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EXISTENCE AND STABILITY OF SOLUTIONS FOR A COUPLED HADAMARD TYPE SEQUENCE FRACTIONAL DIFFERENTIAL SYSTEM ON GLUCOSE GRAPHS*

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Abstract Chemical graph theory is an interdisciplinary mathematics and chemistry discipline that obtains mathematical information about the structure of target compounds and is an important research branch in theoretical pharmacology and nanomedicine. This paper study a coupled Hadamard type sequential fractional differential system on glucose graphs and establishes the Ulam's stability and existence of the system solutions. Furthermore, we examine examples against different background graphs and provide approximate graphs of the solutions. The novelty of this paper is that the origin of each edge is not fixed in modeling the glucose graphs, and one of the two vertices of the corresponding edge can be arbitrarily chosen as the origin to build the system and give the approximate graphs of the solutions using iterative methods and numerical simulation.

Keywords Chemical graph theory, fractional differential system, glucose graphs, Ulam's stability, numerical simulation.

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1. Introduction

Fractional calculus has been developed as an extension of integer order calculus for more than three hundred years. As detailed in [4, 12, 13, 19], fractional derivative can more accurately describe the real processes associated with the memory and genetic properties of various materials. Due to this, fractional differential equations are used in a wide range of disciplines, such as fluid flow, control theory of dynamical systems, biology, physics, and more [5, 6, 8, 32].

Coupled systems of fractional differential equations have also been investigated by many authors. Such system appear naturally in many real words situations, for

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examplem, the research investigated coupled implicit fractional integral-differential equations with Riemann-Liouville derivative in the literature [23], the existence and uniqueness of the projection model are investigated using the immovable point theorem. In [36], the authors study the stochastic resonance of two coupled fractional harmonic oscillators with a dichotomous variable mass. The average behaviors of the two oscillators are completely synchronized, and the analytical formulation of the output amplitude gain is determined. The literature [7, 14, 15, 35, 38, 39] and citations contain recent research results on fractional differential equations.

Graph theory is a mathematical discipline that investigates graphs and networks. It is frequently regarded as a branch of combinatorial mathematics. It started in 1736 with Euler's first paper on graph theory, which solved the famous "the Seven Bridges of Konigsberg Problem", establishing Euler as the father of graph theory.

A network is a graph, such as transportation route maps, telephone line networks, computer network extensions, molecular bodies in medicine and biology, and so on [1, 21]. Building mathematical models on graphs is also appealing and since mathematical models can be represented graphically. In fact, the boundary value problems on a graph is defined as a problem consisting of a set of differential equations on a given graph and certain node boundary conditions. The origin of differential equations on metric graphs is related to Lumer [16], who pioneered the application of differential equations to graph theory in the 1980s by exploring solution of evolutionary equations on ramification spaces and under different operator rules. Nicaise [20] studied the propagation of nerve impulses. In 1989, Zavgorodnii and Pokornyi [37] considered linear differential equations on geometric graphs where the solution of the differential equations are coordinated internally. In 2008, Gordeziani et al [9] solved the differential equations on graphs by the double-sweep method and proposed a numerical method. We can refer to [11, 17, 33] for more applications of differential equations on metric graphs to flexible structures composed of strings, beams and plates, quantum graphs.

Star graph $G = (V, E)$ consists of a finite set of nodes or vertices $V(G) = \{v_0, v_1, ..., v_k\}$ and a set of edges $E(G) = \{e_1 = \overrightarrow{v_1 v_0}, e_2 = \overrightarrow{v_2 v_0}, ..., e_k = \overrightarrow{v_k v_0}\}$ connecting these nodes, where v_0 is the joint point and e_i is the length of l_i the edge connecting the nodes v_i and v_0, i.e. $l_i = |\overrightarrow{v_i v_0}|$.

Graph theory, which lies at the intersection of mathematics and chemistry, is also applied in chemistry to study molecules. Chemical graph theory represents a compound's molecular structure as a graph, with each atom represented by a vertex and the chemical bonds between the atoms represented by edges between the vertices. Informed by the aforementioned research and related literature, we consider investigating the existence and stability of solution of nonlinearly coupled Hadamard-type sequence of fractional differential systems on glucose graphs Figure 1. Glucose is the primary energy donor of living organisms and the energy source and metabolic intermediate of living cells.

The carbon, oxygen, and hydroxyl atoms are used as the graph's vertices, and the chemical bonds forming their edges, to model the molecular structure of glucose. To facilitate the study, the vertices of the glucose graph are labeled as 0 or 1, and the length of each edge is fixed at e ($|\overrightarrow{e_i}| = e,\ i = 1, 2, ..., 19$) Figure 2. As a consequence, the orientation of each vertex is determined by the orientation of its corresponding edge. The labels of the beginning and ending vertices are taken into account as values 0 and 1, respectively, as we move along any edge. In contrast to the method used to analyze star graphs, the origin of each edge is not fixed and changes when the direction of movement along the edge is changed. Therefore, our study does not require a specific transformation to regulate the length of each edge, and it is also possible to arbitrarily choose one of the two vertices of the corresponding edge as the origin for the construction of the system.

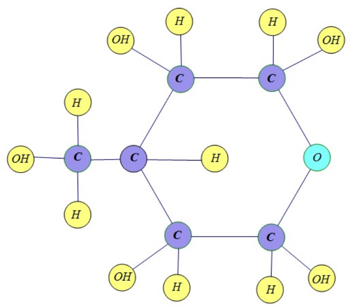

Figure 1. Molecular structure of glucose.

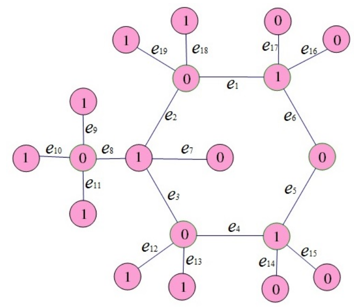

Figure 2. Glucose graphs with vertices 0 or 1.

In order to better explore the application of the fractional differential system on star graphs, here we briefly review some related results in the existing literature.

Graef et al [10] studied the fractional differential system on star graphs as follows:

$$\begin{cases} -D_{0+}^{\alpha}\mathfrak{u}_i = \varpi_{\mathrm{i}}\mathfrak{f}_i(x,\mathfrak{u}_i), \ 0<x<l_i, \ i=1,2, \\ \mathfrak{u}_1(0)=\mathfrak{u}_2(0), \ \mathfrak{u}_1(l_1)=\mathfrak{u}_2(l_2), \\ D_{0+}^{\beta}\mathfrak{u}_1(l_1)+D_{0+}^{\beta}\mathfrak{u}_2(l_2)=0, \end{cases}$$

where $D_{0+}^{\alpha}, D_{0+}^{\beta}$ are the Riemann-Liouville fractional derivative operator, $1<\alpha\le 2$, $0<\beta<\alpha$, $\varpi_i\in C[0,1]$, $i=1,2$ with $\varpi_i(x)\neq 0$ on $[0,l_i]$ and $\mathfrak{f}_i\in C([0,1]\times\mathbb{R},\mathbb{R})$, $i=1,2$. The authors proved that the existence and uniqueness results by using Banach contraction principle and Schauder fixed point theorem.

There is very little research on fractional differential equations on graphs, to the best of our knowledge, for example, in reference [18], the authors investigate the nonlinear Caputo fractional boundary value problem on the star graphs, and obtain the existence and uniqueness of the solutions to the boundary value problem by using the fixed point theory. Zhang et al [40] studied the fractional boundary value problem on star graphs, and obtained the existence and uniqueness results of solutions to boundary value problem by fixed piont theory. In addition, different types of Ulam type stability results of the proposed problems are also discussed. Note that in the literature [2, 3, 24, 25, 31, 34], attention was mainly focused on the existence of solutions to the fractional differential equations. Naturally, it is interesting and necessary to study the existence and stability of solutions to fractional differential equations on graphs. And our modeling of differential equations on each glucose edge may be used in a variety of domains. For example, in organic chemistry, each set of solutions funcitions (u_i, v_i) on any edge might represent parameters such as bounded energy, binding strength, bound power, and the like.

Motivated by the references [2, 3, 10, 18, 22, 24–26, 31, 34, 40], in this paper, we consider the existence and stability of solutions to the following nonlinear boundary value problem on graphs of the form

$$\begin{cases} \left({}^H D^q + k^H D^{q-1}\right) u_i(t) = f_i\left(t, u_i(t), v_i(t), {}^H D^\alpha v_i(t)\right), \\ 1 < q \le 2,\ 0 < \alpha < 1,\ k > 0,\ t \in [1, e], \\ \left({}^H D^p + k^H D^{p-1}\right) v_i(t) = g_i\left(t, v_i(t), u_i(t), {}^H D^\delta u_i(t)\right), \\ 1 < p \le 2,\ 0 < \delta < 1, \\ u_i(1) = 0, \quad v_i(1) = 0, \\ \lambda_1 u_i(e) - \lambda_2 \int_1^e \frac{v_i(s)}{s} ds = K_1,\ K_1,\ K_2 > 0, \\ \mu_1 v_i(e) - \mu_2 \int_1^e \frac{u_i(s)}{s} ds = K_2,\ i = 1, 2, ..., 19, \end{cases} \tag{1.1}$$

where ${}^H D^{(\cdot)}$ denote the Hadamard fractional derivative, $f_i,\ g_i : [1, e] \times \mathbb{R}^3 \to \mathbb{R}$ are given continuous functions and $\lambda_i, \mu_i (i = 1, 2)$ are real constants. The Ulam's stability and existence of the system solutions to fractional differential system (1.1) are established. Furthermore, we examine examples against various background graphs and provide approximate graphs of the solutions. The interesting of this paper is that the origin of each edge is nor fixed in modeling the glucose graphs, and one of the two vertices of the corresponding edges can be arbitrarily chosen as the origin to build the system and given the approximate graphs of the solutions using iterative methods and numerical simulation.

The outline of the paper is as follows, in Section 2, some basic definitions and related lemmas are given. The existence of uniqueness of solutions to the system of fractional differential system (1.1) under some assumptions are proved in Section 3. In Section 4 suitable conditions are constructed so that Ulam's stability is satisfied in system (1.1). Some examples and perform numerical simulations on the examples are given in the last section.

2. Preliminaries

We introduce notations and definitions of fractional calculus.

Definition 2.1 ([3,13]). The Hadamard fractional integral of order $q \in \mathbb{C}$, $\Re(q) > 0$, for a function $g \in L^p[a, b]$, $0 \le a \le t \le b \le \infty$, is defined as

$$I^q_{a^+} g(t) = \frac{1}{\Gamma(q)} \int_a^t \left(\log \frac{t}{s}\right)^{q-1} \frac{g(s)}{s} ds,$$
$$I^q_{b^-} g(t) = \frac{1}{\Gamma(q)} \int_t^b \left(\log \frac{t}{s}\right)^{q-1} \frac{g(s)}{s} ds.$$

Definition 2.2 ([3,13]). Let $[a, b] \subset \mathbb{R}$, $\delta = t\frac{d}{dt}$ and $AC^n_\delta[a, b] = \{g : [a, b] \to \mathbb{R} : \delta^{n-1}(g(t)) \in AC[a, b]\}$. The Hadamard derivative of fractional order q for a function $g \in AC^n_\delta[a, b]$ is defined as

$$D^q_{a^+} g(t) = \delta^n (I^{n-q}_{a^+})(t) = \frac{1}{\Gamma(n-q)} \Big(t\frac{d}{dt}\Big)^n \int_a^t \big(\log \frac{t}{s}\big)^{n-q-1} \frac{g(s)}{s} ds,$$
$$D^q_{b^-} g(t) = (-\delta)^n (I^{n-q}_{b^-})(t) = \frac{1}{\Gamma(n-q)} \Big(-t\frac{d}{dt}\Big)^n \int_t^b \big(\log \frac{t}{s}\big)^{n-q-1} \frac{g(s)}{s} ds,$$

where $n-1<q<n$, $n=[q]+1$ and $[q]$ denotes the integer part of the real number q and $\log(\cdot)=\log_e(\cdot)$.

Theorem 2.1. [14] *Let M be a closed convex and nonempty subset of a Banach space X. Let A, B be the operators such that*

(i) $Ax+By \in M$ whenever $x, y \in M$;

(ii) B is contraction mapping;

(iii) A is compact and continuous.
Then there exist $z \in M$ such that $z=Az+Bz$.

Lemma 2.1. *Let $h_i(t), z_i(t) \in AC([1,e],\mathbb{R}), i=1,2,...,19$, then the solution of sequential fractional differential equations:*

$$\begin{cases} \left({}^H D^q + k{}^H D^{q-1}\right) u_i(t) = h_i(t), \quad t \in [1,e], \\ \left({}^H D^p + k{}^H D^{p-1}\right) v_i(t) = z_i(t), \\ u_i(1)=0, \quad v_i(1)=0, \\ \lambda_1 u_i(e) - \lambda_2 \int_1^e \frac{v_i(s)}{s} ds = K_1, \\ \mu_1 v_i(e) - \mu_2 \int_1^e \frac{u_i(s)}{s} ds = K_2, \quad i=1,2,...,19, \end{cases} \tag{2.1}$$

is given by

$$\begin{aligned} u_i(t) =& \frac{1}{\Lambda}\Big(t^{-k}\int_1^t s^{k-1}(\log s)^{q-2}ds\Big)\Big\{B_2\Big[K_1 \\ &+ \frac{\lambda_2}{\Gamma(p-1)}\int_1^e s^{-k-1}\Big(\int_1^s m^{k-1}\Big(\int_1^m \Big(\log\frac{m}{r}\Big)^{p-2}\frac{z_i(r)}{r}dr\Big)dm\Big)ds \\ &- \frac{\lambda_1 e^{-k}}{\Gamma(q-1)}\int_1^e s^{k-1}\Big(\int_1^s \Big(\log\frac{s}{r}\Big)^{q-2}\frac{h_i(r)}{r}dr\Big)ds\Big] \\ &+ A_2\Big[\frac{\mu_1 e^{-k}}{\Gamma(p-1)}\int_1^e s^{k-1}\Big(\int_1^s \big(\log\frac{s}{r}\big)^{p-2}\frac{z_i(r)}{r}dr\Big)ds - K_2 \\ &- \frac{\mu_2}{\Gamma(q-1)}\int_1^e s^{-k-1}\Big(\int_1^s m^{k-1}\Big(\int_1^m \big(\log\frac{m}{r}\big)^{q-2}\frac{h_i(r)}{r}dr\Big)dm\Big)ds\Big]\Big\} \\ &+ \frac{t^{-k}}{\Gamma(q-1)}\int_1^t s^{k-1}\Big(\int_1^s \big(\log\frac{s}{r}\big)^{q-2}\frac{h_i(r)}{r}dr\Big)ds, \end{aligned} \tag{2.2}$$

and

$$\begin{aligned} v_i(t) =& \frac{1}{\Lambda}\Big(t^{-k}\int_1^t s^{k-1}(\log s)^{p-2}ds\Big)\Big\{A_1\Big[K_2 \\ &+ \frac{\mu_2}{\Gamma(q-1)}\int_1^e s^{-k-1}\Big(\int_1^s m^{k-1}\Big(\int_1^m \Big(\log\frac{m}{r}\Big)^{q-2}\frac{h_i(r)}{r}dr\Big)dm\Big)ds \\ &- \frac{\mu_1 e^{-k}}{\Gamma(p-1)}\int_1^e s^{k-1}\Big(\int_1^s \Big(\log\frac{s}{r}\Big)^{p-2}\frac{z_i(r)}{r}dr\Big)ds\Big] \\ &+ B_1\Big[\frac{\lambda_1 e^{-k}}{\Gamma(q-1)}\int_1^e s^{k-1}\Big(\int_1^s \Big(\log\frac{s}{r}\Big)^{q-2}\frac{h_i(r)}{r}dr\Big)ds - K_1 \end{aligned} \tag{2.3}$$

$$-\frac{\lambda_2}{\Gamma(p-1)}\int_1^e s^{-k-1}\Big(\int_1^s m^{k-1}\Big(\int_1^m \Big(\log\frac{m}{r}\Big)^{p-2}\frac{z_i(r)}{r}dr\Big)dm\Big)ds\Big]\Big\}$$
$$+\frac{t^{-k}}{\Gamma(p-1)}\int_1^t s^{k-1}\Big(\int_1^s \Big(\log\frac{s}{r}\Big)^{p-2}\frac{z_i(r)}{r}dr\Big)ds,$$

where

$$\Lambda = A_1B_2 - A_2B_1 \neq 0, \tag{2.4}$$
$$A_1 = \lambda_1 e^{-k}\int_1^e s^{k-1}(\log s)^{q-2}ds, \tag{2.5}$$
$$A_2 = -\lambda_2\int_1^e s^{-k-1}\Big(\int_1^s r^{k-1}(\log r)^{p-2}dr\Big)ds, \tag{2.6}$$
$$B_1 = -\mu_2\int_1^e s^{-k-1}\Big(\int_1^s r^{k-1}(\log r)^{q-2}dr\Big)ds, \tag{2.7}$$
$$B_2 = \mu_1 e^{-k}\int_1^e s^{k-1}(\log s)^{p-2}ds. \tag{2.8}$$

Proof. As argued in [12], the general solution of the system (2.1) can be written as

$$u_i(t) = a_0^{(i)}t^{-k} + a_1^{(i)}t^{-k}\int_1^t s^{k-1}(\log s)^{q-2}ds + t^{-k}\int_1^t s^{k-1\,H}I^{q-1}h_i(s)ds,$$
$$v_i(t) = b_0^{(i)}t^{-k} + b_1^{(i)}t^{-k}\int_1^t s^{k-1}(\log s)^{p-2}ds + t^{-k}\int_1^t s^{k-1\,H}I^{p-1}z_i(s)ds.$$

Where $a_0^{(i)}, b_0^{(i)}, a_1^{(i)}, b_1^{(i)} (i = 1, 2, ..., 19)$ are unknown arbitrary constants. Using the initial conditions $u_i(1) = 0$ and $v_i(1) = 0$ implies that

$$a_0^{(i)} = b_0^{(i)} = 0,$$

which leads to

$$u_i(t) = a_1^{(i)}t^{-k}\int_1^t s^{k-1}(\log s)^{q-2}ds + t^{-k}\int_1^t s^{k-1\,H}I^{q-1}h_i(s)ds,$$
$$v_i(t) = b_1^{(i)}t^{-k}\int_1^t s^{k-1}(\log s)^{p-2}ds + t^{-k}\int_1^t s^{k-1\,H}I^{p-1}z_i(s)ds.$$

Using the nonlocal integral boundary conditions

$$\lambda_1 u_i(e) - \lambda_2\int_1^e \frac{v_i(s)}{s}ds = K_1, \quad \mu_1 v_i(e) - \mu_2\int_1^e \frac{u_i(s)}{s}ds = K_2,$$

we obtain

$$A_1a_1^{(i)} + A_2b_1^{(i)} = N_1^{(i)}, \quad B_1a_1^{(i)} + B_2b_1^{(i)} = N_2^{(i)}, \tag{2.9}$$

where A_i and $B_i (i = 1, 2)$ are respectively given by (2.5)-(2.8), and

$$N_1^{(i)} = K_1 + \frac{\lambda_2}{\Gamma(p-1)}\int_1^e s^{-k-1}\Big(\int_1^s m^{k-1}\Big(\int_1^m \Big(\log\frac{m}{r}\Big)^{p-2}\frac{z_i(r)}{r}dr\Big)dm\Big)ds$$
$$-\frac{\lambda_1 e^{-k}}{\Gamma(q-1)}\int_1^e s^{k-1}\Big(\int_1^s \Big(\log\frac{s}{r}\Big)^{q-2}\frac{h_i(r)}{r}dr\Big)ds,$$

$$N_2^{(i)} =K_2 + \frac{\mu_2}{\Gamma(q-1)} \int_1^e s^{-k-1}\Big(\int_1^s m^{k-1}\Big(\int_1^m \Big(\log \frac{m}{r}\Big)^{q-2} \frac{h_i(r)}{r} dr\Big) dm\Big) ds$$
$$- \frac{\mu_1 e^{-k}}{\Gamma(p-1)} \int_1^e s^{k-1}\Big(\int_1^s \Big(\log \frac{s}{r}\Big)^{p-2} \frac{z_i(r)}{r} dr\Big) ds.$$

Solving the system (2.9), we find that

$$a_1^{(i)} =\frac{1}{\Lambda}\Big\{ B_2\Big[K_1 + \frac{\lambda_2}{\Gamma(p-1)} \int_1^e s^{-k-1}\Big(\int_1^s m^{k-1}\Big(\int_1^m \Big(\log \frac{m}{r}\Big)^{p-2} \frac{z_i(r)}{r} dr\Big) dm\Big) ds$$
$$- \frac{\lambda_1 e^{-k}}{\Gamma(q-1)} \int_1^e s^{k-1}\Big(\int_1^s \Big(\log \frac{s}{r}\Big)^{q-2} \frac{h_i(r)}{r} dr\Big) ds\Big]$$
$$+ A_2\Big[\frac{\mu_1 e^{-k}}{\Gamma(p-1)} \int_1^e s^{k-1}\Big(\int_1^s \Big(\log \frac{s}{r}\Big)^{p-2} \frac{z_i(r)}{r} dr\Big) ds - K_2$$
$$- \frac{\mu_2}{\Gamma(q-1)} \int_1^e s^{-k-1}\Big(\int_1^s m^{k-1}\Big(\int_1^m \Big(\log \frac{m}{r}\Big)^{q-2} \frac{h_i(r)}{r} dr\Big) dm\Big) ds\Big]\Big\},$$

and

$$b_1^{(i)} =\frac{1}{\Lambda}\Big\{ A_1\Big[K_2 + \frac{\mu_2}{\Gamma(q-1)} \int_1^e s^{-k-1}\Big(\int_1^s m^{k-1}\Big(\int_1^m \Big(\log \frac{m}{r}\Big)^{q-2} \frac{h_i(r)}{r} dr\Big) dm\Big) ds$$
$$- \frac{\mu_1 e^{-k}}{\Gamma(p-1)} \int_1^e s^{k-1}\Big(\int_1^s \Big(\log \frac{s}{r}\Big)^{p-2} \frac{z_i(r)}{r} dr\Big) ds\Big]$$
$$+ B_1\Big[\frac{\lambda_1 e^{-k}}{\Gamma(q-1)} \int_1^e s^{k-1}\Big(\int_1^s \Big(\log \frac{s}{r}\Big)^{q-2} \frac{h_i(r)}{r} dr\Big) ds - K_1$$
$$- \frac{\lambda_2}{\Gamma(p-1)} \int_1^e s^{-k-1}\Big(\int_1^s m^{k-1}\Big(\int_1^m \Big(\log \frac{m}{r}\Big)^{p-2} \frac{z_i(r)}{r} dr\Big) dm\Big) ds\Big]\Big\}.$$

Substituting the values of $a_1^{(i)}$ and $b_1^{(i)}$ in (2.9), we get the desired solution (2.2)-(2.3). The proof is complete. □

The following lemma contains certain estimates that we need in the sequel.

Lemma 2.2. *For $F \in C([1,e],\mathbb{R})$ with $\|F\| = \sup_{t\in[1,e]} |F(t)|$, we have*

(i) $\Big| t^{-k} \int_1^t s^{k-1} (\log s)^{q-2} ds \Big| \le \frac{1}{q-1}$,

(ii) $\Big| t^{-k} \int_1^t s^{k-1}\Big(\int_1^s \big(\log \frac{s}{r}\big)^{q-2} \frac{F(r)}{r} dr\Big) ds\Big| \le \frac{\|F\|}{q(q-1)}$.

Proof. Note that
$$\int_1^t (\log s)^{q-2} \frac{1}{s} ds \le \frac{1}{q-1}.$$

Since that $s^k \le t^k$ for $1 \le s \le t \le e$, then

$$\Big| t^{-k} \int_1^t s^{k-1}\Big(\int_1^s \Big(\log \frac{s}{r}\Big)^{q-2} \frac{F(r)}{r} dr\Big) ds\Big|$$
$$\le \sup_{t\in[1,e]} \Big| t^{-k} \int_1^t s^{k-1}\Big(\int_1^s \Big(\log \frac{s}{r}\Big)^{q-2} \frac{F(r)}{r} dr\Big) ds\Big|$$
$$\le \|F\| \sup_{t\in[1,e]} \Big| t^{-k} \int_1^t s^{k-1} \frac{(\log s)^{q-1}}{q-1} ds\Big|$$

$$\leq \frac{\|F\|}{q(q-1)}.$$

□

3. Main results

Let

$$X_i = \{u_i : u_i \in C([1,e],\mathbb{R}),\ {}^H D^\delta u_i \in C([1,e],\mathbb{R})\},$$

and

$$Y_i = \{v_i : v_i \in C([1,e],\mathbb{R}),\ {}^H D^\alpha v_i \in C([1,e],\mathbb{R})\},$$

then $(X_i, \|\cdot\|_{X_i})$ and $(Y_i, \|\cdot\|_{Y_i})$ $(i = 1,2,...,19)$ are Banach space respectively endowed with the norms

$$\|u_i\|_{X_i} = \|u_i\| + \|{}^H D^\delta u_i\| = \sup_{t\in[1,e]} |u_i(t)| + \sup_{t\in[1,e]} |{}^H D^\delta u_i(t)|,$$

and

$$\|v_i\|_{Y_i} = \|v_i\| + \|{}^H D^\alpha v_i\| = \sup_{t\in[1,e]} |v_i(t)| + \sup_{t\in[1,e]} |{}^H D^\alpha v_i(t)|.$$

Take $X = (X_1, X_2, ..., X_{19})$ and $Y = (Y_1, Y_2, ..., Y_{19})$ denote the spaces respectively equipped with the norms

$$\|u\|_X = \|(u_1,u_2,...,u_{19})\|_X = \sum_{i=1}^{19} \|u_i\|_{X_i},\ (u_1,u_2,...,u_{19}) \in X,$$
$$\|v\|_Y = \|(v_1,v_2,...,v_{19})\|_Y = \sum_{i=1}^{19} \|v_i\|_{Y_i},\ (v_1,v_2,...,v_{19}) \in Y.$$

According to the basic theory of functional analysis, we obtain that $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ are Banach space. In turn, the product space $(X_i \times Y_i, \|\cdot\|_{X_i\times Y_i})$ is a Banach space endowed with the norms $\|(u_i,v_i)\|_{X_i\times Y_i} = \|u_i\|_{X_i} + \|v_i\|_{Y_i}$ for $(u_i,v_i) \in X_i \times Y_i$. Similarly, let $(X \times Y, \|\cdot\|_{X\times Y})$ also be a Bannach space with norm $\|(u,v)\|_{X\times Y} = \sum_{i=1}^{19} \|(u_i,v_i)\|_{X_i\times Y_i}$.

Define operators $T : X\times Y \to X\times Y$ and $T_i : X_i \times Y_i \to X_i \times Y_i$ are respectively represented as follows:

$$T(u,v)(t) = \left(T_1(u,v)(t), T_2(u,v)(t), ..., T_{19}(u,v)(t)\right),$$
$$T_i(u,v)(t) = \big(T_i^{(1)}(u_i,v_i)(t), T_i^{(2)}(u_i,v_i)(t)\big),\ i = 1,2,...,19,$$

where

$$\begin{aligned} &T_i^{(1)}(u_i,v_i)(t)\\ =&\frac{1}{\Lambda}\Big(t^{-k}\int_1^t s^{k-1}(\log s)^{q-2}ds\Big)\Big\{B_2\Big[K_1 + \frac{\lambda_2}{\Gamma(p-1)}\int_1^e s^{-k-1}\Big(\int_1^s m^{k-1}\\ &\times\Big(\int_1^m \big(\log\frac{m}{r}\big)^{p-2} g_i\big(r, v_i(r), u_i(r), {}^H D^\delta u_i(r)\big)\frac{1}{r}dr\Big)dm\Big)ds\\ &-\frac{\lambda_1 e^{-k}}{\Gamma(q-1)}\int_1^e s^{k-1}\Big(\int_1^s \big(\log\frac{s}{r}\big)^{q-2} f_i\big(r, u_i(r), v_i(r), {}^H D^\alpha v_i(r)\big)\frac{1}{r}dr\Big)ds\Big]\end{aligned}$$

$$
\begin{aligned}
&+A_2\Big[\frac{\mu_1 e^{-k}}{\Gamma(p-1)}\int_1^e s^{k-1}\Big(\int_1^s \big(\log\frac{s}{r}\big)^{p-2} g_i\big(r, v_i(r), u_i(r), {}^H D^\delta u_i(r)\big)\frac{1}{r}dr\Big)ds\\
&-K_2-\frac{\mu_2}{\Gamma(q-1)}\int_1^e s^{-k-1}\Big(\int_1^s m^{k-1}\Big(\int_1^m \big(\log\frac{m}{r}\big)^{q-2}\\
&\times f_i\big(r, u_i(r), v_i(r), {}^H D^\alpha v_i(r)\big)\frac{1}{r}dr\Big)dm\Big)ds\Big]\Big\}\\
&+\frac{t^{-k}}{\Gamma(q-1)}\int_1^t s^{k-1}\Big(\int_1^s \big(\log\frac{s}{r}\big)^{q-2} f_i\big(r, u_i(r), v_i(r), {}^H D^\alpha v_i(r)\big)\frac{1}{r}dr\Big)ds,
\end{aligned}
$$

and

$$
\begin{aligned}
&T_i^{(2)}(u_i, v_i)(t)\\
=&\frac{1}{\Lambda}\Big(t^{-k}\int_1^t s^{k-1}(\log s)^{p-2}ds\Big)\Big\{A_1\Big[K_2+\frac{\mu_2}{\Gamma(q-1)}\int_1^e s^{-k-1}\Big(\int_1^s m^{k-1}\\
&\times\Big(\int_1^m \big(\log\frac{m}{r}\big)^{q-2} f_i\big(r, u_i(r), v_i(r), {}^H D^\alpha v_i(r)\big)\frac{1}{r}dr\Big)dm\Big)ds\\
&-\frac{\mu_1 e^{-k}}{\Gamma(p-1)}\int_1^e s^{k-1}\Big(\int_1^s \big(\log\frac{s}{r}\big)^{p-2} g_i\big(r, v_i(r), u_i(r), {}^H D^\delta u_i(r)\big)\frac{1}{r}dr\Big)ds\Big]\\
&+B_1\Big[\frac{\lambda_1 e^{-k}}{\Gamma(q-1)}\int_1^e s^{k-1}\Big(\int_1^s \big(\log\frac{s}{r}\big)^{q-2} f_i\big(r, u_i(r), v_i(r), {}^H D^\alpha v_i(r)\big)\frac{1}{r}dr\Big)ds\\
&-K_1-\frac{\lambda_2}{\Gamma(p-1)}\int_1^e s^{-k-1}\Big(\int_1^s m^{k-1}\Big(\int_1^m \big(\log\frac{m}{r}\big)^{p-2}\\
&\times g_i\big(r, v_i(r), u_i(r), {}^H D^\delta u_i(r)\big)\frac{1}{r}dr\Big)dm\Big)ds\Big]\Big\}\\
&+\frac{t^{-k}}{\Gamma(p-1)}\int_1^t s^{k-1}\Big(\int_1^s \big(\log\frac{s}{r}\big)^{p-2} g_i\big(r, v_i(r), u_i(r), {}^H D^\delta u_i(r)\big)\frac{1}{r}dr\Big)ds.
\end{aligned}
$$

Assume that the following conditions hold:

(H_1) $f_i, g_i : [1,e]\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}, i = 1,2,...,19$ be continuous functions and there exist nonnegative functions $l_i(t), q_i(t)\in C[1,e]$ such that

$$|f_i(t,x_1,y_1,z_1)-f_i(t,x_2,y_2,z_2)|\le l_i(t)(|x_1-x_2|+|y_1-y_2|+|z_1-z_2|),$$

$$|g_i(t,x_1,y_1,z_1)-g_i(t,x_2,y_2,z_2)|\le q_i(t)(|x_1-x_2|+|y_1-y_2|+|z_1-z_2|),$$

where $t\in[1,e], x_i, y_i\in\mathbb{R}$;

(H_2) Let $\|l_i\| = \sup_{t\in[1,e]}|l_i(t)|$ and $\|q_i\| = \sup_{t\in[1,e]}|q_i(t)|, i = 1,2,...,19$;

(H_3) $a_i = \sup_{t\in[1,e]}|f_i(t,0,0,0)| < \infty$, $b_i = \sup_{t\in[1,e]}|g_i(t,0,0,0)| < \infty, i = 1,2,...,19.$

For computational convenience, we also set the following quantities:

$$
\begin{aligned}
G_1 &= \frac{1}{|\Lambda|(q-1)}\Big(\frac{\lambda_1|B_2|}{\Gamma(q+1)}+\frac{\mu_2|A_2|}{\Gamma(q+2)}+\frac{|\Lambda|}{q\Gamma(q-1)}\Big),\\
G_2 &= \frac{1}{|\Lambda|(q-1)}\Big(\frac{\lambda_2|B_2|}{\Gamma(p+2)}+\frac{\mu_1|A_2|}{\Gamma(p+1)}\Big),
\end{aligned}
$$

$$\begin{aligned}
G_3 &= \frac{1}{|\Lambda|(p-1)}\Big(\frac{\mu_2|A_1|}{\Gamma(q+2)} + \frac{\lambda_1|B_1|}{\Gamma(q+1)}\Big),\\
G_4 &= \frac{1}{|\Lambda|(p-1)}\Big(\frac{\mu_1|A_1|}{\Gamma(p+1)} + \frac{\lambda_2|B_1|}{\Gamma(p+2)} + \frac{|\Lambda|}{p\Gamma(p-1)}\Big),\\
S_1 &= \frac{1}{|\Lambda|(q-1)}\Big(\frac{\lambda_1|B_2|}{\Gamma(q+1)} + \frac{\mu_2|A_2|}{\Gamma(q+2)}\Big),\\
S_2 &= \frac{1}{|\Lambda|(p-1)}\Big(\frac{\mu_1|A_1|}{\Gamma(p+1)} + \frac{\lambda_2|B_1|}{\Gamma(p+2)}\Big),\\
M_1 &= \frac{1}{|\Lambda|(q-1)}\Big(K_1|B_2| + K_2|A_2|\Big),\\
M_2 &= \frac{1}{|\Lambda|(p-1)}\Big(K_2|A_1| + K_1|B_1|\Big).
\end{aligned}$$

In the following, the main results on the existence of solution to the fractional differential system which we studied are listed.

Theorem 3.1. *Assume that* (H1) *hold, then system* (1.1) *has a unique solution on* $[1, e]$, *if*

$$\sum_{i=1}^{19} \Delta_i < 1, \tag{3.1}$$

where

$$\Delta_i = \frac{(1+\Gamma(1-\delta))\,(\|l_i\|G_1 + \|q_i\|G_2)}{\Gamma(1-\delta)} + \frac{(1+\Gamma(1-\alpha))\,(\|l_i\|G_3 + \|q_i\|G_4)}{\Gamma(1-\alpha)}.$$

Proof. Define

$$c_i \geq \max\left\{\frac{2\,(\Gamma(1-\delta)+1)\,(|a_i|G_1 + |b_i|G_2 + M_1)}{(\Gamma(1-\delta) - 2\,(\Gamma(1-\delta)+1))\,(\|l_i\|G_1 + \|q_i\|G_2)},\right.$$
$$\left.\frac{2\,(\Gamma(1-\alpha)+1)\,(|a_i|G_3 + |b_i|G_4 + M_2)}{(\Gamma(1-\alpha) - 2\,(\Gamma(1-\alpha)+1))\,(\|l_i\|G_3 + \|q_i\|G_4)}\right\}$$

and

$$c \geq \sum_{i=1}^{19} c_i.$$

Let $TB_r \subset B_r$ be a cone define by

$$B_r = \{(u, v) \in X \times Y : \|(u, v)\|_{X\times Y} \leq c\},$$

where $(u, v) = \big((u_1, v_1), (u_2, v_2), ..., (u_{19}, v_{19})\big) \in X \times Y$.

Let

$$B_{r_i} = \{(u_i, v_i) \in X_i \times Y_i : \|(u_i, v_i)\|_{X_i\times Y_i} \leq c_i\},$$

for $(u_i, v_i) \in B_{r_i}$, we have

$$\begin{aligned}
&\left|f_i\left(t, u_i(t), v_i(t), ^H D^\alpha v_i(t)\right)\right|\\
\leq &\left|f_i\left(t, u_i(t), v_i(t), ^H D^\alpha v_i(t)\right) - f_i(t, 0, 0, 0)\right| + |f_i(t, 0, 0, 0)|\\
\leq &l_i(t)\left(|u_i(t)| + |v_i(t)| + \left|^H D^\alpha v_i(t)\right|\right) + |a_i|
\end{aligned}$$

$$
\begin{aligned}
&\leq l_i(t)\|(u_i,v_i)\|_{X_i\times Y_i}+|a_i|\\
&\leq \|l_i\|c_i+|a_i|.
\end{aligned}
$$

Similarly, we obtain

$$
\left|g_i\left(t,v_i(t),u_i(t),^H D^\delta u_i(t)\right)\right|\leq q_i(t)\|(u_i,v_i)\|_{X_i\times Y_i}+|b_i|\leq \|q_i\|c_i+|b_i|.
$$

Then

$$
\begin{aligned}
&\left|T_i^{(1)}(u_i,v_i)(t)\right|\\
\leq&\ \frac{1}{|\Lambda|}\Big(t^{-k}\int_1^t s^{k-1}(\log s)^{q-2}ds\Big)\Big\{\big|B_2\big|\Big[K_1\\
&+\frac{\lambda_2}{\Gamma(p-1)}\int_1^e s^{-k-1}\Big(\int_1^s m^{k-1}\Big(\int_1^m\big(\log\frac{m}{r}\big)^{p-2}\\
&\times\big|g_i\big(r,v_i(r),u_i(r),^H D^\delta u_i(r)\big)\big|\frac{1}{r}dr\Big)dm\Big)ds\\
&+\frac{\lambda_1 e^{-k}}{\Gamma(q-1)}\int_1^e s^{k-1}\Big(\int_1^s\big(\log\frac{s}{r}\big)^{q-2}\big|f_i\big(r,u_i(r),v_i(r),^H D^\alpha v_i(r)\big)\big|\frac{1}{r}dr\Big)ds\Big]\\
&+\big|A_2\big|\Big[\frac{\mu_1 e^{-k}}{\Gamma(p-1)}\int_1^e s^{k-1}\Big(\int_1^s\big(\log\frac{s}{r}\big)^{p-2}\big|g_i\big(r,v_i(r),u_i(r),^H D^\delta u_i(r)\big)\big|\frac{1}{r}dr\Big)ds\\
&+K_2+\frac{\mu_2}{\Gamma(q-1)}\int_1^e s^{-k-1}\Big(\int_1^s m^{k-1}\Big(\int_1^m\big(\log\frac{m}{r}\big)^{q-2}\\
&\times\big|f_i\big(r,u_i(r),v_i(r),^H D^\alpha v_i(r)\big)\big|\frac{1}{r}dr\Big)dm\Big)ds\Big]\Big\}\\
&+\frac{t^{-k}}{\Gamma(q-1)}\int_1^t s^{k-1}\Big(\int_1^s\big(\log\frac{s}{r}\big)^{q-2}\big|f_i\big(r,u_i(r),v_i(r),^H D^\alpha v_i(r)\big)\big|\frac{1}{r}dr\Big)ds\\
\leq&\ \frac{1}{|\Lambda|(q-1)}\Big(\frac{\lambda_1\big|B_2\big|}{\Gamma(q+1)}+\frac{\mu_2\big|A_2\big|}{\Gamma(q+2)}+\frac{|\Lambda|}{q\Gamma(q-1)}\Big)\big|f_i\big(t,u_i(t),v_i(t),^H D^\alpha v_i(t)\big)\big|\\
&+\frac{1}{|\Lambda|(q-1)}\Big(\frac{\lambda_2\big|B_2\big|}{\Gamma(p+2)}+\frac{\mu_1\big|A_2\big|}{\Gamma(p+1)}\Big)\big|g_i\big(t,v_i(t),u_i(t),^H D^\delta u_i(t)\big)\big|\\
&+\frac{1}{|\Lambda|(q-1)}\big(K_1\big|B_2\big|+K_2\big|A_2\big|\big)\\
\leq&\ G_1\big(\|l_i\|c_i+|a_i|\big)+G_2\big(\|q_i\|c_i+|b_i|\big)+M_1\\
=&\ \big(\|l_i\|G_1+\|q_i\|G_2\big)c_i+|a_i|G_1+|b_i|G_2+M_1,
\end{aligned}
$$

and

$$
\begin{aligned}
&\left|{}^H D^\delta T_i^{(1)}(u_i,v_i)(t)\right|\\
\leq&\ \frac{1}{\Gamma(1-\delta)}\Big(t\frac{d}{dt}\Big)\int_1^t\big(\log\frac{t}{s}\big)^{-\delta}\frac{\big|T_i^{(1)}(u_i,v_i)(s)\big|}{s}ds\\
\leq&\ \frac{\big(\|l_i\|G_1+\|q_i\|G_2\big)c_i+|a_i|G_1+|b_i|G_2+M_1}{\Gamma(1-\delta)}.
\end{aligned}
$$

Hence

$$
\big\|T_i^{(1)}(u_i,v_i)(t)\big\|_{X_i}
$$

$$\begin{aligned}
&\leq \left\|T_i^{(1)}(u_i,v_i)(t)\right\| + \left\|{}^H D^{\delta} T_i^{(1)}(u_i,v_i)(t)\right\| \\
&\leq \Big(\frac{\Gamma(1-\delta)+1}{\Gamma(1-\delta)}\Big)\big(\big(\|l_i\|G_1+\|q_i\|G_2\big)c_i+|a_i|G_1+|b_i|G_2+M_1\big) \\
&\leq \frac{c_i}{2}.
\end{aligned}$$

In the same way, we have,

$$\left|T_i^{(2)}(u_i,v_i)(t)\right| \leq \big(\|l_i\|G_3+\|q_i\|G_4\big)c_i+|a_i|G_3+|b_i|G_4+M_2,$$

and

$$\begin{aligned}
&\left|{}^H D^{\alpha} T_i^{(2)}(u_i,v_i)(t)\right| \\
&\leq \frac{\big(\|l_i\|G_3+\|q_i\|G_4\big)c_i+|a_i|G_3+|b_i|G_4+M_2}{\Gamma(1-\alpha)}.
\end{aligned}$$

Next, we get

$$\begin{aligned}
&\left\|T_i^{(2)}(u_i,v_i)(t)\right\|_{Y_i} \\
&\leq \left\|T_i^{(2)}(u_i,v_i)(t)\right\| + \left\|{}^H D^{\alpha} T_i^{(2)}(u_i,v_i)(t)\right\| \\
&\leq \Big(\frac{\Gamma(1-\alpha)+1}{\Gamma(1-\alpha)}\Big)\big(\big(\|l_i\|G_3+\|q_i\|G_4\big)c_i+|a_i|G_3+|b_i|G_4+M_2\big) \\
&\leq \frac{c_i}{2}.
\end{aligned}$$

Hence

$$\left\|T_i(u,v)(t)\right\|_{X_i\times Y_i} = \left\|T_i^{(1)}(u_i,v_i)(t)\right\|_{X_i} + \left\|T_i^{(2)}(u_i,v_i)(t)\right\|_{Y_i} \leq c_i,$$

which implies $TB_{r_i} \subset B_{r_i}$. Thus

$$\left\|T(u,v)(t)\right\|_{X\times Y} = \sum_{i=1}^{19}\left\|T_i(u,v)(t)\right\|_{X_i\times Y_i} \leq \sum_{i=1}^{19} c_i \leq c,$$

this show that $TB_r \subset B_r$.

Now we prove that the operator T is a contraction. For $(u_i,u_i^{'}),(v_i,v_i^{'}) \in B_{r_i}$ and for each $t\in[1,e]$ we have

$$\begin{aligned}
&\left|T_i^{(1)}(u_i,v_i)(t) - T_i^{(1)}(u_i^{'},v_i^{'})(t)\right| \\
&\leq \frac{1}{|\Lambda|}\Big(t^{-k}\int_1^t s^{k-1}(\log s)^{q-2}ds\Big)\Big\{\frac{|B_2|\lambda_2}{\Gamma(p-1)}\int_1^e s^{-k-1}\Big(\int_1^s m^{k-1}\Big(\int_1^m \big(\log\frac{m}{r}\big)^{p-2} \\
&\quad\times\big|g_i\big(r,v_i(r),u_i(r),{}^H D^{\delta}u_i(r)\big) - g_i\big(r,v_i^{'}(r),u_i^{'}(r),{}^H D^{\delta}u_i^{'}(r)\big)\big|\frac{1}{r}dr\Big)dm\Big)ds \\
&\quad+\frac{|B_2|\lambda_1 e^{-k}}{\Gamma(q-1)}\int_1^e s^{k-1}\Big(\int_1^s \big(\log\frac{s}{r}\big)^{q-2}\big|f_i\big(r,u_i(r),v_i(r),{}^H D^{\alpha}v_i(r)\big) \\
&\quad-f_i\big(r,u_i^{'}(r),v_i^{'}(r),{}^H D^{\alpha}v_i^{'}(r)\big)\big|\frac{1}{r}dr\Big)ds + \frac{|A_2|\mu_1 e^{-k}}{\Gamma(p-1)}\int_1^e s^{k-1}\Big(\int_1^s \big(\log\frac{s}{r}\big)^{p-2} \\
&\quad\times\big|g_i\big(r,v_i(r),u_i(r),{}^H D^{\delta}u_i(r)\big) - g_i\big(r,v_i^{'}(r),u_i^{'}(r),{}^H D^{\delta}u_i^{'}(r)\big)\big|\frac{1}{r}dr\Big)ds
\end{aligned}$$

$$
\begin{aligned}
&+\frac{|A_2|\mu_2}{\Gamma(q-1)}\int_1^e s^{-k-1}\Big(\int_1^s m^{k-1}\Big(\int_1^m \big(\log\frac{m}{r}\big)^{q-2}\big|f_i\big(r,u_i(r),v_i(r),{}^H D^\alpha v_i(r)\big)\\
&-f_i\big(r,u_i^{'}(r),v_i^{'}(r),{}^H D^\alpha v_i^{'}(r)\big)\big|\frac{1}{r}dr\Big)dm\Big)ds\Big\}+\frac{t^{-k}}{\Gamma(q-1)}\int_1^t s^{k-1}\Big(\int_1^s \big(\log\frac{s}{r}\big)^{q-2}\\
&\times\big|f_i\big(r,u_i(r),v_i(r),{}^H D^\alpha v_i(r)\big)-f_i\big(r,u_i^{'}(r),v_i^{'}(r),{}^H D^\alpha v_i^{'}(r)\big)\big|\frac{1}{r}dr\Big)ds\\
\le\ & \frac{1}{|\Lambda|(q-1)}\Big(\frac{\lambda_1|B_2|}{\Gamma(q+1)}+\frac{\mu_2|A_2|}{\Gamma(q+2)}+\frac{|\Lambda|}{q\Gamma(q-1)}\Big)\|l_i\|\big(\|u_i-u_i^{'}\|_{X_i}+\|v_i-v_i^{'}\|_{Y_i}\big)\\
&+\frac{1}{|\Lambda|(q-1)}\Big(\frac{\lambda_2|B_2|}{\Gamma(p+2)}+\frac{\mu_1|A_2|}{\Gamma(p+1)}\Big)\|q_i\|\big(\|u_i-u_i^{'}\|_{X_i}+\|v_i-v_i^{'}\|_{Y_i}\big)\\
=\ & \big(\|l_i\|G_1+\|q_i\|G_2\big)\big(\|u_i-u_i^{'}\|_{X_i}+\|v_i-v_i^{'}\|_{Y_i}\big),
\end{aligned}
$$

$$
\begin{aligned}
&\big|{}^H D^\delta T_i^{(1)}(u_i,v_i)(t)-{}^H D^\delta T_i^{(1)}(u_i^{'},v_i^{'})(t)\big|\\
\le\ & \frac{1}{\Gamma(1-\delta)}\Big(t\frac{d}{dt}\Big)\int_1^t \big(\log\frac{t}{s}\big)^{-\delta}\big|T_i^{(1)}(u_i,v_i)(s)-T_i^{(1)}(u_i^{'},v_i^{'})(t)\big|\frac{1}{s}ds\\
\le\ & \frac{\big(\|l_i\|G_1+\|q_i\|G_2\big)\big(\|u_i-u_i^{'}\|_{X_i}+\|v_i-v_i^{'}\|_{Y_i}\big)}{\Gamma(1-\delta)}.
\end{aligned}
$$

It can be obtained from the above inequalities

$$
\begin{aligned}
&\big\|T_i^{(1)}(u_i,v_i)(t)-T_i^{(1)}(u_i^{'},v_i^{'})(t)\big\|_{X_i}\\
=\ & \big\|T_i^{(1)}(u_i,v_i)(t)-T_i^{(1)}(u_i^{'},v_i^{'})(t)\big\|+\big\|{}^H D^\delta T_i^{(1)}(u_i,v_i)(t)-{}^H D^\delta T_i^{(1)}(u_i^{'},v_i^{'})(t)\big\|\\
=\ & \Big(1+\frac{1}{\Gamma(1-\delta)}\Big)\big(\|l_i\|G_1+\|q_i\|G_2\big)\big(\|u_i-u_i^{'}\|_{X_i}+\|v_i-v_i^{'}\|_{Y_i}\big).
\end{aligned}
$$

Similarly one can find that

$$
\begin{aligned}
&\big|T_i^{(2)}(u_i,v_i)(t)-T_i^{(2)}(u_i^{'},v_i^{'})(t)\big|\\
&\le\big(\|l_i\|G_3+\|q_i\|G_4\big)\big(\|u_i-u_i^{'}\|_{X_i}+\|v_i-v_i^{'}\|_{Y_i}\big),
\end{aligned}
$$

and

$$
\begin{aligned}
&\big|{}^H D^\alpha T_i^{(2)}(u_i,v_i)(t)-{}^H D^\alpha T_i^{(2)}(u_i^{'},v_i^{'})(t)\big|\\
&\le\big(1+\frac{1}{\Gamma(1-\alpha)}\big)\big(\|l_i\|G_3+\|q_i\|G_4\big)\big(\|u_i-u_i^{'}\|_{X_i}+\|v_i-v_i^{'}\|_{Y_i}\big).
\end{aligned}
$$

Hence

$$
\begin{aligned}
&\big\|T_i^{(2)}(u_i,v_i)(t)-T_i^{(2)}(u_i^{'},v_i^{'})(t)\big\|_{Y_i}\\
=\ & \big\|T_i^{(2)}(u_i,v_i)(t)-T_i^{(2)}(u_i^{'},v_i^{'})(t)\big\|+\big\|{}^H D^\alpha T_i^{(2)}(u_i,v_i)(t)-{}^H D^\alpha T_i^{(2)}(u_i^{'},v_i^{'})(t)\big\|\\
=\ & \Big(1+\frac{1}{\Gamma(1-\alpha)}\Big)\big(\|l_i\|G_3+\|q_i\|G_4\big)\big(\|u_i-u_i^{'}\|_{X_i}+\|v_i-v_i^{'}\|_{Y_i}\big).
\end{aligned}
$$

Therefore, we get the following formula:

$$
\big\|T(u,v)(t)-T(u^{'},v^{'})(t)\big\|_{X\times Y}
$$

$$
\begin{aligned}
&= \sum_{i=1}^{19} \left\| T_i(u_i, v_i)(t) - T_i(u_i^{'}, v_i^{'})(t) \right\|_{X_i \times Y_i} \\
&\le \sum_{i=1}^{19} \Bigg[\frac{\Big(1+\Gamma(1-\delta)\Big)\big(\|l_i\|G_1 + \|q_i\|G_2\big)}{\Gamma(1-\delta)} + \frac{\Big(1+\Gamma(1-\alpha)\Big)\big(\|l_i\|G_3 + \|q_i\|G_4\big)}{\Gamma(1-\alpha)} \Bigg] \\
&\quad \times \big(\|u-u^{'}\|_X + \|v-v^{'}\|_Y\big) \\
&= \sum_{i=1}^{19} \Delta_i \big(\|u-u^{'}\|_X + \|v-v^{'}\|_Y\big).
\end{aligned}
$$

As $\sum_{i=1}^{19} \Delta_i < 1$, we obtain that T is contraction operator. It follows from the Banach contraction principle that operator T has a unique fixed point in B_r, so the system (1.1) exist a unique solution on $[1, e]$. The proof is complete. □

Our next existence result is based on the krasnoselskii fixed-point theorem.

Theorem 3.2. *Assume that* $f_i, g_i : [1, e] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}, i = 1, 2, ..., 19$ *are continues functions satisfying assumption* (H_1). *In addition we suppose that there exist two positive constants* L_i *and* Q_i, *such that*

$$
\left| f_i\left(t, u_i(t), v_i(t), {}^H D^{\alpha} v_i(t)\right) \right| < L_i, \ \left| g_i\left(t, v_i(t), u_i(t), {}^H D^{\delta} u_i(t)\right) \right| < Q_i,
$$

where $t \in [1, e], (u_i, v_i) \in X_i \times Y_i, i = 1, 2, ..., 19.$

If

$$
\sum_{i=1}^{19} \Theta_i < 1,
$$

where

$$
\Theta_i = \frac{(\Gamma(1-\delta)+1)\left(\|l_i\|S_1 + \|q_i\|G_2\right)}{\Gamma(1-\delta)} + \frac{(\Gamma(1-\alpha)+1)\left(\|l_i\|G_3 + \|q_i\|S_2\right)}{\Gamma(1-\alpha)} < 1,
$$

then the system (1.1) has at least one solution on $[1, e]$.

Proof.

$$
B_\varepsilon = \{(u, v) \in X \times Y : \|(u, v)\|_{X \times Y} \le \varepsilon\},
$$

where $(u, v) = \big((u_1, v_1), (u_2, v_2), ..., (u_{19}, v_{19})\big) \in X \times Y$ and ε is

$$
\begin{aligned}
\varepsilon = \sum_{i=1}^{19} \Bigg\{ &\frac{(\Gamma(1-\delta)+1)\left(L_i G_1 + Q_i G_2 + M_1\right)}{\Gamma(1-\delta)} \\
&+ \frac{(\Gamma(1-\alpha)+1)\left(L_i G_3 + Q_i G_4 + M_2\right)}{\Gamma(1-\alpha)} \Bigg\}.
\end{aligned}
$$

It is know that B_ε is closed, bounded, convex and nonempty suset of product Banach space $X \times Y = (X_1 \times Y_1, X_2 \times Y_2, ..., X_{19} \times Y_{19})$.

Now we define some operators as follows

$$
\begin{aligned}
&T_{i,1}^{(1)}(u_i, v_i)(t) \\
&= \frac{1}{\Lambda}\Big(t^{-k}\int_1^t s^{k-1}(\log s)^{q-2} ds\Big)\Big\{B_2\Big[K_1 + \frac{\lambda_2}{\Gamma(p-1)}\int_1^e s^{-k-1}\Big(\int_1^s m^{k-1}
\end{aligned}
$$

$$
\begin{aligned}
&\times\Big(\int_1^m \Big(\log\frac{m}{r}\Big)^{p-2} g_i\big(r, v_i(r), u_i(r), {}^H D^\delta u_i(r)\big)\frac{1}{r}dr\Big)dm\Big)ds\\
&-\frac{\lambda_1 e^{-k}}{\Gamma(q-1)}\int_1^e s^{k-1}\Big(\int_1^s \big(\log\frac{s}{r}\big)^{q-2} f_i\big(r, u_i(r), v_i(r), {}^H D^\alpha v_i(r)\big)\frac{1}{r}dr\Big)ds\Big]\\
&+A_2\Big[\frac{\mu_1 e^{-k}}{\Gamma(p-1)}\int_1^e s^{k-1}\Big(\int_1^s \big(\log\frac{s}{r}\big)^{p-2} g_i\big(r, v_i(r), u_i(r), {}^H D^\delta u_i(r)\big)\frac{1}{r}dr\Big)ds\\
&-K_2-\frac{\mu_2}{\Gamma(q-1)}\int_1^e s^{-k-1}\Big(\int_1^s m^{k-1}\Big(\int_1^m \big(\log\frac{m}{r}\big)^{q-2}\\
&\times f_i\big(r, u_i(r), v_i(r), {}^H D^\alpha v_i(r)\big)\frac{1}{r}dr\Big)dm\Big)ds\Big]\Big\},
\end{aligned}
$$

$$
\begin{aligned}
&T_{i,2}^{(1)}(u_i, v_i)(t)\\
=&\frac{t^{-k}}{\Gamma(q-1)}\int_1^t s^{k-1}\Big(\int_1^s \big(\log\frac{s}{r}\big)^{q-2} f_i\big(r, u_i(r), v_i(r), {}^H D^\alpha v_i(r)\big)\frac{1}{r}dr\Big)ds,
\end{aligned}
$$

$$
\begin{aligned}
&T_{i,1}^{(2)}(u_i, v_i)(t)\\
=&\frac{1}{\Lambda}\Big(t^{-k}\int_1^t s^{k-1}(\log s)^{p-2}ds\Big)\Big\{A_1\Big[K_2+\frac{\mu_2}{\Gamma(q-1)}\int_1^e s^{-k-1}\Big(\int_1^s m^{k-1}\\
&\times\Big(\int_1^m \big(\log\frac{m}{r}\big)^{q-2} f_i\big(r, u_i(r), v_i(r), {}^H D^\alpha v_i(r)\big)\frac{1}{r}dr\Big)dm\Big)ds\\
&-\frac{\mu_1 e^{-k}}{\Gamma(p-1)}\int_1^e s^{k-1}\Big(\int_1^s \big(\log\frac{s}{r}\big)^{p-2} g_i\big(r, v_i(r), u_i(r), {}^H D^\delta u_i(r)\big)\frac{1}{r}dr\Big)ds\Big]\\
&+B_1\Big[\frac{\lambda_1 e^{-k}}{\Gamma(q-1)}\int_1^e s^{k-1}\Big(\int_1^s \big(\log\frac{s}{r}\big)^{q-2} f_i\big(r, u_i(r), v_i(r), {}^H D^\alpha v_i(r)\big)\frac{1}{r}dr\Big)ds\\
&-K_1-\frac{\lambda_2}{\Gamma(p-1)}\int_1^e s^{-k-1}\Big(\int_1^s m^{k-1}\Big(\int_1^m \big(\log\frac{m}{r}\big)^{p-2}\\
&\times g_i\big(r, v_i(r), u_i(r), {}^H D^\delta u_i(r)\big)\frac{1}{r}dr\Big)dm\Big)ds\Big]\Big\},
\end{aligned}
$$

and

$$
\begin{aligned}
&T_{i,2}^{(2)}(u_i, v_i)(t)\\
=&\frac{t^{-k}}{\Gamma(q-1)}\int_1^t s^{k-1}\Big(\int_1^s \big(\log\frac{s}{r}\big)^{p-2} g_i\big(r, v_i(r), u_i(r), {}^H D^\delta u_i(r)\big)\frac{1}{r}dr\Big)ds,
\end{aligned}
$$

for all $t\in[1,e]$, $(u,v)=\big((u_1,v_1),(u_2,v_2),...,(u_{19},v_{19})\big)\in B_\varepsilon$.

Let

$$
\begin{aligned}
&T(u,v)(t)=\big(T^{(1)}(u,v)(t), T^{(2)}(u,v)(t)\big),\\
&T^{(1)}(u,v)(t)=(T_1^{(1)}(u,v)(t), T_2^{(1)}(u,v)(t)),\\
&T^{(2)}(u,v)(t)=(T_1^{(2)}(u,v)(t), T_2^{(2)}(u,v)(t)),\\
&T_1^{(1)}(u,v)(t)=\big(T_{1,1}^{(1)}(u_1,v_1)(t), T_{2,1}^{(1)}(u_2,v_2)(t),...,T_{19,1}^{(1)}(u_{19},v_{19})(t)\big),\\
&T_2^{(1)}(u,v)(t)=\big(T_{1,2}^{(1)}(u_1,v_1)(t), T_{2,2}^{(1)}(u_2,v_2)(t),...,T_{19,2}^{(1)}(u_{19},v_{19})(t)\big),
\end{aligned}
$$

$$T_1^{(2)}(u,v)(t)=\big(T_{1,1}^{(2)}(u_1,v_1)(t),T_{2,1}^{(2)}(u_2,v_2)(t),...,T_{19,1}^{(2)}(u_{19},v_{19})(t)\big),$$

and

$$T_2^{(2)}(u,v)(t)=\big(T_{1,2}^{(2)}(u_1,v_1)(t),T_{2,2}^{(2)}(u_2,v_2)(t),...,T_{19,2}^{(2)}(u_{19},v_{19})(t)\big).$$

Setting $x=(x_i^{'},x_i^{''}),y=(y_i^{'},y_i^{''})\in B_\varepsilon, i=1,2,...,19$, we then have

$$\begin{aligned}
&\big|T_{i,1}^{(1)}(x_i^{'},x_i^{''})(t)+T_{i,2}^{(1)}(y_i^{'},y_i^{''})(t)\big|\\
\le&\frac{1}{|\Lambda|}\Big(t^{-k}\int_1^t s^{k-1}(\log s)^{q-2}ds\Big)\Big\{\big|B_2\big|\Big[K_1\\
&+\frac{\lambda_2}{\Gamma(p-1)}\int_1^e s^{-k-1}\Big(\int_1^s m^{k-1}\Big(\int_1^m\big(\log\frac{m}{r}\big)^{p-2}\\
&\times\big|g_i\big(r,x_i^{''}(r),x_i^{'}(r),{}^H D^\delta x_i^{'}(r)\big)\big|\frac{1}{r}dr\Big)dm\Big)ds\\
&+\frac{\lambda_1 e^{-k}}{\Gamma(q-1)}\int_1^e s^{k-1}\Big(\int_1^s\big(\log\frac{s}{r}\big)^{q-2}\big|f_i\big(r,x_i^{'}(r),x_i^{''}(r),{}^H D^\alpha x_i^{''}(r)\big)\big|\frac{1}{r}dr\Big)ds\Big]\\
&+\big|A_2\big|\Big[\frac{\mu_1 e^{-k}}{\Gamma(p-1)}\int_1^e s^{k-1}\Big(\int_1^s\big(\log\frac{s}{r}\big)^{p-2}\big|g_i\big(r,x_i^{''}(r),x_i^{'}(r),{}^H D^\delta x_i^{'}(r)\big)\big|\frac{1}{r}dr\Big)ds\\
&+K_2+\frac{\mu_2}{\Gamma(q-1)}\int_1^e s^{-k-1}\Big(\int_1^s m^{k-1}\Big(\int_1^m\big(\log\frac{m}{r}\big)^{q-2}\\
&\times\big|f_i\big(r,x_i^{'}(r),x_i^{''}(r),{}^H D^\alpha x_i^{''}(r)\big)\big|\frac{1}{r}dr\Big)dm\Big)ds\Big]\Big\}\\
&+\frac{t^{-k}}{\Gamma(q-1)}\int_1^t s^{k-1}\Big(\int_1^s\big(\log\frac{s}{r}\big)^{q-2}\big|f_i\big(r,y_i^{'}(r),y_i^{''}(r),{}^H D^\alpha y_i^{''}(r)\big)\big|\frac{1}{r}dr\Big)ds\\
\le&\frac{L_i}{|\Lambda|(q-1)}\Big(\frac{\lambda_1\big|B_2\big|}{\Gamma(q+1)}+\frac{\mu_2\big|A_2\big|}{\Gamma(q+2)}+\frac{|\Lambda|}{q\Gamma(q-1)}\Big)\\
&+\frac{Q_i}{|\Lambda|(q-1)}\Big(\frac{\lambda_2\big|B_2\big|}{\Gamma(p+2)}+\frac{\mu_1\big|A_2\big|}{\Gamma(p+1)}\Big)\\
&+\frac{1}{|\Lambda|(q-1)}\Big(K_1\big|B_2\big|+K_2\big|A_2\big|\Big)\\
=&L_iG_1+Q_iG_2+M_1,
\end{aligned}$$

$$\begin{aligned}
&\big|{}^H D^\delta T_{i,1}^{(1)}(x_i^{'},x_i^{''})(t)+{}^H D^\delta T_{i,2}^{(1)}(y_i^{'},y_i^{''})(t)\big|\\
\le&\frac{1}{\Gamma(1-\delta)}\Big(t\frac{d}{dt}\Big)\int_1^t\big(\log\frac{t}{s}\big)^{-\delta}\big|T_{i,1}^{(1)}(x_i^{'},x_i^{''})(t)\big|\frac{1}{s}ds\\
&+\frac{1}{\Gamma(1-\delta)}\Big(t\frac{d}{dt}\Big)\int_1^t\big(\log\frac{t}{s}\big)^{-\delta}\big|T_{i,2}^{(1)}(y_i^{'},y_i^{''})(t)\big|\frac{1}{s}ds\\
\le&\frac{1}{\Gamma(1-\delta)}\big(L_iG_1+Q_iG_2+M_1\big).
\end{aligned}$$

We can also get

$$\big|T_{i,1}^{(2)}(x_i^{'},x_i^{''})(t)+T_{i,2}^{(2)}(y_i^{'},y_i^{''})(t)\big|\le L_iG_3+Q_iG_4+M_2,$$

$$\big|{}^H D^\delta T_{i,1}^{(2)}(x_i^{'},x_i^{''})(t)+{}^H D^\delta T_{i,2}^{(2)}(y_i^{'},y_i^{''})(t)\big|\le\frac{1}{\Gamma(1-\alpha)}\left(L_iG_3+Q_iG_4+M_2\right).$$

Hence

$$\left\|\left(T_1^{(1)},T_1^{(2)}\right)+\left(T_2^{(1)},T_2^{(2)}\right)\right\|_{X\times Y}=\sum_{i=1}^{19}\left(\left\|\left(T_{i,1}^{(1)},T_{i,1}^{(2)}\right)+\left(T_{i,2}^{(1)},T_{i,2}^{(2)}\right)\right\|_{X_i\times Y_i}\right)<\varepsilon,$$

which implies $\left(T_1^{(1)},T_1^{(2)}\right)x+\left(T_2^{(1)},T_2^{(2)}\right)y\in B_\varepsilon$. Therefore, condition (i) of Theorem 2.1 is satisfied. For $(u_i,v_i),(u_i^{'},v_i^{'})\in B_\varepsilon$, we have

$$\begin{aligned}
&\left|T_{i,1}^{(1)}(u_i,v_i)(t)-T_{i,1}^{(1)}(u_i^{'},v_i^{'})(t)\right|\\
\leq\ &\frac{1}{|\Lambda|(q-1)}\Big(\frac{\lambda_1|B_2|}{\Gamma(q+1)}+\frac{\mu_2|A_2|}{\Gamma(q+2)}\Big)\|l_i\|\big(\|u_i-u_i^{'}\|_{X_i}+\|v_i-v_i^{'}\|_{Y_i}\big)\\
&+\frac{1}{|\Lambda|(q-1)}\Big(\frac{\lambda_2|B_2|}{\Gamma(p+2)}+\frac{\mu_1|A_2|}{\Gamma(p+1)}\Big)\|q_i\|\big(\|u_i-u_i^{'}\|_{X_i}+\|v_i-v_i^{'}\|_{Y_i}\big)\\
=\ &\big(\|l_i\|S_1+\|q_i\|G_2\big)\big(\|u_i-u_i^{'}\|_{X_i}+\|v_i-v_i^{'}\|_{Y_i}\big),
\end{aligned}$$

$$\begin{aligned}
&\left|{}^HD^\delta T_{i,1}^{(1)}(u_i,v_i)(t)-{}^H D^\delta T_{i,1}^{(1)}(u_i^{'},v_i)^{'}(t)\right|\\
\leq\ &\frac{\big(\|l_i\|S_1+\|q_i\|G_2\big)}{\Gamma(1-\delta)}\big(\|u_i-u_i^{'}\|_{X_i}+\|v_i-v_i^{'}\|_{Y_i}\big).
\end{aligned}$$

Moreover, by the similar way we obtain

$$\begin{aligned}
&\left|T_{i,1}^{(2)}(u_i,v_i)(t)-T_{i,1}^{(2)}(u_i^{'},v_i^{'})(t)\right|\\
\leq\ &\big(\|l_i\|G_3+\|q_i\|S_2\big)\big(\|u_i-u_i^{'}\|_{X_i}+\|v_i-v_i^{'}\|_{Y_i}\big),
\end{aligned}$$

$$\begin{aligned}
&\left|{}^HD^\alpha T_{i,1}^{(2)}(u_i,v_i)(t)-{}^H D^\alpha T_{i,1}^{(2)}(u_i^{'},v_i^{'})(t)\right|\\
\leq\ &\frac{\big(\|l_i\|G_3+\|q_i\|S_2\big)}{\Gamma(1-\alpha)}\big(\|u_i-u_i^{'}\|_{X_i}+\|v_i-v_i^{'}\|_{Y_i}\big).
\end{aligned}$$

Hence

$$\begin{aligned}
&\left\|\left(T_1^{(1)},T_1^{(2)}\right)(u,v)(t)+\left(T_1^{(1)},T_1^{(2)}\right)(u^{'},v^{'})(t)\right\|_{X\times Y}\\
=&\sum_{i=1}^{19}\left(\left\|\left(T_{i,1}^{(1)},T_{i,1}^{(2)}\right)(u_i,v_i)(t)+\left(T_{i,1}^{(1)},T_{i,1}^{(2)}\right)(u_i^{'},v_i)^{'}(t)\right\|_{X_i\times Y_i}\right)\\
=&\sum_{i=1}^{19}\Big\{\frac{(\Gamma(1-\delta)+1)\big(\|l_i\|S_1+\|q_i\|G_2\big)}{\Gamma(1-\delta)}+\frac{(\Gamma(1-\alpha)+1)\big(\|l_i\|G_3+\|q_i\|S_2\big)}{\Gamma(1-\alpha)}\Big\}\\
&\times\big(\|u_i-u_i^{'}\|_{X_i}+\|v_i-v_i^{'}\|_{Y_i}\big)\\
<&\sum_{i=1}^{19}\Theta_i\big(\|u_i-u_i^{'}\|_{X_i}+\|v_i-v_i^{'}\|_{Y_i}\big).
\end{aligned}$$

According to $\sum_{i=1}^{19}\Theta_i<1$, we obtain operator $\left(T_1^{(1)},T_1^{(2)}\right)$ is a contraction on B_ε. Then the condition (ii) of Theorem 2.1 is satisfied. Finally, we show that the operator $\left(T_2^{(1)},T_2^{(2)}\right)$ satisfied the condition (iii) of Theorem 2.1. For each $(u,v)=\big((u_1,v_1),(u_2,v_2),...,(u_{19},v_{19})\big)\in B_\varepsilon$, one has

$$\left|T_{i,2}^{(1)}(u_i,v_i)(t)\right|$$

$$
\begin{aligned}
&\leq \frac{t^{-k}}{\Gamma(q-1)}\int_1^t s^{k-1}\Big(\int_1^s \big(\log\frac{s}{r}\big)^{q-2}\big|f_i\big(r,u_i(r),v_i(r),{}^H D^{\alpha}v_i(r)\big)\big|\frac{1}{r}dr\Big)ds\\
&\leq \frac{L_i}{\Gamma(q+1)},
\end{aligned}
$$

$$
\big|{}^H D^{\alpha}T_{i,2}^{(1)}(u_i,v_i)(t)\big| \leq \frac{L_i}{\Gamma(1-\delta)\Gamma(q+1)},\ \big|T_{i,2}^{(2)}(u_i,v_i)(t)\big| \leq \frac{Q_i}{\Gamma(p+1)},
$$

and

$$
\big|{}^H D^{\alpha}T_{i,2}^{(2)}(u_i,v_i)(t)\big| \leq \frac{Q_i}{\Gamma(1-\alpha)\Gamma(p+1)}.
$$

Then

$$
\begin{aligned}
&\big\|\big(T_2^{(1)},T_2^{(2)}\big)(u,v)(t)\big\|_{X\times Y}\\
=&\sum_{i=1}^{19}\Big(\big\|\big(T_{i,2}^{(1)},T_{i,2}^{(2)}\big)(u_i,v_i)(t)\big\|_{X_i\times Y_i}\Big)\\
=&\sum_{i=1}^{19}\Big\{\frac{L_i(\Gamma(1-\delta)+1)}{\Gamma(1-\delta)\Gamma(q+1)}+\frac{Q_i(\Gamma(1-\alpha)+1)}{\Gamma(1-\alpha)\Gamma(p+1)}\Big\},
\end{aligned}
$$

which implies that $\big(T_2^{(1)},T_2^{(2)}\big)$ are uniformly bounded on the subset B_{ε}. Next we show that the operator $\big(T_2^{(1)},T_2^{(2)}\big)$ are equi-continuous. Let $t\in[1,e]$ with $t_1<t_2$. Then we have

$$
\begin{aligned}
&\big|T_{i,2}^{(1)}(u_i,v_i)(t_2)-T_{i,2}^{(1)}(u_i,v_i)(t_1)\big|\\
\leq&\frac{L_i}{\Gamma(q-1)}t_2^{-k}\int_1^{t_2} s^{k-1}\Big(\int_1^s\big(\log\frac{s}{r}\big)^{q-2}\frac{1}{r}dr\Big)ds\\
&-\frac{L_i}{\Gamma(q-1)}t_1^{-k}\int_1^{t_1} s^{k-1}\Big(\int_1^s\big(\log\frac{s}{r}\big)^{q-2}\frac{1}{r}dr\Big)ds\\
=&\frac{L_i\big|t_1^k-t_2^k\big|}{\Gamma(q-1)t_1^kt_2^k}\int_1^{t_1} s^{k-1}\Big(\int_1^s\big(\log\frac{s}{r}\big)^{q-2}\frac{1}{r}dr\Big)ds\\
&+\frac{L_it_2^{-k}}{\Gamma(q-1)}\int_{t_1}^{t_2} s^{k-1}\Big(\int_1^s\big(\log\frac{s}{r}\big)^{q-2}\frac{1}{r}dr\Big)ds.
\end{aligned}
$$

Hence $\big|T_{i,2}^{(1)}(u_i,v_i)(t_2)-T_{i,2}^{(1)}(u_i,v_i)(t_1)\big|\to 0$ as $t_2\to t_1$. On the other hand, since we have

$$
\begin{aligned}
&\lim_{t_2\to t_1}\big|{}^H D^{\delta}T_{i,2}^{(1)}(u_i,v_i)(t_2)-{}^H D^{\delta}T_{i,2}^{(1)}(u_i,v_i)(t_1)\big|\to 0,\\
&\lim_{t_2\to t_1}\big|T_{i,2}^{(2)}(u_i,v_i)(t_2)-T_{i,2}^{(2)}(u_i,v_i)(t_1)\big|\to 0,\\
&\lim_{t_2\to t_1}\big|{}^H D^{\delta}T_{i,2}^{(2)}(u_i,v_i)(t_2)-{}^H D^{\delta}T_{i,2}^{(2)}(u_i,v_i)(t_1)\big|\to 0,
\end{aligned}
$$

thus

$$
\begin{aligned}
&\big\|\big(T_2^{(1)},T_2^{(2)}\big)(u,v)(t_2)-\big(T_2^{(1)},T_2^{(2)}\big)(u,v)(t_1)\big\|_{X\times Y}\\
=&\sum_{i=1}^{19}\Big(\big\|\big(T_{i,2}^{(1)},T_{i,2}^{(2)}\big)(u_i,v_i)(t_2)-\big(T_{i,2}^{(1)},T_{i,2}^{(2)}\big)(u_i,v_i)(t_1)\big\|_{X_i\times Y_i}\Big)
\end{aligned}
$$

$\to 0(t_2 \to t_1)$.

Above all the operator $\left(T_2^{(1)}, T_2^{(2)}\right)$ is equicontinuous. By applying the continuity of functions f_i and g_i on $[1,e]\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}$, we can conclude that operator $\left(T_2^{(1)}, T_2^{(2)}\right)$ is continuous. The Arzela-Ascoli Theorem implies that operator $\left(T_2^{(1)}, T_2^{(2)}\right)$ is compact on B_ε. According to the Theorem 2.1, there exist $z=(z_1,z_2)\in B_\varepsilon$, such that $z = \left(T_1^{(1)}, T_1^{(2)}\right)z + \left(T_2^{(1)}, T_2^{(2)}\right)z$. Therefore, problem (1.1) has at least one solution on $[1,e]$. This completes the proof. $\square$

4. Ulam-Hyers stability

Let $\xi_i, \zeta_i > 0$, $f_i, g_i : [1,e]\times\mathbb{R}\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}^+, (i = 1, 2, ..., 19)$ be continuous functions and $\varphi_i(t), \psi_i(t) : [1,e] \to \mathbb{R}^+$ are nondecreasing continuous functions, consider the following inequalities:

$$\begin{cases} \left| \left({}^H D^q + k^H D^{q-1}\right) u_i(t) - f_i\left(t, u_i(t), v_i(t), {}^H D^\alpha v_i(t)\right) \right| \le \xi_i, \\ \left| \left({}^H D^p + k^H D^{p-1}\right) v_i(t) - g_i\left(t, v_i(t), u_i(t), {}^H D^\delta u_i(t)\right) \right| \le \zeta_i, \\ t\in[1,e],\ i = 1, 2, ..., 19, \end{cases} \tag{4.1}$$

$$\begin{cases} \left| \left({}^H D^q + k^H D^{q-1}\right) u_i(t) - f_i\left(t, u_i(t), v_i(t), {}^H D^\alpha v_i(t)\right) \right| \le \varphi_i(t)\xi_i, \\ \left| \left({}^H D^p + k^H D^{p-1}\right) v_i(t) - g_i\left(t, v_i(t), u_i(t), {}^H D^\delta u_i(t)\right) \right| \le \psi_i(t)\zeta_i, \\ t\in[1,e], i = 1, 2, ..., 19, \end{cases} \tag{4.2}$$

$$\begin{cases} \left| \left({}^H D^q + k^H D^{q-1}\right) u_i(t) - f_i\left(t, u_i(t), v_i(t), {}^H D^\alpha v_i(t)\right) \right| \le \varphi_i(t), \\ \left| \left({}^H D^p + k^H D^{p-1}\right) v_i(t) - g_i\left(t, v_i(t), u_i(t), {}^H D^\delta u_i(t)\right) \right| \le \psi_i(t), \\ t\in[1,e], i = 1, 2, ..., 19. \end{cases} \tag{4.3}$$

Definition 4.1. Couple system (1.1) is called Ulam-Hyers stable, if there are constants $d_{f_1,f_2,...,f_{19}} > 0$ and $p_{g_1,g_2,...,g_{19}} > 0$ that make each $\xi = \xi(\xi_1, \xi_2, ..., \xi_{19}) > 0$ and $\zeta = \zeta(\zeta_1, \zeta_2, ..., \zeta_{19}) > 0$, each solution $(u,v) = \big((u_1,v_1),(u_2,v_2),...,(u_{19},v_{19})\big) \in X\times Y$ of the inequality (4.1), there exists a solution $(u^*,v^*) = \big((u_1^*,v_1^*),(u_2^*,v_2^*), ..., (u_{19}^*,v_{19}^*)\big) \in X\times Y$ of (1.1) with

$$\|(u,v)-(u^*,v^*)\|_{X\times Y} \le d_{f_1,f_2,...,f_{19}}\xi + p_{g_1,g_2,...,g_{19}}\zeta,\ t\in[1,e].$$

Definition 4.2. Couple system (1.1) is called generalized Ulam-Hyers stable if there exist functions $\omega_{f_1,f_2,...,f_{19}} \in C(\mathbb{R}^+,\mathbb{R}^+)$ and $\varpi_{g_1,g_2,...,g_{19}} \in C(\mathbb{R}^+,\mathbb{R}^+)$ that satisfy with $\omega_{f_1,f_2,...,f_{19}}(0) = 0$ and $\varpi_{g_1,g_2,...,g_{19}}(0) = 0$, such that for each $\xi = \xi(\xi_1,\xi_2,...,\xi_{19}) > 0$ and $\zeta = \zeta(\zeta_1,\zeta_2,...,\zeta_{19}) > 0$, for each solution $(u,v) = \big((u_1,v_1), (u_2,v_2),...,(u_{19},v_{19})\big) \in X\times Y$ of the inequality (4.1), there exists a solution $(u^*,v^*) = \big((u_1^*,v_1^*),(u_2^*,v_2^*),...,(u_{19}^*,v_{19}^*)\big) \in X\times Y$ of (1.1) with

$$\|(u,v)-(u^*,v^*)\|_{X\times Y} \le \omega_{f_1,f_2,...,f_{19}}(\xi) + \varpi_{g_1,g_2,...,g_{19}}(\zeta),\ t\in[1,e].$$

Definition 4.3. Couple system (1.1) is called Ulam-Hyers-Rassias stable with respect to $\varphi = \varphi(\varphi_1,\varphi_2,...,\varphi_{19}) \in C([1,e],\mathbb{R}^+)$ and $\psi = \psi(\psi_1,\psi_2,...,\psi_{19}) \in$

$C([1,e],\mathbb{R}^+)$, if there exist constants $d_{f_1,f_2,...,f_{19},\varphi} > 0$ and $p_{g_1,g_2,...,g_{19},\psi} > 0$, such that for every $\xi = \xi(\xi_1, \xi_2, ..., \xi_{19}) > 0$ and $\zeta = \zeta(\zeta_1, \zeta_2, ..., \zeta_{19}) > 0$, for each solution $(u,v) = \big((u_1,v_1),(u_2,v_2),...,(u_{19},v_{19})\big) \in X \times Y$ of the inequality (4.2), there exists a solution $(u^*,v^*) = \big((u_1^*,v_1^*),(u_2^*,v_2^*),...,(u_{19}^*,v_{19}^*)\big) \in X \times Y$of (1.1) with

$$\|(u,v)-(u^*,v^*)\|_{X\times Y} \le d_{f_1,f_2,...,f_{19},\varphi}\xi\varphi(t) + p_{g_1,g_2,...,g_{19},\psi}\zeta\psi(t),\ t\in[1,e].$$

Definition 4.4. Couple system (1.1) is called generalized Ulam-Hyers-Rassias stable with respect to $\varphi = \varphi(\varphi_1,\varphi_2,...,\varphi_{19}) \in C([1,e],\mathbb{R}^+)$ and $\psi = \psi(\psi_1,\psi_2,...,\psi_{19}) \in C([1,e],\mathbb{R}^+)$, if there exist constants $d_{f_1,f_2,...,f_{19},\varphi} > 0$ and $p_{g_1,g_2,...,g_{19},\psi} > 0$, such that for each solution $(u,v) = \big((u_1,v_1),(u_2,v_2),...,(u_{19},v_{19})\big) \in X \times Y$ of the inequality (4.3), there exists a solution $(u^*,v^*) = \big((u_1^*,v_1^*),(u_2^*,v_2^*),...,(u_{19}^*,v_{19}^*)\big) \in X \times Y$ of (1.1) with

$$\|(u,v)-(u^*,v^*)\|_{X\times Y} \le d_{f_1,f_2,...,f_{19},\varphi}\varphi(t) + p_{g_1,g_2,...,g_{19},\psi}\psi(t),\ t\in[1,e].$$

Remark 4.1. Let function $(u,v) = \big((u_1,v_1),(u_2,v_2),...,(u_{19},v_{19})\big) \in X \times Y$ satisfies inequality (4.1) if and only if there exist functions $\phi_i, \theta_i \in C([1,e],\mathbb{R})(i = 1,2,...,19)$ such that

(i) $|\phi_i(t)| < \xi_i,\ \ |\theta_i(t)| < \zeta_i,\ t\in[1,e],\ i=1,2,...,19;$

(ii) $\left({}^HD^q + k^HD^{q-1}\right)u_i(t) = f_i\left(t,u_i(t),v_i(t),{}^HD^\alpha v_i(t)\right) + \phi_i(t),$
$\left({}^HD^p + k^HD^{p-1}\right)v_i(t) = g_i\left(t,v_i(t),u_i(t),{}^HD^\delta u_i(t)\right) + \theta_i(t),$
$t\in[1,e],\ i=1,2,...,19.$

Remark 4.2. Let function $(u,v) = \big((u_1,v_1),(u_2,v_2),...,(u_{19},v_{19})\big) \in X \times Y$ satisfies inequality (4.2) if and only if there exist functions $\phi_i, \theta_i \in C([1,e],\mathbb{R})(i = 1,2,...,19)$ such that

(i) $|\phi_i(t)| < \varphi_i(t)\xi_i,\ \ |\theta_i(t)| < \psi_i(t)\zeta_i,\ t\in[1,e],\ i=1,2,...,19;$

(ii) $\left({}^HD^q + k^HD^{q-1}\right)u_i(t) = f_i\left(t,u_i(t),v_i(t),{}^HD^\alpha v_i(t)\right) + \phi_i(t),$
$\left({}^HD^p + k^HD^{p-1}\right)v_i(t) = g_i\left(t,v_i(t),u_i(t),{}^HD^\delta u_i(t)\right) + \theta_i(t),$
$t\in[1,e],\ i=1,2,...,19.$

Remark 4.3. Let function $(u,v) = \big((u_1,v_1),(u_2,v_2),...,(u_{19},v_{19})\big) \in X \times Y$ satisfies inequality (4.3) if and only if there exist functions $\phi_i, \theta_i \in C([1,e],\mathbb{R})(i = 1,2,...,19)$ such that

(i) $|\phi_i(t)| < \varphi_i(t),\ \ |\theta_i(t)| < \psi_i(t),\ t\in[1,e],\ i=1,2,...,19;$

(ii) $\left({}^HD^q + k^HD^{q-1}\right)u_i(t) = f_i\left(t,u_i(t),v_i(t),{}^HD^\alpha v_i(t)\right) + \phi_i(t),$
$\left({}^HD^p + k^HD^{p-1}\right)v_i(t) = g_i\left(t,v_i(t),u_i(t),{}^HD^\delta u_i(t)\right) + \theta_i(t),$
$t\in[1,e],\ i=1,2,...,19.$

Lemma 4.1. *Assume that* $(u,v) = \big((u_1,v_1),(u_2,v_2),...,(u_{19},v_{19})\big) \in X \times Y$ *is the solution of the inequalities* (4.1). *Then, the following inequalities hold:*

$$\begin{aligned}
&|u_i(t) - \bar{u}_i(t)| \le G_1\xi_i + G_2\zeta_i,\\
&|v_i(t) - \bar{v}_i(t)| \le G_3\xi_i + G_4\zeta_i,\\
&\left|{}^HD^\delta u_i(t) - {}^HD^\delta \bar{u}_i(t)\right| \le \frac{G_1\xi_i + G_2\zeta_i}{\Gamma(1-\delta)},
\end{aligned}$$

and

$$\left|{}^H D^\alpha v_i(t) - {}^H D^\alpha \bar{v}_i(t)\right| \le \frac{G_3\xi_i + G_4\zeta_i}{\Gamma(1-\alpha)},$$

where

$$\begin{aligned}
\bar{u}_i(t) =& \frac{1}{\Lambda}\Big(t^{-k}\int_1^t s^{k-1}(\log s)^{q-2}ds\Big)\Big\{B_2\Big[K_1 \\
&+ \frac{\lambda_2}{\Gamma(p-1)}\int_1^e s^{-k-1}\Big(\int_1^s m^{k-1}\Big(\int_1^m \big(\log\frac{m}{r}\big)^{p-2}\frac{z_i(r)}{r}dr\Big)dm\Big)ds \\
&- \frac{\lambda_1 e^{-k}}{\Gamma(q-1)}\int_1^e s^{k-1}\Big(\int_1^s \big(\log\frac{s}{r}\big)^{q-2}\frac{h_i(r)}{r}dr\Big)ds\Big] \\
&+ A_2\Big[\frac{\mu_1 e^{-k}}{\Gamma(p-1)}\int_1^e s^{k-1}\Big(\int_1^s \big(\log\frac{s}{r}\big)^{p-2}\frac{z_i(r)}{r}dr\Big)ds - K_2 \\
&- \frac{\mu_2}{\Gamma(q-1)}\int_1^e s^{-k-1}\Big(\int_1^s m^{k-1}\Big(\int_1^m \big(\log\frac{m}{r}\big)^{q-2}\frac{h_i(r)}{r}dr\Big)dm\Big)ds\Big]\Big\} \\
&+ \frac{t^{-k}}{\Gamma(q-1)}\int_1^t s^{k-1}\Big(\int_1^s \big(\log\frac{s}{r}\big)^{q-2}\frac{h_i(r)}{r}dr\Big)ds,
\end{aligned}$$

and

$$\begin{aligned}
\bar{v}_i(t) =& \frac{1}{\Lambda}\Big(t^{-k}\int_1^t s^{k-1}(\log s)^{p-2}ds\Big)\Big\{A_1\Big[K_2 \\
&+ \frac{\mu_2}{\Gamma(q-1)}\int_1^e s^{-k-1}\Big(\int_1^s m^{k-1}\Big(\int_1^m \big(\log\frac{m}{r}\big)^{q-2}\frac{h_i(r)}{r}dr\Big)dm\Big)ds \\
&- \frac{\mu_1 e^{-k}}{\Gamma(p-1)}\int_1^e s^{k-1}\Big(\int_1^s \big(\log\frac{s}{r}\big)^{p-2}\frac{z_i(r)}{r}dr\Big)ds\Big] \\
&+ B_1\Big[\frac{\lambda_1 e^{-k}}{\Gamma(q-1)}\int_1^e s^{k-1}\Big(\int_1^s \big(\log\frac{s}{r}\big)^{q-2}\frac{h_i(r)}{r}dr\Big)ds - K_1 \\
&- \frac{\lambda_2}{\Gamma(p-1)}\int_1^e s^{-k-1}\Big(\int_1^s m^{k-1}\Big(\int_1^m \big(\log\frac{m}{r}\big)^{p-2}\frac{z_i(r)}{r}dr\Big)dm\Big)ds\Big]\Big\} \\
&+ \frac{t^{-k}}{\Gamma(p-1)}\int_1^t s^{k-1}\Big(\int_1^s \big(\log\frac{s}{r}\big)^{p-2}\frac{z_i(r)}{r}dr\Big)ds,
\end{aligned}$$

and here

$$\begin{aligned}
h_i(t) &= f_i\left(t, u_i(t), v_i(t), {}^H D^\alpha v_i(t)\right),\ i = 1,2,...,19, \\
z_i(t) &= g_i\left(t, v_i(t), u_i(t), {}^H D^\delta u_i(t)\right),\ i = 1,2,...,19.
\end{aligned}$$

Proof. Since (u, v) is the solution of (4.1), then by Remark (4.1), (u_i, v_i) is the solution of the following problem:

$$\begin{cases}
\left({}^H D^q + k{}^H D^{q-1}\right)u_i(t) = f_i\left(t, u_i(t), v_i(t), {}^H D^\alpha v_i(t)\right) + \phi_i(t),\ t\in[1,e], \\
\left({}^H D^p + k{}^H D^{p-1}\right)v_i(t) = g_i\left(t, v_i(t), u_i(t), {}^H D^\delta u_i(t)\right) + \theta_i(t), \\
u_i(1) = 0, \quad v_i(1) = 0, \\
\lambda_1 u_i(e) - \lambda_2\int_1^e \frac{v_i(s)}{s}ds = K_1, \\
\mu_1 v_i(e) - \mu_2\int_1^e \frac{u_i(s)}{s}ds = K_2, \quad i = 1,2,...,19.
\end{cases} \tag{4.4}$$

For simplicity of presentation, we let

$$h_i(t) = f_i\left(t, u_i(t), v_i(t), {}^H D^{\alpha} v_i(t)\right), \ i = 1, 2, ..., 19,$$
$$z_i(t) = g_i\left(t, v_i(t), u_i(t), {}^H D^{\delta} u_i(t)\right), \ i = 1, 2, ..., 19.$$

Then, by Lemma (2.1), the solution of (4.4) can be given in the following form:

$$\begin{aligned}
u_i(t) =& \frac{1}{\Lambda}\Big(t^{-k}\int_1^t s^{k-1}(\log s)^{q-2}ds\Big)\Big\{B_2\Big[K_1 \\
&+ \frac{\lambda_2}{\Gamma(p-1)}\int_1^e s^{-k-1}\Big(\int_1^s m^{k-1}\Big(\int_1^m \big(\log\frac{m}{r}\big)^{p-2}\frac{z_i(r)+\theta_i(r)}{r}dr\Big)dm\Big)ds \\
&- \frac{\lambda_1 e^{-k}}{\Gamma(q-1)}\int_1^e s^{k-1}\Big(\int_1^s \big(\log\frac{s}{r}\big)^{q-2}\frac{h_i(r)+\phi_i(r)}{r}dr\Big)ds\Big] \\
&+ A_2\Big[\frac{\mu_1 e^{-k}}{\Gamma(p-1)}\int_1^e s^{k-1}\Big(\int_1^s \big(\log\frac{s}{r}\big)^{p-2}\frac{z_i(r)+\theta_i(r)}{r}dr\Big)ds - K_2 \\
&- \frac{\mu_2}{\Gamma(q-1)}\int_1^e s^{-k-1}\Big(\int_1^s m^{k-1}\Big(\int_1^m \big(\log\frac{m}{r}\big)^{q-2} \\
&\times \frac{h_i(r)+\phi_i(r)}{r}dr\Big)dm\Big)ds\Big]\Big\} \\
&+ \frac{t^{-k}}{\Gamma(q-1)}\int_1^t s^{k-1}\Big(\int_1^s \big(\log\frac{s}{r}\big)^{q-2}\frac{h_i(r)+\phi_i(r)}{r}dr\Big)ds,
\end{aligned}$$

and

$$\begin{aligned}
v_i(t) =& \frac{1}{\Lambda}\Big(t^{-k}\int_1^t s^{k-1}(\log s)^{p-2}ds\Big)\Big\{A_1\Big[K_2 \\
&+ \frac{\mu_2}{\Gamma(q-1)}\int_1^e s^{-k-1}\Big(\int_1^s m^{k-1}\Big(\int_1^m \big(\log\frac{m}{r}\big)^{q-2}\frac{h_i(r)+\phi_i(r)}{r}dr\Big)dm\Big)ds \\
&- \frac{\mu_1 e^{-k}}{\Gamma(p-1)}\int_1^e s^{k-1}\Big(\int_1^s \big(\log\frac{s}{r}\big)^{p-2}\frac{z_i(r)+\theta_i(r)}{r}dr\Big)ds\Big] \\
&+ B_1\Big[\frac{\lambda_1 e^{-k}}{\Gamma(q-1)}\int_1^e s^{k-1}\Big(\int_1^s \big(\log\frac{s}{r}\big)^{q-2}\frac{h_i(r)+\phi_i(r)}{r}dr\Big)ds - K_1 \\
&- \frac{\lambda_2}{\Gamma(p-1)}\int_1^e s^{-k-1}\Big(\int_1^s m^{k-1}\Big(\int_1^m \big(\log\frac{m}{r}\big)^{p-2}\frac{z_i(r)+\theta_i(r)}{r}dr\Big)dm\Big)ds\Big]\Big\} \\
&+ \frac{t^{-k}}{\Gamma(p-1)}\int_1^t s^{k-1}\Big(\int_1^s \big(\log\frac{s}{r}\big)^{p-2}\frac{z_i(r)+\theta_i(r)}{r}dr\Big)ds.
\end{aligned}$$

Then, we deduce that

$$\begin{aligned}
&\big|u_i(t) - \bar{u}_i(t)\big| \\
=&\Big|\frac{1}{\Lambda}\Big(t^{-k}\int_1^t s^{k-1}(\log s)^{q-2}ds\Big) \\
&\times\Big\{B_2\Big[\frac{\lambda_2}{\Gamma(p-1)}\int_1^e s^{-k-1}\Big(\int_1^s m^{k-1}\Big(\int_1^m \big(\log\frac{m}{r}\big)^{p-2}\frac{\theta_i(r)}{r}dr\Big)dm\Big)ds \\
&+ \frac{\lambda_1 e^{-k}}{\Gamma(q-1)}\int_1^e s^{k-1}\Big(\int_1^s \big(\log\frac{s}{r}\big)^{q-2}\frac{\phi_i(r)}{r}dr\Big)ds\Big]
\end{aligned}$$

$$
\begin{aligned}
&+ A_2\Big[\frac{\mu_1 e^{-k}}{\Gamma(p-1)}\int_1^e s^{k-1}\Big(\int_1^s \big(\log\frac{s}{r}\big)^{p-2}\frac{\theta_i(r)}{r}dr\Big)ds \\
&+ \frac{\mu_2}{\Gamma(q-1)}\int_1^e s^{-k-1}\Big(\int_1^s m^{k-1}\Big(\int_1^m \big(\log\frac{m}{r}\big)^{q-2}\frac{\phi_i(r)}{r}dr\Big)dm\Big)ds\Big]\Big\} \\
&+ \frac{t^{-k}}{\Gamma(q-1)}\int_1^t s^{k-1}\Big(\int_1^s \big(\log\frac{s}{r}\big)^{q-2}\frac{\phi_i(r)}{r}dr\Big)ds\Big| \\
\leq& \frac{1}{|\Lambda|(q-1)}\Big(\frac{\lambda_1|B_2|}{\Gamma(q+1)}+\frac{\mu_2|A_2|}{\Gamma(q+2)}+\frac{|\Lambda|}{q\Gamma(q-1)}\Big)\xi_i \\
&+ \frac{1}{|\Lambda|(q-1)}\Big(\frac{\lambda_2|B_2|}{\Gamma(p+2)}+\frac{\mu_1|A_2|}{\Gamma(p+1)}\Big)\zeta_i \\
=& G_1\xi_i + G_2\zeta_i,
\end{aligned}
$$

$$
\begin{aligned}
&\big|{}^H D^\delta u_i(t) -{}^H D^\delta \bar{u}_i(t)\big| \\
\leq& \frac{1}{\Gamma(1-\delta)}\Big(t\frac{d}{dt}\Big)\int_1^t \big(\log\frac{t}{s}\big)^{-\delta}\big|u_i(t)-\bar{u}_i(t)\big|\frac{1}{s}ds \\
\leq& \frac{G_1\xi_i + G_2\zeta_i}{\Gamma(1-\delta)}.
\end{aligned}
$$

By the similar way we obtain

$$
\begin{aligned}
\big|v_i(t)-\bar{v}_i(t)\big| &\leq G_3\xi_i + G_4\zeta_i, \\
\big|{}^H D^\delta u_i(t) -{}^H D^\delta \bar{u}_i(t)\big| &\leq \frac{G_3\xi_i + G_4\zeta_i}{\Gamma(1-\delta)}.
\end{aligned}
$$

The proof is cmpleted. □

Lemma 4.2. *Assume that* $(u,v) = \big((u_1,v_1),(u_2,v_2),...,(u_{19},v_{19})\big) \in X\times Y$ *is the solution of the inequalities* (4.2). *Then, the following inequalities hold:*

$$
\begin{aligned}
&\big|u_i(t)-\bar{u}_i(t)\big| \\
\leq& \frac{1}{|\Lambda|(q-1)}\Big(\frac{\lambda_1|B_2|(q+1)+\mu_2|A_2|}{\Gamma(q+2)}\varphi_i(e)+\frac{|\Lambda|}{q\Gamma(q-1)}\varphi_i(t)\Big)\xi_i + G_2\psi_i(e)\zeta_i,
\end{aligned}
$$

$$
\begin{aligned}
&\big|{}^H D^\delta u_i(t) -{}^H D^\delta \bar{u}_i(t)\big| \\
\leq& \frac{1}{|\Lambda|(q-1)\Gamma(1-\delta)}\Big(\frac{\lambda_1|B_2|(q+1)+\mu_2|A_2|}{\Gamma(q+2)}\varphi_i(e)+\frac{|\Lambda|}{q\Gamma(q-1)}\varphi_i(t)\Big)\xi_i \\
&+ \frac{G_2\psi_i(e)}{\Gamma(1-\delta)}\zeta_i.
\end{aligned}
$$

And

$$
\begin{aligned}
&|v_i(t)-\bar{v}_i(t)| \\
\leq& \frac{1}{|\Lambda|(p-1)}\Big(\frac{\mu_1|A_1|(p+1)+\lambda_2|B_1|}{\Gamma(p+2)}\psi_i(e)+\frac{|\Lambda|}{p\Gamma(p-1)}\psi_i(t)\Big)\zeta_i + G_3\varphi_i(e)\xi_i,
\end{aligned}
$$

$$
\big|{}^H D^\delta v_i(t) -{}^H D^\delta \bar{v}_i(t)\big|
$$

$$\leq \frac{1}{|\Lambda|(p-1)\Gamma(1-\alpha)}\Big(\frac{\mu_1\big|A_1\big|(p+1)+\lambda_2\big|B_1\big|}{\Gamma(p+2)}\psi_i(e)+\frac{|\Lambda|}{p\Gamma(p-1)}\psi_i(t)\Big)\zeta_i + \frac{G_3\varphi_i(e)}{\Gamma(1-\alpha)}\xi_i.$$

Proof. Using the some argument as in the proof of Lemma 4.1 and Remark 4.2, we can conclude that

$$|u_i(t)-\bar{u}_i(t)| = \frac{1}{|\Lambda|(q-1)}\Big(\frac{\lambda_1\big|B_2\big|(q+1)+\mu_2\big|A_2\big|}{\Gamma(q+2)}\varphi_i(e)+\frac{|\Lambda|}{q\Gamma(q-1)}\varphi_i(t)\Big)\xi_i + G_2\psi_i(e)\zeta_i,$$

$$\big|{}^H D^\delta u_i(t) -{}^H D^\delta \bar{u}_i(t)\big| \leq \frac{1}{|\Lambda|(q-1)\Gamma(1-\delta)}\Big(\frac{\lambda_1\big|B_2\big|(q+1)+\mu_2\big|A_2\big|}{\Gamma(q+2)}\varphi_i(e)+\frac{|\Lambda|}{q\Gamma(q-1)}\varphi_i(t)\Big)\xi_i + \frac{G_2\psi_i(e)}{\Gamma(1-\delta)}\zeta_i,$$

and

$$|v_i(t)-\bar{v}_i(t)| \leq \frac{1}{|\Lambda|(p-1)}\Big(\frac{\mu_1\big|A_1\big|(p+1)+\lambda_2\big|B_1\big|}{\Gamma(p+2)}\psi_i(e)+\frac{|\Lambda|}{p\Gamma(p-1)}\psi_i(t)\Big)\zeta_i + G_3\varphi_i(e)\xi_i,$$

$$\big|{}^H D^\delta v_i(t) -{}^H D^\delta \bar{v}_i(t)\big| \leq \frac{1}{|\Lambda|(p-1)\Gamma(1-\alpha)}\Big(\frac{\mu_1\big|A_1\big|(p+1)+\lambda_2\big|B_1\big|}{\Gamma(p+2)}\psi_i(e)+\frac{|\Lambda|}{p\Gamma(p-1)}\psi_i(t)\Big)\zeta_i + \frac{G_3\varphi_i(e)}{\Gamma(1-\alpha)}\xi_i.$$

This completes the proof of Lemma 4.2. □

For computational convenience, we set

$$E_1 = \frac{\Gamma(1-\alpha)\left(\Gamma(1-\delta)+1\right)G_1+\Gamma(1-\delta)\left(\Gamma(1-\alpha)+1\right)G_3}{\Gamma(1-\delta)\Gamma(1-\alpha)},$$

$$E_2 = \frac{\Gamma(1-\alpha)\left(\Gamma(1-\delta)+1\right)G_2+\Gamma(1-\delta)\left(\Gamma(1-\alpha)+1\right)G_4}{\Gamma(1-\delta)\Gamma(1-\alpha)},$$

$$E_3 = \frac{\big(\Gamma(1-\delta)+1\big)}{|\Lambda|(q-1)\Gamma(1-\delta)}\Big(\frac{\lambda_1\left|B_2\right|(q+1)+\mu_2\left|A_2\right|}{\Gamma(q+2)} + \frac{|\Lambda|}{q\Gamma(q-1)}+\frac{|\Lambda|(q-1)\Gamma(1-\delta)\left(\Gamma(1-\alpha)+1\right)}{\Gamma(1-\alpha)\left(\Gamma(1-\delta)+1\right)}G_3\Big),$$

$$E_4 = \frac{\big(\Gamma(1-\alpha)+1\big)}{|\Lambda|(p-1)\Gamma(1-\alpha)}\Big(\frac{\mu_1\left|A_2\right|(p+1)+\lambda_2\left|B_1\right|}{\Gamma(p+2)} + \frac{|\Lambda|}{p\Gamma(p-1)}+\frac{|\Lambda|(p-1)\Gamma(1-\alpha)\big(\Gamma(1-\delta)+1\big)}{\big(\Gamma(1-\alpha)+1\big)\Gamma(1-\delta)}G_2\Big).$$

Theorem 4.1. *Assume that to condition* (H1) *and Theorem* 3.1 *hold. Then the solution of the system* (1.1) *is Hyers-Ulam stable.*

Proof. Let $(u,v) = \big((u_1,v_1),(u_2,v_2),...,(u_{19},v_{19})\big) \in X\times Y$ be solution of the inequalities given by

$$\begin{cases} \big|\left({}^{H}D^{q}+k^{H}D^{q-1}\right)u_i(t)-f_i\left(t,u_i(t),v_i(t),{}^{H}D^{\alpha}v_i(t)\right)\big|\le \xi_i,\\ t\in[1,e],\ \ i=1,2,...,19,\\ \big|\left({}^{H}D^{p}+k^{H}D^{p-1}\right)v_i(t)-g_i\left(t,v_i(t),u_i(t),{}^{H}D^{\delta}u_i(t)\right)\big|\le \zeta_i,\\ t\in[1,e],\ \ i=1,2,...,19,\end{cases}$$

and $(u^*,v^*) = \big((u_1^*,v_1^*),(u_2^*,v_2^*),...,(u_{19}^*,v_{19}^*)\big) \in X\times Y$ be the solution of the following system

$$\begin{cases} \left({}^{H}D^{q}+k^{H}D^{q-1}\right)u_i^*(t)=f_i\left(t,u_i^*(t),v_i^*(t),{}^{H}D^{\alpha}v_i^*(t)\right),\ t\in[1,e],\\ \left({}^{H}D^{p}+k^{H}D^{p-1}\right)v_i^*(t)=g_i\left(t,v_i^*(t),u_i^*(t),{}^{H}D^{\delta}u_i^*(t)\right),\\ u_i^*(1)=0,\quad v_i^*(1)=0,\\ \lambda_1u_i^*(e)-\lambda_2\displaystyle\int_1^e\frac{v_i^*(s)}{s}ds=K_1,\\ \mu_1v_i^*(e)-\mu_2\displaystyle\int_1^e\frac{u_i^*(s)}{s}ds=K_2,\quad i=1,2,...,19.\end{cases}\tag{4.5}$$

Then by Lemma 2.1, Theorem 3.1 and system (4.5) has a unique solution that can be written as

$$\begin{aligned} &u_i^*(t)\\ =&\frac{1}{\Lambda}\Big(t^{-k}\int_1^t s^{k-1}(\log s)^{q-2}ds\Big)\Big\{B_2\Big[K_1\\ &+\frac{\lambda_2}{\Gamma(p-1)}\int_1^e s^{-k-1}\Big(\int_1^s m^{k-1}\Big(\int_1^m\big(\log\frac{m}{r}\big)^{p-2}\\ &\times g_i\big(r,v_i^*(r),u_i^*(r),{}^{H}D^{\delta}u_i^*(r)\big)\frac{1}{r}dr\Big)dm\Big)ds\\ &-\frac{\lambda_1e^{-k}}{\Gamma(q-1)}\int_1^e s^{k-1}\Big(\int_1^s\big(\log\frac{s}{r}\big)^{q-2}f_i\big(r,u_i^*(r),v_i^*(r),{}^{H}D^{\alpha}v_i^*(r)\big)\frac{1}{r}dr\Big)ds\Big]\\ &+A_2\Big[\frac{\mu_1e^{-k}}{\Gamma(p-1)}\int_1^e s^{k-1}\Big(\int_1^s\big(\log\frac{s}{r}\big)^{p-2}g_i\big(r,v_i^*(r),u_i^*(r),{}^{H}D^{\delta}u_i^*(r)\big)\frac{1}{r}dr\Big)ds\\ &-K_2-\frac{\mu_2}{\Gamma(q-1)}\int_1^e s^{-k-1}\Big(\int_1^s m^{k-1}\Big(\int_1^m\big(\log\frac{m}{r}\big)^{q-2}\\ &\times f_i\big(r,u_i^*(r),v_i^*(r),{}^{H}D^{\alpha}v_i^*(r)\big)\frac{1}{r}dr\Big)dm\Big)ds\Big]\Big\}\\ &+\frac{t^{-k}}{\Gamma(q-1)}\int_1^t s^{k-1}\Big(\int_1^s\big(\log\frac{s}{r}\big)^{q-2}f_i\big(r,u_i^*(r),v_i^*(r),{}^{H}D^{\alpha}v_i^*(r)\big)\frac{1}{r}dr\Big)ds,\end{aligned}$$

and

$$\begin{aligned} &v_i^*(t)\\ =&\frac{1}{\Lambda}\Big(t^{-k}\int_1^t s^{k-1}(\log s)^{p-2}ds\Big)\Big\{A_1\Big[K_2\end{aligned}$$

$$
\begin{aligned}
&+\frac{\mu_2}{\Gamma(q-1)}\int_1^e s^{-k-1}\Big(\int_1^s m^{k-1}\Big(\int_1^m \big(\log\frac{m}{r}\big)^{q-2}\\
&\times f_i\big(r,u_i^*(r),v_i^*(r),^H D^\alpha v_i^*(r)\big)\frac{1}{r}dr\Big)dm\Big)ds\\
&-\frac{\mu_1 e^{-k}}{\Gamma(p-1)}\int_1^e s^{k-1}\Big(\int_1^s \big(\log\frac{s}{r}\big)^{p-2} g_i\big(r,v_i^*(r),u_i^*(r),^H D^\delta u_i^*(r)\big)\frac{1}{r}dr\Big)ds\Big]\\
&+B_1\Big[\frac{\lambda_1 e^{-k}}{\Gamma(q-1)}\int_1^e s^{k-1}\Big(\int_1^s \big(\log\frac{s}{r}\big)^{q-2} f_i\big(r,u_i^*(r),v_i^*(r),^H D^\alpha v_i^*(r)\big)\frac{1}{r}dr\Big)ds\\
&-K_1-\frac{\lambda_2}{\Gamma(p-1)}\int_1^e s^{-k-1}\Big(\int_1^s m^{k-1}\Big(\int_1^m \big(\log\frac{m}{r}\big)^{p-2}\\
&\times g_i\big(r,v_i^*(r),u_i^*(r),^H D^\delta u_i^*(r)\big)\frac{1}{r}dr\Big)dm\Big)ds\Big]\Big\}\\
&+\frac{t^{-k}}{\Gamma(p-1)}\int_1^t s^{k-1}\Big(\int_1^s \big(\log\frac{s}{r}\big)^{p-2} g_i\big(r,v_i^*(r),u_i^*(r),^H D^\delta u_i^*(r)\big)\frac{1}{r}dr\Big)ds.
\end{aligned}
$$

For simplicity of presentation, here we let

$$
\begin{aligned}
&h_i(t)=f_i\left(t,u_i(t),v_i(t),^H D^\alpha v_i(t)\right)-f_i\left(t,u_i^*(t),v_i^*(t),^H D^\alpha v_i^*(t)\right),\\
&z_i(t)=g_i\left(t,v_i(t),u_i(t),^H D^\delta u_i(t)\right)-g_i\left(t,v_i^*(t),u_i^*(t),^H D^\delta u_i^*(t)\right).
\end{aligned}
$$

Now, by Lemma 4.1, for $t\in[1,e]$, we have

$$
\begin{aligned}
&\big|u_i(t)-u_i^*(t)\big|\\
\le&\big|u_i(t)-\bar{u}_i(t)\big|+\big|\bar{u}_i(t)-u_i^*(t)\big|\\
\le&G_1\xi_i+G_2\zeta_i+\frac{1}{|\Lambda|}\Big(t^{-k}\int_1^t s^{k-1}(\log s)^{q-2}ds\Big)\Big\{\big|B_2\big|\\
&\times\Big[\frac{\lambda_2}{\Gamma(p-1)}\int_1^e s^{-k-1}\Big(\int_1^s m^{k-1}\Big(\int_1^m \big(\log\frac{m}{r}\big)^{p-2}\frac{\big|z_i(r)\big|}{r}dr\Big)dm\Big)ds\\
&+\frac{\lambda_1 e^{-k}}{\Gamma(q-1)}\int_1^e s^{k-1}\Big(\int_1^s \big(\log\frac{s}{r}\big)^{q-2}\frac{\big|h_i(r)\big|}{r}dr\Big)ds\Big]\\
&+\big|A_2\big|\Big[\frac{\mu_1 e^{-k}}{\Gamma(p-1)}\int_1^e s^{k-1}\Big(\int_1^s \big(\log\frac{s}{r}\big)^{p-2}\frac{\big|z_i(r)\big|}{r}dr\Big)ds\\
&+\frac{\mu_2}{\Gamma(q-1)}\int_1^e s^{-k-1}\Big(\int_1^s m^{k-1}\Big(\int_1^m \big(\log\frac{m}{r}\big)^{q-2}\frac{\big|h_i(r)\big|}{r}dr\Big)dm\Big)ds\Big]\Big\}\\
&+\frac{t^{-k}}{\Gamma(q-1)}\int_1^t s^{k-1}\Big(\int_1^s \big(\log\frac{s}{r}\big)^{q-2}\frac{\big|h_i(r)\big|}{r}dr\Big)ds\\
\le&G_1\xi_i+G_2\zeta_i+\frac{1}{|\Lambda|(q-1)}\Big(\frac{\lambda_1\big|B_2\big|}{\Gamma(q+1)}\\
&+\frac{\mu_2\big|A_2\big|}{\Gamma(q+2)}+\frac{|\Lambda|}{q\Gamma(q-1)}\Big)\|l_i\|\big(\|u_i-u_i^*\|_{X_i}+\|v_i-v_i^*\|_{Y_i}\big)\\
&+\frac{1}{|\Lambda|(q-1)}\Big(\frac{\lambda_2\big|B_2\big|}{\Gamma(p+2)}+\frac{\mu_1\big|A_2\big|}{\Gamma(p+1)}\Big)\|q_i\|\big(\|u_i-u_i^*\|_{X_i}+\|v_i-v_i^*\|_{Y_i}\big)\\
=&G_1\xi_i+G_2\zeta_i+\big(\|l_i\|G_1+\|q_i\|G_2\big)\big(\|u_i-u_i^*\|_{X_i}+\|v_i-v_i^*\|_{Y_i}\big),
\end{aligned}
$$

$$
\begin{aligned}
&\left|{}^H D^{\delta}u_i(t) -{}^H D^{\delta}u_i^*(t)\right|\\
\le& \left|{}^H D^{\delta}u_i(t) -{}^H D^{\delta}\bar{u}_i(t)\right| + \left|{}^H D^{\delta}\bar{u}_i(t) -{}^H D^{\delta}u_i^*(t)\right|\\
\le& \frac{G_1\xi_i + G_2\zeta_i}{\Gamma(1-\delta)} + \frac{1}{\Gamma(1-\delta)}\Big(t\frac{d}{dt}\Big)\int_1^t \big(\log\frac{t}{s}\big)^{-\delta}\frac{\left|{}^H D^{\delta}\bar{u}_i(t) -{}^H D^{\delta}u_i^*(t)\right|}{s}ds\\
\le& \frac{G_1\xi_i + G_2\zeta_i}{\Gamma(1-\delta)} + \frac{\|l_i\|G_1 + \|q_i\|G_2}{\Gamma(1-\delta)}\big(\|u_i-u_i^*\|_{X_i} + \|v_i - v_i^*\|_{Y_i}\big).
\end{aligned}
$$

Hence

$$
\begin{aligned}
&\left\|u_i(t) - u_i^*(t)\right\|_{X_i}\\
\le& \frac{\big(\Gamma(1-\delta)+1\big)}{\Gamma(1-\delta)}\big(G_1\xi_i + G_2\zeta_i\big)\\
&+\frac{\big(\Gamma(1-\delta)+1\big)}{\Gamma(1-\delta)}\big(\|l_i\|G_1 + \|q_i\|G_2\big)\big(\|u_i-u_i^*\|_{X_i} + \|v_i - v_i^*\|_{Y_i}\big).
\end{aligned}
$$

In a similar way, we have

$$
\begin{aligned}
|v_i(t) - v_i^*(t)| &\le |v_i(t) - \bar{v}_i(t)| + |\bar{v}_i(t) - v_i^*(t)|\\
&\le G_3\xi_i + G_4\zeta_i + \big(\|l_i\|G_3 + \|q_i\|G_4\big)\left(\|u_i-u_i^*\|_{X_i} + \|v_i - v_i^*\|_{Y_i}\right),
\end{aligned}
$$

and

$$
\begin{aligned}
&\left|{}^H D^{\delta}v_i(t) -{}^H D^{\alpha}v_i^*(t)\right|\\
\le& \frac{G_3\xi_i + G_4\zeta_i}{\Gamma(1-\alpha)} + \frac{\|l_i\|G_3 + \|q_i\|G_4}{\Gamma(1-\alpha)}\left(\|u_i-u_i^*\|_{X_i} + \|v_i - v_i^*\|_{Y_i}\right).
\end{aligned}
$$

Then, we obtain

$$
\begin{aligned}
&\|v_i(t) - v_i^*(t)\|_{Y_i}\\
\le& \frac{\big(\Gamma(1-\alpha)+1\big)}{\Gamma(1-\alpha)}\big(G_3\xi_i + G_4\zeta_i\big)\\
&+\frac{\big(\Gamma(1-\alpha)+1\big)}{\Gamma(1-\alpha)}\big(\|l_i\|G_3 + \|q_i\|G_4\big)\left(\|u_i-u_i^*\|_{X_i} + \|v_i - v_i^*\|_{Y_i}\right).
\end{aligned}
$$

Which gives

$$
\begin{aligned}
&\|(u_i,v_i)(t) - (u_i^*,v_i^*)(t)\|_{X_i\times Y_i}\\
\le& \frac{\Gamma(1-\alpha)\left(\Gamma(1-\delta)+1\right)G_1 + \Gamma(1-\delta)\left(\Gamma(1-\alpha)+1\right)G_3}{\Gamma(1-\delta)\Gamma(1-\alpha)}\xi_i\\
&+\frac{\Gamma(1-\alpha)\left(\Gamma(1-\delta)+1\right)G_2 + \Gamma(1-\delta)\left(\Gamma(1-\alpha)+1\right)G_4}{\Gamma(1-\delta)\Gamma(1-\alpha)}\zeta_i\\
&+\left[\frac{\left(1+\Gamma(1-\delta)\right)\left(\|l_i\|G_1 + \|q_i\|G_2\right)}{\Gamma(1-\delta)} + \frac{\left(1+\Gamma(1-\alpha)\right)\left(\|l_i\|G_3 + \|q_i\|G_4\right)}{\Gamma(1-\alpha)}\right]\\
&\times\left(\|u_i-u_i^*\|_{X_i} + \|v_i - v_i^*\|_{Y_i}\right),
\end{aligned}
$$

$$
\|(u,v)(t) - (u^*,v^*)(t)\|_{X\times Y}
$$

$$
\begin{aligned}
&= \sum_{i=1}^{19} \|(u_i, v_i)(t) - (u_i^*, v_i^*)(t)\|_{X_i \times Y_i} \\
&\le \sum_{i=1}^{19} \Big(\frac{\Gamma(1-\alpha)\left(\Gamma(1-\delta)+1\right) G_1 + \Gamma(1-\delta)\left(\Gamma(1-\alpha)+1\right) G_3}{\Gamma(1-\delta)\Gamma(1-\alpha)} \xi_i \\
&\quad + \frac{\Gamma(1-\alpha)\left(\Gamma(1-\delta)+1\right) G_2 + \Gamma(1-\delta)\left(\Gamma(1-\alpha)+1\right) G_4}{\Gamma(1-\delta)\Gamma(1-\alpha)} \zeta_i \\
&\quad + \Big[\frac{\left(1+\Gamma(1-\delta)\right)\left(\|l_i\| G_1 + \|q_i\| G_2\right)}{\Gamma(1-\delta)} + \frac{\left(1+\Gamma(1-\alpha)\right)\left(\|l_i\| G_3 + \|q_i\| G_4\right)}{\Gamma(1-\alpha)} \Big] \\
&\quad \times \left(\|u_i - u_i^*\|_{X_i} + \|v_i - v_i^*\|_{Y_i} \right) \Big).
\end{aligned}
$$

Hence

$$
\|(u,v)(t) - (u^*, v^*)(t)\|_{X \times Y} \le \frac{\sum_{i=1}^{19} \left(E_1 \xi + E_2 \zeta\right)}{1 - \sum_{i=1}^{19} \Delta_i}. \tag{4.6}
$$

Thus, we have derived that system (1.1) is Ulam-Hyers stable. □

Remark 4.4. Making $\omega_{f_1,f_2,...,f_k}(\xi) = \frac{E_1}{1-\sum_{i=1}^{19}\Delta_i}\xi$ and $\varpi_{f_1,f_2,...,f_k}(\zeta) = \frac{E_2}{1-\sum_{i=1}^{19}\Delta_i}\zeta$ in (4.6). We have $\omega_{f_1,f_2,...,f_k}(0) = 0$ and $\varpi_{f_1,f_2,...,f_k}(0) = 0$. Then, by Definition 4.2, we deduce that the fractional differential system (1.1) is generalized Ulam-Hyers stable.

Theorem 4.2. *Assume that to* (H1) *and Theorem* 3.1 *hold. Then the solution of the system* (2.1) *is Hyers-Ulam-Rassias stable.*

Proof. Define $\varrho(t) = \max_{t\in[1,e]}\{\varphi_i(t)\}$ and $\sigma(t) = \max_{t\in[1,e]}\{\psi_i(t)\}$, $i = 1, 2, ..., 19$.

$(u, v) = \big((u_1, v_1), (u_2, v_2), ..., (u_{19}, v_{19})\big) \in X \times Y$ be any solution of the inequalities (4.2) and $(u^*, v^*) = \big((u_1^*, v_1^*), (u_2^*, v_2^*), ..., (u_{19}^*, v_*^{19})\big) \in X \times Y$ be the unique solution of the system (1.1). Now, by using Lemma 4.2 and proceeding as in the proof of Theorem 4.1, we have

$$
\begin{aligned}
&|u_i(t) - u_i^*(t)| \\
&\le |u_i(t) - \bar{u}_i(t)| + |\bar{u}_i(t) - u_i^*(t)| \\
&\le \frac{1}{|\Lambda|(q-1)} \Big(\frac{\lambda_1 |B_2|(q+1) + \mu_2 |A_2|}{\Gamma(q+2)} \varphi_i(e) + \frac{|\Lambda|}{q\Gamma(q-1)} \varphi_i(t) \Big) \xi_i \\
&\quad + G_2 \psi_i(e) \zeta_i + \left(\|l_i\| G_1 + \|q_i\| G_2 \right) \left(\|u_i - u_i^*\|_{X_i} + \|v_i - v_i^*\|_{Y_i} \right),
\end{aligned}
$$

$$
\begin{aligned}
&\left| {}^H D^\delta u_i(t) - {}^H D^\delta u_i^*(t) \right| \\
&\le \frac{1}{|\Lambda|(q-1)\Gamma(1-\delta)} \Big(\frac{\lambda_1 |B_2|(q+1) + \mu_2 |A_2|}{\Gamma(q+2)} \varphi_i(e) + \frac{|\Lambda|}{q\Gamma(q-1)} \varphi_i(t) \Big) \xi_i \\
&\quad + \frac{G_2 \psi_i(e)}{\Gamma(1-\delta)} \zeta_i + \frac{\|l_i\| G_1 + \|q_i\| G_2}{\Gamma(1-\delta)} \left(\|u_i - u_i^*\|_{X_i} + \|v_i - v_i^*\|_{Y_i} \right),
\end{aligned}
$$

$$
\begin{aligned}
&|v_i(t) - v_i^*(t)| \\
&\le \frac{1}{|\Lambda|(p-1)} \Big(\frac{\mu_1 |A_1|(p+1) + \lambda_2 |B_1|}{\Gamma(p+2)} \psi_i(e) + \frac{|\Lambda|}{p\Gamma(p-1)} \psi_i(t) \Big) \zeta_i
\end{aligned}
$$

$$+G_3\varphi_i(e)\xi_i + (\|l_i\|G_3 + \|q_i\|G_4)\left(\|u_i - u_i^*\|_{X_i} + \|v_i - v_i^*\|_{Y_i}\right),$$

$$\begin{aligned}
&\left|{}^H D^\alpha v_i(t) -{}^H D^\alpha v_i^*(t)\right| \\
\leq\ & \frac{1}{|\Lambda|(p-1)\Gamma(1-\alpha)}\Big(\frac{\mu_1\big|A_1\big|(p+1)+\lambda_2\big|B_1\big|}{\Gamma(p+2)}\psi_i(e) + \frac{|\Lambda|}{p\Gamma(p-1)}\psi_i(t)\Big)\zeta_i \\
&+\frac{G_3\varphi_i(e)}{\Gamma(1-\alpha)}\xi_i + \frac{\|l_i\|G_3 + \|q_i\|G_4}{\Gamma(1-\alpha)}\left(\|u_i - u_i^*\|_{X_i} + \|v_i - v_i^*\|_{Y_i}\right).
\end{aligned}$$

Then, we get

$$\begin{aligned}
&\|u_i(t) - u_i^*(t)\|_{X_i} \\
\leq\ & \frac{\big(\Gamma(1-\delta)+1\big)}{|\Lambda|(q-1)\Gamma(1-\delta)}\Big(\frac{\lambda_1\left|B_2\right|(q+1)+\mu_2\left|A_2\right|}{\Gamma(q+2)}\varphi_i(e) + \frac{|\Lambda|}{q\Gamma(q-1)}\varphi_i(t)\Big)\xi_i \\
&+\frac{\big(\Gamma(1-\delta)+1\big)}{\Gamma(1-\delta)}\big(\|l_i\|G_1 + \|q_i\|G_2\big)\left(\|u_i - u_i^*\|_{X_i} + \|v_i - v_i^*\|_{Y_i}\right) \\
&+\frac{\big(\Gamma(1-\delta)+1\big)}{\Gamma(1-\delta)}G_2\psi_i(e)\zeta_i,
\end{aligned}$$

and

$$\begin{aligned}
&\|v_i(t) - v_i^*(t)\|_{Y_i} \\
\leq\ & \frac{\big(\Gamma(1-\alpha)+1\big)}{|\Lambda|(p-1)\Gamma(1-\alpha)}\Big(\frac{\mu_1\left|A_1\right|(p+1)+\lambda_2\left|B_1\right|}{\Gamma(p+2)}\psi_i(e) + \frac{|\Lambda|}{p\Gamma(p-1)}\psi_i(t)\Big)\zeta_i \\
&+\frac{\big(\Gamma(1-\alpha)+1\big)}{\Gamma(1-\alpha)}\big(\|l_i\|G_3 + \|q_i\|G_4\big)\left(\|u_i - u_i^*\|_{X_i} + \|v_i - v_i^*\|_{Y_i}\right) \\
&+\frac{\big(\Gamma(1-\alpha)+1\big)}{\Gamma(1-\alpha)}G_3\varphi_i(e)\xi_i.
\end{aligned}$$

Hence

$$\begin{aligned}
&\|(u,v)(t) - (u^*,v^*)(t)\|_{X\times Y} \\
=\ & \sum_{i=1}^{19}\|(u_i,v_i)(t) - (u_i^*,v_i^*)(t)\|_{X_i\times Y_i} \\
\leq\ & \sum_{i=1}^{19}\Bigg(\frac{\big(\Gamma(1-\delta)+1\big)}{|\Lambda|(q-1)\Gamma(1-\delta)}\Big(\frac{\lambda_1\left|B_2\right|(q+1)+\mu_2\left|A_2\right|}{\Gamma(q+2)}\varphi_i(e) \\
&+\frac{|\Lambda|}{q\Gamma(q-1)}\varphi_i(t) + \frac{|\Lambda|(q-1)\Gamma(1-\delta)\left(\Gamma(1-\alpha)+1\right)}{\Gamma(1-\alpha)\left(\Gamma(1-\delta)+1\right)}G_3\varphi_i(e)\Big)\xi_i \\
&+\frac{\big(\Gamma(1-\alpha)+1\big)}{|\Lambda|(p-1)\Gamma(1-\alpha)}\Big(\frac{\mu_1\left|A_1\right|(p+1)+\lambda_2\left|B_1\right|}{\Gamma(p+2)}\psi_i(e) + \frac{|\Lambda|}{p\Gamma(p-1)}\psi_i(t) \\
&+\frac{|\Lambda|(p-1)\Gamma(1-\alpha)\big(\Gamma(1-\delta)+1\big)}{\big(\Gamma(1-\alpha)+1\big)\Gamma(1-\delta)}G_2\psi_i(e)\Big)\zeta_i
\end{aligned}$$

$$+\Delta_i\big(\|u_i-u_i^*\|_{X_i}+\|v_i-v_i^*\|_{Y_i}\big)\Bigg),$$

$$\|(u,v)(t)-(u^*,v^*)(t)\|_{X\times Y}\leq\frac{\sum_{i=1}^{19}\big(E_3\varrho(t)\xi+E_4\sigma(t)\zeta\big)}{1-\sum_{i=1}^{19}\Delta_i}. \tag{4.7}$$

Therefore, we arrive at the conclusion that the system (1.1) is stable for Ulam-Hyers-Rassias. □

Remark 4.5. Taking $\xi=\zeta=1$ in (4.7), then by Definition 4.4, we conclude that the system (1.1) is stable for generalized Ulam-Hyers-Rassias .

5. Numerical simulation

In this section, an example is provided to illustrate the flexibility of these criteria and approximate graphs of the solution are given with iterative methods and numerical simulations.

Example 5.1. The glucose graph we studied in the system (1.1) can be extended to other types of graphs. For example, star graphs and chord bipartite graphs provide a theoretical basis for physics, computer networks and other fields. Here we only discuss the fractional differential system on the star graphs ($i=1,2$) Figure 11. We mainly study the existence of solutions of differential equations on each edge of the star graphs and the stability of Ulam.

$$\begin{cases}\left({}^HD^{\frac{3}{2}}+2{}^HD^{\frac{1}{2}}\right)u_1(t)=\dfrac{1}{30t}\left(\sin(u_1(t))+\sin(v_1(t))+\left|{}^HD^{\frac{1}{2}}v_1(t)\right|\right)+12+t,\\ \left({}^HD^{\frac{15}{8}}+2{}^HD^{\frac{7}{8}}\right)v_1(t)=\dfrac{\sin(u_1(t))+\cos\left({}^HD^{\frac{1}{3}}u_1(t)\right)+|v_1(t)|}{2t^2+15}+\cos(t+1),\\ \left({}^HD^{\frac{3}{2}}+2{}^HD^{\frac{1}{2}}\right)u_2(t)=\dfrac{1}{(1+t)^5}\left(\dfrac{|u_2(t)|}{1+|u_2(t)|}+\dfrac{|v_2(t)|}{1+|v_2(t)|}+\sin\left|{}^HD^{\frac{1}{2}}v_2(t)\right|\right)\\ \qquad +1,\\ \left({}^HD^{\frac{15}{8}}+2{}^HD^{\frac{7}{8}}\right)v_2(t)=\dfrac{1}{(5+t)^2}\left(\dfrac{|v_2(t)|}{1+|v_2(t)|}+\dfrac{|u_2(t)|}{1+|u_2(t)|}+\cos\left|{}^HD^{\frac{1}{3}}u_2(t)\right|\right)\\ \qquad +t,\\ u_1(1)=0,\quad v_1(1)=0,\\ u_2(1)=0,\quad v_2(1)=0,\\ 4u_1(e)-0.2\displaystyle\int_1^e\frac{v_1(s)}{s}ds=3,\\ 4u_2(e)-0.2\displaystyle\int_1^e\frac{v_2(s)}{s}ds=3,\\ 2v_1(e)-0.1\displaystyle\int_1^e\frac{u_1(s)}{s}ds=5,\\ 2v_2(e)-0.1\displaystyle\int_1^e\frac{u_2(s)}{s}ds=5,\ i=1,2,...,19.\end{cases} \tag{5.1}$$

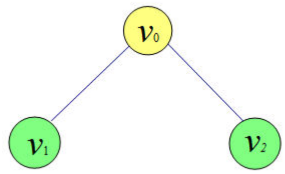

Figure 3. A sketch of the star graphs

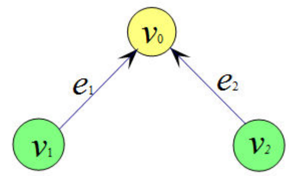

Figure 4. A sketch of the directed star graphs

Corresponding to the system (1.1), we obtain

$$p = \frac{15}{8},\ q = \frac{3}{2},\ k = 2,\ \alpha = \frac{1}{2},\ \delta = \frac{1}{3},\ \lambda_1 = 4,$$

$$\lambda_2 = 0.2,\ \mu_1 = 2, \mu_2 = 0.1,\ K_1 = 3,\ K_2 = 5.$$

We establish coordinate systems with v_1 and v_2 as coordinate origin respectively on the star graphs with 2 edges Figure 12, where (u_1, v_1) is a set of coupled solution of system (5.1) on $\overrightarrow{v_1 v_0}$, $t \in [1, e]$. By the same token, (u_2, v_2) is a set of coupled solutions of system (5.1) on $\overrightarrow{v_2 v_0}$, $t \in [1, e]$.

Let

$$l_1 = l_2 = |e_1| = |e_2| = e.$$

For $t \in [1, e]$ and $u, v, z, u_1, v_1, z_1 \in \mathbb{R}$, it is clear that, we have

$$|f_1(t, u_1, v_1, z_1) - f_1(t, u, v, z)| \leq \frac{1}{30t}(|u_1 - u| + |v_1 - v| + |z_1 - z|),$$

$$|f_2(t, u_1, v_1, z_1) - f_1(t, u, v, z)| \leq \frac{1}{(1+t)^5}(|u_1 - u| + |v_1 - v| + |z_1 - z|),$$

$$|g_1(t, u_1, v_1, z_1) - g_1(t, u, v, z)| \leq \frac{1}{2t^2 + 15}(|u_1 - u| + |v_1 - v| + |z_1 - z|),$$

$$|f_2(t, u_1, v_1, z_1) - f_1(t, u, v, z)| \leq \frac{1}{(5+t)^2}(|u_1 - u| + |v_1 - v| + |z_1 - z|),$$

so condition (H_1) holds, we have

$$l_1(t) = \frac{1}{30t},\ l_2(t) = \frac{1}{(1+t)^5},\ q_1(t) = \frac{1}{2t^2 + 15},\ q_2(t) = \frac{1}{(5+t)^2},$$

$$A_1 = 2.56,\ A_2 = -0.2306,\ B_1 = -0.1364,\ B_2 = 0.9334,$$

$$G_1 = 3.1403,\ A_2 = 0.2496,\ B_1 = 0.2363,\ B_2 = 1.9651,$$

$$\sum_{i=1}^{2} \Delta_i = 0.6802 < 1.$$

It follows from Theorem 3.1 that system (5.1) has a unique solution on $[1, e]$. In addition,we know that

$$S_1 = 2.3881,\ S_2 = 1.4057,$$

we can get $\sum_{i=1}^{2}\Theta_i = 0.5200 < 1$.

It follows from Theorem 3.2 that system (5.1) has at least one solution on $[1,e]$. System 1.1 is not only Ulam-Hyers stable, but also generalized Ulam-Hyers stable, since that satisfies condition (H_1) and Theorem 3.1. Furthermore, the Ulam-Hyers-Rassias and generalized Ulam-Hyers-Rassias stable are satisfied. Finally, the fractional differential system (5.1) approximate solution and simulate iterative process curve are provided using the iterative method and numerical simulation. Following is our iteration sequence with initial value $u_{1,0} = v_{i,0} = 0$, $i = 1, 2$,

$$\begin{aligned}
u_{i,n+1}(t) =& a_1^{(i)} t^{-k}\int_1^t s^{k-1}(\log s)^{q-2}ds \\
&+ t^{-k}\int_1^t s^{k-1\,H}I^{q-1} f_i\left(s, u_{i,n}(s), v_{i,n}(s), {}^H D^{\alpha} v_{i,n}(s)\right)ds, \\
v_{i,n+1}(t) =& b_1^{(i)} t^{-k}\int_1^t s^{k-1}(\log s)^{p-2}ds \\
&+ t^{-k}\int_1^t s^{k-1\,H}I^{p-1} g_i\left(s, v_{i,n}(s), u_{i,n}(s), {}^H D^{\delta} u_{i,n}\right)ds,
\end{aligned}$$

$$\begin{aligned}
{}^H D^{\delta} u_{i,n+1}(t) =& \frac{1}{\Gamma(1-\delta)}\Big(t\frac{d}{dt}\Big)\int_1^t \big(\log\frac{t}{s}\big)^{-\delta} f_i \\
&\times \left(s, u_{i,n}(s), v_{i,n}(s), {}^H D^{\alpha} v_{i,n}(s)\right)\frac{1}{s}ds,
\end{aligned}$$

$$\begin{aligned}
{}^H D^{\alpha} v_{i,n+1}(t) =& \frac{1}{\Gamma(1-\alpha)}\Big(t\frac{d}{dt}\Big)\int_1^t \big(\log\frac{t}{s}\big)^{-\alpha} g_i \\
&\times \left(s, v_{i,n}(s), u_{i,n}(s), {}^H D^{\alpha} u_{i,n}(s)\right)\frac{1}{s}ds.
\end{aligned}$$

Figure 5 and Figure 7 show the iterative process of a group of coupled solution (u_1, v_1) of system (5.1) on $\overrightarrow{v_1v_0}$, and Figure 6 and Figure 8 are the approximate graphs solved on $\overrightarrow{v_1v_0}$ after 20 iterations. Figure 9 and Figure 11 show the iterative process of a group of coupled solution (u_2, v_2) of system (5.1) on $\overrightarrow{v_2v_0}$, and Figure 10 and Figure 12 are the approximate graphs solved on $\overrightarrow{v_2v_0}$ after 20 iterations.

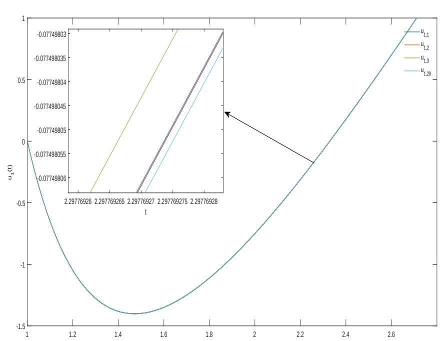

Figure 5. Iterative process of u_1

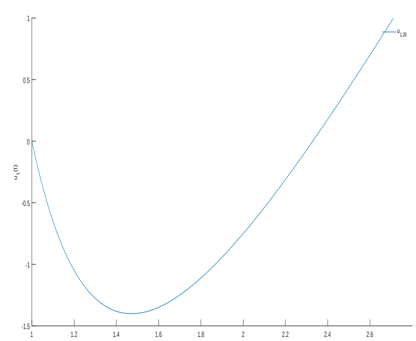

Figure 6. Approximate solution of u_1

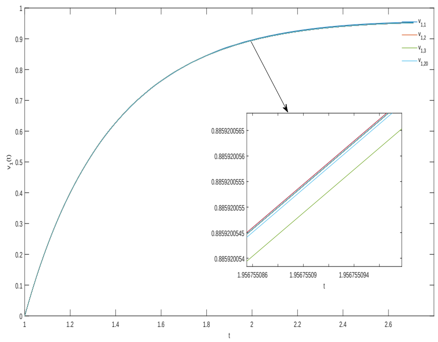

Figure 7. Iterative process of v_1

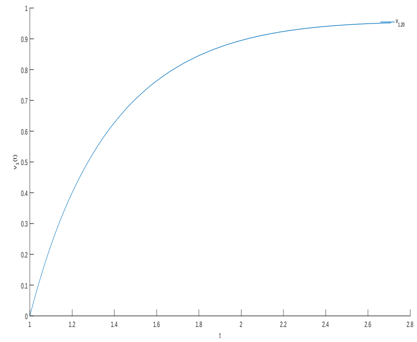

Figure 8. Approximate solution of v_1

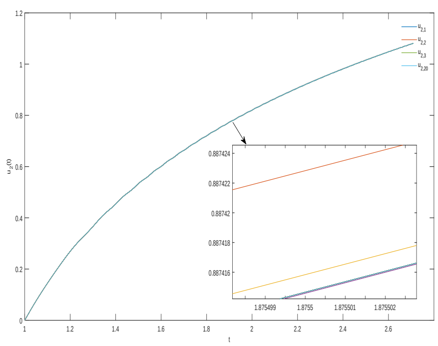

Figure 9. Iterative process of u_2

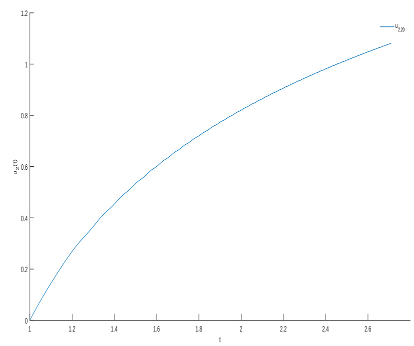

Figure 10. Approximate solution of u_2

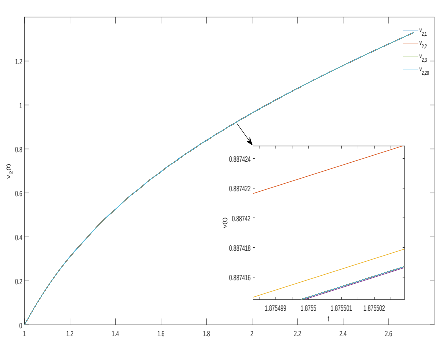

Figure 11. Iterative process of v_2

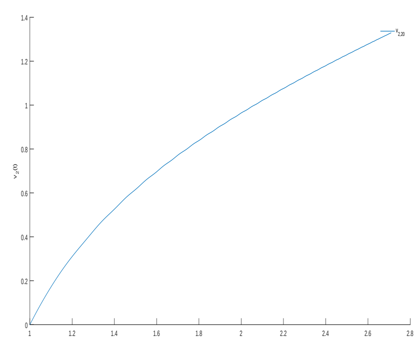

Figure 12. Approximate solution of v_2

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