Journal of Applied Analysis and Computation
Volume 14, Number 2, April 2024, 886–910

Website:http://www.jaac-online.com
DOI:10.11948/20230189

NUMERICAL AND MATHEMATICAL ANALYSIS OF A NONLINEAR SCHRÖDINGER PROBLEM WITH MOVING ENDS

Daniele C. R. Gomes[1,†], Mauro A. Rincon[1], Maria Darci G. da Silva[2] and Gladson O. Antunes[3]

Abstract Existence and uniqueness of solution in $\mathbb{R}^n$ for a nonlinear Shrödinger equation in a domain with moving ends and an optimal algorithm to obtain an approximate numerical solution for the two-dimensional space. The linearized Crank-Nicolson-Galerkin method is proposed to achieve high-performance computing. Numerical examples for the two-dimensional domain are presented to confirm the theoretical analysis and numerical results. Numerical errors associated with the linear, quadratic and cubic base are displayed.

Keywords Nonlinear Schrödinger problem, noncylindrical domain, existence and uniqueness, linearized Crank-Nicolson-Galerkin method, numerical simulation.

MSC(2010) 35K55, 65M06, 65M60, 35R37.

1. Introduction

Let $T > 0$ and Ω be an open bounded set of $\mathbb{R}^n$, with a regular boundary Γ of class C^2. We represent by $k(t)$ a real function defined on the set of all non-negative real numbers $[0, \infty)$. Consider the subset $\Omega_t = \{x \in \mathbb{R}^n; x = k(t)y, y \in \Omega\} \subset \mathbb{R}^n$, for each $t \in [0, T]$. Let $\widehat{Q}$ be the non-cylindrical domain of $\mathbb{R}^{n+1}$ with regular one-sided boundary $\widehat{\Sigma}$ defined by

$$\widehat{Q} = \bigcup_{0<t<T} \{\Omega_t \times \{t\}\}, \quad \widehat{\Sigma} = \bigcup_{0<t<T} \{\Gamma_t \times \{t\}\}, \quad \text{where} \quad \Gamma_t = \partial\Omega_t.$$

Consider the following nonlinear Schrödinger problem:

$$\begin{cases} u'(x,t) - i\Delta u(x,t) + |u(x,t)|^\rho \, u(x,t) = \hat{f}(x,t) \ \text{ in } \ \hat{Q}, \\ u(x,t) = 0 \ \text{ on } \ \hat{\Sigma}, \\ u(x,0) = u_0(x) \ \text{ in } \ \Omega_0. \end{cases} \qquad (i^2 = -1). \tag{1.1}$$

[†]The corresponding author.
[1]Instituto de Computação, Universidade Federal do Rio de Janeiro, RJ, Brazil
[2]Instituto de Matemática, Universidade Federal do Rio de Janeiro, RJ, Brazil
[3]Escola de Matemática, Universidade Federal do Estado do Rio de Janeiro, RJ, Brazil
Email: daniele.rocha@ppgi.ufrj.br(D. C. R. Gomes), rincon@ic.ufrj.br(M. A. Rincon), darci@im.ufrj.br(M. D. G. da Silva), gladson.antunes@uniriotec.br(G. O. Antunes)

We will denote the derivative with respect to time t as the prime $(')$.

Let us consider the diffeomorphism, see for example [21], $\mathcal{T} : \widehat{Q} \to Q$ defined by $\mathcal{T}(x,t) = (y,t)$. Then the change of variables $u(x,t) = v(y,t)$ transforms the problem (1.1) into the following equivalent one

$$\begin{cases} v'(y,t) - \dfrac{k'}{k} y_j \dfrac{\partial v(y,t)}{\partial y_j} - i\dfrac{1}{k^2}\Delta v(y,t) + |v(y,t)|^{\rho} v(y,t) \\ = f(y,t) \text{ in } Q, \qquad\qquad\qquad\qquad (i^2 = -1), \\ v(y,t) = 0 \quad \text{on} \quad \Sigma, \\ v(y,0) = v_0 \quad \text{in} \quad \Omega, \end{cases} \tag{1.2}$$

where $f(y,t) = \hat{f}(k(t)y,t)$ and when (x,t) varies in $\widehat{Q}$ the point (y,t), with $y = x/k(t)$, varies in $Q = \Omega \times (0,T)$.

For the analysis of problem (1.2), consider the following hypotheses:

(**H1**) $k \in W^{2,\infty}_{loc}([0,\infty[); k(t) \geq k_0 > 0, \ \forall t \geq 0;$

(**H2**) $0 \leq \rho < \infty$ if $n = 1,2$ and $0 \leq \rho \leq \dfrac{2}{n-2}$ if $n \geq 3$.

The nonlinear Schrödinger equation represents many physical configurations and their applications, such as surface gravity waves, superconductivity, nonlinear optics, and Bose-Einstein condensation. See, for example, the works of [9,11,14,18,27] and references, therein.

The study of Schrödinger's nonlinear equations, both in terms of theoretical and numerical analysis and their applications, becomes a very important subject in applied and computational mathematics.

One of the first theoretical results about the existence and uniqueness of the solution for the equation $u' - i\Delta u + |u|^{\rho}\, u = f$, for cylindrical domains, was obtained by Lions [19]. For semilinear Schrödinger equations, we refer to [7] and its references for an important complete theoretical mathematical analysis.

An interesting numerical analysis of the convergence rates of various numerical approximation schemes using the Fourier transform was set out in [16]. Error estimate in Sobolev space for Crank-Nicolson approximations of a nonlinear Schrödinger equation, with a nonlinearity of the type $\Upsilon(u) = |u|^2 u$ and Dirichlet boundary conditions, was established in [15]. From a numerical point of view, it is also important to mention the following works: [4, 13, 22, 30].

W. Strauss [26] showed that if ρ (the exponent of the nonlinear term) is large enough, then a substantial class of solutions are asymptotically free, that is, it proved that the only asymptotically free solution was identically zero when $0 < \rho \leq 1/n$ for $n \geq 2$, and when $0 < p \leq 1$ for $n = 1$. After that, in 1984, J. Barab [3] extended the theorem of W. Strauss and proved that the only smooth asymptotically free solution was identically zero when $n \geq 1$ and $0 < \rho \leq 2/n$.

For a numerical investigation on the asymptotically free solution of the Schrödinger equations, using the finite difference methods we can mention [25]. An analysis of semi-implicit compact finite difference methods for the nonlinear Schrödinger equation perturbed by the wave operator was studied in [31].

In the unbounded domain setting, a finite element approximation for the one-dimensional time-dependent Schrödinger equation was considered in [17].

We include, in the references at the end of this paper, some works relating to non-cylindrical mixed problems for others models and related arguments, such as: [5, 10, 20].

In [12] the authors investigated, in the one-dimensional case, the existence and uniqueness of solution and presented the numerical simulation to a nonlinear Schrödinger equation with moving boundary. In order to obtain the approximate solution, the Crank-Nicolson method was used in temporal discretization and consequently resulted in a non-linear algebraic system. To ensure the quadratic convergence order, Newton's method was used, which has a high computational cost.

In this paper, we obtain theoretical results of existence and uniqueness of solution to a nonlinear Schrödinger equation with moving boundary, in the n-dimensional case. To apply Faedo-Galerkin method and compactness results, we need transform the original problem (1.1) into an equivalent one (1.2) defined in a cylindrical domain. We also present numerical simulation for the two-dimensional case, obtained applying the linearized method of Crank-Nicolson-Galerkin which guarantees the quadratic convergence order, with low computational cost.

The plan of this paper is as follows. In Section 2, we analyze the mathematical theoretical aspects related to the existence and uniqueness of solution to problems (1.1) and (1.2). In order to make this article as complete as possible, we include all the details regarding to the estimates, passage to the limit and uniqueness. Section 3 is devoted to numerical solution, in this section we develop a numerical method for the system (1.2). The change of variables previously defined allows us to return to the original problem and study the behavior of the numerical solution with different types of boundaries. We use the finite element method for a spatial discretization and a linearized Crank-Nicolson-Galerkin method for a temporal discretization. In Section 4, we present numerical examples in order to verify the efficiency and feasibility of the algorithm developed in the Section 3. Finally, we compare the numerical results obtained with the expected results from the theoretical analysis.

2. Existence and uniqueness

Theorem 2.1. *We assume that the hypotheses (H1) and (H2) are satisfied. Let us consider the initial data* $u_0 \in H_0^1(\Omega_0)$ *and* $\widehat{f} \in L^2(0,T;H_0^1(\Omega_t))$, *then there exists a function* $u : \widehat{Q} \longrightarrow \mathbb{C}$ *such that*

1. $u \in L^\infty(0,T;H_0^1(\Omega_t)) \cap L^p(0,T;L^p(\Omega_t))$, *with* $p = \rho + 2$;
2. $u' \in L^{p'}(0,T;H^{-1}(\Omega_t))$, *with* $p' = \frac{\rho+2}{\rho+1}$;
3. $u' - i\Delta u + |u|^\rho u = \hat{f}$ *in* $L^{\frac{\rho+2}{\rho+1}}(0,T;H^{-1}(\Omega_t))$;
4. $u(0) = u_0$ *in* Ω_0.

Due to diffeomorphism $\mathcal{T}$, we know that u is solution of the problem (1.1) given by Theorem (2.1) if, and only if, v is solution of the problem (1.2) given by the following theorem:

Theorem 2.2. *We assume that the hypotheses (H1) and (H2) are satisfied. Let us consider the initial data* $v_0 \in H_0^1(\Omega)$ *and* $f \in L^2(0,T;H_0^1(\Omega))$, *then there exists a function* $v : Q \to \mathbb{C}$, *such that*

1. $v \in L^\infty(0,T;H_0^1(\Omega)) \cap L^p(0,T;L^p(\Omega))$, *with* $p = \rho + 2$;
2. $v' \in L^{p'}(0,T;H^{-1}(\Omega))$, *with* $p' = \frac{\rho+2}{\rho+1}$;
3. $v' - \dfrac{k'}{k} y_j \dfrac{\partial v}{\partial y_j} - i\dfrac{1}{k^2}\Delta v + |v|^\rho v = f$ *in* $L^{\frac{\rho+2}{\rho+1}}(0,T;H^{-1}(\Omega))$;
4. $v(0) = v_0$ *in* Ω.

The problem (1.2) is defined in the cylindrical domain. Thus, we can prove the Theorem 2.2 using appropriate techniques for this type of domain.

We use the standard notation for Sobolev spaces and their norms. In particular, let $(\cdot,\cdot)$, $|\cdot|$ and $((\cdot,\cdot))$, $\|\cdot\|$ be respectively the scalar products and the norms in $L^2(\Omega)$ and $H_0^1(\Omega)$.

Proof of Theorem 2.2. (Existence of solutions) The proof will be done by Faedo-Galerkin method. In fact, let $\{w_j\}_{j\in\mathbb{N}}$ be a Hilbertian basis of $L^2(\Omega)$, where each w_j, $j \in \mathbb{N}$, is the unique solution of the following spectral problem

$$\begin{cases} -\Delta w_j = \lambda_j w_j & \text{in} \quad \Omega, \\ w_j = 0 & \text{on} \quad \Gamma. \end{cases} \tag{2.1}$$

Therefore, $\forall j, w_j \in W = H^k(\Omega) \cap H_0^1(\Omega), \forall k \in \mathbb{N}$. We are looking for a function $v_m(y,t) = \displaystyle\sum_{j=1}^{m} g_{jm}(t) w_j(y)$ in $V_m = Span\{w_1, w_2, ..., w_m\}$ solution of the approximate problem

$$\begin{cases} (v_m'(t), w_j) - \dfrac{k'}{k}(y \cdot \nabla v_m(t), w_j) + i\dfrac{1}{k^2}(\nabla v_m(t), \nabla w_j) + (|v_m(t)|^\rho v_m(t), w_j) \\ =(f(t), w_j) \quad \text{in} \quad Q, \forall j = 1,2,\ldots,m, \\ v_m(y,t) = 0 \text{ on } \Sigma, \\ v_m(0) = v_{0m} \quad \text{in} \quad \Omega. \end{cases} \tag{2.2}$$

This approximate system (2.2) has a local solution v_m in the interval $[0,t_m]$, where $0 < t_m < T$, and its extension to the interval $[0,T]$, $T > 0$, is a consequence of *a priori* estimates established as follows.

First Estimate. Multiplying both sides of (2.2) by $\overline{g_{jm}(t)}$, conjugate of $g_{jm}(t)$, and adding in $j = 1,2,\ldots,m$, we obtain

$$\begin{aligned} &(v_m'(t), v_m(t)) - \frac{k'}{k}(y \cdot \nabla v_m(t), v_m(t)) + i\frac{1}{k^2}(\nabla v_m(t), \nabla v_m(t)) \\ &\quad + (|v_m(t)|^\rho v_m(t), v_m(t)) \\ &=(f(t), v_m(t)). \end{aligned} \tag{2.3}$$

Note that $Re\Big(i\dfrac{1}{k^2}(\nabla v_m(t), \nabla v_m(t))\Big) = 0$, thus, considering the real part of (2.3), we get

$$\begin{aligned} &Re(v_m'(t), v_m(t)) - Re\Big(\frac{k'}{k}(y\cdot\nabla v_m(t), v_m(t))\Big) \\ &+ Re(|v_m(t)|^\rho v_m(t), v_m(t)) \\ =&Re(f(t), v_m(t)). \end{aligned} \tag{2.4}$$

By Green's formula and Gauss's Lemma, it follows

$$(y\cdot\nabla v_m(t), v_m(t)) = \int_\Omega y_j \frac{\partial v_m}{\partial y_j}\overline{v_m}\ d\Omega = -\overline{\int_\Omega y_j \frac{\partial v_m}{\partial y_j}\overline{v_m}\ d\Omega} - n|v_m(t)|^2.$$

Therefore,

$$Re\Big(\frac{k'}{k}(y\cdot\nabla v_m(t), v_m(t))\Big) = -\frac{1}{2}\frac{nk'}{k}|v_m(t)|^2. \tag{2.5}$$

We also have

$$Re(|v_m(t)|^\rho v_m(t), v_m(t)) = Re\int_\Omega |v_m|^\rho v_m\ \overline{v_m}\ d\Omega = \int_\Omega |v_m|^{\rho+2}\ d\Omega. \tag{2.6}$$

Observing that $Re(v_m'(t), v_m(t)) = \dfrac{1}{2}\dfrac{d}{dt}|v_m(t)|^2$, and substituting (2.5) and (2.6) in (2.4), we obtain

$$\frac{1}{2}\frac{d}{dt}|v_m(t)|^2 + \frac{1}{2}\frac{nk'}{k}|v_m(t)|^2 + \int_\Omega |v_m|^{\rho+2}\ d\Omega \le |f(t)||v_m(t)|.$$

Integrating from 0 to $t\in[0, t_m[$ the last inequality, we get

$$\begin{aligned} &|v_m(t)|^2 + \int_0^t \frac{nk'(s)}{k(s)}|v_m(s)|^2\ ds + 2\int_0^t\int_\Omega |v_m(y,s)|^{\rho+2}\ d\Omega\ ds \\ \le&|v_m(0)|^2 + \int_0^t |f(s)|^2\ ds + \int_0^t |v_m(s)|^2\ ds. \end{aligned}$$

From the above inequality, as $\displaystyle\int_0^t\int_\Omega |v_m(y,s)|^{\rho+2}\ d\Omega\ ds \ge 0$, we can conclude

$$|v_m(t)|^2 \le |v_m(0)|^2 + \int_0^T |f(s)|^2\ ds + \int_0^t \Big[\frac{n|k'(s)|}{k(s)} + 1\Big]|v_m(s)|^2\ ds.$$

From Gronwall's inequality, $\forall t\in[0, t_m[$, we obtain

$$|v_m(t)|^2 \le C_0\ \exp\Big(\int_0^t \frac{n|k'(s)|}{k(s)} + 1\ ds\Big) \le C_0\ \exp\Big(\int_0^T \frac{n|k'(s)|}{k(s)} + 1\ ds\Big),$$

with $C_0 = |v_m(0)|^2 + \displaystyle\int_0^T |f(s)|^2\ ds$.

Therefore,

$$|v_m(t)|^2 + 2\int_0^t \int_\Omega |v_m(y,s)|^{\rho+2}\ d\Omega ds \le C_1, \forall t \in [0,T],$$

where $C_1 = C_0\ \exp\Big[T\Big(\frac{nC(T)}{k_0} + 1\Big)\Big]$.

From the last inequality, it follows the limitations

$$(v_m)_{m\in\mathbb{N}} \text{ is bounded in } L^\infty(0,T;L^2(\Omega)) \tag{2.7}$$

and

$$(v_m)_{m\in\mathbb{N}} \text{ is bounded in } L^{\rho+2}(0,T;L^{\rho+2}(\Omega)). \tag{2.8}$$

Second Estimate. Multiplying both sides of (2.2) by $\lambda_j\overline{g_{jm}(t)}$ and adding from $j = 1,2,\dots,m$, we get

$$\begin{aligned}&(v'_m(t), -\Delta v_m(t)) - \frac{k'}{k}(y\cdot\nabla v_m(t), -\Delta v_m(t))\\&+ i\frac{1}{k^2}(\nabla v_m(t), \nabla(-\Delta v_m(t))) + (|v_m(t)|^\rho v_m(t), -\Delta v_m(t))\\=&(f(t), -\Delta v_m(t)).\end{aligned} \tag{2.9}$$

Observing that $Re\Big(i\frac{1}{k^2}(\nabla v_m(t), \nabla(-\Delta v_m(t)))\Big) = 0$, taking the real part of (2.9), we obtain

$$\begin{aligned}&Re(v'_m(t), -\Delta v_m(t)) - Re\Big(\frac{k'}{k}(y\cdot\nabla v_m(t), -\Delta v_m(t))\Big)\\&+ Re(|v_m(t)|^\rho v_m(t), -\Delta v_m(t))\\=&Re(f(t), -\Delta v_m(t)).\end{aligned} \tag{2.10}$$

By Green's formula, it follows that

$$\begin{aligned}(y\cdot\nabla v_m(t), -\Delta v_m(t)) &= -\int_\Omega y_j\frac{\partial v_m}{\partial y_j}\frac{\partial^2\overline{v_m}}{\partial y_k^2}\ d\Omega\\&=\|v_m(t)\|^2 + \overline{\int_\Omega y_j\frac{\partial v_m}{\partial y_j}\frac{\partial^2\overline{v_m}}{\partial y_k^2}\ d\Omega} - \int_\Gamma y_j\cdot\nu_j\Big|\frac{\partial v_m}{\partial\nu}\Big|^2 d\Gamma.\end{aligned}$$

Therefore,

$$\begin{aligned}&Re\Big(\frac{k'}{k}(y\cdot\nabla v_m(t), -\Delta v_m(t))\Big)\\=&\frac{(2-n)k'}{2k}\|v_m(t)\|^2 - \frac{k'}{2k}\int_\Gamma y_j\cdot\nu_j\Big|\frac{\partial v_m}{\partial\nu}\Big|^2 d\Gamma.\end{aligned} \tag{2.11}$$

Let us analyse the nonlinear term $Re\Big(|v_m(t)|^\rho v_m(t), -\frac{\partial^2 v_m(t)}{\partial y^2}\Big)$. Applying Green's formula, we have the following equality

$$(|v_m(t)|^\rho v_m(t), -\Delta v_m(t))$$

$$
\begin{aligned}
=&\sum_{j=1}^{m}\int_{\Omega}|v_m|^{\rho}v_m\Big(-\frac{\partial^2\overline{v_m}}{\partial y_j^2}\Big)d\Omega\\
=&\frac{\rho}{2}\int_{\Omega}|v_m|^{\rho-2}\Big[|v_m|^2\Big|\frac{\partial\overline{v_m}}{\partial y_j}\Big|^2+\Big(v_m\frac{\partial\overline{v_m}}{\partial y_j}\Big)^2\Big]d\Omega+\int_{\Omega}|v_m|^{\rho}\Big|\frac{\partial v_m}{\partial y_j}\Big|^2d\Omega.
\end{aligned}
$$

Observing that for all $z = x + iy \in \mathbb{C}$, we have $Re(z^2 + |z|^2) = 2(Re(z))^2$, we can rewrite the above equality as follows

$$
\begin{aligned}
&Re(|v_m(t)|^{\rho}v_m(t),-\Delta v_m(t))\\
=&\frac{\rho}{2}\int_{\Omega}|v_m|^{\rho-2}2\Big(Re\Big(v_m\frac{\partial\overline{v_m}}{\partial y_j}\Big)\Big)^2d\Omega+\int_{\Omega}|v_m|^{\rho}\Big|\frac{\partial v_m}{\partial y_j}\Big|^2d\Omega.
\end{aligned}
$$

Substituting (2.11) in (2.10), noting that $Re(|v_m(t)|^{\rho}v_m(t), -\Delta v_m(t)) \geq 0$ and

$$
Re\Big(v_m'(t),-\Delta v_m(t)\Big)=\frac{1}{2}\frac{d}{dt}\|v_m(t)\|^2,
$$

we obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|v_m(t)\|^2-\frac{(2-n)k'}{2k}\|v_m(t)\|^2+\frac{k'}{2k}\int_{\Gamma}y_j\cdot\nu_j\Big|\frac{\partial v_m}{\partial\nu}\Big|^2d\Gamma\\
\leq&Re(f(t),-\Delta v_m(t)).
\end{aligned}
\tag{2.12}
$$

The term

$$
\frac{k'}{2k}\int_{\Gamma}y_j\cdot\nu_j\Big|\frac{\partial v_m}{\partial\nu}\Big|^2d\Gamma,
$$

presented on the left side of (2.12), brings an important difficulty in order to obtain estimates. To overcome such difficulty, we need the following proposition:

Proposition 2.1. *The approximate solution $v_m(t)$ satisfies the identity*

$$
\begin{aligned}
&\frac{k'}{k}\int_{\Gamma}y\cdot\nu\left|\frac{\partial v_m}{\partial\nu}\right|^2d\Gamma\\
=&k'kIm\frac{d}{dt}(v_m(t),y\cdot\nabla v_m(t))\\
&+n(k')^2Im(v_m(t),y\cdot\nabla v_m(t))+\frac{2k'}{k}\|v_m(t)\|^2\\
&+2k'kIm(P_m[|v_m(t)|^{\rho}v_m(t)],y\cdot\nabla v_m(t))-nk'kIm(f(t),v_m(t))\\
&-2k'kIm(P_mf(t),y\cdot\nabla v_m(t)),
\end{aligned}
\tag{2.13}
$$

for all $0 \leq t \leq t_m$, where P_m denotes the orthogonal projection from $L^2(\Omega)$ into $V_m \subset L^2(\Omega)$.

Applying the previous proposition, which will be proved later, we can modify

(2.12) and obtain

$$
\begin{aligned}
&\frac{d}{dt}\Big\{\|v_m(t)\|^2 + k'kIm(v_m(t), y\cdot\nabla v_m(t))\Big\}\\
&+\frac{nk'}{k}\Big\{\|v_m(t)\|^2 + k'kIm(v_m(t), y\cdot\nabla v_m(t))\Big\}\\
\le&[(k')^2 + kk'']Im(v_m(t), y\cdot\nabla v_m(t)) \qquad (2.14)\\
&-2k'kIm(P_m[|v_m(t)|^\rho v_m(t)], y\cdot\nabla v_m(t))\\
&+2k'kIm(P_m f(t), y\cdot\nabla v_m(t)) + nk'kIm(f(t), v_m(t))\\
&+2Re(f(t), -\Delta v_m(t)).
\end{aligned}
$$

If we define

$$h(t) = \|v_m(t)\|^2 + k'(t)k(t)Im(v_m(t), y\cdot\nabla v_m(t)),$$

then we can rewrite (2.14) as

$$h'(t) + \theta(t)h(t) \le r(t), \quad 0 \le t \le T, \qquad (2.15)$$

where $\theta(t) = (nk')/k$ and $r(t)$ is the right side of (2.14).

Solving (2.15) and observing that $\exp\Big(\int_0^t \theta(s)ds\Big) = \Big[\frac{k(t)}{k(0)}\Big]^n$, we obtain

$$h(t) \le \Big[\frac{k(0)}{k(t)}\Big]^n h(0) + (k(t))^{-n}\int_0^t (k(s))^n r(s)ds.$$

From our initial hypotheses and the first estimate, it follows that

$$\|v_m(t)\|^2 \le c_0 + c_1\int_0^t \|v_m(s)\|^2 ds,$$

with c_0 and c_1 being positive constants.

From Gronwall's inequality we obtain, for all $t \in [0,T]$,

$$\|v_m(t)\|^2 \le c_0 e^{c_1 t} \le c_0 e^{c_1 T}.$$

Consequently, we conclude

$$(v_m)_{m\in\mathbb{N}} \text{ is bounded in } L^\infty(0,T;H_0^1(\Omega)). \qquad (2.16)$$

Proof of Proposition 2.1. Let us consider the orthogonal projection P_m from $L^2(\Omega)$ into $V_m \subset L^2(\Omega)$, such that $P_m v = \sum_{j=1}^m (v, w_j)w_j$, $\forall v \in L^2(\Omega)$.

Multiplying the approximate equation (2.2) by w_j, adding from $j = 1, 2, \ldots, m$ and taking P_m on both sides, we obtain

$$v'_m - \frac{k'}{k}P_m[y\cdot\nabla v_m] - i\frac{1}{k^2}\Delta v_m + P_m[|v_m|^\rho v_m] = P_m f. \qquad (2.17)$$

Taking the inner product on both sides of (2.17) with $y \cdot \nabla v_m$, we get

$$\begin{aligned} &(v_m'(t), y \cdot \nabla v_m(t)) - \frac{k'}{k}(P_m[y \cdot \nabla v_m(t)], y \cdot \nabla v_m(t)) \\ &- i\frac{1}{k^2}(\Delta v_m(t), y \cdot \nabla v_m(t)) + (P_m[|v_m(t)|^\rho v_m(t)], y \cdot \nabla v_m(t)) \\ =&(P_m f(t), y \cdot \nabla v_m(t)). \end{aligned} \tag{2.18}$$

Now, since $P_m^2 = P_m$, it follows that

$$\begin{aligned} (P_m[y \cdot \nabla v_m(t)], y \cdot \nabla v_m(t)) =&(P_m^2[y \cdot \nabla v_m(t)], y \cdot \nabla v_m(t)) \\ =&(P_m[y \cdot \nabla v_m(t)], P_m[y \cdot \nabla v_m(t)]) \\ =&|P_m[y \cdot \nabla v_m(t)]|^2 \in \mathbb{R}. \end{aligned}$$

Therefore, taking the imaginary part on both sides of (2.18) and observing that $Im(iz) = Re(z)$, we conclude that

$$\begin{aligned} &Im(v_m'(t), y \cdot \nabla v_m(t)) - Re\Big(\frac{1}{k^2}(\Delta v_m(t), y \cdot \nabla v_m(t))\Big) \\ &+ Im(P_m[|v_m(t)|^\rho v_m(t)], y \cdot \nabla v_m(t)) \\ =&Im(P_m f(t), y \cdot \nabla v_m(t)). \end{aligned}$$

Integrating the equality above from 0 to T we get

$$\begin{aligned} &\int_0^T Im(v_m'(t), y \cdot \nabla v_m(t))dt + \int_0^T \frac{1}{k^2} Re(-\Delta v_m(t), y \cdot \nabla v_m(t))dt \\ &+ \int_0^T Im(P_m[|v_m(t)|^\rho v_m(t)], y \cdot \nabla v_m(t))dt \\ =&\int_0^T Im(P_m f(t), y \cdot \nabla v_m(t))dt. \end{aligned} \tag{2.19}$$

Now, we will analyse the two first terms on the left side of (2.19).

• Analysis of the term $\displaystyle\int_0^T Im(v_m'(t), y \cdot \nabla v_m(t))dt$

We have that

$$\begin{aligned} &\int_0^T (v_m'(t), y \cdot \nabla v_m(t))dt \\ =&\int_0^T \frac{d}{dt}(v_m(t), y \cdot \nabla v_m(t))dt - \int_0^T (v_m(t), y \cdot \nabla v_m'(t))dt. \end{aligned} \tag{2.20}$$

From Gauss' Lemma, we get

$$\int_\Omega y_j v_m \frac{\partial \overline{v_m'}}{\partial y_j}\, d\Omega + \int_\Omega y_j \frac{\partial v_m}{\partial y_j} \overline{v_m'}\, d\Omega + \int_\Omega n v_m \overline{v_m'}\, d\Omega = 0. \tag{2.21}$$

Substituting (2.21) in (2.20), we obtain

$$\begin{aligned} &\int_0^T (v_m'(t), y\cdot\nabla v_m(t))dt - \int_0^T (y\cdot\nabla v_m(t), v_m'(t))dt \\ &= \int_0^T \frac{d}{dt}(v_m(t), y\cdot\nabla v_m(t))dt + n\int_0^T (v_m(t), v_m'(t))dt. \end{aligned} \tag{2.22}$$

Note that $-i(z-\overline{z}) = 2Im(z)$. Therefore, we can rewrite (2.22) as

$$\begin{aligned} &\int_0^T 2Im(v_m'(t), y\cdot\nabla v_m(t))dt \\ &= -i\Big[\int_0^T \frac{d}{dt}(v_m(t), y\cdot\nabla v_m(t))dt + n\int_0^T (v_m(t), v_m'(t))dt\Big]. \end{aligned} \tag{2.23}$$

Taking the real part of the equation (2.23) and noting that $Re(iz) = -Imz$, we arrive at

$$\begin{aligned} &\int_0^T 2Im(v_m'(t), y\cdot\nabla v_m(t))dt \\ &= \int_0^T Im\frac{d}{dt}(v_m(t), y\cdot\nabla v_m(t))dt + n\int_0^T Im(v_m(t), v_m'(t))dt. \end{aligned} \tag{2.24}$$

We observe that

$$n\int_0^T Im(v_m(t), v_m'(t)) = n\int_0^T Im(-v_m'(t), v_m(t))dt. \tag{2.25}$$

From (2.17), we get

$$-v_m' = -\frac{k'}{k}P_m[y\cdot\nabla v_m] - i\frac{1}{k^2}\Delta v_m + P_m[|v_m|^\rho v_m] - P_m f.$$

In this way, we can rewrite it as

$$\begin{aligned} &nIm(-v_m'(t), v_m(t)) \\ =&\frac{-nk'}{k}Im(P_m[y\cdot\nabla v_m(t)], v_m(t)) \\ &+ \frac{n}{k^2}Re(-\Delta v_m(t), v_m(t)) + nIm(P_m[|v_m(t)|^\rho v_m(t)], v_m(t)) \\ &- nIm(P_m f(t), v_m(t)). \end{aligned} \tag{2.26}$$

Using the properties of P_m and the Green's formula, (2.26) becomes

$$\begin{aligned} nIm(-v_m'(t), v_m(t)) =&\frac{-nk'}{k}Im(y\cdot\nabla v_m(t), v_m(t)) + \frac{n}{k^2}|\nabla v_m(t)|^2 \\ &+ nIm(|v_m(t)|^\rho v_m(t), v_m(t)) - nIm(f(t), v_m(t)). \end{aligned}$$

Now, notice that

$$(|v_m(t)|^\rho v_m(t), v_m(t)) = \int_\Omega |v_m|^\rho v_m\overline{v_m}d\Omega = \int_\Omega |v_m|^{\rho+2}d\Omega \in \mathbb{R}.$$

Therefore, $Im(|v_m(t)|^\rho v_m(t), v_m(t)) = 0$.
Thus, we obtain

$$\begin{aligned} nIm(-v'_m(t), v_m(t)) =& \frac{-nk'}{k} Im(y \cdot \nabla v_m(t), v_m(t)) + \frac{n}{k^2}\|v_m(t)\|^2 \\ &- nIm(f(t), v_m(t)). \end{aligned} \tag{2.27}$$

Applying (2.27) in (2.25), we get

$$\begin{aligned} &n\int_0^T Im(v_m(t), v'_m(t)) \\ =&L - \int_0^T \frac{nk'}{k} Im(y \cdot \nabla v_m(t), v_m(t))dt \\ &+ \int_0^T \frac{n}{k^2}\|v_m(t)\|^2 dt - n\int_0^T Im(f(t), v_m(t))dt. \end{aligned} \tag{2.28}$$

Finally, making use of (2.28) in (2.24), we conclude

$$\begin{aligned} &\int_0^T Im(v'_m(t), y \cdot \nabla v_m(t))dt \\ =&\frac{1}{2}\int_0^T Im\Big[\frac{d}{dt}(v_m(t), y \cdot \nabla v_m(t))\Big]dt - \int_0^T \frac{nk'}{2k} Im(y \cdot \nabla v_m(t), v_m(t))dt \\ &+ \int_0^T \frac{n}{2k^2}\|v_m(t)\|^2 dt - \frac{n}{2}\int_0^T Im(f(t), v_m(t))dt. \end{aligned} \tag{2.29}$$

• Analysis of the term $\displaystyle\int_0^T \frac{1}{k^2} Re(-\Delta v_m(t), y \cdot \nabla v_m(t))dt$

From Gauss' Lemma, we get

$$\begin{aligned} &(-\Delta v_m(t), y \cdot \nabla v_m(t)) \\ =&\Big(\frac{\partial v_m}{\partial y_j}, \frac{\partial}{\partial y_j}\Big[y_k \frac{\partial v_m}{\partial y_k}\Big]\Big) - \int_\Gamma \frac{\partial v_m}{\partial \nu} y_k \frac{\partial \overline{v_m}}{\partial y_k} d\Gamma \\ =&\Big(\frac{\partial v_m}{\partial y_j}, \delta_k^j \frac{\partial v_m}{\partial y_k}\Big) + \Big(\frac{\partial v_m}{\partial y_j}, y_k \frac{\partial}{\partial y_j}\Big[\frac{\partial v_m}{\partial y_k}\Big]\Big) - \int_\Gamma \frac{\partial v_m}{\partial \nu} y_k \frac{\partial \overline{v_m}}{\partial y_k} d\Gamma, \end{aligned}$$

and, therefore,

$$\begin{aligned} &Re(-\Delta v_m(t), y \cdot \nabla v_m(t)) \\ =&\|v_m(t)\|^2 + \int_\Omega y_k Re\Big\{\frac{\partial v_m}{\partial y_j} \frac{\partial}{\partial y_k}\Big[\frac{\partial \overline{v_m}}{\partial y_j}\Big]\Big\} d\Omega - \int_\Gamma y \cdot \nu \Big|\frac{\partial v_m}{\partial \nu}\Big|^2 d\Gamma. \end{aligned}$$

From Gauss' Lemma, we get

$$\int_\Omega y_k Re\Big\{\frac{\partial v_m}{\partial y_j} \frac{\partial}{\partial y_k}\Big[\frac{\partial \overline{v_m}}{\partial y_j}\Big]\Big\} d\Omega = -\frac{n}{2}\|v_m(t)\|^2 + \frac{1}{2}\int_\Gamma y \cdot \nu \Big|\frac{\partial v_m}{\partial \nu}\Big|^2 d\Gamma.$$

Thus,

$$\int_0^T \frac{1}{k^2} Re(-\Delta v_m(t), y\cdot\nabla v_m(t))dt$$
$$= \int_0^T \frac{1}{k^2}\|v_m(t)\|^2 dt - \frac{n}{2}\int_0^T \frac{1}{k^2}\|v_m(t)\|^2 dt - \frac{1}{2}\int_0^T \frac{1}{k^2}\int_\Gamma y\cdot\nu\Big|\frac{\partial v_m}{\partial \nu}\Big|^2 d\Gamma\ dt. \tag{2.30}$$

Making use of (2.29) and (2.30) in (2.19) follows that

$$\int_0^T Im\frac{d}{dt}(v_m(t), y\cdot\nabla v_m(t))dt - n\int_0^T \frac{k'}{k} Im(y\cdot\nabla v_m(t), v_m(t))dt$$
$$- n\int_0^T Im(f(t), v_m(t))dt + \int_0^T \frac{2}{k^2}\|v_m(t)\|^2 dt$$
$$- \int_0^T \frac{1}{k^2}\int_\Gamma y\cdot\nu\Big|\frac{\partial v_m}{\partial \nu}\Big|^2 d\Gamma\ dt + 2\int_0^T Im(P_m[|v_m(t)|^\rho v_m(t)], y\cdot\nabla v_m(t))dt$$
$$=2\int_0^T Im(P_m f(t), y\cdot\nabla v_m(t))dt.$$

Note that the above identity is true $\forall t\in[0,T]$. Then, considering $T=t$, taking the derivative with respect to t on both sides of the equation and multiplying by $k'k$, we arrive at

$$\frac{k'}{k}\int_\Gamma y\cdot\nu\Big|\frac{\partial v_m}{\partial \nu}\Big|^2 d\Gamma$$
$$=k'kIm\frac{d}{dt}(v_m(t), y\cdot\nabla v_m(t)) + n(k')^2 Im(v_m(t), y\cdot\nabla v_m(t)) + \frac{2k'}{k}\|v_m(t)\|^2$$
$$+ 2k'kIm(P_m[|v_m(t)|^\rho v_m(t)], y\cdot\nabla v_m(t)) - nk'kIm(f(t), v_m(t))$$
$$- 2k'kIm(P_m f(t), y\cdot\nabla v_m(t)).$$

□

Passage to the limit. By the limitations (2.7), (2.8), (2.16), we can extract a subsequence, still denoted by $(v_m)_{m\in\mathbb{N}}$, such that

$$v_m \overset{*}{\rightharpoonup} v \text{ in } L^\infty(0,T;L^2(\Omega)), \tag{2.31}$$
$$v_m \rightharpoonup v \text{ in } L^{\rho+2}(0,T;L^{\rho+2}(\Omega)), \tag{2.32}$$
$$v_m \overset{*}{\rightharpoonup} v \text{ in } L^\infty(0,T;H_0^1(\Omega)). \tag{2.33}$$

Let us consider, once again, $L^2(\Omega) = V_m \oplus V_m^\perp$ and P_m the orthogonal projection from $L^2(\Omega)$ into V_m. We remember that P_m is bounded and self-adjoint,

$$P_m w = \sum_{j=1}^m (w, w_j)w_j,\ \forall w\in L^2(\Omega) \text{ and } P_m w = w \quad \forall w\in V_m.$$

From (2.17) we obtain that,

$$v_m' = \frac{k'}{k} P_m[y \cdot \nabla v_m] + i \frac{1}{k^2} \Delta v_m - P_m[|v_m|^\rho v_m] + P_m f.$$

From the last equality, thanks to the limitations previously obtained and the properties of the projection operator, we can conclude that

$$(v_m')_{m \in \mathbb{N}} \text{ is bounded in } L^{\frac{\rho+2}{\rho+1}}(0,T; H^{-1}(\Omega)). \tag{2.34}$$

From the compactness theorem due to Lions and the limitations in (2.7) and (2.34) we obtain the following strong convergence,

$$v_m \longrightarrow v \text{ in } L^2(0,T; L^2(\Omega)). \tag{2.35}$$

Thus, with the convergences obtained above, we can pass to the limit in the approximate problem. In fact, applying the approximate equation to $\theta \in \mathcal{D}(0,T)$ we get

$$\begin{aligned} &\int_0^T (v_m(t), w_j)\theta' \, dt - \int_0^T \frac{k'}{k} (y \cdot \nabla v_m(t), w_j)\theta \, dt \\ &+ \int_0^T i \frac{1}{k^2} (\nabla v_m(t), \nabla w_j)\theta \, dt + \int_0^T (|v_m(t)|^\rho v_m(t), w_j)\theta \, dt \\ = &\int_0^T (f(t), w_j)\theta \, dt. \end{aligned}$$

Using the convergences obtained above we can take the limit when $m \to +\infty$, j fixed, and obtain

$$\begin{aligned} &\frac{d}{dt}(v_m(t), w) - \frac{k'}{k}(y \cdot \nabla v_m(t), w) + i \frac{1}{k^2}(\nabla v_m(t), \nabla w) + (|v_m(t)|^\rho v_m(t), w) \\ &= (f(t), w), \end{aligned}$$

in the sense of $\mathcal{D}'(0,T)$, for each $w \in H_0^1(\Omega)$.

Since $v' \in L^{\frac{\rho+2}{\rho+1}}(0,T; H^{-1}(\Omega))$ and

$$v' = \frac{k'}{k} y \nabla v + i \frac{1}{k^2} \Delta v - |v|^\rho v + f,$$

it follows that

$$v' - \frac{k'}{k} y \nabla v - i \frac{1}{k^2} \Delta v + |v|^\rho v = f \quad \text{in} \quad L^{\frac{\rho+2}{\rho+1}}(0,T; H^{-1}(\Omega)).$$

Note that, by the regularity obtained for v and v', we have that $v(0) = v_0$ makes sense using the standard method.

Uniqueness of solution. Let us consider v_1 and v_2 solutions of (1.2). If we define $z = v_2 - v_1$, we know that $z \in L^\infty(0,T;H_0^1(\Omega)) \cap L^{\rho+2}(0,T;L^{\rho+2}(\Omega))$, $z' \in L^{\frac{\rho+2}{\rho+1}}(0,T;H^{-1}(\Omega))$ and $\forall w \in H_0^1(\Omega) \cap L^{\rho+2}(\Omega)$, z satisfies

$$\langle z'(t), w\rangle_{H^{-1}(\Omega)\times H_0^1(\Omega)} - \left(\frac{k'}{k} y \cdot \nabla z(t), w\right) + i\frac{1}{k^2}\langle -\Delta z(t), w\rangle_{H^{-1}(\Omega)\times H_0^1(\Omega)}$$
$$+ \langle |v_2(t)|^\rho v_2(t) - |v_1(t)|^\rho v_1(t), w\rangle_{L^{\frac{\rho+2}{\rho+1}}(\Omega)\times L^{\rho+2}(\Omega)}$$
$$=0.$$

If we take $w = z(t)$, we get

$$\langle z'(t), z(t)\rangle_{H^{-1}(\Omega)\times H_0^1(\Omega)} - \left(\frac{k'}{k} y \cdot \nabla z(t), z(t)\right) + i\frac{1}{k^2}\langle -\Delta z(t), z(t)\rangle_{H^{-1}(\Omega)\times H_0^1(\Omega)}$$
$$+ \langle |v_2(t)|^\rho v_2(t) - |v_1(t)|^\rho v_1(t), z(t)\rangle_{L^{\frac{\rho+2}{\rho+1}}(\Omega)\times L^{\rho+2}(\Omega)}$$
$$=0.$$

Taking the real part of the above equation, it follows that

$$\frac{1}{2}\frac{d}{dt}\langle z(t), z(t)\rangle_{H^{-1}(\Omega)\times H_0^1(\Omega)} + \frac{nk'}{k}|z(t)|^2$$
$$+ Re\langle |v_2(t)|^\rho v_2(t) - |v_1(t)|^\rho v_1(t), z(t)\rangle_{L^{\frac{\rho+2}{\rho+1}}(\Omega)\times L^{\rho+2}(\Omega)}$$
$$=0.$$

We observe that,

$$\langle |v_2(t)|^\rho v_2(t) - |v_1(t)|^\rho v_1(t), z(t)\rangle_{L^{\frac{\rho+2}{\rho+1}}(\Omega)\times L^{\rho+2}(\Omega)}$$
$$= \int_\Omega \{|v_2|^\rho v_2 - |v_1|^\rho v_1\} \cdot \{\overline{v_2} - \overline{v_1}\}\, d\Omega$$
$$= \int_\Omega |v_2|^{\rho+2} + |v_1|^{\rho+2} - |v_2|^\rho v_2 \overline{v_1} - |v_1|^\rho v_1 \overline{v_2}\, d\Omega.$$

Therefore,

$$Re\langle |v_2(t)|^\rho v_2(t) - |v_1(t)|^\rho v_1(t), z(t)\rangle_{L^{\frac{\rho+2}{\rho+1}}(\Omega)\times L^{\rho+2}(\Omega)}$$
$$= \int_\Omega |v_2|^{\rho+2} + |v_1|^{\rho+2}\, d\Omega - Re\int_\Omega |v_2|^\rho v_2\overline{v_1} + |v_1|^\rho v_1 \overline{v_2}\, d\Omega$$
$$\geq \int_\Omega |v_2|^{\rho+2} + |v_1|^{\rho+2}\, d\Omega - \int_\Omega |v_2|^\rho |v_2||v_1| + |v_1|^\rho |v_1||v_2|\, d\Omega$$
$$= \int_\Omega |v_2|^{\rho+1}(|v_2| - |v_1|)\, d\Omega - \int_\Omega |v_1|^{\rho+1}(|v_2| - |v_1|)\, d\Omega$$

$$= \int_{\Omega} (|v_2|^{\rho+1} - |v_1|^{\rho+1})(|v_2| - |v_1|)\, d\Omega$$

$$\geq 0.$$

Thus,

$$\frac{d}{dt}\langle z(t), z(t)\rangle_{H^{-1}(\Omega)\times H_0^1(\Omega)} \leq \frac{2n|k'|}{k}|z(t)|^2 \text{ q. s. em }]0,T[.$$

Integrating from 0 to $t \in [0,T]$, on both sides of the above inequality, we obtain

$$\langle z(t), z(t)\rangle_{H^{-1}(\Omega)\times H_0^1(\Omega)} \leq \langle z(0), z(0)\rangle_{H^{-1}(\Omega)\times H_0^1(\Omega)} + \int_0^t \frac{2n|k'(s)|}{k(s)}|z(s)|^2 ds.$$

From Gronwall's inequality, noting that $z(0) = 0$, we get

$$0 \leq |z(t)|^2 \leq 0, \ \forall\, t \in [0,T].$$

Therefore, $|z(t)|^2 = 0$ and, thus, $v_2 = v_1$. □

3. Approximate numerical solution

In this section, we will develop an algorithm for the system(1.2), using the finite element method for a spatial discretization and a linearized Crank-Nicolson-Galerkin method for a temporal discretization. The algorithm was initially used in [2] for parabolic equations and in [28] for hyperbolic equations. For each discrete time step, a linear system is obtained preserving a quadratic order of convergence in time. Solving the linear system, the numerical solution of the problem with fixed boundary is determined and then we return to the original variables to study the behavior of the solution with different types of boundary. We use linear, quadratic and cubic Lagrange polynomials as basis functions to obtain the approximate numerical solution of the non-linear Schrödinger problem, in the two-dimensional case.

3.1. Iterative method

This subsection focuses finite element method to discretize the spatial variable and the Crank-Nicolson-Galerkin method to discretize the temporal variable.

Consider $\{\mathcal{T}_h\}$ a family of polygonalization $\mathcal{T}_h = \{K\}$ of Ω, satisfying the standart condition, see for instance [8]. For a given integer $p \geq 1$, we introduce the finite element space

$$V_h(\Omega) = \{q_h \in C^0(\overline{\Omega}); q_h\,|_K \in P_p(K),\ \forall K \in \mathcal{T}_h, q_h = 0 \text{ on } \Gamma\},$$

where $P_p(K)$ is the set of polynomials on K of degree less than or equal to p. Thus, $V_h(\Omega)$ denotes the space of piecewise continuous polynomial functions of degree p. More specifically, in this paper we will use Lagrange polynomials as basis functions with degree $p = 1, p = 2$ and $p = 3$.

Now, we present a linearized modification to obtain an approximate solution in discrete time. This finite difference method preserves the quadratic convergence order in time, a consequence of the Crank-Nicolson-Galerkin method. For simplicity, we will use the following helper function on the nonlinear term: $g : \mathbb{C} \to \mathbb{C}, s \mapsto g(s) = |s|^\rho s$.

Algorithm of the Crank-Nicolson-Galerkin method.

Let $0 = t_0 < t_1 < \cdots < t_N = T,\ \forall N \in \mathbb{N}$, a uniform discretization of the interval $[0,T]$ and $\Delta t = T/N$ the length of each time interval and consider $U^n \in V_h(\Omega)$ be an approximation of $u(t_n)$, with $t_n = n\Delta t$, $0 \le n \le N$.

The solution numeric consists of determining $\{U^n\}_{n=0}^N$ in $V_h(\Omega)$, such that, to $n \in \{2, \dots, N\}$,

$$\begin{aligned}&(\overline{\partial} U^n, w) - \frac{k'(t_{n-\frac{1}{2}})}{k(t_{n-\frac{1}{2}})}(y \cdot \nabla \widehat{U}^n, w) + i\frac{1}{k^2(t_{n-\frac{1}{2}})}(\nabla \widehat{U}^n, \nabla w) + (g(\widetilde{U}^n), w)\\ =&(f_{n-\frac{1}{2}}, w), \quad \forall w \in V_h(\Omega),\end{aligned} \tag{3.1}$$

where

$$\overline{\partial} U^n = \frac{U^n - U^{n-1}}{\Delta t},\ \widehat{U}^{\,n} = \frac{U^n + U^{n-1}}{2},\ \widetilde{U}^{\,n} = \frac{3U^{n-1} - U^{n-2}}{2}, \tag{3.2}$$

$U^n = U(t_n)$ and $t_{n-\frac{1}{2}} = (t_n + t_{n-1})/2$ is the midpoint in each interval.

We note that this method requires a mechanism to determine U^1. Thus, we consider a single-step corrective predictor method, through two problems defined below.

Let $U^{1,0} \in V_h(\Omega)$ be a predictive approximation of $v(t_1)$. Taking $n = 1$ in the equation (3.1) and replacing $\widetilde{U}^1$ by U^0 in the nonlinear term, we arrived at

$$\begin{aligned}&\Big(\frac{U^{1,0} - U^0}{\Delta t}, w\Big) - \frac{k'(t_{\frac{1}{2}})}{k(t_{\frac{1}{2}})}\Big(y \cdot \nabla\Big(\frac{U^{1,0} + U^0}{2}\Big), w\Big)\\ &+ i\frac{1}{k^2(t_{\frac{1}{2}})}\Big(\nabla\Big(\frac{U^{1,0} + U^0}{2}\Big), \nabla w\Big) + (g(U^0), w)\\ =&(f(t_{\frac{1}{2}}), w),\end{aligned} \tag{3.3}$$

$\forall w \in V_h(\Omega)$. Now let $U^1 \in V_h(\Omega)$ be a corrective approximation of $v(t_1)$. Considering once more the aquation (3.1) at $n = 1$ and replacing $\widetilde{U}^1$ by $\dfrac{U^{1,0} + U^0}{2}$ in the non-linear term, we get $\forall w \in V_h(\Omega)$,

$$\begin{aligned}&(\overline{\partial} U^1, w) - \frac{k'(t_{\frac{1}{2}})}{k(t_{\frac{1}{2}})}(y \cdot \nabla \widehat{U}^1, w) + i\frac{1}{k^2(t_{\frac{1}{2}})}(\nabla \widehat{U}^1, \nabla w)\\ &+ \Big(g\Big(\frac{U^{1,0} + U^0}{2}\Big), w\Big) + \Big(g\Big(\frac{U^{1,0} + U^0}{2}\Big), w\Big)\\ =&(f(t_{\frac{1}{2}}), w).\end{aligned} \tag{3.4}$$

As $U^n, U^{1,0}$ and $U^1 \in V_h(\Omega)$, we get

$$U^n = \sum_{j=1}^m c_j^n \varphi_j, \quad U^{1,0} = \sum_{j=1}^m c_j^{1,0} \varphi_j \ \text{ and } \ U^1 = \sum_{j=1}^m c_j^1 \varphi_j, \tag{3.5}$$

where $\{\varphi_j\}_{j=1}^m$ are polynomials of the base of $V_h(\Omega)$. Thus, we can rewrite the numerical problem as a matrix system.

Denoting the matrices:

$$A = [a_{jk}] = (\varphi_j, \varphi_k), \quad B = [b_{jk}] = \Big(y_j \frac{\partial \varphi_j}{\partial y}, \varphi_k\Big), \quad D = [d_{jk}] = \Big(\frac{\partial \varphi_j}{\partial y}, \frac{\partial \varphi_k}{\partial y}\Big),$$

$$R(\widetilde{c}^{\,n}) = [r_{jk}(\widetilde{c}^{\,n})] = \Big(g\Big(\sum_{i=1}^{m} \widetilde{c}_j^{\,n} \varphi_j\Big), \varphi_k\Big), \quad R(c^0) = \Big(g\Big(\sum_{i=1}^{m} c_j^0 \varphi_j\Big), \varphi_k\Big),$$

$$R(\widehat{c}^{\,1,0}) = \Big(g\Big(\sum_{i=1}^{m} \widehat{c}_j^{\,1,0} \varphi_j\Big), \varphi_k\Big), \quad F^{n-\frac{1}{2}} = [f_{jk}(t_{n-\frac{1}{2}})] = (f(t_{n-\frac{1}{2}}), \varphi_k),$$

$$F^{\frac{1}{2}} = [f_{jk}(t_{\frac{1}{2}})] = (f(t_{\frac{1}{2}}), \varphi_k),$$

where $j,k \in \{1,\ldots,m\}$ and $c^n = (c_1^n, \ldots, c_m^n)^T$, $c^{1,0} = (c_1^{1,0}, \ldots, c_m^{1,0})^T$, $c^1 = (c_1^1, \ldots, c_m^1)^T$.

Therefore, substituting (3.5) in (3.1), (3.3) and (3.4), respectively, and taking $w = \varphi_k$, for $k = 1, \ldots, m$, we get

$$\begin{cases} A\overline{\partial} c^n - \dfrac{k'(t_{n-\frac{1}{2}})}{k(t_{n-\frac{1}{2}})} B\widehat{c}^{\,n} + i\dfrac{1}{k^2(t_{n-\frac{1}{2}})} D\widehat{c}^{\,n} + R(\widetilde{c}^{\,n}) = F^{n-\frac{1}{2}}, \\ A\overline{\partial} c^{1,0} - \dfrac{k'(t_{\frac{1}{2}})}{k(t_{\frac{1}{2}})} B\widehat{c}^{\,1,0} + i\dfrac{1}{k^2(t_{\frac{1}{2}})} D\widehat{c}^{\,1,0} + R(c^0) = F^{\frac{1}{2}}, \\ A\overline{\partial} c^1 - \dfrac{k'(t_{\frac{1}{2}})}{k(t_{\frac{1}{2}})} B\widehat{c}^{\,1} + i\dfrac{1}{k^2(t_{\frac{1}{2}})} D\widehat{c}^{\,1} + R(\widehat{c}^{\,1,0}) = F^{\frac{1}{2}}, \end{cases} \tag{3.6}$$

where $C = (c^1, c^2, \cdots, c^N)$ is the coefficient vector to be determined.

Rearranging the corresponding terms of (3.6), for each $n \in \{2, \ldots, N\}$ and using the notation (3.2), we get the following linear algebraic system of first order

$$\begin{cases} M^{n-\frac{1}{2}} c^n = L^{n-\frac{1}{2}} c^{n-1} - R\Big(\dfrac{3c^{n-1} - c^{n-2}}{2}\Big) + F^{n-\frac{1}{2}}, \\ M^{\frac{1}{2}} c^{1,0} = L^{\frac{1}{2}} c^0 - R(c^0) + F^{\frac{1}{2}}, \\ M^{\frac{1}{2}} c^1 = L^{\frac{1}{2}} c^0 - R\Big(\dfrac{c^{1,0} + c^0}{2}\Big) + F^{\frac{1}{2}}, \end{cases} \tag{3.7}$$

where

$$M^s = \frac{A}{\Delta t} - \frac{k'(t_s)}{2k(t_s)} B + i\frac{1}{2k^2(t_s)} D \text{ and } L^s = \frac{A}{\Delta t} + \frac{k'(t_s)}{2k(t_s)} B - i\frac{1}{2k^2(t_s)} D.$$

Before solving the system (3.7) we need to define the initial data.

Let $U^0 = P_h v(t_0)$, where P_h is an elliptic projection operator, or Ritz, given by $P_h : H_0^1(\Omega) \to V_h(\Omega)$, $v \mapsto P_h v$, satisfying

$$(\nabla P_h v - \nabla v, \nabla \varphi) = 0, \ \forall \varphi \in V_h(\Omega).$$

Thus, it is possible to determine c^0 and therefore display the numerical solution at each time step.

4. Numerical experiments

In this section, we perform some numerical experiments and validate the theoretical results established in the previous sections. We shall carry our numerical experiments for the completely discrete scheme, that is, using the linearized Crank-Nicolson-Galerkin method.

In this perspective, we use linear, quadratic and cubic Lagrange polynomials as the basis of the finite element space to study the numerical error behavior for the two-dimensional case with different boundaries. The examples taken were chosen to explore the hypotheses involved and their influence on the problem under study. The results are displayed graphically and the numerical implementation was performed in Matlab language.

Although we did not carry out a theoretical study of error estimation in this article, we found related works that exposed the error analysis showing consistency in their results. For non-linear problems, see for example [6, 28, 29, 32], and with moving border, see [1, 23, 24].

In view of this, we calculate the error between the numerical solution and the exact solution in the norm:

$$\max_{0\leq n\leq N}|U^n - v(t_n)|_{L^2(\Omega)}.$$

Using piecewise linear functions and the linearized Crank-Nicolson Galerkin method, we expect to find, at best, an order of convergence $\mathcal{O}(h^{p+1} + \Delta t^2)$, where p is the degree of the polynomial of the base of the finite element space.

The domain for numerical simulations for the two-dimensional case is given by $\Omega = [0, 1] \times [0, 1]$ and the diameter of each element is given by

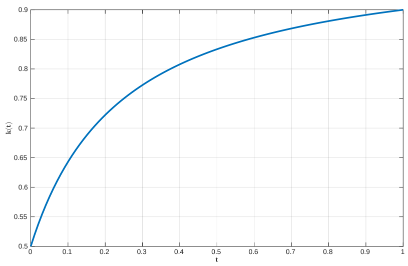
$$h = \sqrt{(1/N_{y_1})^2 + (1/N_{y_2})^2},$$

where N_{y_1} and N_{y_2} are the number of elements on the axis y_1 and y_2, respectively. We consider for a uniform mesh of squares of sides $1/N_{y_1}$ e $1/N_{y_2}$, so that $N_{y_1} = N_{y_2} \in \{2^2, 2^3, \ldots\}$. We take, for $i = 1, 2, \ldots$, discretizations for space h_i, and for the time $\Delta t = {h_i}^{(p+1)/2}$ to estimate the convergence rate. So, for every h_i, we get that the error is given approximately by $E_i \approx h_i^{p+1}$, with $p + 1$ being the optimal convergence order. In addition, the convergence rate is given by $p + 1 = \log_2\Big(E_i/E_{i+1}\Big)$.

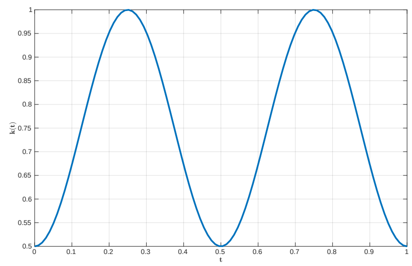
The error graphs will be displayed in logarithmic scale and the triangles that will be arranged will allow us to measure the slope of each curve. In this way, it will be possible to obtain the related convergence orders for each chosen base. In all numerical simulations we are considering the final time $T = 1$ and we take $\rho = 3$ in the nonlinear term. Note that the results are similar when we vary this parameter in the simulations.

Furthermore, the numerical solution is displayed in the original variables of the problem (1.1) to visualize the influence of the frontier over time. Therefore, to $t \in [0, 1]$, we consider two types of borders:

$$\text{boundary 1: } k(t) = \frac{8t + 1}{8t + 2} \text{ and boundary 2: } k(t) = \frac{3 - \cos(4\pi t)}{4}.$$



(a) Function $k(t)$ of the boundary 1.



(b) Function $k(t)$ of the boundary 2.

In the examples that will be presented, we replace the solution v in the system (1.2) to construct the function f and the initial conditions.

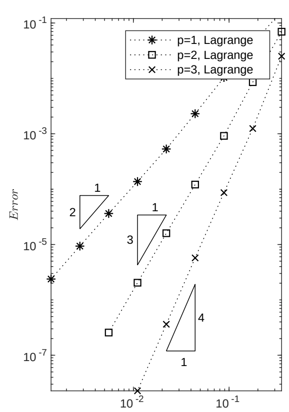
Example 4.1. In this example we consider the solution

$$v(y_1, y_2, t) = \sin(\pi y_1)\sin(\pi y_2)\cos(\pi t) + i\sin(\pi y_1)\sin(\pi y_2)\cos(\pi t).$$

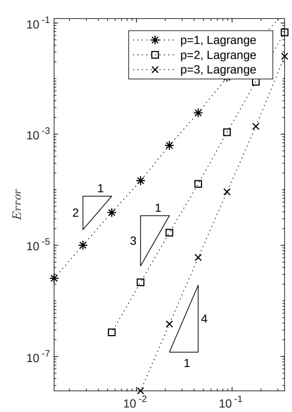
The Figure 1 presents the order of convergence of the error associated with the Example 4.1 for the borders 1 and 2 in the linear, quadratic and cubic Lagrange bases. The results are parallel with the side of the triangle that represents the expected order of convergence for each base, that is, $\mathcal{O}(h^{p+1})$.

Since the real and imaginary part of this example are the same, we display the numerical solution for the real part only. Furthermore, for each time step, the numerical solution $\{U^n\}_{n=0}^N$ is a surface. In this way, we chose some time steps to observe the behavior of the solution in front of the border.

The Figures 2, 3 show the numerical solution for Example 4.1 with the boundaries 1 and 2, respectively, with $\Delta t = 1/7$, $h = \sqrt{(1/4)^2 + (1/4)^2}$ and Lagrangian quadratic basis.



(c) Error associated with Example 4.1 with boundary 1.



(d) Error associated with Example 4.1 with boundary 2.

Figure 1. Numerical error associated with Example 4.1 on Lagrange linear, quadratic and cubic bases.

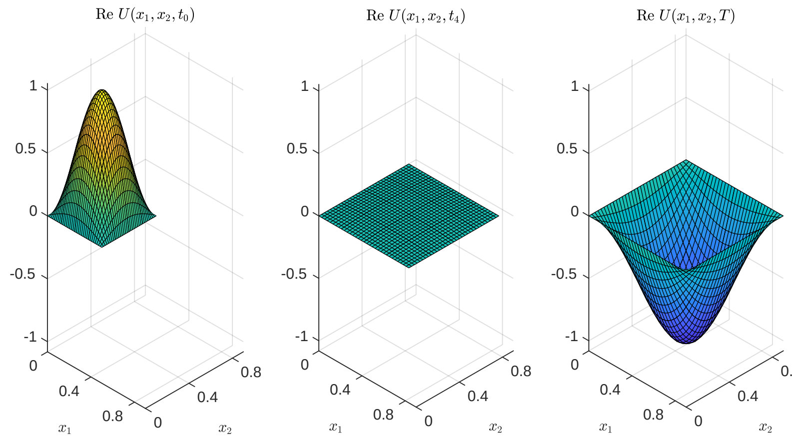


Figure 2. Numerical solution of the Example 4.1 with boundary 1 and Lagrangian quadratic basis.

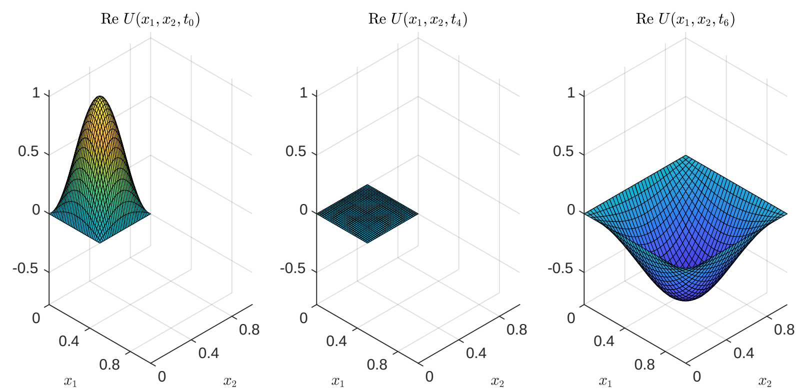


Figure 3. Numerical solution of the Example 4.1 with boundary 2 and Lagrangian quadratic basis.

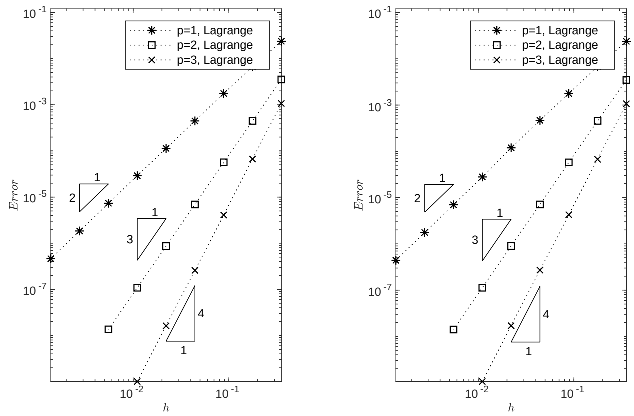
Example 4.2. In this example we consider the solution

$$v(y_1, y_2, t) = (\sin(\pi y_1)\sin(\pi y_2) - i(y_1^2 - y_1 + y_2^2 - y_2))\exp(-y_1^2 - y_2^2 - t).$$

The Figure 4 presents the order of convergence of the error associated with the 2 example for the 1 and 2 boundaries on the Lagrange linear, quadratic and cubic bases. The results are also parallel with the side of the triangle that represents the expected order of convergence for each base, that is, $\mathcal{O}(h^{p+1})$.

Since the real and imaginary part of this example are different, we display the separate numerical solution for each part. At each time step, the numerical solution $\{U^n\}_{n=0}^N$ is also a surface, so we choose some time steps to observe the behavior of the solution against the boundary.

The Figures 5, 6 show the real part of the numerical solution for Example 4.2 with the boundaries 1 and 2, respectively. The Figures 7, 8 show the imaginary part of the numerical solution on a logarithmic scale for the Example 4.2 with the boundaries 1 and 2, respectively. In all cases we consider $\Delta t = 1/7$, $h = \sqrt{(1/4)^2 + (1/4)^2}$ and Lagrangian quadratic basis.



(a) Error associated with Example 4.2 with boundary 1.

(b) Error associated with Example 4.2 with boundary 2.

Figure 4. Numerical error associated with Example 4.2 on Lagrange linear, quadratic and cubic bases.

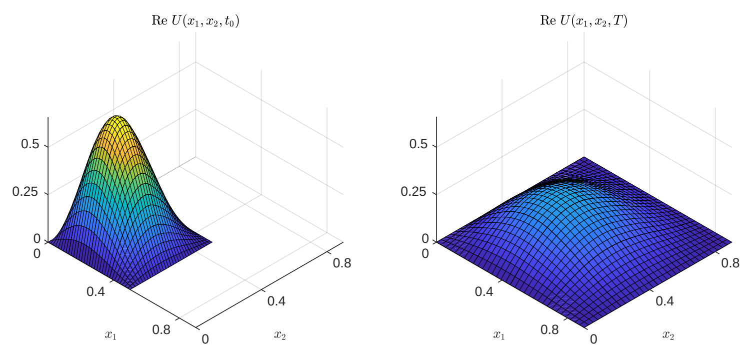


Figure 5. Real part of the numerical solution of the Example 4.2 with boundary 1 and Lagrangian quadratic basis.

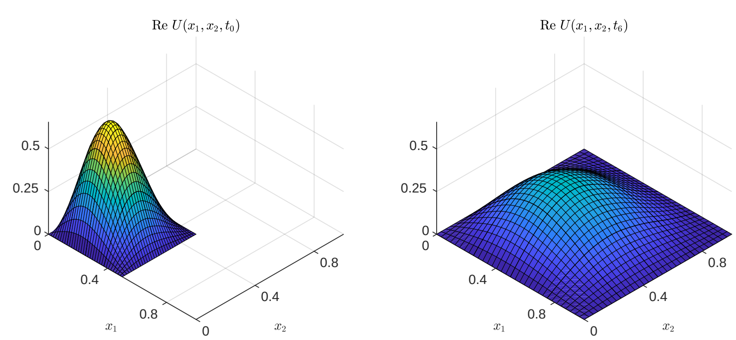


Figure 6. Real part of the numerical solution of the Example 4.2 with boundary 2 and Lagrangian quadratic basis.

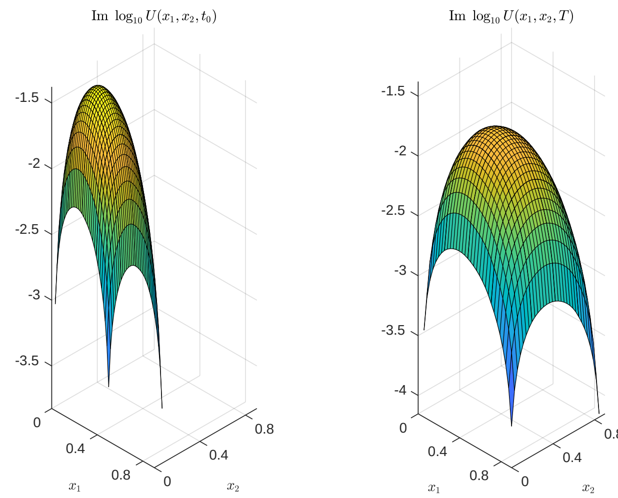


Figure 7. Imaginary part of the numerical solution of the Example 4.2 with boundary 1 and Lagrangian quadratic basis.

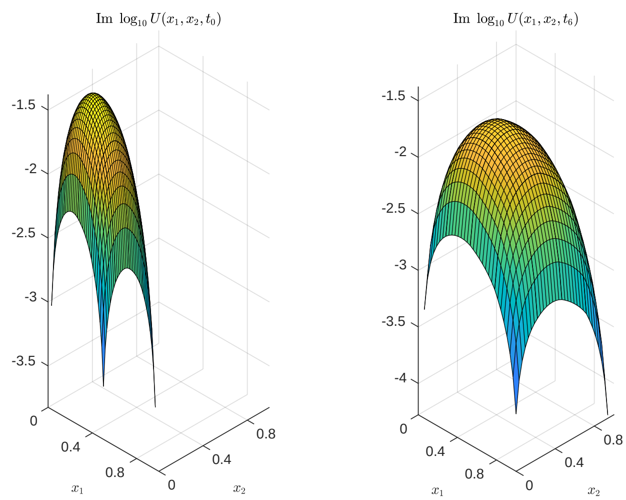


Figure 8. Imaginary part of the numerical solution of the Example 4.2 with boundary 2 and Lagrangian quadratic basis.

5. Conclusion

In this paper, we prove the existence and uniqueness of solution of the nonlinear Schrödinger problem with moving boundary (1.1) from problem (1.2), for the n-dimensional case. We also performed a numerical simulation, for the two dimensional case, where it was possible to observe the influence of the boundary moving in the study.

The proposition 2.1 considers functions for the boundary $k(t)$ decreasing, expanding our options in the study of the problem. The restriction of the growing frontier to guarantee theoretical results can be observed in works such as [20] and [24].

In numerical simulations for nonlinear problems, works such as [12] and [24] solve the resulting system of using Newton's method. The linearization method used in this work preserves the quadratic convergence order in time without the need to use Newton's method in numerical simulations. Thus, the computational cost was reduced and the execution of the simulations was optimized. It was also possible to estimate the order of numerical convergence and observe the behavior of the numerical error in the norm $L^2(\Omega)$. In the examples studied, we validate the optimal order, that is, $\mathcal{O}(h^{p+1})$, where p is the degree of the polynomial basis, and the competent error as we refine a mesh.

The linearization method used in this work preserves the quadratic convergence order in time without the need of using Newton's Method in numerical simulations.

Acknowledgements

We would like to recognize the valuable suggestions made by the late professor Luiz Adauto da Justa Medeiros in order to obtain the theoretical results in this present work and for his constant support whenever possible.

References

[1] R. M. P. Almeida, J. C. M. Duque, J. Ferreira and R. J. Robalo, *Finite element schemes for a class of nonlocal parabolic systems with moving boundaries*, Journal of Applied Numerical Mathematics, 2018, 127, 226–248. DOI: 10.1016/j.apnum.2018.01.007.

[2] G. A. Baker, *Error estimates for finite element methods for second order hyperbolic equations*, SIAM Journal on Numerical Analysis, 1976, 13(4), 564–576. DOI: 10.2307/2156246.

[3] J. E. Barab, *Nonexistence of asymptotically free solutions for a nonlinear Schrodinger equation*, Journal of Mathematical Physics, 1984, 25, 3270–3273. DOI: 10.1063/1.526074.

[4] C. Besse, *A relaxation scheme for the nonlinear Schrodinger equation*, SIAM Journal on Numerical Analysis, 2004, 42(3), 934–952. DOI: 10.1137/S0036142901396521.

[5] P. Cannarsa, G. da Prato, and J. P. Zolesio, *The damped wave equation in a moving domain*, Journal of Differential Equations, 1990, 85(1), 1–16.

[6] X. Carvajal, M. Panthee and M. Scialom, *On well-posedness of the third-order nonlinear Schrodinger equation with time-dependent coefficients*, Communications in Contemporary Mathematics, 2015, 17(04). DOI: 10.1142/S021919971450031X.

[7] T. Cazenave, *Semilinear Schrodinger Equations*, Courant Lecture Notes in Mathematics, AMS, New York, 2003, 10.

[8] F. G. Ciarlet, *The Finite Element Method for Elliptic Problems*, Classics in applied mathematics, SIAM, Paris, France, 2002, 40.

[9] R. Cipolatti, J. L. Gondar and C. Trallero-Giner, *Bose-Einstein condensates in optical lattices: Mathematical analysis and analytical approximate formulae*, Physica D: Nonlinear Phenomena, Amsterdam, 2012, 241(7), 755–763. DOI: 10.1016/j.physd.2011.12.012.

[10] H. R. Clark, M. A. Rincon and R. D. Rodrigues, *Beam equation with weak-internal damping in domain with moving boundary*, Journal of Applied Numerical Mathematics, 2003, 47(2), 139–157.

[11] F. Dalfovo and S. Giorgini, *Theory of Bose-Einstein condensation in trapped gases*, Reviews of Modern Physics, 1999, 71(3), 463–512.

[12] D. C. R. Gomes, M. A. Rincon, M. D. G. Silva and G. O. Antunes, *Theoretical and computational analysis of a nonlinear Schrödinger problem with moving boundary*, Advances in Computational Mathematics, Springer US, 2019, 45(4), 981–1004. DOI: 10.1007/s10444-018-9643-3.

[13] D. F. Griffiths, R. Mitchell and J. L. Morris, *A numerical study of the nonlinear Schrödinger equation*, Computer Methods in Applied Mechanics and Engineering, Springer US, 1984, 45, 177–205. DOI: 10.1016/0045-7825(84)90156-7.

[14] D. Hartree, *The wave mechanics of an atom with a non-Coulomb central field. Part I. Theory and methods*, Proc. Camb. Philos. Soc., 1968, (24), 89–132.

[15] P. Henning and J. Wärnegärd, *A note on optimal H^1-error estimates for Crank-Nicolson approximations to the nonlinear Schrödinger equation*, BIT Numerical Mathematics, 2020. DOI: 10.1007/s10543-020-00814-3.

[16] L. I. Ignat and E. Zuazua, *Convergence rates for dispersive approximation schemes to nonlinear Schrodinger equations*, Journal de Mathématiques Pures et Appliquées, 2012, 98(5), 479–517.

[17] J. Jina and X. Wu, *Analysis of finite element method for one-dimensional time-dependent Schrodinger equation on unbounded domain*, Journal of Computational and Applied Mathematics, 2008, 220, 240–256.

[18] P. L. Kelley, *Self-focusing of optical beams*, Phys. Rev. Lett., 1965, 15(26), 1005–1008. DOI: 10.1103/PhysRevLett.15.1005.

[19] J. L. Lions, *Quelques Méthodes de Résolutions des Problémes aux Limites non Linéaires*, Dunod, Paris, 1969.

[20] I.-S. Liu and M. A. Rincon, *Effect of moving boundaries on the vibrating elastic string*, Applied Numerical Mathematics, 2003, 47(2), 159–172.

[21] L. A. Medeiros, J. Límaco and S. B. Menezes, *Vibrations of elastic strings: mathematical aspects*, Journal of Computational Analysis and applications, 2002, 4, 211–263.

[22] Y. Mei and R. An, *Error estimates of second-order BDF Galerkin finite element methods for a coupled nonlinear Schrödinger system*, Computers & Mathematics with Applications, 2022, 122, 117–125. DOI: 10.1016/j.camwa.2022.07.018.

[23] M. A. Rincon and M. O. Lacerda, *Error analysis of thermal equation with moving ends*, Journal of Mathematics and Computers in Simulation, 2010, 80(11), 2200–2215.

[24] M. A. Rincon, I.-S. Liu, W. R. Huarcaya and B. A. Carmo, *Numerical analysis for a nonlinear model of elastic strings with moving ends*, Applied Numerical Mathematics, 2019, 135, 146–164. DOI: 10.1016/j.apnum.2018.08.014.

[25] M. Sepúlveda and O. Vera, *Numerical methods for a coupled nonlinear Schrodinger system*, Bol. Soc. Esp. Mat. Apl., 2008, 43, 97–104.

[26] W. A. Strauss, *Nonlinear scattering theory*, Scattering Theory in Mathematical Physics, 1974, 53–78. DOI: 10.1007/978-94-010-2147-0_3.

[27] K. E. Strecker, G. B. Partridge, A. G. Truscott and R. G. Hulet, *Bright matter wave solitons in Bose-Einstein condensates*, New Journal of Physics, 2003, 5(73), 1–8.

[28] V. Thomée, *Galerkin finite element methods for parabolic problems*, Springer, Springer Series in Computational Mathematics, Rio de Janeiro, 2006, 25(2).

[29] D. A. Vigo, J. C. Xavier and M. A. Rincon, *Analysis and computation of a nonlinear Korteweg-de Vries system*, BIT Numerical Mathematics, 2016, 56(3), 1069–1099. DOI: 10.1007/s10543-015-0589-2.

[30] J. Wang, *A new error analysis of Crank-Nicolson Galerkin FEMs for a generalized nonlinear Schrodinger equation*, Journal of Scientific Computing, 2014, 60, 390–407. DOI: 10.1007/s10915-013-9799-4.

[31] T. Wang, *Uniform point-wise error estimates of semi-implicit compact finite difference methods for the nonlinear Schrodinger equation perturbed by wave operator*, Journal of Computational and Applied Mathematics, 2015, 422(1), 286–308.

[32] M. L. Wheeler, *A priori L_2 error estimates for Galerkin approximations to parabolic partial differential equations*, SIAM Journal on Numerical Analysis, 1973, 10(4), 723–759.