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NUMERICAL SIMULATION FOR THE FRACTIONAL-ORDER SMOKING MODEL USING A SPECTRAL COLLOCATION METHOD BASED ON THE GEGENBAUER WAVELET POLYNOMIALS

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Abstract Smoking is a social trend that is prevalent around the world, particularly in places of learning and at some significant events. The World Health Organization (WHO) defines smoking as the most important preventable cause of disease and the third major cause of death in humans. So, in this paper, we present an effective simulation to study the solution behavior of the Liouville-Caputo fractional-order smoking model by using a presumably new approximation technique that is based on the Gegenbauer wavelet polynomials (GWPs). We use the spectral collocation method based on the properties of GWPs. This procedure converts the given model into a system of algebraic equations. We satisfy the efficiency and accuracy of the given procedure by evaluating the residual error function. The results obtained are then compared with the results obtained by using the fourth-order Runge-Kutta method. Our results show that the implemented technique provides an easy and efficient tool to simulate the solution of such smoking models.

Keywords Smoking model, Liouville-Caputo fractional derivative, Gegenbauer wavelet polynomials, spectral collocation method, fourth-order Runge-Kutta method.

MSC(2010) 34A12, 41A30, 47H10, 65N20.

1. Introduction

Nowadays, smoking is one of the foremost health problems in the world. According to the WHO's report on the smoking epidemic [34], smoking kills numerous people in their most active life. More than 5 million deaths in the world are caused due to the effect of smoking on different organs of the human body, which may increase to up to 8 million people per year by 2030 [33]. The chance of heart attack is 70% more in smokers than that among persons who do not smoke. Smokers have a 10% higher incidence rate of lung cancer than nonsmokers. The lifespan of smokers is 10 to 13

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years shorter than that of non-smokers. Researchers tried to help control smoking by securing the life expectancy of an individual. To give the best illustration of the cigarette smoking phenomena, several researchers have tried to study many different effective smoking models. Presumably, for the first time in the year 1997, a mathematical model was outlined for smoking by dividing the total population into three different classes: potential smokers, chain smokers, and permanently quit smokers [8]. More recently, in the year 2007, Ham [11] documented the different stages and procedures of smoking among students through a survey in different vocational-technical schools in Korea. In 2008, Liu et al. [20] improved Ham's model in [11] in order to represent a new temporarily quit smokers class. An extension of the model by presenting a new occasional smokers class and offering a dynamical interaction in an integer-order model was presented in [11]. Several other authors, too, have presented the smoking models in integer and fractional orders [2, 7, 35].

Fractional calculus (FC) is considered a branch of mathematical analysis, which is capable of dealing with the modelings and analyzing many real-life situations [14–16, 19]. FC analysis has drawn increasing attention in the study of biological models, where the scaling power law of fractional order appears universally as an empirical description of such complex phenomena [12, 23]. One of these mathematical equations is used in the smoking model, which is one of the most fundamental equations that study a social trend that is prevalent around the world. A modification, which was added to the smoking model, was introduced to give us the fractional-order smoking model. Many research papers studied studied this biological model, see for example, [3].

Gegenbauer wavelet polynomials (GWPs) have been used to solve the time-fractional derivative system of the Burger's equations [21], nonlinear fractional-order delay differential equations [13], fractional-order initial and boundary value problems [22], Fisher-Kolmogorov equation [4], generalized Kuramoto-Sivashinsky equation [5], time-fractional KdV-Burger-Kuramoto equation [25], variable-order fractional differential equations [32]. To the best of our knowledge, this article presents the first application of the spectral collocation approach based on GWPs for solving the fractional-order smoking model ([10,17,30,31] for some recent developments based upon the collocation approach). When the method is applied, the system of differential equations is converted into a system of algebraic equations. The unknown coefficients of the series solution are then obtained by solving the system and thereby the approximate solution of the original equation is found.

The spectral collocation method (SCM) has some advantages for handling this class of problems in which the Gegenbauer coefficients for the solution can exist very easily after using the numerical programs. For this reason, this method is much faster than the other methods. They are widely used because of their good properties in the approximation of functions. Also, this method is a numerical technique with high accuracy and fast convergence and it is easy to use in finite and infinite domains for different problems. In addition, the domain discretization and approximation of the nonlinear terms are not necessary for this method, which is an important advantage [1, 18].

The reader may find it helpful to understand the mathematical modeling process by reading through several significant studies, such as an efficient operation matrix method for solving fractal-fractional differential equations (FFDEs) with generalized Caputo-type fractional-fractal derivative [29]; a highly accurate artificial neural networks scheme for solving higher multi-order FFDEs based on the

generalized Caputo derivative [27]; solving FFDEs by using an operational matrix of derivatives via Hilfer fractal-fractional derivative sense [28]; and a detailed study of a fractal-fractional transmission dynamical model of viral infectious disease with vaccination [26].

The rest of the paper is organized as follows: In Section 2, we present some definitions and concepts concerning fractional derivatives, the Gegenbauer wavelet polynomials, and other related ideas. Through Section 3, we give the implementation of the proposed method. In Section 4, we present a numerical simulation of the proposed model under study. Finally, the conclusions are in Section 5.

2. Preliminaries

We choose to divide this section into the following subsections.

2.1. Fractional integration and fractional derivative

Many definitions for the fractional-order integration and fractional-order differentiation are available in the literature [30]. The most important ones are those that are utilized in the development of fractional calculus theory are the Riemann-Liouville and Liouville-Caputo fractional derivatives, which are defined as follows.

Definition 2.1. The Riemann-Liouville fractional integral I^ν of order ν for a given function $\psi(t)$ is defined as follows [30]:

$$I^\nu \psi(t) = \frac{1}{\Gamma(\nu)} \int_0^t (t-\tau)^{\nu-1} \psi(\tau)\, d\tau, \qquad t > 0; \quad \nu \in \mathbb{R}^+, \tag{2.1}$$

where $\Gamma(\cdot)$ is the gamma function.

The operator I^ν possesses the following properties:

$$I^\nu I^\gamma \psi(t) = I^{\nu+\gamma}\, \psi(t), \qquad \nu,\ \gamma > 0, \tag{2.2}$$

and

$$I^\nu t^m = \frac{\Gamma(m+1)}{\Gamma(m+\nu+1)} t^{m+\nu}. \tag{2.3}$$

This operator is linear, that is,

$$I^\nu (c_1\, f_1(t) + c_2\, f_2(t)) = c_1\, I^\nu\, (f_1(t)) + c_2\, I^\nu\, (f_2(t)), \tag{2.4}$$

for some constants c_1 and c_2.

Definition 2.2. The Riemann-Liouville fractional derivative ${}^{\mathrm{RL}}D^\nu$ of order ν for a given function $\psi(t)$ is defined as follows [24]:

$${}^{\mathrm{RL}}D^\nu \psi(t) = \frac{d^m}{dt^m} \left(I^{m-\nu} \psi(t) \right), \qquad m-1 < \nu \leqq m; \quad m \in \mathbb{N}, \tag{2.5}$$

where, as usual, $\mathbb{N}$ denotes the set of natural numbers.

When modeling some real-world issues, the Riemann-Liouville definition has some drawbacks [24, 30]. However, the Liouville-Caputo definition was meant to address such issues. We utilize it as described in the following definition.

Definition 2.3. [30] In the Liouville-Caputo sense, the fractional derivative ${}^{\mathrm{LC}}D^{\nu}$ of a function $\psi(t)$ is defined as follows:

$$ {}^{\mathrm{LC}}D^{\nu}\psi(t) = \frac{1}{\Gamma(n-\nu)}\int_0^t \frac{\psi^{(n)}(\tau)}{(t-\tau)^{\nu-n+1}}d\tau, \qquad n-1<\nu<n; \quad n\in\mathbb{N}. \tag{2.6}$$

The Liouville-Caputo fractional derivative ${}^{\mathrm{LC}}D^{\nu}$ possesses the following properties:

$$ {}^{\mathrm{LC}}D^{\nu}C = 0, \qquad C \text{ is a constant},$$

and

$$ {}^{\mathrm{LC}}D^{\nu}t^{\theta} = \frac{\Gamma(\theta+1)}{\Gamma(\theta+1-\nu)}t^{\theta-\nu}, \qquad \theta\in\mathbb{N}\cup\{0\}; \quad \theta \geqq \lceil\nu\rceil, \tag{2.7}$$

where $\lceil\nu\rceil$ denotes the ceil function. Also, this operator is linear:

$$ {}^{\mathrm{LC}}D^{\nu}(c_1\, f_1(t) + c_2\, f_2(t)) = c_1\, {}^{\mathrm{LC}}D^{\nu}\left(f_1(t)\right) + c_2\, {}^{\mathrm{LC}}D^{\nu}\left(f_2(t)\right), \tag{2.8}$$

for some constants c_1 and c_2.

2.2. The Gegenbauer wavelet polynomials

The wavelets are created by translation and dilation of the mother wavelet $\phi(t)$. If the translation parameter τ and dilation parameter γ change continuously in $\mathbb{R}$, we get the continuous wavelets of the following form [5]:

$$ \phi_{\gamma,\tau}(t) = |\gamma|^{-1/2}\,\phi\left(\frac{t-\tau}{\gamma}\right), \qquad \gamma,\tau\in\mathbb{R}; \qquad \gamma\neq 0.$$

If we take the limit of these two parameters as $\gamma\to\gamma_0^{-\ell}$ and $\tau\to\epsilon\,\tau_0\,\gamma_0^{-\ell}$, where $\gamma_0>1$, $\tau_0>0$, $\ell>0$ and $\epsilon>0$, we get the discrete wavelets of the form given below [22]:

$$ \phi_{\ell,\epsilon}(t) = \gamma_0^{\ell/2}\,\phi(\gamma_0^{\ell}\,t-\epsilon\,\tau_0),$$

which provides a basis in $L^2(\mathbb{R})$. If $\gamma_0=2$ and $\tau_0=1$, then we get the orthonormal basis as follows [4]:

$$ \phi_{\ell,\epsilon}(t) = 2^{\ell/2}\phi(2^{\ell}\,t-\epsilon).$$

2.3. Properties of Gegenbauer polynomials

The nth-order Gegenbauer polynomials, $G_n^{\kappa}(t)$, are obtained by using the formula [32]:

$$ G_{n+1}^{\kappa}(t) = \frac{1}{n+1}\big(2(n+\kappa)\,t\,G_n^{\kappa}(t) - (n+2\kappa-1)\,G_{n-1}^{\kappa}(t)\big), \qquad n\in\mathbb{N},$$

where $G_0^{\kappa}(t)=1$ and $G_1^{\kappa}(t)=2\kappa\,t$, $\kappa>-0.5$ being the known ultraspherical parameter. Different values of κ yield different wavelets. When $\kappa=1/2$, $\kappa=0$ and $\kappa=1$, we have Legendre wavelets, and the first and second kinds of Chebyshev wavelets, respectively. These polynomials are orthogonal on $[-1,1]$:

$$ \int_{-1}^{1}\tilde{w}(t)\,G_i^{\kappa}(t)\,G_j^{\kappa}(t)\,dt = L_i^{\kappa}\,\delta_{ij},$$

where $\tilde{w}(t) = \frac{1}{(1-t^2)^{0.5-\kappa}}$ is the weight function, and δ is the Kronecker delta function. We use the following for normalization:

$$L_i^{\kappa} = \frac{\pi\, 2^{1-2\kappa}\, \Gamma(i+2\kappa)}{i!\,(i+\kappa)(\Gamma(\kappa))^2}.$$

2.4. Formulation of the Gegenbauer wavelets

The Gegenbauer wavelets are defined on $[0,1]$ as follows [25]:

$$\phi_{\epsilon,n}^{\kappa}(t) = \begin{cases} \dfrac{1}{\sqrt{L_n^{\kappa}}}\, 2^{\ell/2}\, G_n^{\kappa}(2^{\ell}t - 2\,\epsilon + 1), & \frac{2\,\epsilon-2}{2^{\ell}} \leqq t \leqq \frac{2\,\epsilon}{2^{\ell}}, \\ 0, & \text{otherwise}, \end{cases}$$

where $\epsilon = 1, 2, \cdots, 2^{\ell-1}$, $n = 0, 1, \cdots, N-1$ for $N \geqq 1$, and $\ell \in \mathbb{N}$ is the level of resolution.

The first functions of the GWPs calculated by taking $\ell = 1$, $N = 4$ and $\kappa = 30$ are given below:

$$\begin{aligned} &\phi_{1,0}^{30}(t) = 2.49122, \qquad \phi_{1,1}^{30}(t) = 39.2318\,t - 19.6159, \\ &\phi_{1,2}^{30}(t) = 447.481\,t^2 - 447.481\,t + 110.066, \\ &\phi_{1,3}^{30}(t) = 4264.91\,t^3 - 6397.36\,t^2 + 3148.7\,t - 508.124. \end{aligned}$$

The unknown function $\theta(t)$ can be expanded by the Gegenbauer wavelets as follows:

$$\theta(t) = \sum_{i=1}^{\infty} \sum_{j=0}^{\infty} c_{ij}\, \phi_{ij}(t),$$

where c_{ij} are Gegenbauer wavelet coefficients in the form $c_{ij} =< \theta(t), \phi_{ij}(t) >$ such that $< .,. >$ refers the inner product. Then we can approximate $\theta(t)$ by truncating this series as follows:

$$\theta(t) \cong \sum_{i=1}^{2^{\ell-1}} \sum_{j=0}^{N-1} c_{ij}\, \phi_{ij}(t). \tag{2.9}$$

3. Numerical implementation

In this section, we give an outline of the implementation of the proposed method to solve the smoking model in its fractional form. We can utilize mathematical modeling to prevent the spread of tobacco smoking because it has been used as a significant tool for pandemic grasp in recent decades [9]. The susceptible-exposed-infected-recovered (SEIR) model is a general model. In the present study, to obtain a better understanding of the qualitative analysis as well as the numerical analysis of the proposed model, the revised form of the smoking model in the sense of Liouville-Caputo fractional derivative is considered, and described as follows [3]:

$$^{\mathrm{LC}}D^{\nu}\psi_1(t) = \sigma - \ell_1\,\psi_1(t)\psi_2(t) + \gamma\,\psi_4(t) - \beta\,\psi_1(t), \tag{3.1}$$

$$^{\mathrm{LC}}D^{\nu}\psi_2(t) = \ell_1\,\psi_1(t)\psi_2(t) - \ell_2\,\psi_2(t)\psi_3(t) - (\sigma_1+\beta)\,\psi_2(t), \tag{3.2}$$

$$^{\mathrm{LC}}D^{\nu}\psi_3(t) = \ell_2\,\psi_2(t)\psi_3(t) - (\theta+\sigma_2+\beta)\,\psi_3(t), \tag{3.3}$$

$$^{\mathrm{LC}}D^{\nu}\psi_4(t) = \theta\,\psi_3(t) - (\alpha+\beta+\gamma)\,\psi_4(t), \tag{3.4}$$

$$^{\mathrm{LC}}D^{\nu}\psi_5(t) = \alpha\,\psi_4(t) - \beta\,\psi_5(t), \tag{3.5}$$

with the following initial conditions:

$$\psi_k(0) = \psi_k^0, \qquad k = 1,2,3,4,5. \tag{3.6}$$

In this model, the total population is divided into five classes, where ψ_1, ψ_2, ψ_3, ψ_4, and ψ_5 are the susceptible smokers, the snuffing (ingestion) class, irregular smokers, regular smokers, and quit smokers, respectively, at a given time t. Also, the parameters used in the model (3.1)-(3.6) are described as follows:

1. σ is the frequency of recruitment (birth or migration);
2. ℓ_1 is the rate of the vulnerable population transitions to the snuffing class;
3. ℓ_2 is the rate of snuffing becomes an irregular smokers;
4. θ is the rate of irregular smokers turning to a regular smoker;
5. α, β, γ are the rates of the departing, natural death, and recovery, respectively;
6. σ_1, σ_2 are the rates of the snuffing class deaths because of smoking, and the death due to smoking, respectively.

Let us approximate the unknown functions $\psi_p(t)$ in terms of the GWPs, by $\psi_{p,ij}(t)$, $p = 1,2,3,4,5$, as follows:

$$\psi_{p,ij}(t) = \sum_{i=1}^{2^{\ell-1}} \sum_{j=0}^{m-1} c_{ij}^{p}\,\phi_{ij}(t). \tag{3.7}$$

Next, by using the property that:

$$^{\mathrm{LC}}D^{\nu} = {}^{\mathrm{LC}}D^{s}I^{s-\nu},$$

for $s-1 < \nu < s$, the double sum in (3.7) can be differentiated and integrated term by term along the interval of convergence for fixed ℓ and m. We can thus obtain the following approximation of the fractional derivative $^{\mathrm{LC}}D^{\nu}\,\psi_{p,ij}(t)$:

$$^{\mathrm{LC}}D^{\nu}\psi_{p,ij}(t) = \sum_{i=1}^{2^{\ell-1}} \sum_{j=0}^{2^{\ell-1}} c_{ij}^{p}\; D^{s}I^{s-\nu}\left[\phi_{ij}(t)\right], \qquad p = 1,2,\cdots,5. \tag{3.8}$$

Upon substituting from (3.7) and (3.8) into the system (3.1)-(3.5), we get:

$$\begin{aligned} &\sum_{i=1}^{2^{\ell-1}} \sum_{j=0}^{m-1} c_{ij}^{1}\,D^{s}I^{s-\nu}\left[\phi_{ij}(t)\right] \\ =&\sigma - \ell_1 \left(\sum_{i=1}^{2^{\ell-1}} \sum_{j=0}^{m-1} c_{ij}^{1}\,\phi_{ij}(t)\right)\left(\sum_{i=1}^{2^{\ell-1}} \sum_{j=0}^{m-1} c_{ij}^{2}\,\phi_{ij}(t)\right) \\ &+\gamma \left(\sum_{i=1}^{2^{\ell-1}} \sum_{j=0}^{m-1} c_{ij}^{4}\,\phi_{ij}(t)\right) - \beta \left(\sum_{i=1}^{2^{\ell-1}} \sum_{j=0}^{m-1} c_{ij}^{4}\,\phi_{ij}(t)\right), \end{aligned} \tag{3.9}$$

$$\sum_{i=1}^{2^{\ell-1}}\sum_{j=0}^{m-1} c_{ij}^{2}\, D^{s} I^{s-\nu}\left[\phi_{ij}(t)\right] \tag{3.10}$$
$$=\ell_1 \left(\sum_{i=1}^{2^{\ell-1}}\sum_{j=0}^{m-1} c_{ij}^{1}\,\phi_{ij}(t)\right)\left(\sum_{i=1}^{2^{\ell-1}}\sum_{j=0}^{m-1} c_{ij}^{2}\,\phi_{ij}(t)\right)$$
$$-\ell_2 \left(\sum_{i=1}^{2^{\ell-1}}\sum_{j=0}^{m-1} c_{ij}^{2}\,\phi_{ij}(t)\right)\left(\sum_{i=1}^{2^{\ell-1}}\sum_{j=0}^{m-1} c_{ij}^{3}\,\phi_{ij}(t)\right)$$
$$-(\sigma_1+\beta)\left(\sum_{i=1}^{2^{\ell-1}}\sum_{j=0}^{m-1} c_{ij}^{2}\,\phi_{ij}(t)\right),$$

$$\sum_{i=1}^{2^{\ell-1}}\sum_{j=0}^{m-1} c_{ij}^{3}\, D^{s} I^{s-\nu}\left[\phi_{ij}(t)\right] \tag{3.11}$$
$$=\ell_2 \left(\sum_{i=1}^{2^{\ell-1}}\sum_{j=0}^{m-1} c_{ij}^{2}\,\phi_{ij}(t)\right)\left(\sum_{i=1}^{2^{\ell-1}}\sum_{j=0}^{m-1} c_{ij}^{3}\,\phi_{ij}(t)\right)$$
$$-(\theta+\sigma_2+\beta)\left(\sum_{i=1}^{2^{\ell-1}}\sum_{j=0}^{m-1} c_{ij}^{3}\,\phi_{ij}(t)\right),$$

$$\sum_{i=1}^{2^{\ell-1}}\sum_{j=0}^{m-1} c_{ij}^{4}\, D^{s} I^{s-\nu}\left[\phi_{ij}(t)\right] \tag{3.12}$$
$$=\theta\left(\sum_{i=1}^{2^{\ell-1}}\sum_{j=0}^{m-1} c_{ij}^{3}\,\phi_{ij}(t)\right)-(\alpha+\beta+\gamma)\left(\sum_{i=1}^{2^{\ell-1}}\sum_{j=0}^{m-1} c_{ij}^{4}\,\phi_{ij}(t)\right),$$

$$\sum_{i=1}^{2^{\ell-1}}\sum_{j=0}^{m-1} c_{ij}^{5}\, D^{s} I^{s-\nu}\left[\phi_{ij}(t)\right] \tag{3.13}$$
$$=\alpha\left(\sum_{i=1}^{2^{\ell-1}}\sum_{j=0}^{m-1} c_{ij}^{4}\,\phi_{ij}(t)\right)-\beta\left(\sum_{i=1}^{2^{\ell-1}}\sum_{j=0}^{m-1} c_{ij}^{5}\,\phi_{ij}(t)\right).$$

By the collocation of these last equations (3.9)-(3.13) at

$$t_r=\frac{(2r-1)h}{2^{\ell}N},$$

where $r=1,2,\cdots,2^{\ell-1}m$ and $0\leqq t_i\leqq h$, we are led to a system of algebraic equations in the Gegenbauer wavelet coefficients $c_{ij}^{p},\ p=1,2,3,4,5,\ i=0,2,\cdots,2^{\ell-1}m,\ j=0,1,\cdots,m-1$. Also, upon substituting Eq.(3.7) into (3.6), the initial conditions (3.6) will be converted to the following algebraic equations:

$$\sum_{i=1}^{2^{\ell-1}}\sum_{j=0}^{m-1} c_{ij}^{p}\,\phi_{ij}(0)=\psi_{p}^{0},\qquad p=1,2,3,4,5. \tag{3.14}$$

We now use the Newton iteration method for solving the nonlinear system consisting of the equations (3.9)-(3.14) for the unknowns $c_{ij}^{p},\ p=1,2,\ldots,5,\ i=$

$0, 2, \cdots, 2^{\ell-1} m,\ j = 0, 1, \cdots, m-1$. This, in turn, leads us to formulate the approximate solution by substitution in the form (3.7).

4. Numerical simulation

In this section, we proceed to verify the accuracy and quality of the given scheme by presenting a numerical simulation on a test example in the interval $[0, 50]$, where we address the system (3.1)-(3.6) with different values of ν, m, β and different values of the initial solutions. But, in all figures, we take the same values of the following parameters:

$$\sigma = 0.1, \quad \ell_1 = \gamma = \sigma_1 = \sigma_2 = 0.003, \quad \ell_2 = \beta = 0.002, \quad \theta = \alpha = 0.05.$$

We consider the following two cases of the initial conditions:

1. **Case 1:** $\psi_1^0 = 40, \quad \psi_2^0 = 30, \quad \psi_3^0 = 20, \quad \psi_4^0 = 10, \quad \psi_5^0 = 5;$
2. **Case 2:** $\psi_1^0 = 75, \quad \psi_2^0 = 60, \quad \psi_3^0 = 45, \quad \psi_4^0 = 30, \quad \psi_5^0 = 15.$

We present a comparison between the results obtained by the proposed method with those results that are obtained by using the fourth-order Runge-Kutta method (RK4). We also evaluate the residual error function (REF) [6] to estimate the accuracy and quality of the proposed scheme.

The obtained numerical results for the studied model by applying the proposed technique are illustrated in Figures 1-6.

1. In Figure 1, we give the behavior of the approximate solution via distinct values of $\nu = 1.0, 0.95, 0.85, 0.75$, with $m = 6$, and the initial conditions in Case 1.
2. In Figure 2, we give the behavior of the approximate solution via distinct values of $\nu = 0.95, 0.85, 0.75, 0.65$, with $m = 6$, and the initial conditions in Case 2.
3. In Figures 3 and 4, we present a comparison between the results obtained by the proposed method with those results obtained by using the RK4 at ($\nu = 1$) with $m = 6$ and the initial conditions in Case 1, and Case 2 respectively.
4. In Figure 5, we compute and plot the REF of the approximate solution at $\nu = 0.97$, with different values of $m = 5, 9$, and the initial conditions in Case 1.
5. In Figure 6, we study the effect of the rate of natural death on the system, with different values of $\beta = 0.002, 0.003, 0.004, 0.005$; at $\nu = 0.93$, $m = 6$, and the initial conditions in Case 2.

Through these results, we note that the behavior of the numerical solution resulting from the application of the proposed method depends on the values of ν, m and β, and this confirms that the proposed method is suitable for solving the proposed model in its fractional-order form with the Liouville-Caputo fractional derivative. From Figure 5, we can regulate the precision of the error and lower it by adding more terms from the approximation solution series, or by raising m. In addition, the proposed approach remarkably improves the results as well as the efficiency of the method.

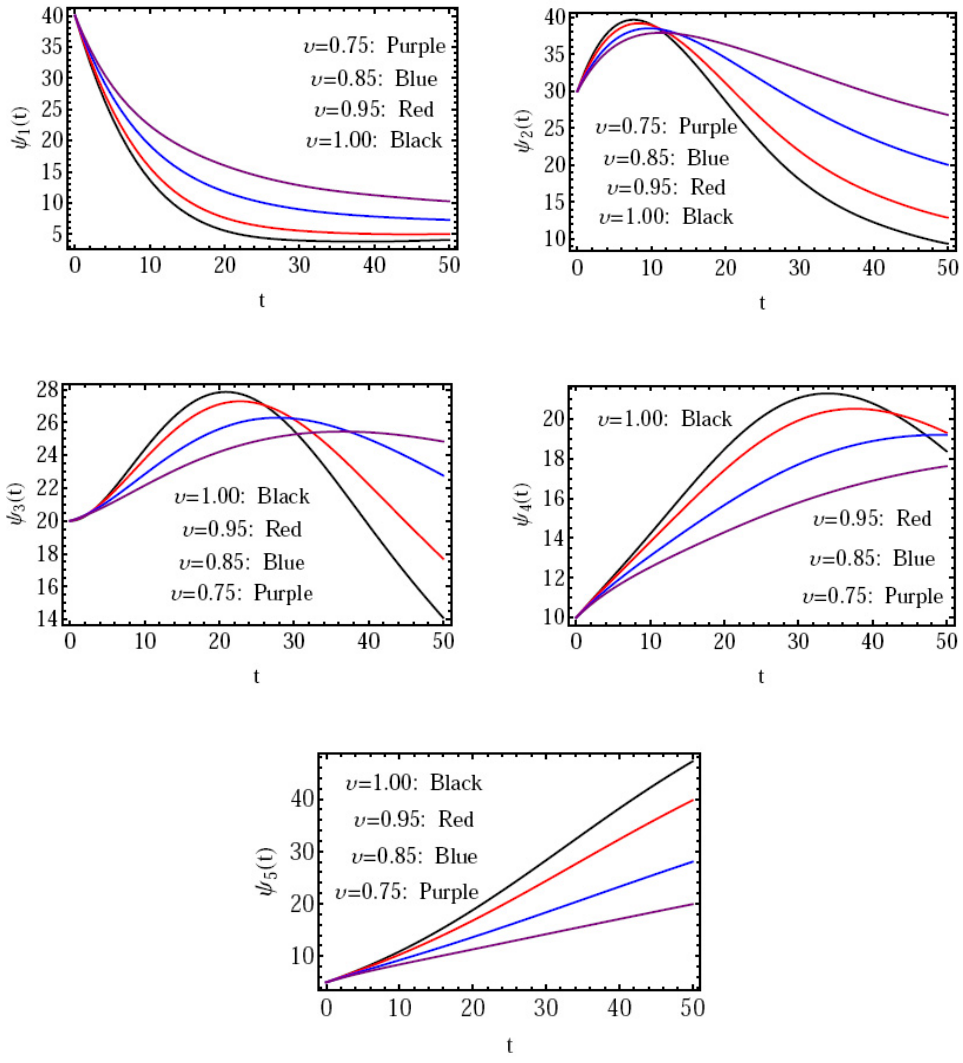

Figure 1. The approximate solution $\psi_i(t)$, $i = 1(1)5$ against distinct values of ν with small initial values.

5. Conclusions

The main goal of this work is to investigate the dynamical behavior of the smoking mathematical model with the help of the Liouville-Caputo fractional derivative operator by using of the tools and techniques of fractional calculus. Through this work, the numerical solutions of the mathematical model under study were computed with different values of the fractional-order ν, approximation-order m, and residual error function, and initial conditions. And then we have confirmed that

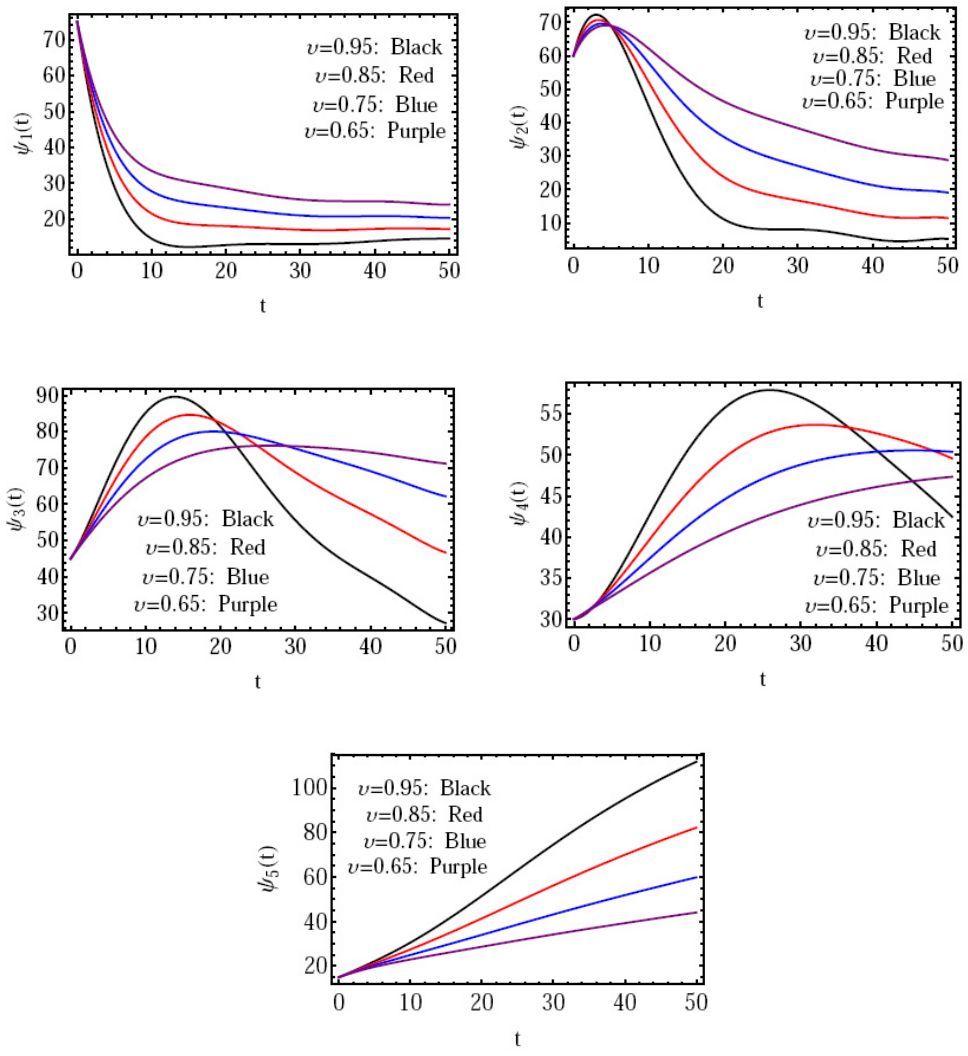

Figure 2. The approximate solution $\psi_i(t)$, $i = 1(1)5$ against distinct values of ν with large initial values.

the proposed technique is remarkably suitable to study this mathematical model effectively. In addition, we can control the accuracy of the error and reduce it by including additional terms from the approximate solution series, that is, by increasing m. Finally, we also concluded that the Liouville-Caputo fractional derivative operator is better suited for numerical simulations for the mathematical model under study in this article. The results obtained in graphical form are comparable to the results obtained by using the RK4 method. Our results also show that the proposed technique is accurate as well as computationally efficient. In our future investigation, we plan to deal with the same model, but with another type of frac-

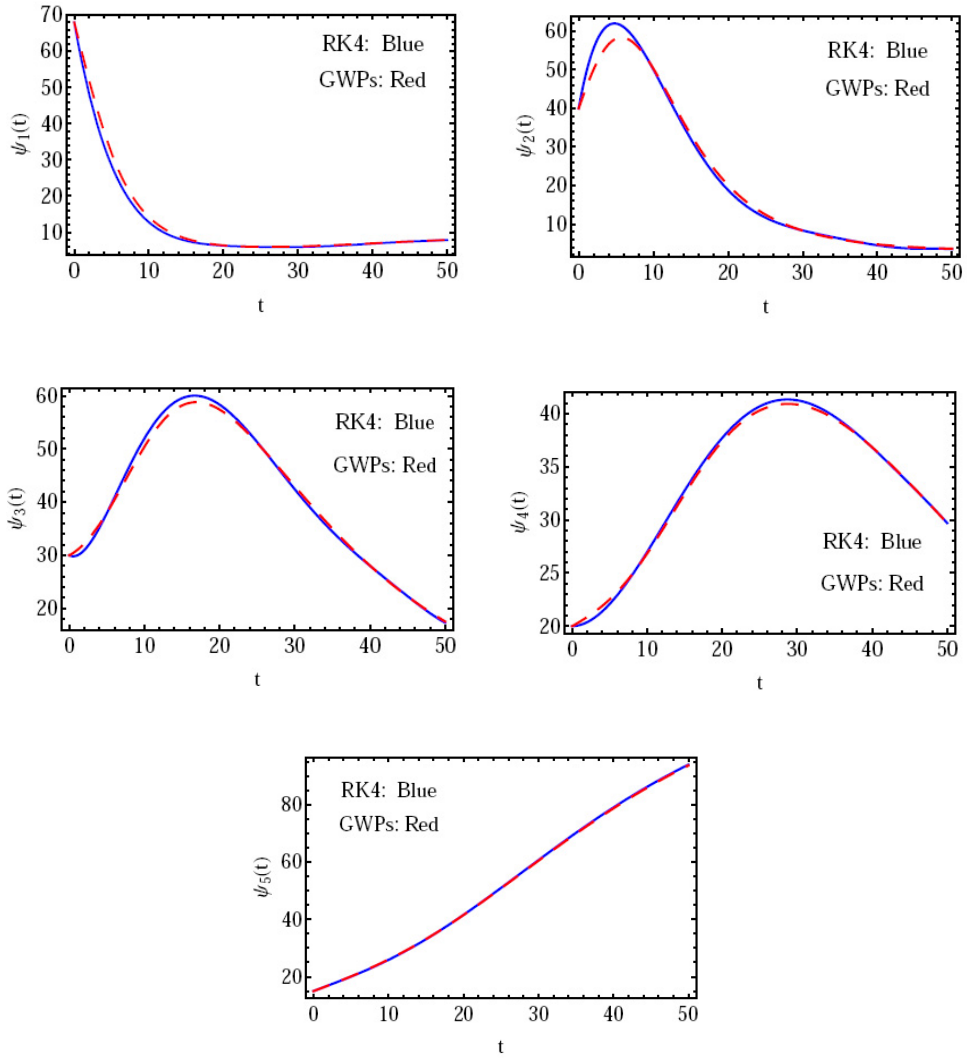

Figure 3. The solution $\psi_i(t)$, $i = 1(1)5$ by GWPs and RK4 methods $\nu = 1$ with small initial values.

tional derivative or another type of polynomials as a generalization of our present study.

6. Conflicts of interest

The authors declared that have no conflicts of interest.

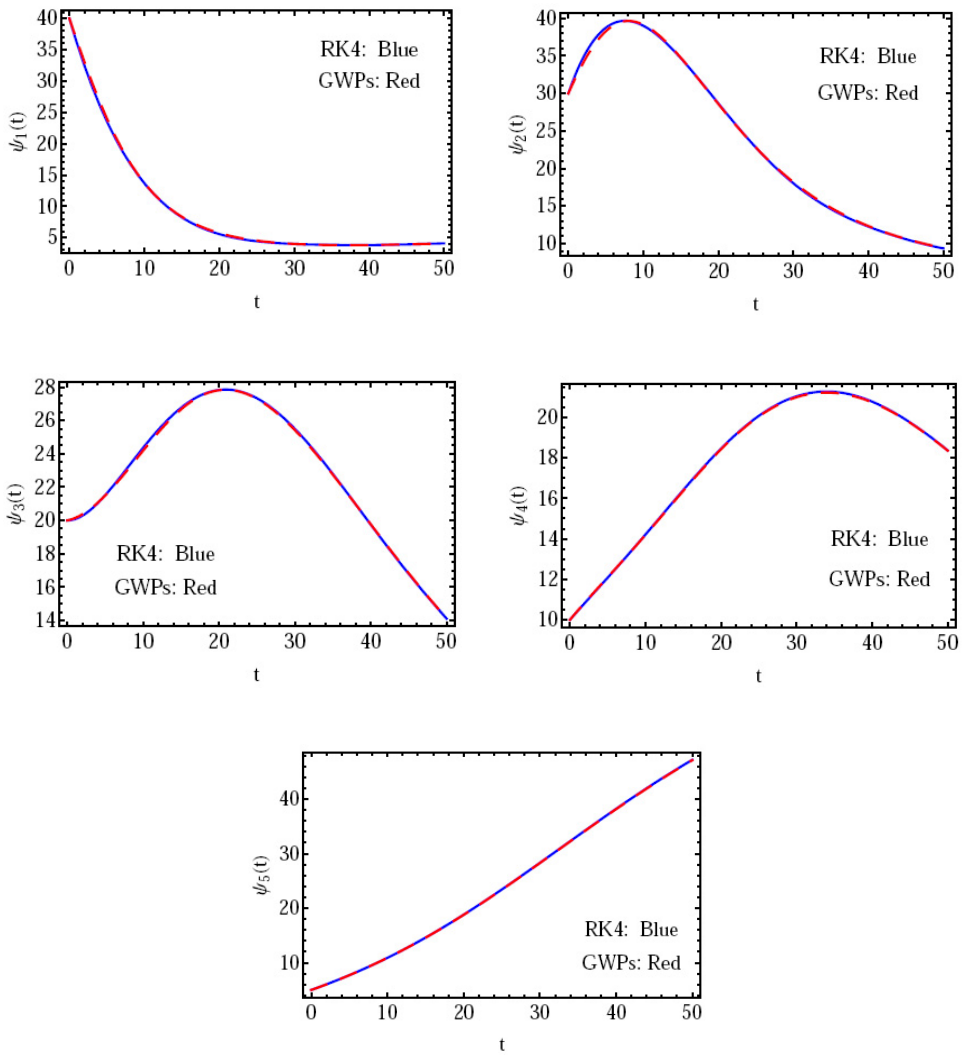

Figure 4. The solution $\psi_i(t)$, $i = 1(1)5$ by GWPs and RK4 methods $\nu = 1$ with large initial values.

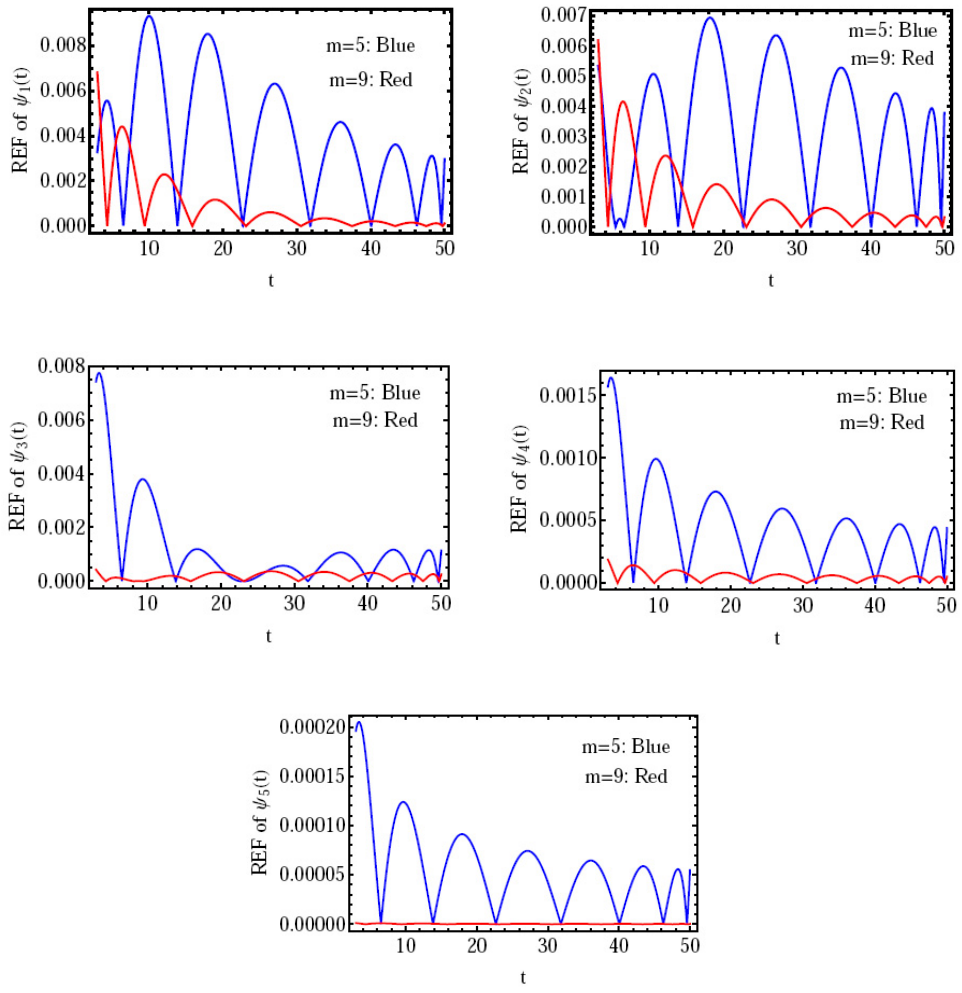

Figure 5. The REF of $\psi_i(t)$, $i = 1(1)5$ against distinct values of m.

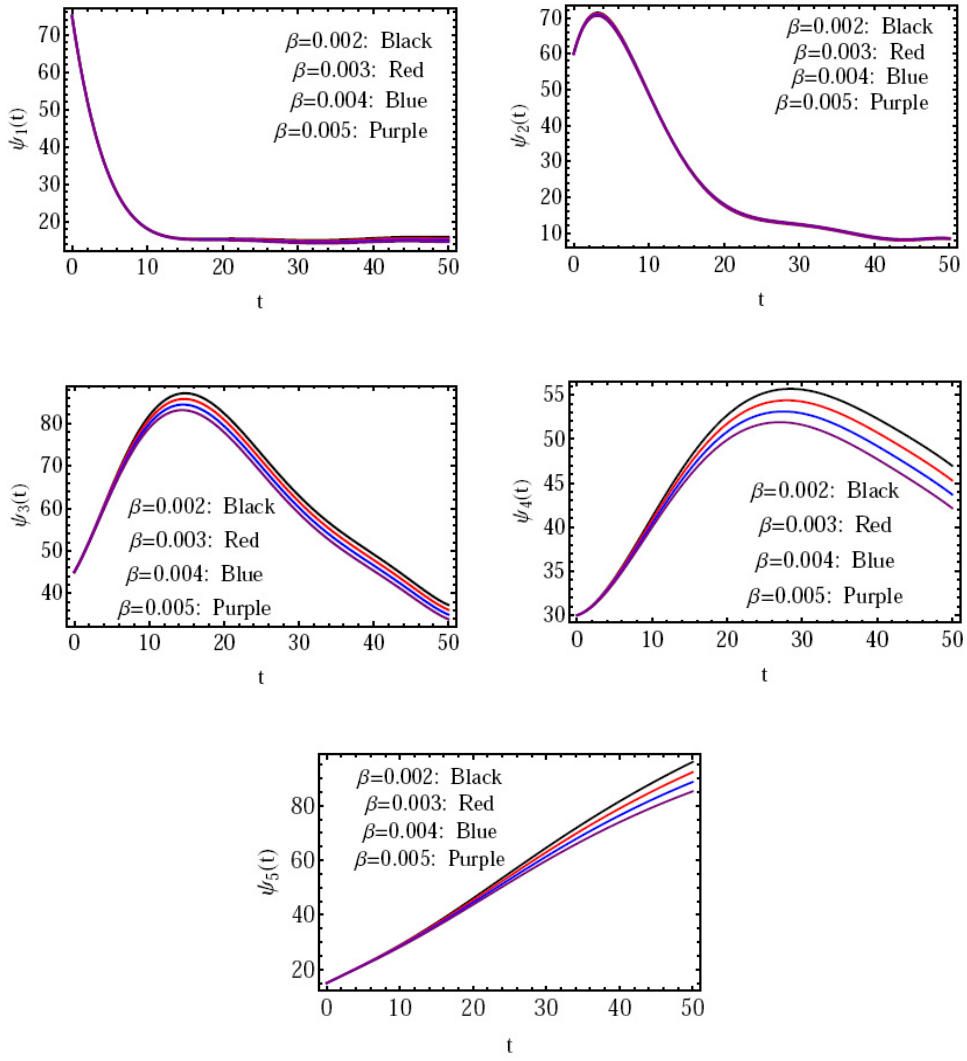

Figure 6. The effect of β on the approximate solution $\psi_i(t)$, $i = 1(1)5$.

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