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ANALYTICAL INTEGRABILITY OF PERTURBATIONS OF DEGENERATE QUADRATIC SYSTEMS*

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Abstract We consider analytic perturbations of quadratic homogeneous differential systems having an isolated singularity at the origin. Here we characterize the analytically integrable perturbations of quadratic homogeneous systems of the form $(\dot{x}, \dot{y})^T = f_1(P_1, Q_1)^T$ with $f_1(x, y)$ a non-zero linear homogeneous polynomial and $P_1(x, y), Q_1(x, y)$ non-zero linear homogeneous polynomials without common factors. We prove that all systems are orbitally equivalent to their quasi-homogeneous leading terms with respect to a certain type but not necessarily to the homogeneous leading terms. This result completes the previous results for the analytic perturbations of irreducible quadratic systems with analytic first integral which are orbitally equivalent to the homogeneous leading term, i.e. all are homogenizable.

Keywords Analytic integrability, quadratic differential systems, degenerate singular points, orbitally equivalence.

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1. Introduction

One of the main problems in qualitative theory of differential systems in the plane is the integrability problem which consists in determine when a differential system has a first integral of certain functional class defined in a neighborhood of a singular point. In [22] was proved the existence of a map that transforms any integrable system into a linear one. This result was generalized in [27] to n-dimensional systems. In both cases the differential system is orbitally equivalent to the linear differential system in a full Lebesgue measure subset of the domain of definition of the

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differential system, which implies that the orbital equivalence is not defined at the singular point. Here we study when the integrable system is orbitally equivalent to the linear part or leading part of the differential system in a neighborhood of the singular point.

The main goal of this paper is to solve the problem for differential systems which are perturbations of quadratic systems. Hence, in general, we consider the problem of having a homogeneous polynomial planar differential system

$$(\dot{x}, \dot{y})^T = \mathbf{F}_n(x, y) = (P_n(x, y), Q_n(x, y))^T, \tag{1.1}$$

with P_n and Q_n homogeneous polynomials of degree n, and we are interested to know whether an analytic perturbation of $\mathbf{F}_n$

$$(\dot{x}, \dot{y})^T = \mathbf{F} := \mathbf{F}_n + \text{h.o.t.} \tag{1.2}$$

has an analytic first integral at the origin, which is a singular point of the system. Indeed if the origin is not singular point, $\mathbf{F}(\mathbf{0}) \neq \mathbf{0}$, from flow box theorem [15, Cauchy-Arnold Theorem], the vector field is locally analytically integrable. Therefore, we will assume that the origin is an isolated singular point of system (1.2).

We say that an analytic vector field is *homogenizable* if it is orbitally equivalent to a homogeneous polynomial vector field, *i.e.* the system $\dot{\mathbf{x}} = \mathrm{d}\mathbf{x}/\mathrm{d}t = \mathbf{F}_n(\mathbf{x}) + \text{h.o.t.}$ by means of a near-identity change of variable $\mathbf{x} = \phi(\mathbf{y})$ and a formal reparameterization of the time $\mathrm{d}t/\mathrm{d}\tau = \eta(\mathbf{x})$, with $\eta(\mathbf{0}) = 1$, it is transformed into $\mathbf{y}' = \mathrm{d}\mathbf{y}/\mathrm{d}\tau = \mathbf{F}_n(\mathbf{y})$. We recall that the polynomial integrability of $\mathbf{F}_n \not\equiv 0$ is a necessary condition of analytical integrability of system (1.2).

For $n = 1$, assuming that the origin is an isolated singular point of $\mathbf{F}_1$, from [24,29,32] we have that $\mathbf{F}_1$+h.o.t. nondegenerate monodromic points and saddles are analytically integrable at the origin if, and only if, they are homogenizable (in this case, *linearizable*). A unified method to compute necessary and sufficient conditions of analytic integrability for such singular points using the blow-up method is given in [20, 21]. The most studied systems whose origin is a resonant saddle are the Lotka-Volterra systems, see [16, 17, 19, 23, 25, 26, 31] and references therein. We recall that if the origin is an isolated singular point, then the nodes and saddle-nodes points are not analytically integrable. Recently in [5] it is proved that an isolated nilpotent singular point is analytically integrable if, and only if, its lowest-degree quasi-homogeneous term is integrable and the complete vector field is orbitally equivalent to its lowest-degree quasi-homogeneous term.

For $n = 2$, that is, $\mathbf{F}_1$ is zero and $\mathbf{F}_2$ is non-zero, assuming that the origin is an isolated singular point of $\mathbf{F}_2$, in [7] has been proved that system (1.2) is analytically integrable if, and only if, it is homogenizable.

Finally, for $n = 3$, $\mathbf{F}_1 = \mathbf{F}_2 = \mathbf{0}$ and $\mathbf{F}_3$ is non-zero, assuming that the origin is an isolated singular point of $\mathbf{F}_3$, in [8, 9] was proved a similar result.

However, in general, not any analytically integrable vector field with zero linear part (a *degenerate singular point*) is orbitally equivalent to its first homogeneous term. Indeed the result for $n \geq 4$ is not satisfied. For example, the Hamiltonian system

$$\dot{x} = x(4y^3 - x^3 - 3x^2y^2), \quad \dot{y} = y(4x^3 - y^3 + 3x^2y^2),$$

it is analytically integrable, and it is non-orbitally equivalent to its leading term $\dot{x} = x(4y^3 - x^3), \dot{y} = y(4x^3 - y^3)$, see Theorem 3.20 of [9]. We note that although the leading term is irreducible, the system is non-orbitally equivalent to its homogeneous leading term. Other families of systems non-orbitally equivalent to their leading term can be seen in [1, 4, 10], see also references therein. In fact in [4] it is proved the existence of analytically integrable vector fields that are non-orbitally equivalent to their quasi-homogeneous leading terms.

If we assume that the origin of $\mathbf{F}_n$ is a not isolated singular point, there are few results about the analytical integrability of the vector fields $\mathbf{F}_n + \cdots$. Only the case $n = 1$ has been solved. If the origin of $\mathbf{F}_1$ is a saddle-node point, the perturbations of a saddle-node point are analytically integrable if, and only if, the vector field is reducible, i.e. the system has a curve of singular points, see [18]. If the origin is a non isolated nilpotent singular point of $\mathbf{F}_n$, in [2] has been proved that the analytically integrable perturbations of a nilpotent singular point are orbitally equivalent to a polynomially integrable quasi-homogeneous vector field.

We focus our study on the analytic integrability of the following system whose origin is an isolated singular point,

$$(\dot{x}, \dot{y})^T = f_1(P_1, Q_1)^T + \Big(\sum_{i+j\geq 3} a_{ij}x^iy^j, \sum_{i+j\geq 3} b_{ij}x^iy^j\Big)^T, \tag{1.3}$$

with $f_1(x, y)$ a non-zero homogeneous polynomial of degree one, $P_1(x, y), Q_1(x, y)$ non-zero homogeneous polynomials of degree one without common factors and $a_{ij}, b_{ij} \in \mathbb{R}$. Note that the quadratic part of the vector field is degenerate.

Here, we solve the analytic integrability problem of system (1.3). We prove that for some cases, the vector field has an analytic first integral if, and only if, it is homogenizable, *i.e.* it is orbitally equivalent to $f_1(P_1, Q_1)$. Nevertheless, for other cases, the vector field is analytically integrable if, and only if, there exists a certain type $\mathbf{t} = (t_1, t_2) \in \mathbb{N}^2$, with $\mathbf{t} \neq (1, 1)$ (the case $\mathbf{t} = (1, 1)$ is the homogeneous expansion of the vector field) such that the vector field (1.3) is orbitally equivalent to its quasi-homogeneous leading vector field with respect to the type $\mathbf{t}$.

1.1. Invariant curves and first integrals of vector fields

We deal with a vector field $\mathbf{F} = (P, Q)^T$ where P, Q are analytic functions at the origin with $P(\mathbf{0}) = Q(\mathbf{0}) = 0$. Throughout the paper, we denote the differential operator associated to the vector field $\mathbf{F}$ by F, that is, $F := P\partial_x + Q\partial_y$. We recall the concept of invariant curve and its associated cofactor.

Definition 1.1. A function $C \in \mathbf{C}[[x, y]]$ (algebra of formal power series in (x, y) over $\mathbf{C}$), with $C(\mathbf{0}) = 0$, is an invariant curve at the origin of the vector field $\mathbf{F}$, if there exists $K \in \mathbf{C}[[x, y]]$, cofactor of C, such that $F(C) = KC$. Moreover, if $K \equiv 0$, then the vector field $\mathbf{F}$ is formally integrable and C is a first integral of $\mathbf{F}$. If $K = \mathrm{div}(\mathbf{F})$ (divergence of $\mathbf{F}$), then C is an inverse integrating factor of $\mathbf{F}$.

We remark that if $C_1, \ldots, C_m$ are invariant curves of a vector field $\mathbf{F}$, then $C_1^{n_1} \cdots C_m^{n_m}$ is also an invariant curve of $\mathbf{F}$ whose cofactor is $n_1K_1 + \cdots + n_mK_m$, where K_j is the cofactor of C_j.

It is worth pointing out that for analytic vector fields, by [28, Theorem A], the existence of a formal first integral is equivalent to the existence of an analytic one. For this reason, when we use Taylor expansions of functions and vector fields, we do not consider convergence problems.

We introduce some notation and concepts. Given $\mathbf{t} = (t_1, t_2)$ with t_1 and t_2 natural numbers without common factors, a scalar function f of two variables is *a quasi-homogeneous function* of type or weight exponent $\mathbf{t}$ and degree j if $f(\varepsilon^{t_1}x, \varepsilon^{t_2}y) = \varepsilon^j f(x, y)$. The vector space of quasi-homogeneous polynomials of type $\mathbf{t}$ and degree j is denoted by $\mathcal{P}_j^{\mathbf{t}}$. A vector field $\mathbf{F} = (P, Q)^T$ is *a quasi-homogeneous vector field* of type $\mathbf{t}$ and degree j if $P \in \mathcal{P}_{j+t_1}^{\mathbf{t}}$ and $Q \in \mathcal{P}_{j+t_2}^{\mathbf{t}}$. We denote the vector space of the quasi-homogeneous polynomial vector fields of type $\mathbf{t}$ and degree j by $\mathcal{Q}_j^{\mathbf{t}}$. An analytic vector field can be expanded into quasi-homogeneous terms of type $\mathbf{t}$ of successive degrees. Thus, the vector field $\mathbf{F} \not\equiv \mathbf{0}$ can be written in the form $\mathbf{F} = \sum_{j \geq r} \mathbf{F}_j$ where $\mathbf{F}_j = (P_{j+t_1}, Q_{j+t_2})^T \in \mathcal{Q}_j^{\mathbf{t}}$ and $\mathbf{F}_r \not\equiv \mathbf{0}$.

Throughout the paper, we write $\mathbf{D}_0^{\mathbf{t}} = (t_1 x, t_2 y)^T \in \mathcal{Q}_0^{\mathbf{t}}$ (dissipative quasi-homogeneous vector field) and $\mathbf{X}_h = (-\partial h/\partial y, \partial h/\partial x)^T$ (Hamiltonian vector field associated to the polynomial h). The following splitting of a quasi-homogeneous vector field plays a main role in our study.

Proposition 1.1. [3, Prop. 2.7] *Every $\mathbf{F}_r \in \mathcal{Q}_r^{\mathbf{t}}$ can be uniquely written as $\mathbf{F}_r = \mathbf{X}_h + \mu \mathbf{D}_0^{\mathbf{t}}$ with $h := \frac{1}{r+|\mathbf{t}|}(\mathbf{D}_0^{\mathbf{t}} \wedge \mathbf{F}_r) \in \mathcal{P}_{r+|\mathbf{t}|}^{\mathbf{t}}$ (product wedge of both vector fields) and $\mu := \frac{1}{r+|\mathbf{t}|} div(\mathbf{F}_r) \in \mathcal{P}_r^{\mathbf{t}}$ (divergence of $\mathbf{F}_r$).*

The following results give an expression of the invariant curves at the origin of a quasi-homogeneous vector field.

Proposition 1.2. *Every quasi-homogeneous polynomial invariant curve at the origin of a quasi-homogeneous vector field $\mathbf{F}_r$ is given by $g_1^{n_1} g_2^{n_2} \dots g_m^{n_m}$ each g_j being an irreducible polynomial invariant curve at the origin of $\mathbf{F}_r$.*

Proof. We suppose that $g = g_1 u$, (g_1 an irreducible quasi-homogeneous polynomial and u a suitable quasi-homogeneous polynomial), it is an invariant curve at the origin of $\mathbf{F}_r$ with K_r cofactor of g. We have that $F_r(g_1 u) = g_1 F_r(u) + u F_r(g_1) = K_r g_1 u$, that is, $g_1(uK_r - F_r(u)) = uF_r(g_1)$. From the irreducibility of g_1, it has two situations: either g_1 is an irreducible invariant curve at the origin of $\mathbf{F}_r$, in such a case, u is also an invariant curve at the origin of $\mathbf{F}_r$ and we repeat the process for u. Or $u = g_1 v$ with v a quasi-homogeneous polynomial, i.e. $g = g_1^2 v$. We now have that $F_r(g_1^2 v) = g_1^2 F_r(v) + 2v g_1 F_r(g_1) = K_r g_1^2 v$. Thus, $g_1(vK_r - F_r(v)) = 2vF_r(g_1)$. Taking into account that the process is a finite process, reasoning in a similar way, the proof is completed. $\square$

Proposition 1.3. *Consider $\mathbf{F}_r \in \mathcal{Q}_r^{\mathbf{t}}$. Any factor of $\mathbf{D}_0^{\mathbf{t}} \wedge \mathbf{F}_r \in \mathcal{P}_{r+|\mathbf{t}|}^{\mathbf{t}}$ is an invariant curve at the origin of $\mathbf{F}_r$. Conversely, any irreducible quasi-homogeneous polynomial invariant curve at the origin of $\mathbf{F}_r$ is a factor of $\mathbf{D}_0^{\mathbf{t}} \wedge \mathbf{F}_r$.*

Moreover, if I is a polynomial first integral of $\mathbf{F}_r$ with $I(\mathbf{0}) = 0$ and $\mathbf{F}_r$ is irreducible, then $I = g_1^{n_1} g_2^{n_2} \cdots g_m^{n_m}$ where $g_1, \dots, g_m$ are all irreducible quasi-homogeneous factors of $\mathbf{D}_0^{\mathbf{t}} \wedge \mathbf{F}_r$ and $n_i > 0$.

Proof. From Proposition 1.1, we know that $\mathbf{F}_r = \mathbf{X}_h + \mu \mathbf{D}_0^{\mathbf{t}}$ with $h = \frac{1}{r+|\mathbf{t}|}(\mathbf{D}_0^{\mathbf{t}} \wedge \mathbf{F}_r)$ and $\mu = \frac{1}{r+|\mathbf{t}|}\mathrm{div}(\mathbf{F}_r)$. We prove that any factor of h is an invariant curve of $\mathbf{F}_r$.

If h is irreducible, the result follows since it is an invariant curve of $\mathbf{F}_r$. Otherwise, let $f \in \mathcal{P}_s^{\mathbf{t}}$ a factor of h, that is, $h = fg$ with g a suitable homogeneous polynomial and $F_r(f) = X_{fg}(f) + \mu D_0^{\mathbf{t}}(f) = f X_g(f) + s\mu f = (X_g(f) + s\mu) f$. Therefore, f is an invariant curve at the origin of $\mathbf{F}_r$.

We see that any irreducible quasi-homogeneous polynomial invariant curve of $\mathbf{F}_r$ is also a factor of h. Indeed, if $f \in \mathcal{P}_s^{\mathbf{t}}$ is an irreducible invariant curve at the origin of $\mathbf{F}_r$ with cofactor K_r then $K_r f = F_r(f) = X_h(f) + \mu D_0^{\mathbf{t}}(f) = X_h(f) + s\mu f$. Thus, $X_h(f) = (K_r - s\mu)f$ and f is an invariant curve at the origin of $\mathbf{X}_h$ and irreducible. So, f divides h.

Last on, if I is a first integral of $\mathbf{F}_r$ with $I(\mathbf{0}) = 0$, it is an invariant curve at the origin of $\mathbf{F}_r$, from Proposition 1.2, a factorization of I is formed by the irreducible factors of h. On the other hand, any first integral satisfying $I(\mathbf{0}) = 0$, is zero on every invariant curve (if $\mathbf{F}_r$ is irreducible, the curves are not curves of singular points). So, $n_i > 0$. □

1.2. Necessary condition of analytical integrability for perturbations of a class of quadratic systems

The following result provides a necessary condition of integrability for an analytic vector field.

Proposition 1.4. [Necessary condition of analytical integrability] *For a type* $\mathbf{t}$ *fixed, we consider* $\mathbf{F} = \mathbf{F}_r + \cdots$ *with* $\mathbf{F}_r \in \mathcal{Q}_r^{\mathbf{t}}$. *If* $\mathbf{F}$ *is formally integrable at the origin, then* $\mathbf{F}_r$ *is polynomially integrable.*

Proof. Let $C = \sum_{j \geq s} C_j,\ C_j \in \mathcal{P}_j^{\mathbf{t}}$ a formal invariant curve of $\mathbf{F}$ with cofactor $K = \sum_{j \geq r} K_j,\ K_j \in \mathcal{P}_j^{\mathbf{t}}$. By the lowest-degree quasi-homogeneous term of the equation $F(C) - KC = 0$, it has that C_s is a polynomial invariant curve at the origin of the polynomial vector field $\mathbf{F}_r$ with cofactor K_r. So, if C is a first integral of $\mathbf{F}$, then $K = 0$ and C_s is a first integral of $\mathbf{F}_r$. □

Remark 1.1. We note that the vector fields $(P_n, Q_n)^T \in \mathcal{H}_n = \{$homogeneous vector fields of degree $n\}$ are quasi-homogeneous vector fields of degree $n-1$ with respect to the type $(1,1)$, *i.e.* $\mathcal{H}_n = \mathcal{Q}_{n-1}^{\mathbf{t}}$ with $\mathbf{t} = (1,1)$. So, following the quasi-homogeneous notation, we emphasize that the quadratic vector fields $(P_2, Q_2)^T \in \mathcal{Q}_1^{\mathbf{t}}$ with $\mathbf{t} = (1,1)$.

Now we proceed with the study of the analytic integrability of irreducible system (1.3), that is the irreducible vector field whose first homogeneous component is $(P_2, Q_2)^T = f_1(P_1, Q_1)^T$ with f_1 linear homogeneous polynomial and P_1, Q_1 linear polynomials without common factors. The following result provides the expression of the lowest-degree component in the case of polynomial integrability of this class of vector fields.

Proposition 1.5. *If system (1.3) is formally integrable then there exist a quasi-homogeneous change of variables and a linear reparameterization of the time such that the vector field is transformed into one of the following vector fields whose quasi-homogeneous expansion with respect to the type* $\mathbf{t} = (t_1, t_2)$ *is given by*
(a) $\mathbf{F} = \mathbf{F}_1 + \cdots$, *with* $\mathbf{t} = (1,1)$ *and*

$$\mathbf{F}_1 = (y + x)(-qx, py))^T,\ \ p, q \in \mathbb{N}. \tag{1.4}$$

(b) $\mathbf{F} = \mathbf{F}_1 + \cdots$, *with* $\mathbf{t} = (1,1)$ *and*

$$\mathbf{F}_1 = x(-qy, px)^T,\ \ p, q \in \mathbb{N}. \tag{1.5}$$

(c.1) $\mathbf{F} = \mathbf{F}_n + \cdots$*, with* $\mathbf{t} = (1, n)$ *and*

$$\mathbf{F}_n = (x(-(q+r)y - (q-r)x^n), py^2 - n(q-r)x^n y - (p + n(q+r))x^{2n})^T. \quad (1.6)$$

(c.2) $\mathbf{F} = \mathbf{F}_n + \cdots$ *with* $\mathbf{t} = (1, n)$ *and*

$$\mathbf{F}_n = (-qxy, py^2 + \sigma(p+nq)x^{2n})^T. \quad (1.7)$$

(c.3) $\mathbf{F} = \mathbf{F}_{2n+1} + \cdots$ *with* $\mathbf{t} = (2, 2n+1)$ *and*

$$\mathbf{F}_{2n+1} = (-qxy, py^2 + \frac{1}{2}(2p + (2n+1)q)x^{2n+1})^T. \quad (1.8)$$

(c.4) $\mathbf{F} = \mathbf{F}_n + \cdots$ *with* $\mathbf{t} = (1, n)$ *and*

$$\mathbf{F}_n = (-qxy - qn_2Ax^{n+1}, py^2 + [(p+nq)n_3 + pn_2]Ax^n y)^T. \quad (1.9)$$

(c.5) $\mathbf{F} = \mathbf{F}_n + \cdots$ *with* $\mathbf{t} = (1, n)$ *and*

$$\mathbf{F}_n = (y + Bx^n)(-qx, py)^T. \quad (1.10)$$

Proof. We first fix the type $\mathbf{t} = (1, 1)$. Let $I = I_M + \cdots$ a formal first integral of $\mathbf{F}$. Equation $F(I) = 0$ for degree $M+1$ is $F_1(I_M) = 0$, i.e. $\mathbf{F}_1$ is polynomially integrable and I_M is a first integral of $\mathbf{F}_1$. We seek the vector fields $\mathbf{F}_1$ satisfying the condition $F_1(I_M) = 0$ (necessary condition of analytical integrability). By Proposition 1.1, $\mathbf{F}_1$ is $\mathbf{F}_1 = \mathbf{X}_h + \mu\mathbf{D}_0$ with $\mathbf{D}_0 = (x, y)^T$, $h \in \mathcal{P}_3^{\mathbf{t}}$ (cubic homogeneous polynomial) and $\mu \in \mathcal{P}_1^{\mathbf{t}}$ (linear homogeneous polynomial).

If the polynomial h is identically zero, we have that $\mathbf{F}_1 = \mu\mathbf{D}_0$ and it is non-formally integrable. Otherwise, h has always a linear factor, $a_1x + b_1y$ with a_1 and b_1 constants, since h is a cubic homogeneous polynomial. We can assume that $h = xp_2$ with p_2 a homogeneous polynomial of degree two, since if $a_1 = 0$ we do the change $(x, y) \to (y, x)$. Otherwise, we do $(x, y) \to (a_1x + b_1y, y)$. We distinguish the following cases according to the factors of the polynomial p_2:

Case (a). Assume $h = ax(x+by)(x+cy)$ with $bc \neq 0$ and $b \neq c$, with a, b, c constants.

The linear change of variables $(x, y) \to ((c-b)x, c(x+by))$ and the linear reparameterization of the time $at = bc(c-b)\tau$, transform $\mathbf{F}_1$ into $\tilde{\mathbf{F}}_1 = \mathbf{X}_{\tilde{h}} + \tilde{\mu}\mathbf{D}_0$ with $\tilde{h} = xy(y-x)$. We write the linear polynomial $\tilde{\mu} = Ax + By$. From Proposition 1.3, if it exists a first integral of $\tilde{\mathbf{F}}_1$, it would have the expression $I_M = x^p y^q (x-y)^r$ with p, q, r non-negative integers. Note that p, q, r are zero if, and only if, $x = 0$, $y = 0$, $y - x = 0$, are curves of singular points of $\tilde{\mathbf{F}}_1$, respectively. By imposing $\tilde{F}_1(I_M) = 0$, we arrive to

$$A = \frac{1}{M}(p - 2q + r), B = \frac{1}{M}(q - 2p + r),$$

with $M = p + q + r$. We distinguish the following cases:

- If $x = 0$ is a curve of singular points, that is $p = 0$, by means of the change $(x, y) \to (x - y, y)$ and the reparameterization of the time $3t = M\tau$ and renaming r by p, the vector field $\tilde{\mathbf{F}}_1$ turns on (1.4).
- If $y = 0$ is a curve of singular points, that is $q = 0$, by performing $3t = -(p+r)\tau$ and changing r by q, we arrive to (1.4).

• If $y-x=0$ is a curve of singular points, that is $r=0$, by means of the change $(x,y)\to(x,-y)$ and the reparemeterization of the time $3t=(p+q)\tau$, $\tilde{\mathbf{F}}_1$ turns on (1.4).

Case (b). Assume $h=ax((x+by)^2+x^2)$ with $b\neq 0$.

The linear change $(x,y)\to(x,x+by)$ and the scaling $at=\tau$, transform $\mathbf{F}_1$ into $\tilde{\mathbf{F}}_1=\mathbf{X}_{\tilde{h}}+\tilde{\mu}\mathbf{D}_0$ with $\tilde{h}=x(x^2+y^2)$. We write $\tilde{\mu}=Ax+By$. Now, by Proposition 1.3, the expression of a first integral of $\tilde{\mathbf{F}}_1$ would be $I_M=x^p(x^2+y^2)^q$ with p,q non-negative integers. By imposing $\tilde{F}_1(I_M)=0$, we have that $A=0, B=\frac{2}{M}(p-q)$, with $M=p+2q$. If $x=0$ is a curve of singular points, that is $p=0$, by performing the reparemeterization of the time $3t=M\tau$, $\tilde{\mathbf{F}}_1$ turns on (1.5).

Case (c). Assume $h=ax(x+by)^2$ with $a,b\neq 0$.

The linear change of variables $(x,y)\to(x,x+by)$ and the scaled of time $at=\tau$, transform $\mathbf{F}_1$ into $\tilde{\mathbf{F}}_1=\mathbf{X}_{\tilde{h}}+\tilde{\mu}\mathbf{D}_0$ with $\tilde{h}=xy^2$. We write $\tilde{\mu}=Ax+By$. From Proposition 1.3, if it exists a first integral of $\tilde{\mathbf{F}}_1$, it would have the expression $I_M=x^py^q$ with p,q non-negative integers. By imposing $\tilde{F}_1(I_M)=0$, we arrive to $A=0, B=\frac{1}{M}(2p-q)$, with $M=p+q$. So, $\tilde{\mathbf{F}}_1=\frac{3}{p+q}y(-qx,py)$. By performing the reparameterization of the time $3t=M\tau$, we have that $\mathbf{F}=y(-qx,py)^T+\cdots$. Let note that the quadratic terms of the vector fields $\mathbf{F}$ do not determine its Newton diagram. For that, we need to provide terms of higher degree.

The analytic perturbations of $\dot{\mathbf{x}}=y(-qx,py)^T,\ p,q\in\mathbb{N}$, can be written as

$$\dot{x}=x(-qy+f_2(x)+yf_1(x,y))+A(y),$$
$$\dot{y}=y(py+g_2(x)+yg_1(x,y))+B(x).$$

If $f_2\equiv 0$ and $g_2\equiv 0$, then $B\not\equiv 0$ since the vector field is irreducible (the origin is an isolated singular point). We write $B(x)=b_{m-1}x^m+\cdots$, with $b_{m-1}\neq 0$. In this case, the quasi-homogeneous expansion of the vector field respect to the type $\mathbf{t}=(2,m)$ is $\mathbf{F}=(-qxy,py^2+b_{m-1}x^m)^T+\cdots$.

We first assume that $m=2n$. Performing the change $(x,y)\to(x,\sqrt{\frac{p+nq}{|b_{2n-1}|}}y)$ and the reparemeterization $t\to\sqrt{\frac{|b_{2n-1}|}{p+nq}}t$, the vector field is transformed into (1.7). It is easy to check that $x^{2p}(y^2+\mathrm{sign}(b_{2n-1})x^{2n})^q$ is a first integral.

For $m=2n+1$, the change $(x,y)\to(\lambda x,y)$ with $\lambda^{2n+1}=\frac{b_{2n}}{2p+(2n+1)q}$, transforms the vector field into (1.8). It is easy to check that $x^{2p}(y^2+x^{2n+1})^q$. is a first integral.

Otherwise, we write $f_2(x)=c_nx^n+\cdots$, $g_2(x)=d_nx^n+\cdots$ with $n=\min\{j,\ c_j^2+d_j^2\neq 0\}$.

If $B\equiv 0$, then the quasi-homogeneous expansion of the vector field respect to the type $\mathbf{t}=(1,n)$ is $\mathbf{F}=\mathbf{F}_n+\cdots$ with $\mathbf{F}_n=(x(-qy+c_nx^n),py^2+d_nx^ny)^T+\cdots$. From [6], if $\mathbf{F}_n$ is polynomially integrable then it can be transformed into (1.9) or (1.10).

If $B(x)=b_{m-1}x^m+\cdots$, with $b_{m-1}\neq 0$, we distinguish three cases:

If $m>2n-1$, then the quasi-homogeneous expansion of the vector field respect to the type $\mathbf{t}=(1,n)$ is $\mathbf{F}=(x(-qy+c_nx^n),py^2+d_nx^ny)^T+\cdots$. So, we arrive to (1.9) or (1.10).

If $m=2n-1$, then the quasi-homogeneous expansion of the vector field respect to the type $\mathbf{t}=(1,n)$ is $\mathbf{F}=\mathbf{F}_n+\cdots$ with $\mathbf{F}_n=(x(-qy+c_nx^n),py^2+d_nx^ny+b_{2n-1}x^{2n})^T$.

If $\mathbf{F}_n$ is reducible, then $-qy + c_n x^n$ is a common factor of both components of $\mathbf{F}_n$. In such a case, $\mathbf{F}_n = (-qy + c_n x^n)(x, -\frac{1}{q^2}(pqy + (pc_n + qd_n)x^n))^T$. The quasi-homogeneous change of variables $(x, y) \to (x, y + \frac{pc_n+qd_n}{(p+nq)q} x^n)$, transforms the vector field into $(y + Bx^n)(-qx, py)^T$ with $B = c_n + \frac{pc_n+qd_n}{p+nq}$, that is, the vector field is (1.10).

We now assume that $\mathbf{F}_n$ is irreducible. Consider $\Delta := (d_n - nc_n)^2 - 4(p + nq)b_{2n-1}$. If $\Delta = 0$, the vector field is not polynomially integrable since the origin is an isolated singular point and the polynomial $h = (x, ny)^T \wedge \mathbf{F}_n = x((p + nq)y^2 + (d_n - nc_n)yx^n + b_{2n-1}x^{2n})$ has multiple factors.

Otherwise, we perform the quasi-homogeneous change of variables $(x, y) \to (x, y + \frac{d_n-nc_n}{2(p+nq)} x^n)$. Next, by means of a scaled and a reparamterization of the time-variable, according the sign of Δ, h can be transformed into $h = x(y^2 + \sigma x^{2n})$ with $\sigma = \pm 1$. So, $\mathbf{F}_n = \mathbf{X}_h + (Ax^n + By)(x, ny)^T$.

For $\sigma = 1$, a first integral is of the form $I_p = x^{n_1}(y^2 + x^{2n})^{n_2}$, with n_1 and n_2 natural numbers. Imposing the equation of integrability, we have that $A = 0$ therefore $b_{2n-1} = 0$.

For $\sigma = -1$, a first integral is of the form $I_p = x^{n_1}(y - x^n)^{n_2}(y + x^n)^{n_3}$, witrh n_1, n_2 and n_3 natural numbers. Imposing the equation of integrability, we have that

$$A = \frac{(2n+1)(n_3 - n_2)}{n_1 + n(n_2 + n_3)}, \qquad B = \frac{2n_1 - n_2 - n_3}{n_1 + n(n_2 + n_3)}.$$

Rescaling the time-variable $t \to \frac{p+nq}{2n+1} t$ and renaming $n_1 = p,\ n_2 = q, n_3 = r$, we have that $\mathbf{F}_n$ is of the form (1.6) and a first integral is $x^p(y - x^n)^q(y + x^n)^r$.

If $m < 2n - 1$, then the quasi-homogeneous expansion of the vector field respect to the type $\mathbf{t} = (2, m)$ is $\mathbf{F} = (-qxy, py^2 + b_{m-1}x^m)^T + \cdots$ and therefore, it can be transformed into (1.7) or (1.8). □

2. Orbital normal form and analytic integrability

We do not consider questions of convergence in the normal forms because, as we mention before, the formal integrability is equivalent to the analytical integrability for the vector fields analyzed, see [28, Theorem A]. Several orbital normal forms of vector fields, whose leader homogeneous term is quadratic, have been provided by several authors, see for example [11–14, 30].

The following orbital normal form of the perturbations of quasi-homogeneous vector fields has been provided in [5].

Proposition 2.1. *Given* $\mathbf{F} = \mathbf{F}_r + \sum_{j\geq 1} \mathbf{F}_{r+j}$ *with* $\mathbf{F}_{r+j} \in \mathcal{Q}^{\mathbf{t}}_{r+j}$. *If* $\mathrm{Ker}\left(\ell^{\mathrm{c}}_{r+j+|\mathbf{t}|}\right) = \{0\}$ *for all* $j \in \mathbb{N}$ *then* $\mathbf{F}$ *is orbitally equivalent to*

$$\mathbf{G} = \mathbf{F}_r + \sum_{j\geq 1} \mathbf{G}_{r+j},\ \text{ with } \mathbf{G}_{r+j} = \mathbf{X}_{\delta_{r+j+|\mathbf{t}|}} + \eta_{r+j}\mathbf{D}^{\mathbf{t}}_0 \in \mathcal{Q}^{\mathbf{t}}_{r+j},$$

where $\delta_{r+j+|\mathbf{t}|} \in \mathrm{Cor}\left(\ell^c_{r+j+|\mathbf{t}|}\right)$ *and* $\eta_{r+j} \in \mathrm{Cor}\left(\ell_{r+j}\right)$, *where* $\ell^c_{r+j+|\mathbf{t}|}$ *is the linear operator (Lie operator of* $\mathbf{F}_r$ *moved)*

$$\ell^{\mathrm{c}}_{r+j+|\mathbf{t}|} : \Delta_{j+|\mathbf{t}|} \longrightarrow \Delta_{r+j+|\mathbf{t}|}, \tag{2.1}$$

$$g_{j+|\mathbf{t}|} \to \mathrm{Proy}_{\Delta_{r+j+|\mathbf{t}|}}(F_r - \tfrac{div(\mathbf{F}_r)}{r+j+|\mathbf{t}|}D_0)(g_{j+|\mathbf{t}|}),$$

the subspaces $\Delta_{j+|\mathbf{t}|}$ *satisfy* $\mathcal{P}_{j+|\mathbf{t}|} = \Delta_{j+|\mathbf{t}|} \bigoplus h\mathcal{P}_{j-r}$ *with* $h = \mathbf{D}_0^{\mathbf{t}} \wedge \mathbf{F}_r$ *(such subspaces must be considered as fixed) and* ℓ_{r+j} *is the Lie-derivative operator of* $\mathbf{F}_r$*, i.e.*

$$\begin{aligned}\ell_{r+j} : \mathcal{P}_j^{\mathbf{t}} &\longrightarrow \mathcal{P}_{r+j}^{\mathbf{t}}, \\ p_j &\to F_r(p_j).\end{aligned} \tag{2.2}$$

We give the following results on formal integrability of the orbital normal forms.

Proposition 2.2. *Consider the irreducible and formal vector field* $\mathbf{F} = \mathbf{F}_r + \mu\mathbf{D}_0^t$*, with* $\mathbf{F}_r = f\mathbf{G} \in \mathcal{Q}_r^{\mathbf{t}}$*,* $\mu = \mu_{r+1} + \cdots$ *and assume that* $f \in \mathcal{P}_s^{\mathbf{t}}$ *is not an invariant curve of* G*. Then,* $\mathbf{F}$ *is not formally integrable.*

Proof. The curve $f(x, y) = 0$ is not a curve of singular points of $\mathbf{F}$ since $\mathbf{F}$ is an irreducible vector field. Moreover, the quasi-homogeneous polynomial f is an invariant curve of $\mathbf{F}$ since

$$F(f) = (fG + \mu D_0^t)(f) = fG(f) + sf\mu = f(G(f) + s\mu).$$

Thus, a primitive first integral of $\mathbf{F}$, if it exists, it would be of the form $I = f^n g$, with n a natural number and g a formal function $g = g_m + \cdots$. So, from $F(f^n g) = 0$, it has that $F_r(f^n g_m) = fG(f^n g_m) = 0$, then $G(f^n g_m) = 0$ for all $(x, y) \in \mathbb{R}^2$ such that $f(x, y) \neq 0$. As $G(f^n g_m)$ is a polynomial and it is null except in a measure-zero set, it has that $G(f^n g_m) \equiv 0$ that is, $f^n g_m$ is a polynomial first integral of $\mathbf{G}$ and $f(\mathbf{0}) = 0$, therefore f would be an invariant curve of $\mathbf{G}$. This fact contradicts the initial hypothesis. $\square$

Proposition 2.3. *Consider the irreducible and formal vector field* $\mathbf{F} = \mathbf{F}_r + \mu\mathbf{D}_0^t$*, with* $\mathbf{F}_r \in \mathcal{Q}_r^{\mathbf{t}}$ *irreducible vector field and* $I_M \in \mathcal{P}_M^{\mathbf{t}}$ *a first integral of* $\mathbf{F}_r$ *and assume that* $\mu = \mu_{r+1} + \cdots$ *with* $\mu_{r+j} \in \mathrm{Cor}\,(\ell_{r+j})$ *and* $\mathrm{Cor}\,(\ell_{r+j+M}) = I_M \mathrm{Cor}\,(\ell_{r+j})$*, for all* j*. Then,* $\mathbf{F}$ *is formally integrable if, and only if,* $\mu \equiv 0$*.*

Proof. We see the necessary condition. Assume that $\mathbf{F}$ is formally integrable and not all the μ_j are zero. Let N be defined by $N = \min\{j > 1 : \mu_j \not\equiv 0\}$. A formal first integral of $\mathbf{F}$ is of the form $I = I_M^l + \sum_{j>Ml} I_j$ with $I_j \in \mathcal{P}_j$. Imposing the integrability condition we have

$$\begin{aligned}0 = (F(I))_{N+Ml} &= (\mu_N D_0)(I_M^l) + F_r(I_{Ml+N-1}) \\ &= Ml\mu_N I_M^l + \ell_{Ml+N}\left(I_{Ml+N-1}\right),\end{aligned}$$

i.e. $\mu_N I_M^l \in \mathrm{Range}\,(\ell_{Ml+N})$. But, by hypothesis, $\mu_N I_M^l \in \mathrm{Cor}\,(\ell_{Ml+N})$ it which is a contradiction. Thus, $\mathbf{F}$ is not formally integrable.

The sufficient condition is trivial since if $\mu = 0$, then $\mathbf{F} = \mathbf{F}_r$ and I_M is a first integral of $\mathbf{F}$. $\square$

We recall that *an inverse integrating factor* of a system is an invariant curve whose cofactor is the divergence of the vector field.

Proposition 2.4. *Consider the irreducible and formal vector field* $\mathbf{F} = \mathbf{F}_r + \mu\mathbf{D}_0^t$*, with* $\mathbf{F}_r \in \mathcal{Q}_r^{\mathbf{t}}$ *irreducible vector field and* $I_M \in \mathcal{P}_M^{\mathbf{t}}$ *a first integral of* $\mathbf{F}_r$ *and assume that* $\mu = \mu_{r+1} + \cdots$ *with* $\mu_{r+j} \in \mathrm{Cor}\,(\ell_{r+j})$ *and* $\mathrm{Cor}\,(\ell_{r+j+M}) = I_M \mathrm{Cor}\,(\ell_{r+j})$*, for all* j*. Then,* $\mathbf{F}$ *is formally integrable if, and only if,* $\mathbf{D}_0^{\mathbf{t}} \wedge \mathbf{F}_r$ *is an inverse integrating factor of* $\mathbf{F}$*.*

Proof. We prove that the condition is necessary. We assume that $\mathbf{F} = \mathbf{F}_r + \cdots$ is formally integrable. From Proposition 2.3, it has that $\mathbf{F} = \mathbf{F}_r$, which has the inverse integrating factor $\mathbf{D}_0^{\mathbf{t}} \wedge \mathbf{F}_r$.

Now we see the sufficiency of the condition. Let $V = V_j + \cdots$ be a formal inverse integrating factor of $\mathbf{F}$, *i.e.* $F(V) = \mathrm{div}(\mathbf{F})V$. Expanding above equation, we have that $F_r(V_j) = \mathrm{div}(\mathbf{F}_r)V_j$, thus V_j is a polynomial inverse integrating factor of $\mathbf{F}_r$. Therefore $V = V_{r+|\mathbf{t}|} + \cdots$ with $V_{r+|\mathbf{t}|} = \mathbf{D}_0^{\mathbf{t}} \wedge \mathbf{F}_r$.

Also, the unique invariant curves at the origin of $\mathbf{F}$ are the factors of $h := \mathbf{D}_0^{\mathbf{t}} \wedge \mathbf{F}_r$. So, we have that $V = hu$ with u a formal function with $u(\mathbf{0}) = 1$. Equation $F(V) - V\mathrm{div}(\mathbf{F}) = 0$ is

$$0 = uF(h) + hF(u) - hu\,\mathrm{div}(\mathbf{F}).$$

Since $F(h) = h\mathrm{div}(\mathbf{F}_r) + \sum_{j>r}(r+|\mathbf{t}|)h\mu_j$ and $\mathrm{div}(\mathbf{F}) = \mathrm{div}(\mathbf{F}_r) + \sum_{j>r}(j+2)\mu_j$, we have that

$$0 = h(F(u) - u\sum_{j>r}(j+2-|\mathbf{t}|-r)\mu_j). \tag{2.3}$$

We see that $\mu_j = 0$ for all $j > r$. Indeed, otherwise, let $j_0 = \min\{j \in \mathbb{N} : \mu_{j+r} \not\equiv 0\}$. As $\mu_{j_0-k+r} = 0$ for $1 \le k \le j_0$, and expanding $u = 1 + \sum_{i\ge 1} u_i$, equation (2.3) to degree $j_0 + r + |\mathbf{t}| + 1$ we get $F_r(u_{j_0}) = j_0\mu_{j_0+r}$, i.e. $\mu_{j_0+r} \in \mathrm{Cor}\,(\ell_{j_0+r})$ and $\mu_{j_0+r} \in \mathrm{Range}\,(\ell_{j_0+r})$. Hence we arrive to $\mu_{j_0+r} = 0$. Therefore, $\mathbf{F} = \mathbf{F}_r$ and I_M is a first integral of $\mathbf{F}$. □

3. Main results

Our purpose is to find a link between orbital equivalence and the analytic integrability of the perturbations of a quadratic system (1.3), $(\dot{x}, \dot{y}) = f_1(P_1, Q_1) + \cdots$, for each case given in Proposition 1.5.

3.1. Case (a)

We first give an orbital normal form for this class of systems.

Proposition 3.1. *A formal orbital normal form for the system* $(\dot{x}, \dot{y}) = (y + x)(-qx, py) + \cdots$*, with* p, q *natural numbers is*

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} -qx(x+y) \\ py(x+y) \end{pmatrix} + \mu(x,y)\begin{pmatrix} x \\ y \end{pmatrix}, \tag{3.1}$$

where $\mu(0,0) = 0$ *and* $\mu(x,y)$ *does not have any linear terms.*

Proof. We fix the type $\mathbf{t} = (1,1)$. In this case, we consider Taylor expansions, $\mathbf{F} = \mathbf{F}_1 + \cdots$ with $\mathbf{F}_1 = (-qxy, px^2)^T \in \mathcal{Q}_1^{\mathbf{t}}$ (let note us that the quasi-homogeneous degree does not coincide with the homogeneous degree), $h := \mathbf{D}_0^{\mathbf{t}} \wedge \mathbf{F}_1 = (p+q)xy(x+y)$ and $\mathrm{div}(\mathbf{F}_1) = (-q+2p)y + (p-2q)x$.

Consider the translated vector field $\tilde{\mathbf{F}}_1^{(k)} = \mathbf{F}_1 - \frac{1}{k+3}\mathrm{div}(\mathbf{F}_1)\mathbf{D}_0^{\mathbf{t}}$. A complementary subspace to $h\mathcal{P}_{k-1}$ and $h\mathcal{P}_k$ are $\Delta_{k+2} = \mathrm{span}\{x^{k+2}, x^{k+1}y\}$ and $\Delta_{k+3} = \mathrm{span}\{x^{k+3}, x^{k+2}y\}$.

The transformed of the basis are

$$\begin{aligned} F_1^{(k)}(x^{k+2}) &= -\tfrac{k+2}{k+3}(p+(k+1)q) + \lambda_1 x^{k+2}y, \\ F_1^{(k)}(x^{k+1}y) &= \tfrac{k+2}{k+3}(p+q)x^{k+1}y + \lambda_2 x^k h, \\ F_1^{(k)}(y^{k+2}) &= \tfrac{k+2}{k+3}((k+1)p+q)y^{k+3} + \lambda_3 x^{k+2}y + \lambda_4 x^{k+3}. \end{aligned}$$

So, the matrix associated to the operator $\ell_{k+3}^{(c)}$ is also not singular, therefore, we have that $\mathrm{Ker}(\ell_{k+3}^{(c)}) = \{0\}$ and $\mathrm{Cor}(\ell_{k+3}^{(c)}) = \{0\}$. □

We give the following result on analytic integrability of the vector fields considered.

Theorem 3.1. *The irreducible vector field* $\mathbf{F} = (-qx(x+y), py(x+y))^T + \cdots$, *with* p, q *natural numbers is not analytically integrable.*

Proof. From Proposition 3.1, the vector field $\mathbf{F}$ is orbitally equivalent to a formal vector field of the form $\mathbf{F}^* = (-qx(x+y), py(x+y)^T + \mu \mathbf{D}_0^{\mathbf{t}}$, with $\mathbf{t} = (1,1)$. On the other hand, $(-qx(x+y), py(x+y)^T = (x+y)(-qx, py)^T$ and $x+y$ is not an invariant curve of $(-qx, py)^T$, therefore from Proposition 2.2, $\mathbf{F}^*$ is not formally integrable and thus $\mathbf{F}$ is not analytically integrable. □

3.2. Case (b)

Proposition 3.2. *A formal orbital normal form for the system* $(\dot{x}, \dot{y}) = x(-qy, px) + \cdots$ *with* p, q *natural numbers is*

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} -qxy \\ px^2 \end{pmatrix} + \mu(x,y)\begin{pmatrix} x \\ y \end{pmatrix}, \tag{3.2}$$

where $\mu(0,0) = 0$ *and* $\mu(x,y)$ *does not have any linear terms.*

Proof. We fix the type $\mathbf{t} = (1,1)$. Therefore, $h := \mathbf{D}_0^{\mathbf{t}} \wedge \mathbf{F}_1 = x(px^2 + qy^2)$ and $\mathrm{div}(\mathbf{F}_1) = -qy$.

We prove that the kerf of the operator ℓ_{k+3}^{c} defined in (2.1) is a trivial set, *i.e.* $\mathrm{Ker}\left(\ell_{k+3}^{c}\right) = \{0\}$ for all $k \in \mathbb{N}$.

We denote $\mathbf{F}_1^{(k)} = \mathbf{F}_1 - \frac{1}{k+3}\mathrm{div}(\mathbf{F}_1)(x,y)^T = \frac{1}{k+3}\begin{pmatrix} -(k+2)qxy \\ (k+3)px^2 + qy^2 \end{pmatrix}$. Consider the following bases for departure and arrival spaces of the operator ℓ_{k+3}^{c}: $\Delta_{k+2} = \mathrm{span}\{x^{k+2}, x^{k+1}y, y^{k+2}\}$ and $\Delta_{k+3} = \mathrm{span}\{x^{k+3}, x^{k+2}y, y^{k+3}\}$. Taking into account that $x^i y^j h = px^{i+3}y^j + qx^{i+1}y^{j+2}$, we have that

$$\begin{aligned} F_1^{(k)}(x^{k+2}) &= -\tfrac{q(k+2)^2}{k+3}x^{k+2}y, \\ F_1^{(k)}(x^{k+1}y) &= \tfrac{p(k+2)^2}{k+3}x^{k+3} - \tfrac{(k^2+3k+1)}{k+3}x^k h, \end{aligned}$$

and $F_1^{(k)}(y^{k+2})$ is

• If $k = 2l+1$,

$$F_1^{(k)}(y^{k+2}) = \tfrac{q(k+2)}{k+3}y^{k+3} + \lambda_1(x,y)h + (-1)^{l+1}(k+2)\tfrac{p^{l+2}}{q^{l+1}}x^{k+3},$$

with $\lambda_1(x,y) = (k+2)h\sum_{j=1}^{l+1}(-1)^{j+1}\frac{p^j}{q^j}x^{2j-1}y^{k-2j+4}$.
• If $k = 2l+2$,

$$F_1^{(k)}(y^{k+2}) = \tfrac{q(k+2)}{k+3}y^{k+3} + \lambda_2(x,y)h + (-1)^{l+1}(k+2)\tfrac{p^{l+2}}{q^{l+1}}x^{k+2}y,$$

with $\lambda_2(x,y) = (k+2)h\sum_{j=1}^{l+1}(-1)^{j+1}\frac{p^j}{q^j}x^{2j-1}y^{k-2j+4}$. So, the determinant of the matrix associated to the operator $\ell_{k+3}^{(c)}$ is $\frac{pq^2(k+2)^5}{(k+3)^3}$, *i.e.* the matrix is not singular, therefore $\mathrm{Ker}(\ell_{k+3}^{(c)}) = \{0\}$ and $\mathrm{Cor}(\ell_{k+3}^{(c)}) = \{0\}$. From Proposition 2.1, the result follows. □

Theorem 3.2. *The irreducible vector field* $\mathbf{F} = (-qxy, px^2)^T + \cdots$, *with* p, q *natural numbers is not analytically integrable.*

Proof. From Proposition 3.2, the vector field $\mathbf{F}$ is orbitally equivalent to a formal vector field of the form $\mathbf{F}^* = (-qxy, px^2)^T + \mu\mathbf{D}_0^{\mathbf{t}}$, with $\mathbf{t} = (1,1)$. On the other hand, $(-qxy, px^2)^T = x(-qy, px)^T$ and x is not an invariant curve of $(-qy, px)^T$, therefore from Proposition 2.2, $\mathbf{F}^*$ is not formally integrable and thus $\mathbf{F}$ is not analytically integrable. □

3.3. Case (c.1)

First we give some auxiliary results. The first one is [5, Lemma 3.22].

Lemma 3.1. *Let* $f \in \mathcal{P}_s^{\mathbf{t}}$ *an irreducible quasi-homogeneous polynomial invariant curve of* $\mathbf{F}_r$, $K_r \in \mathcal{P}_r^{\mathbf{t}}$ *its cofactor and* $k, m \in \mathbb{N}$. *Assume that the vector fields of* $\mathcal{Q}_r^{\mathbf{t}}$, $(k-js)\mathbf{F}_r + jK_r\mathbf{D}_0^{\mathbf{t}}$, $0 \le j \le m-1$, *are irreducible. Then, if* $F_r(p_k) \in \langle f^m \rangle$ *with* $p_k \in \mathcal{P}_k^{\mathbf{t}}$, *it satisfies that* $p_k \in \langle f^m \rangle$.

Lemma 3.2. *We consider the vector field* $\mathbf{F}_n = (x(-(q+r)y - (q-r)x^n), py^2 - n(q-r)x^ny - (p+n(q+r))x^{2n})^T \in \mathcal{Q}_n^{\mathbf{t}}$ *with* $\mathbf{t} = (1,n)$ *and denote* $M := p+nq+nr$, *degree of the quasi-homogeneous polynomial first integral* $I_M = x^p(y-x^n)^q(y+x^n)^r$. *Given* m *a natural number, assume that for every* $k \ge m$, *it satisfies that*

$$M \ne p\frac{k}{j},\ M \ne q\frac{k}{j},\ M \ne r\frac{k}{j},\ j = 1, \ldots, m-1.$$

If $p_k \in \mathcal{P}_k^{\mathbf{t}}$ *such that* $F_n(p_k) \in \langle f_i^m \rangle$, *where* $f_1 = x$, $f_2 = y - x^n$, $f_3 = y + x^n$, *are invariant curves at the origin of* $\mathbf{F}_n$, *then* $p_k \in \langle f_i^m \rangle$, $i = 1,2,3$.

Proof. We prove the case $i = 1$, $(f_1 = x)$, the cases $i = 2,3$ are analogous.

The cofactor of $x \in \mathcal{P}_1^{\mathbf{t}}$ is $K_n = (r-q)(y-x^n)$. From Lemma 3.1, it is enough to prove that for any j with $0 \le j \le m-1$, the vector field $(k-j)\mathbf{F}_n + jK_n\mathbf{D}_0^{\mathbf{t}}$ is irreducible, i.e.

$$-k(q+r)xy - k(q-r)x^{n+1},$$
$$(kp - jM)y^2 - nk(q-r)x^ny + (-k+j)Mx^{2n},$$

are coprime. Analyzing the different factorization of both polynomials, one has that both polynomials are coprime if and only if $M \ne p\frac{k}{j}$, $j = 1, \ldots, m-1$.

For $f_2 = y - x^n$ (and for $f_3 = y + x^n$, respectively) it is easy to prove that the vector fields $(k-js)\mathbf{F}_n + jK_n\mathbf{D}_0^{\mathbf{t}}$, $0 \le j \le m-1$, with K_n, cofactor of f_2 (and f_3, respectively), are irreducible if $M \ne q\frac{k}{j}$ and $M \ne r\frac{k}{j}$. □

The following result provides an orbitally equivalent normal form of the systems (c.1).

Proposition 3.3. *A formal orbital normal form for the system* $\dot{\mathbf{x}} = \mathbf{F}_n + \cdots$ *with* $\mathbf{F}_n = (x(-(q+r)y - (q-r)x^n), py^2 - n(q-r)x^n y - (p + n(q+r))x^{2n})^T$ *is*

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \mathbf{F}_n + \mu(x,y) \begin{pmatrix} x \\ ny \end{pmatrix}, \tag{3.3}$$

with $\mu = \sum_{k>n} \mu_k$, $\mu_k \in \mathrm{Cor}(\ell_k)$, *a complementary subspace to* $\mathrm{Range}(\ell_k)$. *Moreover, it is always possible to choose* $\mathrm{Cor}(\ell_{k+M})$, *a complementary subspace to* $\mathrm{Range}(\ell_{k+M})$, *such that* $\mathrm{Cor}(\ell_{k+M}) = I_M \mathrm{Cor}(\ell_k)$ *with* $I_M = x^p(y-x^n)^q(y+x^n)^r$ *and* $M = p + (q+r)n$.

Proof. We fix the type $\mathbf{t} = (1, n)$. In this case, $h := \mathbf{D}_0^{\mathbf{t}} \wedge \mathbf{F}_n = (p + nq + nr)x(y - x^n)(y + x^n)$ and $\mathrm{div}(\mathbf{F}_n) = (2p + q - r)y + n(r-q)x^n$.

Consider the translated vector field $\tilde{\mathbf{F}}_n^{(k)} = \mathbf{F}_n - \frac{1}{k+2n+1}\mathrm{div}(\mathbf{F}_n)\mathbf{D}_0^{\mathbf{t}}$. We distinguish two cases according to the values of $k > n$:

If $k+1$ is not multiple of n, a complementary subspace to $h\mathcal{P}_{k-n}$ and $h\mathcal{P}_k$ are $\Delta_{k+n+1} = \mathrm{span}\{x^{k+n+1}, x^{k+1}y\}$ and $\Delta_{k+2n+1} = \mathrm{span}\{x^{k+2n+1}, x^{k+n+1}y\}$. We have that

$$\begin{aligned} F_n^{(k)}(x^{k+n+1}) &= \tfrac{k+n+1}{k+2n+1}(k(r-q)x^{k+2n+1} - ((k+2n)(q+r) + 2p)x^{k+n+1}y), \\ F_n^{(k)}(x^{k+1}y) &= \tfrac{k+n+1}{k+2n+1}((-(k+2n)(q+r) - 2p)x^{k+2n+1} + k(r-q)x^{k+n+1}y) \\ &\quad + \lambda x^k h. \end{aligned}$$

The determinant of the matrix associated to the operator $\ell_{k+2n+1}^{(c)}$ is

$$-\tfrac{4(k+n+1)^2}{(k+2n+1)^2}(p + nq + (n+k)r)(p + (n+k)q + nr),$$

therefore the matrix is non-singular, $\mathrm{Ker}(\ell_{k+2n+1}^{(c)}) = \{0\}$ and $\mathrm{Cor}(\ell_{k+2n+1}^{(c)}) = \{0\}$.

If $k+1 = sn$ with s natural number, $\Delta_{(s+1)n} = \mathrm{span}\{x^{(s+1)n}, x^{sn}y, y^{s+1}\}$ and $\Delta_{(s+2)n} = \mathrm{span}\{x^{(s+2)n}, x^{(s+1)n}y, y^{s+2}\}$. The transformed of the basis are

$$\begin{aligned} F_n^{(k)}(x^{(s+1)n}) &= \tfrac{s+1}{s+2}((ns-1)(r-q)x^{s(n+2)} \\ &\quad - ((s(n+2)-1)(q+r) + 2p)x^{n(s+1)}y), \\ F_n^{(k)}(x^{sn}y) &= \tfrac{s+1}{s+2}((-((s+2)n-1)(q+r) - 2p)x^{s(n+2)} + (sn-1)(r-q)x^{n(s+1)}y) \\ &\quad + \lambda_1 x^{ns-1}h, \\ F_n^{(k)}(y^{s+1}) &= \tfrac{s+1}{s+2}(ps + q + r)y^{s+2} + \lambda_2 x^{(s+1)n}y + \lambda_3 x^{(s+2)n}. \end{aligned}$$

The determinant of the matrix associated to the operator $\ell_{(s+2)n}^{(c)}$ is

$$-\tfrac{4(s+1)^3}{(s+2)^3}(p + nq + (s+1)nr)(p + (s+1)nq + nr)(sp + q + r).$$

It is not zero, *i.e.* the matrix associated to the operator $\ell^{(c)}_{(s+2)n}$ is also not singular, therefore $\mathrm{Ker}(\ell^{(c)}_{(s+2)n}) = \{0\}$ and $\mathrm{Cor}(\ell^{(c)}_{(s+2)n}) = \{0\}$. Applying Proposition 2.1, we have that a normal form orbitally equivalent is $\dot{\mathbf{x}} = \mathbf{F}_n + \mu(x, ny)^T$ and $\mu = \sum_{k>n} \mu_k,\ \mu_k \in \mathrm{Cor}(\ell_k)$, a complementary subspace to $\mathrm{Range}(\ell_k)$.

Let prove the second part. First, we see that both subspaces, $\mathrm{Cor}(\ell_{k+M})$ and $I_M \mathrm{Cor}(\ell_k)$, have the same dimension. Indeed, $Ker(\ell_k) = \mathrm{span}\{I_M^l\}$ if $k - n = lM$. Otherwise, $Ker(\ell_k) = \{0\}$. Thus, $\dim(\mathrm{Cor}(\ell_k)) = 2$ if $k = lM$ and $\dim(\mathrm{Cor}(\ell_k)) = 1$, otherwise; i.e. $\dim(\mathrm{Cor}(\ell_k)) = \dim(\mathrm{Cor}(\ell_{k+M}))$.

The proof is completed by showing that $I_M \mathrm{Cor}(\ell_k) \subset \mathrm{Cor}(\ell_{k+M})$ or equivalently that $I_M \mathrm{Cor}(\ell_k) \cap \mathrm{Range}(\ell_{k+M}) = \{0\}$ by *reductio ad absurdum.* Let $p_k \in \mathrm{Cor}\,(\ell_k) \setminus \{0\}$ such that $p_k I_M \in \mathrm{Range}\,(\ell_{k+M})$, then there exists $p_{k+M-n} \in \mathcal{P}_{k+M-n} \setminus \{0\}$ such that $\ell_{k+M}(p_{k+M-n}) = p_k I_M$, that is, $\ell_{k+M}(p_{k+M-n})$ is multiple of I_M. As $p\frac{(k+M-n)}{j} > p\frac{M}{j} > M,\ j = 1, \ldots, p-1$, by applying Lemma 3.2 we have that $p_{k+M-n} \in \langle x^p \rangle \cap \langle (y - x^n)^q \rangle \cap \langle (y + x^n)^r \rangle$, thus $p_{k+M-n} = p_{k-n} I_M$ with $p_{k-n} \in \mathcal{P}_{k-n} \setminus \{0\}$ and consequently

$$p_k I_M = F_n(p_{k+M-n}) = F_n(p_{k-n} I_M) = I_M F_n(p_{k-n}).$$

Hence $p_k = F_n(p_{k-n})$, i.e., $p_k \in \mathrm{Range}\,(\ell_k) \cap \mathrm{Cor}\,(\ell_k)$ which gives a contradiction. $\square$

Theorem 3.3. *Consider the system $\dot{\mathbf{x}} = \mathbf{F} := \mathbf{F}_n + \cdots$ where $\mathbf{F}_n = (x(-(q+r)y - (q-r)x^n), py^2 - n(q-r)x^n y - (p + n(q+r))x^{2n})^T \in \mathcal{Q}_n^{\mathbf{t}}$ with $\mathbf{t} = (1, n)$ and assume that the origin is an isolated singular point of $\mathbf{F}$. Then it is analytically integrable if, and only if, it is orbitally equivalent to $\dot{\mathbf{x}} = \mathbf{F}_n$.*

Moreover, in such a case, an analytic first integral is of the form

$$(x + \cdots)^p (y - x^n + \cdots)^q (y + x^n + \cdots)^r,$$

where the dots mean quasi-homogeneous terms of higher degree than n with respect to the type $\mathbf{t} = (1, n)$.

Proof. We see the sufficiency. The polynomial $I_M = x^p(y - x^n)^q(y + x^n)^r$ is a first integral of $\mathbf{F}_n$ which is transformed into a formal first integral $I = I_M + \cdots$ of $\mathbf{F}$ and from [28, Theorem A], $\mathbf{F}$ is analytically integrable.

We see the necessity of the condition. Applying Proposition 3.3, we can assert that $\mathbf{F}$ is orbital equivalent to $\mathbf{G} = \mathbf{F}_n + \sum_{j>n} \mu_j \mathbf{D}_0^{\mathbf{t}}$ with $\mu_j \in \mathrm{Cor}\,(\ell_j)$.

Let note that $\mathbf{F}$ has an analytic first integral equivalents to $\mathbf{G}$ has a formal first integral. By Proposition 3.3 and Proposition 2.3, $\mathbf{G}$ is formally integrable if and only if $\mathbf{G} = \mathbf{F}_n$, i.e. $\mathbf{F}$ is orbitally equivalent to $\mathbf{F}_n$. $\square$

It has the following result.

Theorem 3.4. *Consider system $\dot{\mathbf{x}} = \mathbf{F} = \mathbf{F}_n + \cdots$ where $\mathbf{F}_n = (x(-(q+r)y - (q-r)x^n), py^2 - n(q-r)x^n y - (p + n(q+r))x^{2n})^T \in \mathcal{Q}_n^{\mathbf{t}}$ with $\mathbf{t} = (1, n)$ and assume that the origin is an isolated singular point of $\mathbf{F}$. It is analytically integrable if, and only if, it has a formal inverse integrating factor of the form $V = x(y^2 - x^{2n}) + \cdots$.*

Proof. We prove that the condition is necessary. We assume that $\mathbf{F} = \mathbf{F}_n + \cdots$ is analytically integrable. From Theorem 3.3, it is orbitally equivalent to system $\dot{\mathbf{x}} = \mathbf{F}_n$, it which has the inverse integrating factor $h := x(y^2 - x^{2n})$. Undoing the change, $\mathbf{F}$ has a formal inverse integrating factor $V = h + \cdots$.

Now we see the sufficiency of the condition. Let $V = h + \cdots$ a formal inverse integrating factor of $\mathbf{F}$. By Proposition 3.3, we can assert that $\mathbf{F}$ is orbital equivalent to $\mathbf{G} = \mathbf{F}_n + \sum_{j>n} \mu_j \mathbf{D}_0^{\mathbf{t}}$ with $\mu_j \in \mathrm{Cor}\,(\ell_j)$. Therefore, $\mathbf{F}$ has a formal inverse integrating factor if, and only if, $\mathbf{G}$ has it too. Moreover, the formal inverse integrating factor W of $\mathbf{G}$ has also the form $W = h + \cdots$. By Proposition 3.3, we can choose $\mathrm{Cor}(\ell_{k+M})$, a complementary subspace to $\mathrm{Range}(\ell_{k+M})$, such that $\mathrm{Cor}(\ell_{k+M}) = I_M \mathrm{Cor}(\ell_k)$. From Proposition 2.4, $\mathbf{G} = \mathbf{F}_n$. So, by applying Proposition 2.3, $\mathbf{F}$ is analytically integrable since it is orbitally equivalent to $\mathbf{F}_n$. □

3.4. Case (c.2)

Lemma 3.3. *We consider the vector field* $\mathbf{F}_n = (-qxy, py^2 + \sigma(p+nq)x^{2n})^T \in \mathcal{Q}_n^{\mathbf{t}}$ *with* $\sigma = \pm 1$, $\mathbf{t} = (1, n)$ *and denote* $M := 2p + 2nq$, *degree of the quasi-homogeneous polynomial first integral* $I_M = x^{2p}(y^2 + \sigma x^{2n})^q$. *Given* m *a natural number, assume that for every* $k \geq m$, *it satisfies that*

$$M \neq 2p\frac{k}{j},\ M \neq q\frac{k}{j},\ j = 1, \ldots, m-1.$$

If $p_k \in \mathcal{P}_k^{\mathbf{t}}$ *such that* $F_n(p_k) \in \langle f_i^m \rangle$, *where* $f_1 = x$, $f_2 = y^2 + \sigma x^{2n}$, *are invariant curves at the origin of* $\mathbf{F}_n$, *then* $p_k \in \langle f_i^m \rangle$, $i = 1, 2$.

Proof. We assume that $\sigma = 1$. We prove the case $i = 1$, $(f_1 = x)$.

The cofactor of $x \in \mathcal{P}_1^{\mathbf{t}}$ is $K_n = -qy$. From Lemma 3.1, it is enough to prove that for any j with $0 \leq j \leq m-1$, the vector field $(k-j)\mathbf{F}_n + jK_n\mathbf{D}_0^{\mathbf{t}}$ is irreducible, i.e.

$$-kqxy, \quad (kp - j(p+nq))y^2 - (k-j)(p+nq)x^{2n},$$

are coprime. Both polynomials are coprime if and only if $M \neq 2p\frac{k}{j}$, $j = 1, \ldots, m-1$.

For $f_2 = y^2 + x^{2n}$, it is easy to prove that the vector fields $(k-2nj)\mathbf{F}_n + jK_n\mathbf{D}_0^{\mathbf{t}}$, $0 \leq j \leq m-1$, with $K_n = 2py$, cofactor of f_2 are irreducible if, and only if, $M \neq q\frac{k}{j}$, $j = 1, \ldots, m-1$.

We assume that $\sigma = -1$. The case $i = 1$, $(f_1 = x)$ is similar.

For $f_2 = y - x^n$, (for $f_3 = y + x^n$, analogously) the cofactor is $K_n = py + (p+nq)x^n$ and the vector field $(k-j)\mathbf{F}_n + jK_n\mathbf{D}_0^{\mathbf{t}}$ is

$$\begin{aligned} &(j(p+nq) - kq)xy + j(p+nq)x^{n+1}, \\ &kpy^2 + nj(p+nq)x^n y + (p+nq)(k-nj)x^{2n}. \end{aligned}$$

The vector fields $(k-2nj)\mathbf{F}_n + jK_n\mathbf{D}_0^{\mathbf{t}}$, $0 \leq j \leq m-1$, with $K_n = 2py$, cofactor of f_2, are irreducible if, and only if, $M \neq q\frac{k}{j}$, $j = 1, \ldots, m-1$. □

Proposition 3.4. *A formal orbital normal form for the system* $\dot{\mathbf{x}} = \mathbf{F}_n + \cdots$ *with* $\mathbf{F}_n = (-qxy, py^2 + \sigma(p+nq)x^{2n})^T$ *with* $\sigma = \pm 1$ *is*

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \mathbf{F}_n + \mu(x,y)\begin{pmatrix} x \\ ny \end{pmatrix}, \tag{3.4}$$

with $\mu = \sum_{k>n} \mu_k,\ \ \mu_k \in \mathrm{Cor}(\ell_k)$, *a complementary subspace to* $\mathrm{Range}(\ell_k)$. *Moreover, it is always possible to choose* $\mathrm{Cor}(\ell_{k+M})$, *a complementary subspace to* $\mathrm{Range}(\ell_{k+M})$, *such that* $\mathrm{Cor}(\ell_{k+M}) = I_M \mathrm{Cor}(\ell_k)$ *with* $I_M = x^{2p}(y^2+\sigma x^{2n})^q$ *and* $M = 2p+2nq$.

Proof. We fix the type $\mathbf{t} = (1, n)$. In this case, $h = (nq+p)x(y^2+\sigma x^{2n})$ and $\mathrm{div}(\mathbf{F}_n) = (2p-q)y$.

Consider the translated vector field $\tilde{\mathbf{F}}_n^{(k)} = \mathbf{F}_n - \frac{1}{k+2n+1}\mathrm{div}(\mathbf{F}_n)\mathbf{D}_0^{\mathbf{t}}$. We distinguish two cases according to the values of $k > n$:

If $k+1$ is not multiple of n, a complementary subspace to $h\mathcal{P}_{k-n}$ and $h\mathcal{P}_k$ are $\Delta_{k+n+1} = \mathrm{span}\{x^{k+n+1}, x^{k+1}y\}$ and $\Delta_{k+2n+1} = \mathrm{span}\{x^{k+2n+1}, x^{k+n+1}y\}$. We have that

$$\begin{aligned} F_n^{(k)}(x^{k+n+1}) &= -\tfrac{k+n+1}{k+2n+1}(2p+(k+2n)q)x^{k+n+1}y, \\ F_n^{(k)}(x^{k+1}y) &= \tfrac{k+n+1}{k+2n+1}(2p+(k+2n)q)\sigma x^{k+2n+1} + \lambda x^k h. \end{aligned}$$

The determinant of the matrix associated to the operator $\ell_{k+2n+1}^{(c)}$ is

$$\tfrac{(k+n+1)^2}{(k+2n+1)^2}(2p+(k+2n)q)^2\sigma.$$

Therefore the matrix is non-singular and $\mathrm{Ker}(\ell_{k+2n+1}^{(c)}) = \{0\}$. Moreover, we have that $\mathrm{Cor}(\ell_{k+2n+1}^{(c)}) = \{0\}$.

If $k+1 = sn$ with s natural number, $\Delta_{(s+1)n} = \mathrm{span}\{x^{(s+1)n}, x^{sn}y, y^{s+1}\}$ and $\Delta_{(s+2)n} = \mathrm{span}\{x^{(s+2)n}, x^{(s+1)n}y, y^{s+2}\}$. The transformed of the basis are

$$\begin{aligned} F_n^{(k)}(x^{(s+1)n}) &= -\tfrac{s+1}{s+2}(2p+((s+2)n-1)q)x^{(s+1)n}y, \\ F_n^{(k)}(x^{sn}y) &= \tfrac{s+1}{s+2}(2p+((s+2)n-1)q)\sigma x^{(s+2)n} + \lambda x^{ns-1}h, \\ F_n^{(k)}(y^{s+1}) &= \tfrac{s+1}{s+2}(q+sp)y^{s+2} + \lambda_2 x^{(s+1)n}y + \lambda_3 x^{(s+2)n}. \end{aligned}$$

The determinant of the matrix associated to the operator $\ell_{(s+2)n}^{(c)}$ is

$$\tfrac{(s+1)^3}{(s+2)^3}(2p+((s+2)n-1)q)^2(sp+q)\sigma.$$

It is not zero, *i.e.,* the matrix associated to the operator $\ell_{(s+2)n}^{(c)}$ is also not singular, therefore $\mathrm{Ker}(\ell_{(s+2)n}^{(c)}) = \{0\}$ and $\mathrm{Cor}(\ell_{(s+2)n}^{(c)}) = \{0\}$. Applying Proposition 2.1, we have that a normal form orbitally equivalent is $\dot{\mathbf{x}} = \mathbf{F}_n + \mu(x, ny)^T$ and $\mu = \sum_{k>n}\mu_k,\ \mu_k \in \mathrm{Cor}(\ell_k)$, a complementary subspace to $\mathrm{Range}(\ell_k)$.

Let prove the second part. Both subspaces, $\mathrm{Cor}(\ell_{k+M})$ and $I_M\mathrm{Cor}(\ell_k)$, have the same dimension. So, the proof is completed by showing that $I_M\mathrm{Cor}(\ell_k) \subset \mathrm{Cor}(\ell_{k+M})$ or equivalently that $I_M\mathrm{Cor}(\ell_k) \cap \mathrm{Range}(\ell_{k+M}) = \{0\}$ by *reductio ad absurdum.* Let $p_k \in \mathrm{Cor}\,(\ell_k)\setminus\{0\}$ such that $p_k I_M \in \mathrm{Range}\,(\ell_{k+M})$, then there exists $p_{k+M-n} \in \mathcal{P}_{k+M-n}\setminus\{0\}$ such that $\ell_{k+M}(p_{k+M-n}) = p_k I_M$, that is, $\ell_{k+M}(p_{k+M-n})$ is multiple of I_M. As $2p\frac{(k+M-n)}{j} > 2p\frac{M}{j} > M,\ j = 1, \ldots, 2p-1;\ q\frac{(k+M-n)}{j} > q\frac{M}{j} > M;\ j = 1, \ldots, q-1$, by applying Lemma 3.3 we have that for $\sigma = 1,\ p_{k+M-n}$

$\in \langle x^{2p}\rangle \cap \langle (y+x^{2n})^q\rangle$ and *for* $\sigma=-1$, $p_{k+M-n} \in \langle x^{2p}\rangle \cap \langle (y-x^n)^q\rangle \cap \langle (y+x^n)^q\rangle$. Thus $p_{k+M-n} = p_{k-n} I_M$ with $p_{k-n} \in \mathcal{P}_{k-n} \setminus \{0\}$ and consequently

$$p_k I_M = F_n(p_{k+M-n}) = F_n(p_{k-n} I_M) = I_M F_n(p_{k-n}).$$

Hence $p_k = F_n(p_{k-n})$, that is, $p_k \in \operatorname{Range}(\ell_k) \cap \operatorname{Cor}(\ell_k)$ which is a contradiction. $\square$

Theorem 3.5. *Consider system* $\dot{\mathbf{x}} = \mathbf{F} = \mathbf{F}_n + \cdots$ *where* $\mathbf{F}_n = (-qxy, py^2 + \sigma(p+nq)x^{2n})^T \in \mathcal{Q}_n^{\mathbf{t}}$ *with* $\sigma = \pm 1$ *and* $\mathbf{t} = (1,n)$ *and assume that the origin is an isolated singular point of* $\mathbf{F}$. *Then it is analytically integrable if, and only if, it is orbitally equivalent to* $\dot{\mathbf{x}} = \mathbf{F}_n$.

Moreover, in such a case, an analytic first integral is of the form

$$(x+\cdots)^{2p}(y^2+\sigma x^{2n}+\cdots)^q,$$

where the dots mean quasi-homogeneous terms of higher degree than n *with respect to the type* $\mathbf{t} = (1,n)$.

Proof. We see the sufficiency. The polynomial $I_M = x^{2p}(y^2+\sigma x^{2n})^q$. is a first integral of $\mathbf{F}_n$ which is transformed into a formal first integral $I = I_M + \cdots$ of $\mathbf{F}$ and from [28, Theorem A], $\mathbf{F}$ is analytically integrable.

We see the necessity of the condition. Applying Proposition 3.4, we can assert that $\mathbf{F}$ is orbital equivalent to $\mathbf{G} = \mathbf{F}_n + \sum_{j>n} \mu_j \mathbf{D}_0^{\mathbf{t}}$ with $\mu_j \in \operatorname{Cor}(\ell_j)$.

Let note that $\mathbf{F}$ has an analytic first integral equivalents to $\mathbf{G}$ has a formal first integral. By Proposition 3.4 and Proposition 2.3, $\mathbf{G}$ is formally integrable if and only if $\mathbf{G} = \mathbf{F}_n$, i.e. $\mathbf{F}$ is orbitally equivalent to $\mathbf{F}_n$. $\square$

Theorem 3.6. *Consider system* $\dot{\mathbf{x}} = \mathbf{F} = \mathbf{F}_n + \cdots$ *where* $\mathbf{F}_n = (-qxy, py^2 + \sigma(p+nq)x^{2n})^T \in \mathcal{Q}_n^{\mathbf{t}}$ *with* $\sigma = \pm 1$ *and* $\mathbf{t} = (1,n)$ *and assume that the origin is an isolated singular point. It is analytically integrable if, and only if, it has a formal inverse integrating factor of the form* $V = x(y^2+\sigma x^{2n}) + \cdots$.

Proof. The proof is analogous to the proof of Theorem 3.4, it is enough to apply Theorem 3.5, Proposition 3.4 and Proposition 2.4. $\square$

3.5. Case (c.3)

Lemma 3.4. *Consider the vector field* $\mathbf{F}_{2n+1} = (-qxy, py^2 + \frac{1}{2}(2p+(2n+1)q)x^{2n+1})^T \in \mathcal{Q}_{2n+1}^{\mathbf{t}}$, $\mathbf{t} = (2, 2n+1)$ *with* p, q *natural numbers and denote* $M := 4p + 2(2n+1)q$, *degree of the quasi-homogeneous polynomial first integral* $I_M = x^{2p}(y^2+x^{2n+1})^q$. *Given* m *a natural number, assume that for every* $k \geq m$, *it satisfies that*

$$M \neq 2p\frac{k}{j},\ M \neq q\frac{k}{j},\ j = 1, \ldots, m-1.$$

If $p_k \in \mathcal{P}_k^{\mathbf{t}}$ *such that* $F_n(p_k) \in \langle f_i^m \rangle$, *where* $f_1 = x$, $f_2 = y^2 + x^{2n+1}$, *are invariant curves at the origin of* $\mathbf{F}_{2n+1}$, *then* $p_k \in \langle f_i^m \rangle$, $i = 1, 2$.

Proof. We prove the case $i = 1$, $(f_1 = x)$.

The cofactor of $x \in \mathcal{P}_1^{\mathbf{t}}$ is $K_{2n+1} = -qy$. From Lemma 3.1, it is enough to prove that for any j with $0 \leq j \leq m-1$, the vector field $(k-2j)\mathbf{F}_{2n+1} + jK_{2n+1}\mathbf{D}_0^{\mathbf{t}}$ is irreducible, i.e.

$$-q(k-j)xy, \quad (kp - \frac{jM}{2})y^2 - \frac{1}{4}(k-2j)Mx^{2n+1},$$

are coprime. Both polynomials are coprime if and only if $M \neq 2p\frac{k}{j}$, $j = 1, \ldots, m-1$.

For $f_2 = y^2 + x^{2n+1}$, The cofactor of $f_2 \in \mathcal{P}^{\mathbf{t}}_{4n+2}$ is $K_{2n+1} = -2py$. From Lemma 3.1, it is enough to prove that for any j with $0 \leq j \leq m-1$, the vector field $(k - 2(4n+2)j)\mathbf{F}_{2n+1} + jK_{2n+1}\mathbf{D}^{\mathbf{t}}_0$ is irreducible, i.e.

$$-(kq - Mj)xy, \qquad kpy^2 - \tfrac{1}{4}(k - 2j - 4nj)Mx^{2n+1},$$

are coprime. Both polynomials are coprime if, and only if, M satisfies $M \neq q\frac{k}{j}$, for $j = 1, \ldots, m-1$. □

We give the following normal form of the vector fields considered.

Proposition 3.5. *A formal orbital normal form for the system* $\dot{\mathbf{x}} = \mathbf{F}_{2n+1} + \cdots$, *with* $\mathbf{F}_{2n+1} = (-qxy, py^2 + \frac{1}{2}(2p + (2n+1)q)x^{2n+1})^T$, *with* p, q *natural numbers is*

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \mathbf{F}_{2n+1} + \mu(x,y) \begin{pmatrix} 2x \\ (2n+1)y \end{pmatrix}, \tag{3.5}$$

with $\mu = \sum_{k>2n+1} \mu_k$, $\mu_k \in \mathrm{Cor}(\ell_k)$, *a complementary subspace to* $\mathrm{Range}(\ell_k)$.

Moreover, it is always possible to choose $\mathrm{Cor}(\ell_{k+M})$, *the complementary subspace to* $\mathrm{Range}(\ell_{k+M})$, *such that* $\mathrm{Cor}(\ell_{k+M}) = I_M \mathrm{Cor}(\ell_k)$ *with* $I_M = x^{2p}(y^2 + x^{2n+1})^q$ *and* $M = 4p + 2(2n+1)q$.

Proof. We fix the type $\mathbf{t} = (2, 2n+1)$. In this case, $h = (2p + (2n+1)q)x(y^2 + x^{2n+1})$ and $\mathrm{div}(\mathbf{F}_{2n+1}) = (2p - q)y$.

Consider the translated vector field $\tilde{\mathbf{F}}^{(k)}_{2n+1} = \mathbf{F}_{2n+1} - \frac{1}{k+4n+4}\mathrm{div}(\mathbf{F}_{2n+1})\mathbf{D}^{\mathbf{t}}_0$.

The sets $\mathcal{P}^{\mathbf{t}}_k$ non-trivial are

$$\mathrm{span}\{x^{k_1+(2n+1)(k_3-j)}y^{k_2+2j},\ k_1 < 2n+1,\ k_2 < 2,\ j = 0, \ldots, k_3\}.$$

We distinguish two cases according to the values of $k > 2n+1$:

If $k + |\mathbf{t}| = (2n+1)k_2 + 2(2n+1)k_3$ with $k_2 \in \{0,1\}$, then we obtain that $\Delta_{k+|\mathbf{t}|} = \mathrm{span}\{x^{(2n+1)k_3}y^{k_2}, y^{k_2+2k_3}\}$ and $\Delta_{k+|\mathbf{t}|+2n+1} = \mathrm{span}\{x^{(2n+1)(k_3+1)}y^{k_2-1}, y^{k_2+2k_3+1}\}$. We have that

$$\ell^{(c)}_{k+4n+4}(x^{(2n+1)k_3}y^{k_2}) = a_1 x^{(2n+1)(k_3+1)}y^{k_2-1},$$

$$\ell^{(c)}_{k+4n+4}(y^{k_2+2k_3}) = a_2 y^{k_2+2k_3+1},$$

with

$$a_1 = \tfrac{k_2+2k_3}{2(k_2+2k_3+1)}((2n+1)q(k_2+2k_3) + 2qn + 4p - q),$$

$$a_2 = \tfrac{k_2+2k_3}{(k_2+2k_3+1)}((k_2+2k_3-1)p + q).$$

Since $a_1a_2 \neq 0$, the matrix associated to the operator $\ell^{(c)}_{k+4n+4}$ is non-singular and $\mathrm{Ker}(\ell^{(c)}_{k+4n+4}) = \{0\}$. Moreover, $\mathrm{Cor}(\ell^{(c)}_{k+4n+4}) = \{0\}$.

If $k + |\mathbf{t}| = 2k_1 + (2n+1)k_2 + 2(2n+1)k_3$ with $k_2 \in \{0,1\}$ and $k_1 < 2n+1$, then $\Delta_{k+|\mathbf{t}|} = \mathrm{span}\{x^{k_1+(2n+1)k_3}y^{k_2}\}$ and $\Delta_{k+|\mathbf{t}|+2n+1} = \mathrm{span}\{x^{k_1+(2n+1)(k_3+1)}y^{k_2-1}\}$. We have that

$$\ell^{(c)}_{k+4n+4}(x^{k_1+(2n+1)k_3}y^{k_2}) = a_3 x^{k_1+(2n+1)(k_3+1)}y^{k_2-1},$$

with

$$a_3 = \frac{(2n+1)(k_2+2k_3)+2k_1}{2((2n+1)(k_2+2k_3+1)+2k_1)}((2n+1)q(k_2+2k_3)+2k_1q+2qn+4p-q).$$

Since $a_3 \neq 0$, we have that $\mathrm{Ker}(\ell^{(c)}_{k+4n+4}) = \{0\}$. Moreover, $\mathrm{Cor}(\ell^{(c)}_{k+4n+4}) = \{0\}$. Applying Proposition 2.1, we have that a normal form orbitally equivalent is $\dot{\mathbf{x}} = \mathbf{F}_{2n+1} + \mu(2x,(2n+1)y)^T$ and $\mu = \sum_{k>2n+1}\mu_k$, $\mu_k \in \mathrm{Cor}(\ell_k)$, a complementary subspace to $\mathrm{Range}(\ell_k)$.

Let prove the second part. Both subspaces, $\mathrm{Cor}(\ell_{k+M})$ and $I_M\mathrm{Cor}(\ell_k)$, have the same dimension. The proof is completed by showing that $I_M\mathrm{Cor}(\ell_k) \subset \mathrm{Cor}(\ell_{k+M})$ or equivalently that $I_M\mathrm{Cor}(\ell_k)\cap\mathrm{Range}(\ell_{k+M}) = \{0\}$ by *reductio ad absurdum*. Let $p_k \in \mathrm{Cor}\,(\ell_k)\setminus\{0\}$ such that $p_kI_M \in \mathrm{Range}\,(\ell_{k+M})$, then there exists $p_{k+M-2n-1} \in \mathcal{P}_{k+M-2n-1}\setminus\{0\}$ such that $\ell_{k+M}(p_{k+M-2n-1}) = p_kI_M$, that is, $\ell_{k+M}(p_{k+M-2n-1})$ is multiple of I_M. As $2p\frac{(k+M-2n-1)}{j} > 2p\frac{M}{j} > M$, $j = 1,\dots,2p-1$; $q\frac{(k+M-2n-1)}{j} > q\frac{M}{j} > M$; $j = 1,\dots,q-1$, by applying Lemma 3.4 we have that $p_{k+M-2n-1} \in \langle x^{2p}\rangle\cap\langle(y+x^{2n+1})^q\rangle$, thus $p_{k+M-2n-1} = p_{k-2n-1}I_M$ with $p_{k-2n-1} \in \mathcal{P}_{k-2n-1}\setminus\{0\}$ and consequently

$$p_kI_M = F_{2n+1}(p_{k+M-2n-1}) = F_{2n+1}(p_{k-2n-1}I_M) = I_MF_{2n+1}(p_{k-2n-1}).$$

Hence $p_k = F_{2n+1}(p_{k-2n-1})$, that is, $p_k \in \mathrm{Range}\,(\ell_k)\cap\mathrm{Cor}\,(\ell_k)$ which gives a contradiction. □

Theorem 3.7. *Consider system* $\dot{\mathbf{x}} = \mathbf{F} = \mathbf{F}_{2n+1} + \cdots$ *where* $\mathbf{F}_{2n+1} = (-qxy, py^2 + \frac{1}{2}(2p+(2n+1)q)x^{2n+1})^T$ *with* p, q *natural numbers and* $\mathbf{t} = (2, 2n+1)$ *and assume that the origin is an isolated singular point of* $\mathbf{F}$*. Then it is analytically integrable if, and only if, it is orbitally equivalent to* $\dot{\mathbf{x}} = \mathbf{F}_{2n+1}$.

Moreover, in such a case, an analytic first integral is of the form

$$(x+\cdots)^{2p}(y+x^{2n+1}+\cdots)^q,$$

where the dots mean quasi-homogeneous terms of higher degree than $2n+1$ *with respect to the type* $\mathbf{t} = (2, 2n+1)$.

Proof. We see the sufficiency. The polynomial $I_M = x^{2p}(y^2+x^{2n+1})^q$ is a first integral of $\mathbf{F}_{2n+1}$ which is transformed into a formal first integral $I = I_M + \cdots$ of $\mathbf{F}$ and from [28, Theorem A], $\mathbf{F}$ is analytically integrable.

We see the necessity of the condition. Applying Proposition 3.5, we can assert that $\mathbf{F}$ is orbital equivalent to $\mathbf{G} = \mathbf{F}_{2n+1} + \sum_{j>2n+1}\mu_j\mathbf{D}_0^{\mathbf{t}}$ with $\mu_j \in \mathrm{Cor}\,(\ell_j)$.

Let note that $\mathbf{F}$ has an analytic first integral equivalents to $\mathbf{G}$ has a formal first integral. By Proposition 3.5 and Proposition 2.3, $\mathbf{G} = \mathbf{F}_{2n+1}$, i.e. $\mathbf{F}$ is orbitally equivalent to $\mathbf{F}_{2n+1}$. □

Theorem 3.8. *Consider system* $\dot{\mathbf{x}} = \mathbf{F} = \mathbf{F}_{2n+1} + \cdots$ *where* $\mathbf{F}_{2n+1} = (-qxy, py^2 + \frac{1}{2}(2p+(2n+1)q)x^{2n+1})^T$ *with* p, q *natural numbers and* $\mathbf{t} = (2, 2n+1)$ *and assume that the origin is an isolated singular point of* $\mathbf{F}$*. It is analytically integrable if, and only if, it has a formal inverse integrating factor of the form* $V = x(y+x^{2n+1}) + \cdots$.

Proof. The proof is analogous to the proof of Theorem 3.4, it is enough to apply Theorem 3.7, Proposition 3.5 and Proposition 2.4. □

The analytic integrability of the systems (c.4) and (c.5) has been studied in [6].

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