}\omega, v_0)| \\
\leq & \int_{\mathbb{R}^n(|v|\geq M)} |v(s, \theta_{-t-1}\omega, v_0)|^2 dx \leq \|v(s, \theta_{-t-1}\omega, v_0)\|^2 \leq C(\omega).
\end{aligned}$$

Choosing $M > \sqrt{C/\hat{\eta}}$, then applying (4.14), we obtain

$$\sup_{\varepsilon\in(0,1]} \sup_{u_0\in B(\theta_{-t-1}\omega)} \mathfrak{M}(\mathbb{R}^n(|v_\varepsilon(s, \theta_{-t-1}\omega, v_0)| \geq M)) \leq \hat{\eta}.$$

This completes the proof. □

Lemma 5.3. *Let $\hat{\eta}$ be a small positive constant. Then for each $0 < \varepsilon \leq 1$ and $B \in \mathcal{D}$ there are two positive random variables $T_1 = T_1(B)$ and $M_1 = M_1(\hat{\eta}, B)$ such that P-a.s. for all $t \geq T_1$*

$$\sup_{\varepsilon\in(0,1]} \sup_{u_0\in B(\theta_{-t-1}\omega)} \int_{\mathbb{R}^n(|u|\geq M_1)} |u_\varepsilon(t, \theta_{-t-1}\omega, u_0)|^2 dx \leq \hat{\eta} \tag{5.10}$$

and

$$\sup_{\varepsilon\in(0,1]} \sup_{u_0\in B(\theta_{-t-1}\omega)} \int_{\mathbb{R}^n(|v|\geq M_1)} |v_\varepsilon(t, \theta_{-t-1}\omega, v_0)|^2 dx \leq \hat{\eta}. \tag{5.11}$$

Proof. Define a subset of $L^2(\mathbb{R}^n)$:

$$N_T(\omega) = \cup_{t\geq T} \cup_{\varepsilon\in(0,1]} \cup_{u_0\in B(\theta_{-t-1}\omega)} \{u_\varepsilon(t, \theta_{-t-1}\omega, u_0)\}. \tag{5.12}$$

Applying Lemma 4.7 at the sample $\theta_{-1}\omega$, we obtain that there exists two positive numbers T and R (independent of ε) such that

$$\int_{|x|\geq R} \psi^2(x)dx < \frac{\hat{\eta}^2}{16} \quad \text{for all } \psi \in N_T(\omega). \tag{5.13}$$

Let $\mathbb{D}_R = \{x \in \mathbb{R}^n; |x| < R\}$. Define the restricted set of $N_T(\omega)$ on $L^2(\mathbb{D}_R)$ by

$$N_{T,R}(\omega) = \{\psi \in L^2(\mathbb{D}_R);\ \exists\ \psi_1 \in N_T(\omega),\ s.t.\ \psi(x) = \psi_1(x) \text{ for } x \in \mathbb{D}_R\}.$$

By Lemma 4.6, there exists a large T such that

$$\|\nabla\psi\|^2 \leq c(1 + r(\theta_{-1}\omega)) \triangleq C(\omega) \quad \text{for all } \psi \in N_{T,R}(\omega).$$

Then, applying the Sobolev compact embedding, we obtain that the restricted set $N_{T,R}(\omega)$ of $N_T(\omega)$ is pre-compact in $L^2(\mathbb{D}_R)$ and thus $N_{T,R}(\omega)$ has a finite $\hat{\eta}/4$-net in $L^2(\mathbb{D}_R)$. Then by (5.13) $N_T(\omega)$ has a finite $\hat{\eta}/2$-net with the centers ψ_j, $(j = 1, 2, \cdots, m)$ in $L^2(\mathbb{R}^n)$. We choose a $\delta > 0$ such that for any set $\mathbb{E} \in \mathbb{R}^n$ with $\mathfrak{M}(\mathbb{E}) < \delta$,

$$\sup_{1\leq j\leq m} \int_{\mathbb{E}} \psi_j^2(x)dx < \frac{\hat{\eta}}{4}. \tag{5.14}$$

Applying Lemma 5.2, there is a $M > 0$ such that $\mathfrak{M}(\mathbb{R}^n(|\psi| \geq M)) < \delta$ for all $\psi \in N_T(\omega)$ if T is large enough. But for every $\psi \in N_T(\omega)$ there is a center ψ_j such that $\|\psi - \psi_j\| \leq \hat{\eta}/2$. Therefore

$$\int_{\mathbb{R}^n(|\psi|\geq M)} \psi^2(x)dx \leq \int_{\mathbb{R}^n(|\psi|\geq M)} (|\psi - \psi_j|^2 + |\psi_j|^2)dx \leq \hat{\eta}^2, \tag{5.15}$$

which implies that the inequality (5.10) holds.

To prove the inequality (5.11), let

$$r_1(\omega) = \|z(\theta_{-1}\omega)\|_\infty = \|h\|_\infty |\phi(\theta_{-1}\omega)|, \tag{5.16}$$

which implies $r_1(\omega)$ is P-a.s. finite. By Lemma 5.2, we can choose $T_1 > T$ and $\tilde{M} \geq M$ (where both T and M are as given above) such that for all $t \geq T_1$,

$$\mathfrak{M}(\mathbb{R}^n(|v(t, \theta_{-t-1}\omega, v_0)| \geq \tilde{M})) < \frac{\hat{\eta}}{r_1^2(\omega) + 1}. \tag{5.17}$$

Let $M_1 = \tilde{M} + r_1$. Because of $u(t, \theta_{-t-1}\omega, u_0) = v(t, \theta_{-t-1}\omega, v_0) + \varepsilon z(\theta_{-1}\omega)$ and $0 < \varepsilon \leq 1$, it follows

$$\mathbb{R}^n(|v| \geq M_1) \subset \mathbb{R}^n(|u| \geq \tilde{M}). \tag{5.18}$$

From (5.10), (5.17) and (5.18), we obtain that for $t \geq T_1$,

$$\begin{aligned}
&\int_{\mathbb{R}^n(|v|\geq M_1)} |v(t, \theta_{-t-1}\omega, v_0)|^2 dx \\
\leq& 2\int_{\mathbb{R}^n(|u|\geq \hat{M})} |u(t, \theta_{-t-1}\omega, u_0)|^2 dx + 2\varepsilon \int_{\mathbb{R}^n(|v|\geq M_1)} |z(\theta_{-1}\omega)|^2 dx \\
\leq& 2\hat{\eta} + 2r_1^2 \mathfrak{M}(\mathbb{R}^n(|v| \geq M_1)) \\
\leq& 4\hat{\eta},
\end{aligned} \tag{5.19}$$

which implies the inequality (5.11) holding. □

Suppose the function f satisfies (1.2). Choosing $M_2 > 0$ such that $-\beta_1 M_2^p + \|\gamma_1(x)\|_\infty < 0$, then we obtain that there exists a $M_2 > 0$ such that for all $x \in \mathbb{R}^n$,

$$f(x,s) < 0 \quad \text{if } s \geq M_2 \quad \text{and } f(x,s) > 0 \quad \text{if } s \leq -M_2. \tag{5.20}$$

The following results show that the system is uniformly asymptotically compact on any finite interval under the p-norm.

Lemma 5.4. *Let $\hat{\eta}$ be a small positive constant. Then for each $0 < \varepsilon \leq 1$ and $B \in \mathcal{D}$ there are two positive random variables $\hat{T} = \hat{T}(\hat{\eta}, B)$ and $\hat{M} = \hat{M}(\hat{\eta}, B)$ such that P-a.s. for all $t \geq T_1$*

$$\sup_{\varepsilon \in (0,1]} \sup_{u_0 \in B(\theta_{-t}\omega)} \int_{\mathbb{R}^n(|u_\varepsilon| \geq \hat{M})} |u_\varepsilon(t, \theta_{-t}\omega, u_0)|^p dx \leq \hat{\eta}. \tag{5.21}$$

Proof. The considered functions are integrable, so one can take a small constant $\delta > 0$ such that, for any set $\mathbb{E} \subset \mathbb{R}^n$ with $\mathfrak{M}(\mathbb{E}) < \delta$,

$$\int_{\mathbb{E}} (g^2 + |g|^p + h^2 + |h|^p + |(-\triangle)^\alpha h|^2 + |\gamma_1| + \gamma_2^2) ds \leq \hat{\eta}. \tag{5.22}$$

By Lemma 5.2, we can choose two random variables $T \geq T_1$ and $M_3 \geq \max\{M_1, M_2\}$ such that for all $s \in [t, t+1]$ with $t \geq T$

$$\sup_{s \in [t,t+1]} \sup_{u_0 \in B(\theta_{-t}\omega)} \mathfrak{M}(\mathbb{R}^n(|v(s, \theta_{-t-1}\omega, v_0)| \geq M_3)) \leq \min\{\hat{\eta}, \delta\}. \tag{5.23}$$

Let

$$\mathcal{Z} = \max_{-1 \leq \tau \leq 0} \|z(\theta_\tau \omega)\|_\infty = \|h\|_\infty \max_{-1 \leq \tau \leq 0} |\phi(\theta_\tau \omega)| \tag{5.24}$$

and $M = M_3 + \mathcal{Z}$. Then both $\mathcal{Z}$ and M are finite for P-a.e. $\omega \in \Omega$ because the Ornstein-Uhlenbeck process $\phi(\theta_\tau \omega)$ is pathwise continuous. Let $t \geq T$ be fixed. Replacing the sample ω by the sample $\theta_{-t-1}\omega$ in (3.5), we know that the process

$$v = v(s) = v(s, \theta_{-t-1}\omega, v_0(\theta_{-t-1}\omega)), \quad s \in [t, t+1]$$

is a solution of the following random differential equation

$$v_t + (-\triangle)^\alpha v + \lambda v = f\big(x, v + \varepsilon z(\theta_{s-t-1}\omega)\big) + g - \varepsilon(-\triangle)^\alpha z(\theta_{s-t-1}\omega). \tag{5.25}$$

Taking the inner product of (5.25) with $(v - M)_+ = \max\{v - M, 0\}$, we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{ds}\|(v-M)_+\|^2 + ((-\triangle)^\alpha v, (v-M)_+) + \lambda(v, (v-M)_+) \\ &= \big(f(x, v + \varepsilon z(\theta_{s-t-1}\omega)), (v-M)_+\big) + \big(g - \varepsilon(-\triangle)^\alpha z(\theta_{s-t-1}\omega), (v-M)_+\big). \end{aligned} \tag{5.26}$$

We estimate every term of (5.26). For the second term and third term on the left-hand side, we have

$$((-\triangle)^\alpha v, (v-M)_+) + \lambda(v, (v-M)_+) \geq 0 + \lambda\|(v-M)_+\|^2 = \lambda\|(v-M)_+\|^2.$$

By (5.22) and (5.23), the last term on the right-hand side of (5.26) can be bounded, for $s \in [t, t+1]$,

$$
\begin{aligned}
&(g - \varepsilon(-\triangle)^\alpha z(\theta_{s-t-1}\omega), (v-M)_+)) \\
\leq\ & \lambda \|(v-M)_+\|^2 + c\int_{\mathbb{R}^n(v\geq M)} (g^2 + |(-\triangle)^\alpha z(\theta_{s-t-1}\omega)|^2) dx \\
\leq\ & \lambda \|(v-M)_+\|^2 + c\hat{\eta} + \max_{-1\leq\tau\leq 0} |\phi(\theta_\tau\omega)|^2 \cdot \int_{\mathbb{R}^n(v\geq M)} |(-\triangle)^\alpha h|^2 dx \\
\leq\ & \lambda \|(v-M)_+\|^2 + C\hat{\eta}.
\end{aligned}
$$

Then (5.26) can be written as

$$
\frac{d}{ds}\|(v-M)_+\|^2 - 2\int_{\mathbb{R}^n(v\geq M)} f(x, v+\varepsilon z(\theta_{s-t-1}\omega))(v-M)dx \leq C\hat{\eta} \tag{5.27}
$$

for all $s \in [t, t+1]$.

Integrating (5.27) with respect to $s \in [t, t+1]$ and applying Lemma 5.3, we obtain

$$
\begin{aligned}
& -\int_t^{t+1}\int_{\mathbb{R}^n(v\geq M)} f(x, v+\varepsilon z(\theta_{s-t-1}\omega))(v-M)dx \\
\leq\ & \frac{1}{2}\|(v(t)-M)_+\|^2 + C\hat{\eta} \leq \int_{\mathbb{R}^n(v\geq M)} v^2(t)dx + C\hat{\eta} \leq C\hat{\eta}.
\end{aligned} \tag{5.28}
$$

Then we prove the first term of (5.28) is positive. In fact, if $s \in [t, t+1]$, then $\|z(\theta_{s-t-1}\omega)\|_\infty \leq \mathcal{Z}$. Hence, if $v \geq M$, then we have $v + \varepsilon z(\theta_{s-t-1}\omega) \geq M - \mathcal{Z} \geq M_2$, which together with (5.20) implies

$$
-f(x, v+\varepsilon z(\theta_{s-t-1}\omega)) > 0, \quad \text{for all } x \in \mathbb{R}^n,\ s \in [t, t+1]. \tag{5.29}
$$

Since $v - M \geq v/2$ on $\mathbb{R}^n(v \geq 2M)$, it follows from (5.4) and (5.29) that for $x \in \mathbb{R}^n(v \geq 2M)$,

$$
\begin{aligned}
& -f(x, v+\varepsilon z(\theta_{s-t-1}\omega))(v-M) \\
\geq\ & -\frac{1}{2} f(x, v+\varepsilon z(\theta_{s-t-1}\omega))v \\
\geq\ & \frac{\beta_1}{2^{p+1}} v^p - c\left(|z(\theta_{s-t-1}\omega)|^2 + |z(\theta_{s-t-1}\omega)|^p + |\gamma_1| + \gamma_2^2\right) \\
\geq\ & \frac{\beta_1}{2^{p+1}} v^p - c\left(h^2 + |h|^p + |\gamma_1| + \gamma_2^2\right).
\end{aligned} \tag{5.30}
$$

Therefore, it follows from (5.22), (5.23), (5.28) and (5.30) that there exists a random number C, independent of $\hat{\eta}$, such that

$$
\int_t^{t+1}\int_{\mathbb{R}^n(v\geq 2M)} v^p(s)dxds < C\hat{\eta} \quad \text{for all } t \geq T. \tag{5.31}
$$

Next, taking the inner product of (5.25) with $(v-2M)_+^{p-1}$, we obtain

$$
\begin{aligned}
& \frac{1}{p}\frac{d}{ds}\|(v-2M)_+\|_p^p + \left((-\triangle)^\alpha v, (v-2M)_+^{p-1}\right) + \lambda(v, (v-2M)_+^{p-1}) \\
=\ & \big(f(x, v+\varepsilon z(\theta_{s-t-1}\omega)), (v-2M)_+^{p-1}\big) + \big(g - \varepsilon(-\triangle)^\alpha z(\theta_{s-t-1}\omega), (v-2M)_+^{p-1})\big).
\end{aligned} \tag{5.32}
$$

We estimate every term of (5.32) as follows. Applying (5.22) and (5.22), the last term on the right-hand side of (5.32) can be bounded, for $s \in [t, t+1]$,

$$\begin{aligned}
&\Big(g-\varepsilon(-\triangle)^\alpha z(\theta_{s-t-1}\omega), (v-2M)_+^{p-1})\Big)\\
\leq&\lambda\|(v-2M)_+\|_p^p+c\int_{\mathbb{R}^n(v\geq 2M)}(|g|^p+|(-\triangle)^\alpha z(\theta_{s-t-1}\omega)|^p)dx\\
\leq&\lambda\|(v-2M)_+\|_p^p+c\int_{\mathbb{R}^n(v\geq 2M)}|g|^p dx\\
&+c\max_{-1\leq\tau\leq 0}|\phi(\theta_\tau\omega)|^2\cdot\int_{\mathbb{R}^n(v\geq 2M)}|(-\triangle)^\alpha h|^2 dx\\
\leq&\lambda\int_{\mathbb{R}^n}v(v-2M)_+^{p-1}dx+C\hat{\eta}.
\end{aligned}$$

By (5.29), the nonlinear term of (5.32) is negative. It is easy to deduce that

$$\Big((-\triangle)^\alpha v, (v-2M)_+^{p-1}\Big)\geq\int_{\mathbb{R}^n(v\geq 2M)}\Big|(-\triangle)^{\alpha/2}(v-2M)^{p/2}\Big|^2 dx\geq 0.$$

Therefore, (5.32) can be rewritten as

$$\frac{d}{ds}\,\|(v-2M)_+\|_p^p\leq C\hat{\eta},\quad\text{for } s\in[t,t+1]\text{ with } t\geq T. \tag{5.33}$$

Integrating (5.33) from s to $t+1$, we infer

$$\|(v(t+1)-2M)_+\|_p^p\leq\|((v(s)-2M)_+\|_p^p+C\hat{\eta}. \tag{5.34}$$

Then integrating (5.34) with respect to s from t to $t+1$ and applying (5.31), we obtain that

$$\begin{aligned}
\|(v(t+1)-2M)_+\|_p^p\leq&\int_t^{t+1}\|((v(s)-2M)_+\|_p^p+C\hat{\eta}\\
\leq&\int_t^{t+1}\int_{\mathbb{R}^n(v\geq 2M)}v^p(s)dxds+C\hat{\eta}\leq C\hat{\eta}.
\end{aligned}$$

It is obvious that $v-2M\geq v/2$ on $\mathbb{R}^n(v\geq 4M)$. Then we get

$$\int_{\mathbb{R}^n(v\geq 4M)}v^p(t+1)dx\leq 2^p\|(v(t+1)-2M)_+\|_p^p\leq C\hat{\eta}.$$

That is, for a random number C (independent of $\hat{\eta}$), we have

$$\int_{\mathbb{R}^n(v\geq 4M)}v^p(t,\theta_{-t}\omega,v_0(\theta_{-t}\omega))dx\leq C\hat{\eta},\quad\text{for all } t\geq T+1. \tag{5.35}$$

Finally, because of $u(t,\theta_{-t}\omega,u_0)=v(t,\theta_{-t}\omega,v_0)+\varepsilon z(\omega)$, we have $\mathbb{R}^n(v\geq 4M+\mathcal{Z})\subset\mathbb{R}^n(v\geq 4M)$. Therefore, it follows from (5.35) that for $t\geq T+1$,

$$\int_{\mathbb{R}^n(u\geq 4M+\mathcal{Z})}u^p(t,\theta_{-t}\omega,v_0(\theta_{-t}\omega))dx$$

$$\leq 2^{p-1}\int_{\mathbb{R}^n(v\geq 4M)} v^p dx + 2^{p-1}\|h\|_\infty |\phi(\omega)| \mathfrak{M}(\mathbb{R}^n(v\geq 4M)) \leq C\hat{\eta}.$$

This implies the inequality (5.21) is true for the non-negative parts. Similarly, taking $(v+M)_-$(resp. $(v+2M)_-^{p-1}$) to replace $(v-M)_-$(resp. $(v-2M)_-^{p-1}$) and repeating the above procedure, we can obtain that for all $t \geq T+1$,

$$\int_{\mathbb{R}^n(u\leq -4M-\mathcal{Z})} |u(t,\theta_{-t}\omega, v_0(\theta_{-t}\omega))|^p dx \leq C\hat{\eta}.$$

Then we deduce the inequality (5.21) holding. □

6. Upper semi-continuity and regularity of (L^2, L^p)-random attractor

In this section, we prove the upper semi-continuous and regularity of (L^2, L^p)-random attractor generated by RDS Ψ_ε with equation (1.1) applying Theorem 2.1.

From Lemma 4.1, we have that Ψ_ε has a closed and random absorbing set in L^2. Applying the uniform estimates of Section 4, we obtain the uniform asymptotic compactness in L^2 (similarly to Lemma 6.1 of [34]). From Lemma 5.4, we obtain that the system is uniformly asymptotically compact in L^p on any finite interval. For the convergence of the system, we firstly prove the convergence at zero point.

We further assume that the nonlinear function f satisfies, for all $x \in \mathbb{R}^n$ and $s \in \mathbb{R}$,

$$\left|\frac{\partial f}{\partial s}(x,s)\right| \leq \beta_4 |s|^{p-2} + \gamma_4(x), \tag{6.1}$$

where $\beta_4 > 0$, $\gamma_4 \in L^\infty(\mathbb{R}^n)$ if $p=2$, and $\gamma_4 \in L^{\frac{p}{p-2}}(\mathbb{R}^n)$ if $p>2$. Under condition (6.1), we will show that, as $\varepsilon \to 0$, the solution of the stochastic equation (1.1) converges to the limiting deterministic equation:

$$\frac{du}{dt} + (-\triangle)^\alpha u + \lambda u = f(x,u) + g(x), \quad x \in \mathbb{R}^n, \quad t > 0. \tag{6.2}$$

Similarly to the proof of Lemma 6.2 in [34], we obtain the following lemma.

Lemma 6.1. *Suppose that* (1.2)–(1.5) *and* (6.1) *hold. Given* $0 < \varepsilon \leq 1$, *let* u^ε *and* u *be the solutions of* (1.1) *and* (6.2) *with the initial data* u_0^ε *and* u_0, *respectively. Then for* P*-a.e.* $\omega \in \Omega$ *and* $t \geq 0$, *we have*

$$\|u^\varepsilon(t,\omega,u_0^\varepsilon) - u(t,u_0)\|^2 \leq ce^{ct}\|u_0^\varepsilon - u_0\|^2 + \varepsilon c e^{ct}\left(r(\omega) + \|u_0^\varepsilon\|^2 + \|u_0\|^2\right),$$

where c *is a positive deterministic constant independent of* ε, *and* $r(\omega)$ *is the tempered function in* (3.2).

Applying Lemma 4.1 and Lemma 6.1, we obtain

Lemma 6.2. *Suppose that* (1.2)–(1.5) *and* (6.1) *hold. Given* $0 < \varepsilon \leq 1$, *let* $g \in H^\alpha(\mathbb{R}^n)$. *Then for* P*-a.s.* $\omega \in \Omega$, *we have*

$$\lim_{\varepsilon\to 0} \|u_\varepsilon(t,\omega,u_{0,\varepsilon}) - u(t,\omega,u_0)\| = 0.$$

Next, we prove the convergence of the systems at nonzero point.

Lemma 6.3. *Suppose that* (1.2)–(1.5) *hold. Let* u_ε *be the solution of* (1.1) *with the initial data* $u_{0,\varepsilon}$. *Suppose* $\|u_{0,\varepsilon} - u_{0,\varepsilon_0}\| \to 0$ *as* $\varepsilon \to \varepsilon_0$ *for* ε_0. *Then for every fixed* $T > 0$, *we have P-a.s.*

$$\lim_{\varepsilon\to\varepsilon_0} \|u_\varepsilon(T,\omega,u_{0,\varepsilon}) - u_{\varepsilon_0}(T,\omega,u_{0,\varepsilon_0})\| = 0.$$

Proof. Applying (3.5), it is easy to obtain that $V_\varepsilon = v_\varepsilon - v_{\varepsilon_0}$ satisfies the following energy equation

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|V_\varepsilon\|^2+\|(-\triangle)^{\frac{\alpha}{2}}V_\varepsilon\|^2+\lambda\|V_\varepsilon\|^2\\ =&\big(f(x,u_\varepsilon)-f(x,u_{\varepsilon_0}),V_\varepsilon\big)+(\varepsilon-\varepsilon_0)\big((-\triangle)^\alpha z(\theta_t\omega),V_\varepsilon\big).\end{aligned}$$

Then by (1.4) we infer

$$\begin{aligned}&\big(f(x,u_\varepsilon)-f(x,u_{\varepsilon_0}),V_\varepsilon\big)\\ =&\left(\frac{\partial f}{\partial s}(x,s)(u_\varepsilon - u_{\varepsilon_0}),V_\varepsilon\right)\\ \le&\beta_3\|V_\varepsilon\|^2 + \beta|\varepsilon-\varepsilon_0||(z(\theta_t\omega),V_\varepsilon)|.\end{aligned}$$

Let $\hat{\eta}$ be a small constant. Then, applying the Young's inequality, we obtain that there exists a $\delta > 0$ such that for all $\varepsilon \in (\varepsilon_0 - \delta, \varepsilon_0 + \delta)$ and $t \in (0,T]$,

$$\frac{d}{dt}\|V_\varepsilon\|^2 \le c\|V_\varepsilon\|^2 + \hat{\eta}.$$

Then, applying Gronwall's inequality, we obtain the desired result. □

Therefore, applying Theorem 2.1, we obtain our main result.

Theorem 6.1. *Assume that* (1.2)-(1.5) *and* (6.1) *hold. Then the associated RDS* Ψ_ε *with* (1.1) *has a unique* (L^2, L^p)*-random attractor* $\mathcal{A}_\varepsilon$ *such that the union* $\mathcal{A}(\omega)$ *over* $(0,1]$ *is precompact in* $L^2 \cap L^p$ *and the family* $\mathcal{A}_\varepsilon$ *is upper semi-continuous at any point* $\varepsilon_0 \in [0,+\infty)$ *under the topology of* $L^2 \cap L^p$. *More precisely*

$$\lim_{\varepsilon\to\varepsilon_0} d(\mathcal{A}_\varepsilon(\omega),\mathcal{A}_{\varepsilon_0}(\omega)) = \lim_{\varepsilon\to\varepsilon_0} d_p(\mathcal{A}_\varepsilon(\omega),\mathcal{A}_{\varepsilon_0}(\omega)) = 0, \quad \textit{for all } \varepsilon_0 > 0$$

and

$$\lim_{\varepsilon\downarrow 0} d(\mathcal{A}_\varepsilon(\omega),\mathcal{A}_0) = \lim_{\varepsilon\to\varepsilon_0} d_p(\mathcal{A}_\varepsilon(\omega),\mathcal{A}_0) = 0.$$

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