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UNIVALENCE CONDITIONS AND RADIUS PROBLEMS FOR HARMONIC DIFFERENTIAL OPERATORS*

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Abstract This article mainly studies univalence condition, the radius problem of fully starlike (fully convex) and uniformly starlike (uniformly convex) for the harmonic mapping differential operator under specific coefficient conditions. Firstly, several criteria for the univalence of harmonic differential operator terms are obtained, followed by the fully starlike and fully convex radius of the harmonic differential operator $D[f] \in \mathcal{K}_H^2(\lambda)$. Next, the radius of uniformly starlike and uniformly convex of the harmonic differential operator is obtained. Finally, the radius of uniformly starlike and uniformly convex of the harmonic mapping convolution differential operator is obtained.

Keywords Harmonic differential operator, univalence, radius of uniformly starlike, radius of uniformly convex, convolution of harmonic mappings.

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1. Introduction

A complex-valued function $f = u+iv$ defined on the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ is harmonic if both u and v are real-valued harmonic functions in $\mathbb{D}$. The canonical decomposition of f is given by $f = h + \overline{g}$, where h and g are analytic functions defined on $\mathbb{D}$, and they are respectively called as the analytic and co-analytic parts of f. Denote by $\mathcal{H}$ the class of all complex-valued harmonic functions f in $\mathbb{D}$ normalized $f(0) = f_z(0) - 1 = 0$. Let $\mathcal{S}_H$ denote the subclass of univalent and sense-preserving harmonic mappings $f = h + \overline{g}$ in $\mathcal{H}$. Moreover, let $\mathcal{S}_H^0$ denote the subclass of functions $f \in \mathcal{S}_H$ with $f_{\overline{z}}(0) = 0$, where

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad \text{and} \quad g(z) = \sum_{n=2}^{\infty} b_n z^n. \tag{1.1}$$

Lewy [15] proved that a harmonic mapping $f = h + \overline{g}$ is locally univalent and sense-preserving in $\mathbb{D}$ if and only if $J_f(z) = |h'(z)|^2 - |g'(z)|^2 > 0$ for $z \in \mathbb{D}$.

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This is equivalent to the condition that $|\omega(z)| = |g'(z)/h'(z)| < 1$. The mapping $\omega(z) = g'(z)/h'(z)$ is known as the dilatation of $f = h + \overline{g}$. The subclasses of $\mathcal{S}_H$ are represented by the symbols $\mathcal{S}_H^*$, $\mathcal{K}_H$ and $\mathcal{C}_H$, which consist of starlike, convex and close-to-convex harmonic mappings, respectively just as $\mathcal{S}^*$, $\mathcal{K}$ and $\mathcal{C}$ are subclasses of $\mathcal{S}$ mapping $\mathbb{D}$ onto their respective domains. Define $\mathcal{S}_H^{*0} = \mathcal{S}_H^* \cap \mathcal{S}_H^0$, $\mathcal{K}_H^0 = \mathcal{K}_H \cap \mathcal{S}_H^0$ and $\mathcal{C}_H^0 = \mathcal{C}_H \cap \mathcal{S}_H^0$.

According to Clunie and Sheil-Small [8], it was conjectured that if $f = h + \overline{g} \in \mathcal{S}_H^0$, then the Taylor coefficients of the series of h and g satisfy the inequalities

$$|a_n| \le \frac{(2n+1)(n+1)}{6} \quad \text{and} \quad |b_n| \le \frac{(2n-1)(n-1)}{6} \qquad (b_1 = 0,\ n \ge 2). \tag{1.2}$$

This coefficient conjecture remains an open problem for the full class $\mathcal{S}_H^0$. However, previous research has validated it for some subclasses of $\mathcal{S}_H^0$, such as typically real harmonic mappings [8], convex in one direction harmonic mappings [23], starlike harmonic mappings [23] and close-to-convex harmonic mappings [24]. The equalities occur if and only if harmonic Koebe mapping $K(z) = H(z) + \overline{G(z)}$, where

$$H(z) = \frac{z - \frac{1}{2}z^2 + \frac{1}{6}z^3}{(1-z)^3} \quad \text{and} \quad G(z) = \frac{\frac{1}{2}z^2 + \frac{1}{6}z^3}{(1-z)^3} \quad (z \in \mathbb{D}).$$

Moreover, they demonstrated that if $f = h + \overline{g} \in \mathcal{K}_H^0$, then

$$|a_n| \le \frac{n+1}{2} \quad \text{and} \quad |b_n| \le \frac{n-1}{2} \quad (b_1 = 0,\ n \ge 2). \tag{1.3}$$

Equality occurs for the half-plane harmonic mapping $L(z) = M(z) + \overline{N(z)}$ given by

$$M(z) = \frac{z - \frac{1}{2}z^2}{(1-z)^2} \quad \text{and} \quad N(z) = \frac{-\frac{1}{2}z^2}{(1-z)^2} \quad (z \in \mathbb{D}). \tag{1.4}$$

The field of complex analysis has shown significant interest in the question of univalence for certain classes of complex mappings. In previous studies, a novel set of analytic univalent mappings was introduced by employing a generalized Sălăgean operator [3], leading to some notable findings regarding univalence and convexity. Recent research, including a survey conducted by Reddy [22], has established sufficient conditions for the univalence of analytic functions utilizing the differential operator.

Abdulhadi [1] generalized the differential operator of harmonic mapping $f(z) = h(z) + \overline{g(z)} \in \mathcal{S}_H^0$, by defining

$$D[f](z) = zf_z(z) - \overline{z}f_{\overline{z}}(z) = h_1(z) + \overline{g_1(z)}, \tag{1.5}$$

and with the dilatation

$$\omega_1(z) = \frac{g_1'(z)}{h_1'(z)}.$$

Similar to the role played by the analytic differential operator $zf'(z)$, the operators $D[f]$ also hold significant importance in harmonic mappings. In 1915, Alexander [2] proved that if $f \in \mathcal{S}$, then $f(z) \in \mathcal{C}$ if and only if $zf'(z) \in \mathcal{S}^*$. An application of it to harmonic mappings was presented by Sheil-Small [23] in 1990, as mentioned below.

Theorem 1.1. *If $f = h + \overline{g} \in \mathcal{H}$ is univalent, and has a starlike range and if H and G are the analytic functions on $\mathbb{D}$ defined by*

$$zH'(z) = h(z),\ zG'(z) = -g(z),\ H(0) = G(0) = 0,$$

then $F = H + \overline{G}$ is univalent and has a convex range.

Theorem 1.2. (Theorem 3, [18]) *Let $f = h+\overline{g}$ be a harmonic mapping in $\mathbb{D}$, where $h \in \mathcal{S}^*(\beta)$ for some $\beta \in (-1/2, 0]$, $g(0) = 0$ and $g'(z) = \omega(z)h'(z)$ in $\mathbb{D}$ for some $\omega : \mathbb{D} \to \mathbb{D}$ satisfying the condition $|\omega(z)| < cos(\beta\pi)$ for $z \in \mathbb{D}$. If H and G are related by*

$$zH'(z) = h(z),\ zG'(z) = -g(z),\ H(0) = G(0) = 0, \tag{1.6}$$

then for each $|\lambda| = 1$, the harmonic function $F_\lambda = H + \lambda\overline{G}$ is sense-preserving and close-to-convex mapping in $\mathbb{D}$.

Theorem 1.3. (Theorem 4, [20]) *Suppose that $f = h + \overline{g}$ is a sense-preserving normalized convex mapping, and Df is sense-preserving in $\mathbb{D}$. If the functions H and G are related by the relations* (1.6)*, then the harmonic function $F = H + \overline{G}$ is univalent sense-preserving and starlike in $\mathbb{D}$.*

As stated in Theorem 1.1, considers the case of a harmonic mapping in $\mathbb{D}$ given by $f = h + \overline{g}$. It provides the insight that F has a convex range under certain conditions. On the other hand, Theorem 1.2 and 1.3 explored similar setting but focused on close-to-convex and starlike range. The comparison of Theorem 1.1 with Theorem 1.2 and Theorem 1.3 illuminates the unique contributions of our study, offering a comprehensive understanding of convex mapping and starlike mapping.

Convexity and starlikeness are the main research objects in geometric function theory of complex variable that every starlike (convex) function maps each disk $\{z : |z| = r < 1\}$ onto a starlike (convex) domain. In [14], David Kalaj et al. discussed the univalent and starlikeness radius of harmonic functions when the coefficients of the series satisfy conditions (1.2) and (1.3). It was discussed in [7] that the failure of the hereditary properties of starlike and convex harmonic mappings gave rise to the concepts of fully starlike of order α and fully convex of order α.

Definition 1.1. A harmonic mapping f of the unit disk $\mathbb{D}$, with $f(0) = 0$ is said to be *fully starlike of order* α $(0 \le \alpha < 1)$ if it maps every circle $|z| = r < 1$ in a one-to-one manner onto a curve that bounds a domain starlike with respect to the origin satisfying

$$\frac{\partial}{\partial\theta}\left(\arg f(re^{i\theta})\right) > \alpha, \quad 0 \le \theta < 2\pi, 0 < r < 1. \tag{1.7}$$

If $\alpha = 0$, then f is *fully starlike.*

Definition 1.2. A harmonic mapping f of the unit disk $\mathbb{D}$, with $f(0) = 0$ is said to be *fully convex of order* α $(0 \le \alpha < 1)$ if it maps every circle $|z| = r < 1$ in a one-to-one manner onto a curve that bounds a domain convex with respect to the origin satisfying

$$\frac{\partial}{\partial\theta}\left(\arg\left\{\frac{\partial}{\partial\theta}f(re^{i\theta})\right\}\right) > \alpha, \quad 0 \le \theta < 2\pi,\ 0 < r < 1. \tag{1.8}$$

If $\alpha = 0$, then f is *fully convex.*

Let $\mathcal{FS}_H^*(\alpha)$ denote the subclass of $\mathcal{S}_H^*$ consisting of *fully starlike of order* α $(0 \le \alpha < 1)$, with $\mathcal{FS}_H^* := \mathcal{FS}_H^*(0)$. Let $\mathcal{FK}_H(\alpha)$ denote the subclass of $\mathcal{K}_H$ consisting of *fully convex of order* α $(0 \le \alpha < 1)$, with $\mathcal{FK}_H := \mathcal{FK}_H(0)$. In 1999, Jahangiri [13] obtained the sufficient conditions that $f \in \mathcal{FS}_H^*(\alpha)$ as follows.

Theorem 1.4. (Theorem 1, [13]) *Let $f = h + \overline{g}$ be given by* (1.1)*. Furthermore, let*

$$\sum_{n=1}^{\infty}\left(\frac{n-\alpha}{1-\alpha}|a_n| + \frac{n+\alpha}{1-\alpha}|b_n|\right) \le 2,$$

where $a_1 = 1, b_1 = 0, 0 \le \alpha < 1$. Then f is harmonic univalent in $\mathbb{D}$ and $f \in \mathcal{FS}_H^*(\alpha)$.

In the original research presented in [21], the following function denoted as

$$C_H^1 = \{f = h + \overline{g} \in \mathcal{S}_H : \operatorname{Re} f_z(z) > |f_{\overline{z}}(z)| \text{ in } \mathbb{D}\}$$

was extensively studied. In [17], the authors obtained new results by varying the parameter values, expanding upon the initial work presented in [21]. Specifically, they derived novel results concerning the following two subclasses of harmonic mappings.

$$\mathcal{K}_H^1(\lambda) = \{f = h + \overline{g} \in \mathcal{H}, \operatorname{Re}(h_\lambda'(z)) > |g_\lambda'(z)|, z \in \mathbb{D}\},$$

where $h_\lambda(z)$=$(h * \phi_\lambda)(z)$, $g_\lambda(z)$=$(g * \phi_\lambda)(z)$, and

$$\phi_\lambda(z) = (1-\lambda)\frac{z}{1-z} + \lambda\frac{z}{(1-z)^2} = \sum_{n=1}^{\infty}(1-\lambda+n\lambda)z^n.$$

Similarly, they defined

$$\mathcal{K}_H^2(\lambda) = \{f = h + \overline{g} \in \mathcal{H}, |h_\lambda'(z) - 1| < 1 - |g_\lambda'(z)|, z \in \mathbb{D}\}.$$

The authors abtained the sufficient condition of $f \in \mathcal{K}_H^2(\lambda)$ as follows.

Theorem 1.5. (Theorem 3.1, [17]) *Let $f = h + \overline{g}$ be given by* (1.1) *with $J_f(0) = 1 - |b_1|^2 > 0$. If*

$$\sum_{n=2}^{\infty}(1-\lambda+\lambda n)n(|a_n| + |b_n|) \le 1 - |b_1| \quad (\lambda \ge 0) \tag{1.9}$$

holds, then $f \in \mathcal{K}_H^2(\lambda)$.

However, Brown [6] indicated that it is not always correct that $f \in \mathcal{S}^*$ maps each disk $|z - z_0| < \rho < 1 - |z_0|$ onto a domain with respect to $f(z_0)$. He did clearly reveal that the result is accurate for each $f \in \mathcal{S}$ and for all sufficiently small disks in $\mathbb{D}$. This stimulates the definition of uniformly starlike mappings, though it was irrelevant to the work of Brown [6]. Section 4 is concerned with the radius problem of uniformly starlikeness and uniformly convexity of $D[f]$. The concepts of uniformly convex mapping class and uniformly starlike mapping class are introduced in [12], as defined below.

Definition 1.3. A locally univalent function $f = h + \overline{g}$ is said to be *uniformly starlike* in the unit disk $\mathbb{D}$, if f is fully starlike in $\mathbb{D}$, and maps every circular arc γ contained in $\mathbb{D}$ with center ζ also in $\mathbb{D}$, to the arc $f(\gamma)$ which is starlike with respect to $f(\zeta)$.

Definition 1.4. A locally univalent function $f = h + \overline{g}$ is said to be *uniformly convex* in the unit disk $\mathbb{D}$, if f is fully convex in $\mathbb{D}$, and maps every circular arc γ contained in $\mathbb{D}$ with center ζ also in $\mathbb{D}$, to the arc $f(\gamma)$ which is convex with respect to $f(\zeta)$.

Let $\mathcal{US}_H^*$ (respectively $\mathcal{US}_H^{*0}$) denote the class of all such functions in $f \in \mathcal{S}_H$ (respectively $f \in \mathcal{S}_H^0$). Clearly $\mathcal{US}_H^* \subset \mathcal{FS}_H^*$. Let $\mathcal{UK}_H$ (respectively $\mathcal{UK}_H^0$) denote the class of all such mappings in $f \in \mathcal{K}_H$ (respectively $f \in \mathcal{K}_H^0$). Some sufficient conditions for mappings in $\mathcal{H}$ belong to $\mathcal{US}_H^{*0}$ and $\mathcal{UK}_H^0$ are respectively established in the following two lemmas.

Lemma 1.1. (Lemma 1.3, [10]) *Let $f = h + \overline{g}$, where h and g are given by* (1.1), *and coefficients satisfy the condition:*

$$\sum_{n=2}^{\infty} n(|a_n| + |b_n|) \le \frac{1}{2}.$$

Then $f \in \mathcal{US}_H^{*0}$.

Lemma 1.2. (Lemma 1.4, [10]) *Let $f = h + \overline{g}$, where h and g are given by* (1.1), *and coefficients satisfy the condition:*

$$\sum_{n=2}^{\infty} n(2n-1)(|a_n| + |b_n|) \le 1.$$

Then $f \in \mathcal{UK}_H^0$.

The convolution of harmonic mappings is a generalized form of convolution of analytic functions. For two harmonic mappings

$$f = h + \overline{g} = z + \sum_{n=2}^{\infty} a_n z^n + \overline{\sum_{n=1}^{\infty} b_n z^n},$$

and

$$F = H + \overline{G} = z + \sum_{n=2}^{\infty} A_n z^n + \overline{\sum_{n=1}^{\infty} B_n z^n},$$

their convolution in a simply connected domain is defined as

$$f * F = h * H + \overline{g * G} = z + \sum_{n=2}^{\infty} a_n A_n z^n + \overline{\sum_{n=1}^{\infty} b_n B_n z^n}.$$

Harmonic convolutions was studied in [5,9,11,16]. In fact, the convolution of two harmonic mappings may not keep their original properties, such as univalence and convexity, which make the convolution of harmonic mappings widely studied. In 2012, Dorff [9] considered the half-plane mapping L in $\mathcal{K}_H^0 \subset \mathcal{K}_H$ as given by (1.4). However, the coefficients of the product $L*L$ turn out to be too large for the product to be in $\mathcal{K}_H$. The image of the unit disk $\mathbb{D}$ under $L * L$ is $\mathbb{C} \setminus (-\infty, -1/4]$, which is not a convex domain. Nevertheless, (Theorem 3, [9]) shows that $L * L \in \mathcal{S}_H^{*0}$. In [19], the starlikeness radius of the family

$$\mathcal{G} = \left\{ f = h + \overline{g} \in \mathcal{H} : |a_n| \le \frac{(n+1)^2}{4} \text{ and } |b_n| \le \frac{(n-1)^2}{4} \text{ for } n \ge 1 \right\}$$

is determined, and it is shown to be $r_0 \approx 0.129831$, which is also the radius of starlikeness of $\mathcal{G}$. Furthermore, the radius of convexity of $\mathcal{G}$ is $s_0 \approx 0.0712543$. This result demonstrates that if $f, g \in \mathcal{K}_H^0$, then $f * g$ is univalent and convex for $|z| < s_0 \approx 0.0712543$.

The remainder of this paper is organized as follows. Section 2 establishes different types of sufficient conditions for univalence of harmonic differential operator. In Section 3, we investigate the fully starlike radius and fully convex radius of differential operators for a family of harmonic mappings $\mathcal{K}_H^2(\lambda)$, where $\lambda \geq 0$. Section 4 delves into the radius of uniformly starlikeness and uniformly convexity for harmonic mappings involving the differential operator. Finally, in Section 5, we explore the radius of uniformly starlikeness and uniformly convexity for differential operators after harmonic convolutions.

2. Univalent criterion

In this section, we establish different types of sufficient conditions for univalence of harmonic mappings involving the differential operator. We need the following lemmas before solving the problems related to univalence of harmonic mappings defined by differential operator $D[f]$. Avkhadiev [4] et al. obtained Becker type univalence conditions for locally univalent harmonic mappings defined in the unit disk.

Lemma 2.1. (Theorem 1, [4]) *Let h and g be mappings holomorphic in the unit disc $\mathbb{D}$ and satisfying the following restrictions: $h'(z) \neq 0$ and $|\omega(z)| < 1$ at any point $z \in \mathbb{D}$, where $\omega(z) = g'(z)/h'(z)$. If*

$$|\omega(z)| + (1 - |z|^2) \left| \frac{zh''(z)}{h'(z)} \right| < 1, \quad \forall z \in \mathbb{D},$$

then the mapping $f = h + \overline{g}$ is univalent on $\mathbb{D}$.

For the class of univalent mappings $f : \mathbb{D} \to \mathbb{C}$, the following Becker's condition holds:

$$\left(1 - |z|^2\right) \left| \frac{f''(z)}{f'(z)} \right| \leq 6, \quad \forall z \in \mathbb{D}.$$

This well-known inequality is a consequence of the classical result by L. Bieberbach for $f \in \mathcal{S}$. Next, F. G. Avkhadiev [4] considered a generalization of Becker's condition to the case of harmonic mappings defined in a half-plane.

Lemma 2.2. (Theorem 2, [4]) *Suppose that the mappings h and g are holomorphic in the half-plane $\Omega_+ = \{z \in \mathbb{C} : x = \mathrm{Re}(z) > 0\}$ and satisfy the conditions: $h'(z) \neq 0$ and $|\omega(z)| < 1$ at any point $z \in \Omega_+$, where $\omega(z) := g'(z)/h'(z)$. If*

$$|\omega(z)| + 2x \left| \frac{h''(z)}{h'(z)} \right| \leq 1, \quad \forall z = x + iy \in \Omega_+,$$

then the mapping $f = h + \overline{g}$ is univalent on $\mathbb{D}$.

Lemma 2.3. (Theorem 5, [4]) *Let h and g be mappings holomorphic in the unit disc $\mathbb{D}$. Suppose that q, m and M are positive constants satisfying the inequalities: $0 \leq q < 1$, $1 < M/m \leq \exp\{\pi(1-q)/2\}$. If*

$$m \leq |h'(z)| \leq M, \quad |g'(z)/h'(z)| \leq q$$

at any point $z \in \mathbb{D}$, then the mapping $f = h + \overline{g}$ is univalent on the domain $\mathbb{D}$.

Now, we obtain the conditions on the coefficients of the harmonic mapping $f = h + \overline{g} \in \mathcal{S}_H^0$ such that $D[f]$ is univalent and $D[f] \in \mathcal{FS}_H^*(\alpha)$.

By the representation of f, the function $F = D[f]$ is harmonic in $\mathbb{D}$. So, we may apply Theorem 1.4 with the function F in place of f and this observation gives the desired result.

Theorem 2.1. *Let $f = h + \overline{g}$ be given by (1.1). Furthermore, let*

$$\sum_{n=1}^{\infty}\left(\frac{n^2-\alpha}{1-\alpha}|a_n| + \frac{n^2+\alpha}{1-\alpha}|b_n|\right) \le 2 \tag{2.1}$$

and $0 \le \alpha < 1$. Then $D[f]$ is harmonic univalent in $\mathbb{D}$ and $D[f] \in \mathcal{FS}_H^*(\alpha)$.

Theorem 2.2. *Let $D[f] = h_1 + \overline{g_1}$ be given by (1.5), and satisfy the following restrictions: $\sum_{n=2}^{\infty} n^2|a_n| < 1$, $h_1'(z) \neq 0$, and $|\omega_1(z)| < 1$ at any point $z \in \mathbb{D}$, where $\omega_1(z) = g_1'(z)/h_1'(z)$. If*

$$\sum_{n=2}^{\infty} n^2(2n-1)|a_n| + \sum_{n=1}^{\infty} n^2|b_n| \le 1, \tag{2.2}$$

then $D[f](z)$ is univalent on $\mathbb{D}$.

Theorem 2.3. *Let $D[f] = h_1 + \overline{g_1}$ be given by (1.5), h_1 and g_1 are holomorphic in the half-plane $\Omega_+ = \{z \in \mathbb{C} : x = \mathrm{Re}(z) > 0\}$ and satisfy the conditions: $\sum_{n=2}^{\infty} n^2|a_n| < 1$, $h_1'(z) \neq 0$ and $|\omega_1(z)| < 1$ at any point $z \in \Omega_+$, where $\omega_1(z) = g_1'(z)/h_1'(z)$. If*

$$\sum_{n=1}^{\infty} n^2|b_n| + \sum_{n=2}^{\infty} n^2\left(2x(n-1)+1\right)|a_n| \le 1, \tag{2.3}$$

then $D[f] = h_1 + \overline{g_1}$ is univalent on $\mathbb{D}$.

Proof. In order to show that $D[f]$ is univalent on $\mathbb{D}$, by Lemma 2.2, it is sufficient to show that

$$|\omega_1(z)| + 2x\left|\frac{h_1''(z)}{h_1'(z)}\right| \le 1, \quad \forall z = x+iy \in \Omega_+, \tag{2.4}$$

substituting $\omega_1(z), h_1'(z), h_1''(z)$ into (2.4), we have

$$\begin{aligned}
&\left|\frac{-\sum_{n=1}^{\infty} n^2 b_n z^{n-1}}{1+\sum_{n=2}^{\infty} n^2 a_n z^{n-1}}\right| + 2x\left|\frac{\sum_{n=2}^{\infty} n^2(n-1)a_n z^{n-2}}{1+\sum_{n=2}^{\infty} n^2 a_n z^{n-1}}\right| \\
\le\ & \frac{\sum_{n=1}^{\infty} n^2|b_n| + 2x\sum_{n=2}^{\infty} n^2(n-1)|a_n|}{1-\sum_{n=2}^{\infty} n^2|a_n|} \\
\le\ & 1.
\end{aligned}$$

The last inequality holds if the assertion (2.3) is hold. Thus in view of Lemma 2.2, $D[f]$ is univalent on $\mathbb{D}$. □

Theorem 2.4. *Let $D[f] = h_1 + \overline{g_1}$ be given by (1.5), h_1 and g_1 be are holomorphic in the unit disc $\mathbb{D}$. Suppose that q, m, and M are positive constants satisfying the* *inequalities:* $0 \le q < 1$, $1 < M/m \le \exp\{\pi(1-q)/2\}$. *If*

$$\sum_{n=2}^{\infty} n^2|a_n| \le \min\{M-1, 1-m\}, \quad q\sum_{n=2}^{\infty} n^2|a_n| + \sum_{n=1}^{\infty} n^2|b_n| \le q \tag{2.5}$$

at any point $z \in \mathbb{D}$*, then the mapping* $D[f] = h_1 + \overline{g_1}$ *is univalent on the domain* $\mathbb{D}$*.*

Proof. In order to show that $D[f]$ is univalent on $\mathbb{D}$, by Lemma 2.3 , it is sufficient to show that

$$m \le |h_1'(z)| \le M, \quad |g_1'(z)/h_1'(z)| \le q.$$

Substituting $h_1'(z), g_1'(z)$ into the above inequality, we have

$$m \le \left|1 + \sum_{n=2}^{\infty} n^2 a_n z^{n-1}\right| \le M, \quad \left|\frac{-\sum_{n=1}^{\infty} n^2 b_n z^{n-1}}{1 + \sum_{n=2}^{\infty} n^2 a_n z^{n-1}}\right| \le q. \tag{2.6}$$

If the conditions (2.5) are satisfied, then the inequalities (2.6) are also satisfied. Therefore, based on Lemma 2.3, it can be concluded that $D[f]$ is univalent on the unit disk $\mathbb{D}$. □

3. Radius of fully starlikeness and fully convexity

The aim of this section is to examine the fully starlikeness and fully convexity radius of differential operator for a subclass of harmonic mappings denoted by $\mathcal{K}_H^2(\lambda)$, where $\lambda \ge 0$.

Theorem 3.1. *Let* $f = h + \overline{g}$ *be given by* (1.1)*, and the coefficients satisfy the conditions*

$$|a_n| \le \frac{(2n+1)(n+1)}{6} \quad \text{and} \quad |b_n| \le \frac{(2n-1)(n-1)}{6} \tag{3.1}$$

for $n \ge 2$*. Then* $D[f] \in \mathcal{K}_H^2(\lambda)$ *and is fully starlike in* $|z| < r_s$*, where* r_s *is the unique positive root of* $P_\lambda(r) = 0$ *in* $(0,1)$ *with*

$$P_\lambda(r) = 1 - 6(3+2\lambda)r + 6(5-7\lambda)r^2 - 2(17+12\lambda)r^3 + (31-2\lambda)r^4 - 12r^5 + 2r^6. \tag{3.2}$$

Proof. According to the condition (3.1) and the harmonic-preserving property of the differential operator $D[f]$, $D[f]$ is harmonic in $\mathbb{D}$. Let $0 < r < 1$. Then it suffices to show that the coefficient of $D[f]_r(z) = r^{-1}D[f](rz)$ satisfies an inequality (1.9), where

$$D[f]_r(z) = \frac{D[f](rz)}{r} = z + \sum_{n=2}^{\infty} n a_n r^{n-1} z^n - \overline{\sum_{n=1}^{\infty} n b_n r^{n-1} z^n}.$$

By hypotheses, $|a_n| + |b_n| \leq \dfrac{2n^2+1}{3}$ and thus, we have

$$\begin{aligned} S &= \sum_{n=2}^{\infty}(1-\lambda+\lambda n)n^2(|a_n|+|b_n|)r^{n-1} \\ &\leq \sum_{n=2}^{\infty}(1-\lambda+\lambda n)\frac{2n^4+n^2}{3}r^{n-1} \\ &= \frac{1}{3}\sum_{n=2}^{\infty}(2\lambda n^3+2(1-\lambda)n^2+\lambda n+(1-\lambda))n^2r^{n-1} \\ &= \frac{-r^6+6r^5+2(\lambda-8)r^4+(24\lambda+14)r^3+(42\lambda-15)r^2+12(\lambda+1)r}{(r-1)^6} \\ &:= T_1, \end{aligned}$$

which is bounded above by 1 if $T_1 \leq 1$, which is equivalent to

$$1-6(3+2\lambda)r+6(5-7\lambda)r^2-2(17+12\lambda)r^3+(31-2\lambda)r^4-12r^5+2r^6 \geq 0.$$

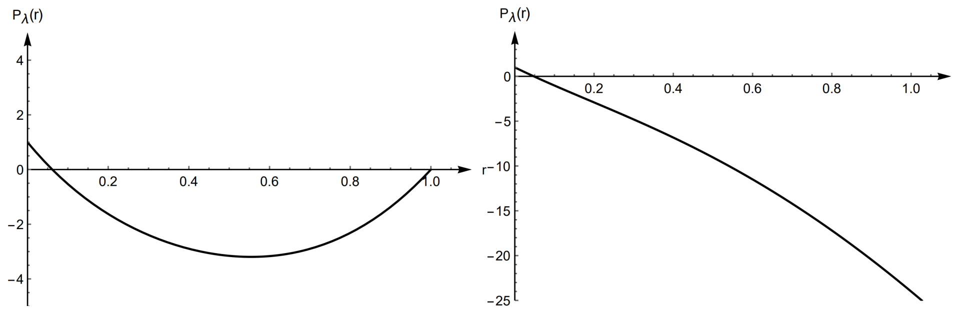
Now we shall show that the polynomial $P_\lambda(r)$ defined by (3.2) has exactly one zero in the interval $(0,1)$ for every $\lambda > 0$. A straightforward calculation shows that

$$\begin{aligned} P_\lambda'(r) &= -6(3+2\lambda)+12(5-7\lambda)r-6(17+12\lambda)r^2+4(31-2\lambda)r^3-60r^4+12r^5 \\ &= -18+60r-102r^2+124r^3-60r^4+12r^5-4\lambda\left(3+21r+18r^2+2r^3\right). \end{aligned}$$

The derivative $P_\lambda'(r)$ decreases in the interval $(0,1)$ as λ increases. Thus, when $\lambda = 0$, $P_\lambda'(r)$ achieves its maximum value. Our analysis begins by examining the range $(0, 0.2)$, assuming $\lambda = 0$. In this instance, we have $P_0'(r) = -18+60r-102r^2+124r^3-60r^4+12r^5$. A straightforward computation reveals that $P_0''(r) = 60-204r+372r^2-240r^3+60r^4 > 0$, demonstrating that $P_0'(r)$ monotonically increases over the interval $(0, 0.2)$. Furthermore, $P_0'(0) = -18 < 0$, and $P_0'(0.2) = -9.18016 < 0$, implying that $P_0'(r) < 0$ in the interval $(0, 0.2)$. This trend persists for all $\lambda > 0$, indicating that $P_\lambda(r)$ monotonically decreases in $(0, 0.2)$. As $P_\lambda(0) = 1 > 0$ and $P_\lambda(0.2) = -1.62611 - 4.2752\lambda < 0$, we conclude that $P_\lambda(r)$ has exactly one zero in the interval $(0, 0.2)$ for every $\lambda > 0$.

Furthermore, it can be observed that $P_\lambda(r)$ decreases as λ increases in $(0,1)$. To analyze this function, we begin by examining the special case when $\lambda = 0$. In this scenario, we solve $P_0'(r) = -18+60r-102r^2+124r^3-60r^4+12r^5 = 0$ to obtain the root $r_0 = 0.554275$ within the interval $(0,1)$. Thus, when $r < 0.554275$, $P_0'(r) < 0$. Consequently, $P_\lambda(r)$ is monotonically decreasing in the interval $[0.2, 0.554275)$ and monotonically increasing on the interval $[0.554275, 1)$. Specifically, when $\lambda = 0$, $P_0(r)$ has a local minimum value of $P_\lambda(0.554275) = -3.19387$. Additionally, since $P_\lambda(0.2) = -1.62611 < 0$ and $P_\lambda(1) = 0$, $P_\lambda(r)$ is negative throughout the interval $[0.2, 1)$. Therefore, we can conclude that $P_\lambda(r)$ has exactly one root in $(0,1)$ for every $\lambda > 0$ (See Figure 1(b)).

Therefore, it can be seen from Theorem 1.5 that for all $0 < r \leq r_s$, $D[f]_r(z) = r^{-1}D[f](rz)$ is close-to-convex (univalent) and fully starlike in $\mathbb{D}$, where r_s is the unique positive root of the equation (3.2) in interval (0,1). In particular, $D[f]$ is close-to-convex (univalent) and fully starlike in $|z| < r_s$. Which completes the proof. □



(a) $\lambda = 0$ (b) $\lambda = 0.3$

Figure 1. The root of $P_\lambda(r)$.

Remark 3.1. The root of $P_\lambda(r)$ for $\lambda \geq 0$ is shown in Figure 1. In Theorem 3.1, if $\lambda = 0$ is taken, the equation (3.2) is simplified as

$$(1-r)(1-17r+13r^2-21r^3+10r^4-2r^5) \geq 0.$$

Consequently, for $r \leq r_s \approx 0.0614313$, $D[f]_r(z) \in \mathcal{FS}_H^* \cap \mathcal{K}_H^2(0)$, where r_s is the unique root of the following equation within the interval $(0,1)$ (See Figure 1(a)).

$$1-17r+13r^2-21r^3+10r^4-2r^5=0.$$

This is consistent with the results in (Theorem 2.2, [17]).

Theorem 3.2. *Let $f = h + \overline{g}$ be given by (1.1), and the coefficients satisfy the conditions*

$$|a_n| \leq \frac{n+1}{2} \quad \textit{and} \quad |b_n| \leq \frac{n-1}{2}$$

for $n \geq 2$. Then $D[f] \in \mathcal{K}_H^2(\lambda)$ and is fully convex in $|z| < r_c$, where r_c is the unique positive root of $Q_\lambda(r) = 0$ in $(0,1)$ with

$$Q_\lambda(r) = 1-(13+8\lambda)r+(23-14\lambda)r^2-(19+2\lambda)r^3+10r^4-2r^5. \tag{3.3}$$

Proof. Obviously, by (3.3), it is easy to observe that

$$n^2(|a_n|+|b_n|) \leq n^3$$

and thus we have

$$\begin{aligned} S &= \sum_{n=2}^{\infty}(1-\lambda+\lambda n)n^2(|a_n|+|b_n|)r^{n-1} \\ &\leq \sum_{n=2}^{\infty}(1-\lambda+\lambda n)n^3 r^{n-1} \\ &= \frac{-r^5+5r^4-(2\lambda+9)r^3-(14\lambda-13)r^2-8(\lambda+1)r}{(r-1)^5} \\ &:= T_2, \end{aligned}$$

which is bounded above by 1 if $T_2 \leq 1$, which is equivalent to

$$1-(13+8\lambda)r+(23-14\lambda)r^2-(19+2\lambda)r^3+10r^4-2r^5 \geq 0.$$

Therefore, it can be seen from Theorem 1.5 that for all $0 < r \leq r_c$, $D[f]_r(z) = r^{-1}D[f](rz)$ is fully convex in $\mathbb{D}$, where r_c is the unique positive root of the equation (3.3) in the interval $(0,1)$. In particular, $D[f]$ is fully convex in $|z| < r_c$. □

Remark 3.2. When setting $\lambda = 0$ in Equation (3.3), Theorem 3.2 yields a simplified form as

$$1 - 13r + 23r^2 - 19r^3 + 10r^4 - 2r^5 = 0. \tag{3.4}$$

Consequently, for $r \leq r_c \approx 0.0903331$, $D[f]_r(z) \in \mathcal{FK}_H \cap \mathcal{K}_H^2(0)$, where r_c is the unique root of equation (3.4) in the interval $(0,1)$.

4. Radius of uniformly starlikeness and uniformly convexity

In this section, we investigate the radius of uniformly starlikeness and uniformly convexity for harmonic differential operator.

Theorem 4.1. *Let $f = h + \overline{g} \in \mathcal{S}_H^0$ be given by (1.1), and the coefficients satisfy the conditions*

$$|a_n| \leq \frac{(2n+1)(n+1)}{6} \quad \textit{and} \quad |b_n| \leq \frac{(2n-1)(n-1)}{6} \tag{4.1}$$

for $n \geq 2$. Then $D[f] \in \mathcal{US}_H^{*0}$ on the disk $|z| < r_{us}$, where $r_{us} \approx 0.035116$ is the unique positive root of $P_{us}(r) = 0$ in $(0,1)$, where

$$P_{us}(r) = -3r^5 + 15r^4 - 32r^3 + 16r^2 - 29r + 1. \tag{4.2}$$

Proof. Let $f = h + \overline{g} \in \mathcal{S}_H^0$ be given by (1.1), and the coefficients satisfy the conditions (4.1) for $n \geq 2$. For $0 < r < 1$, let

$$D[f]_r(z) = \frac{D[f](rz)}{r} = z + \sum_{n=2}^{\infty} na_n r^{n-1} z^n - \overline{\sum_{n=1}^{\infty} nb_n r^{n-1} z^n}, \quad (z \in \mathbb{D}).$$

Consider the sum

$$\begin{aligned} S &= \sum_{n=2}^{\infty} n^2(|a_n| + |b_n|) r^{n-1} \\ &\leq \sum_{n=2}^{\infty} \frac{(2n^2+1)n^2}{3} r^{n-1} \\ &= \frac{r(12 - 3r + 11r^2 - 5r^3 + r^4)}{(1-r)^5} \\ &:= S_1. \end{aligned}$$

From Lemma 1.1, it suffices to show that $S_1 \leq 1/2$, which implies that

$$24r - 6r^2 + 22r^3 - 10r^4 + 2r^5 \leq 1 - 5r + 10r^2 - 10r^3 + 5r^4 - r^5.$$

It is equivalent to

$$-3r^5 + 15r^4 - 32r^3 + 16r^2 - 29r + 1 \geq 0.$$

Therefore $S_1 \leq 1/2$ which holds whenever $P_{us}(r) \geq 0$. It is easy to see that $P_{us}(0) = 1 > 0$ and $P_{us}(1) = -32 < 0$, and hence P_{us} has at least one real root in $(0, 1)$. To show that $P_{us}(r)$ has exactly one real root in $(0, 1)$, it is sufficient to prove that $P_{us}(r)$ is monotonic on $(0, 1)$. A simple calculation shows that

$$P_{us}^{'}(r) = -29 + 32r - 96r^2 + 60r^3 - 15r^4,$$

and

$$P_{us}^{''}(r) = 32 - 192r + 180r^2 - 60r^3.$$

We note that $P_{us}^{''}(r)$ is positive in $(0, 0.2025)$ and is negative in (0.2025,1). Hence, $P_{us}^{'}(r) < P_{us}^{'}(0.2025) \approx -25.98 < 0$. This shows that $P_{us}(r)$ is strictly monotonically decreasing in $(0, 1)$, and hence $P_{us}(r)$ has exactly one real root (say $r_{us} \approx 0.035116$) in $(0, 1)$. □

Obviously, if $f = h+\overline{g} \in \mathcal{S}_H^{*0}$, its coefficient estimate satisfies the equation (4.1). Therefore, Theorem 4.1 readily gives the following corollary.

Corollary 4.1. *Let $f = h + \overline{g} \in \mathcal{S}_H^{*0}$. Then $D[f] \in \mathcal{US}_H^{*0}$ in at least $|z| < r_{us}$, where r_{us} is the real root of (4.2) in $(0, 1)$.*

Using a method similar to Theorem 4.1 and using Lemma 1.2 instead of Lemma 1.1, we can obtain the following results.

Theorem 4.2. *Let $f = h + \overline{g} \in \mathcal{S}_H^0$ be given by (1.1), and the coefficients satisfy the conditions*

$$|a_n| \leq \frac{(2n+1)(n+1)}{6} \quad \text{and} \quad |b_n| \leq \frac{(2n-1)(n-1)}{6}$$

for $n \geq 2$. Then $D[f]$ is univalent and $D[f] \in \mathcal{UK}_H^0$ on the disk $|z| < r_{uc}$, where $|z| < r_{uc} \approx 0.0230996$ is the unique positive real root of $P_{uc}(r) = 0$ in $(0, 1)$, and where

$$P_{uc}(r) = 1 - 42r - 54r^2 - 82r^3 + 27r^4 - 12r^5 + 2r^6. \tag{4.3}$$

Proof. Let $f = h + \overline{g} \in \mathcal{S}_H^0$ be given by (1.1), and the coefficients satisfy the conditions (4.1) for $n \geq 2$. For $0<r<1$, let

$$D[f]_r(z) = \frac{D[f](rz)}{r} = z + \sum_{n=2}^{\infty} n a_n r^{n-1} z^n - \overline{\sum_{n=1}^{\infty} n b_n r^{n-1} z^n}, \ z \in \mathbb{D}.$$

Consider the sum

$$S = \sum_{n=2}^{\infty} n^2(2n-1)(|a_n| + |b_n|)r^{n-1}.$$

Using (4.1) in the above equation, we obtain

$$\begin{aligned} S &\leq \sum_{n=2}^{\infty} \frac{4n^5 - 2n^4 + 2n^3 - n^2}{3} r^{n-1} \\ &= \frac{r(36 + 69r + 62r^2 - 12r^3 + 6r^4 - r^5)}{(1-r)^6} \\ &:= S_2. \end{aligned}$$

From Lemma 1.2, we note that $D[f]_r(z) \in \mathcal{UK}_H^0$ on $\mathbb{D}$ if $S_2 \leq 1$, which implies that

$$36r + 69r^2 + 62r^3 - 12r^4 + 6r^5 - r^6 \leq 1 - 6r + 15r^2 - 20r^3 + 15r^4 - 6r^5 + r^6,$$

which is equivalent to

$$2r^6 - 12r^5 + 27r^4 - 82r^3 - 54r^2 - 42r + 1 \geq 0.$$

Therefore $S_2 \leq 1$, when $P_{uc}(r) \geq 0$. It is easy to see that $P_{uc}(0) = 1 > 0$ and $P_{uc}(1) = -160 < 0$, and hence $P_{uc}(r)$ has at least one real root in $(0,1)$. To show that $P_{uc}(r)$ has exactly one zero in $(0,1)$, it is sufficient to prove that $P_{uc}(r)$ is monotonic on $(0,1)$. A simple calculation shows that

$$P_{uc}^{'}(r) = -42 + r(-108 - 246r + 108r^2 - 60r^3 + 12r^4).$$

Let

$$\xi(r) = -108 - 246r + 108r^2 - 60r^3 + 12r^4.$$

Then

$$\xi^{''}(r) = 216 - 360r + 144r^2.$$

We note that $\xi^{''}(r)$ attains its minimum value at $r = 1$, and $\xi^{''}(r) > \xi^{''}(1) = 0$ for all $r \in (0,1)$. Thus $\xi^{'}(r) \leq \xi^{'}(1) = -162 < 0$, and $\xi(r)$ is strictly monotonically decreasing in $(0,1)$. Thus $\xi(r) < \xi(0) = -108 < 0$. This shows that $P_{uc}^{'}(r) < 0$, and hence $P_{uc}(r)$ has exactly one real root (say $r_{uc} \approx 0.0230996$) in $(0,1)$. □

Corollary 4.2. *Let $f \in \mathcal{S}_H^{*0}$. Then $D[f] \in \mathcal{UK}_H^0$ in at least $|z| < r_{uc}$, where r_{uc} is the real root of* (4.3) *in* $(0,1)$.

Similar to Theorems 4.1 and 2.2, the next two theorems provide a sufficient condition which guarantees a sense-preserving harmonic differential operator to be uniformly starlike(uniformly convex). A slight change in the proof of Theorem 4.1 with the help of Lemma 1.2 yields the radius of uniformly starlike(uniformly convex) for $D[f]$. So the proof is omitted here.

Theorem 4.3. *Let $f = h + \overline{g} \in \mathcal{S}_H^0$ be given by* (1.1) *with $b_1 = g'(0) = 0$, and the coeffcients of the series satisfy the conditions*

$$|a_n| \leq \frac{n+1}{2} \quad \text{and} \quad |b_n| \leq \frac{n-1}{2}.$$

Then $D[f] \in \mathcal{US}_H^{*0}$ on the disk $|z| < r_{us}$, where $r_{us} \approx 0.0520867$ is the unique positive real root of $Q_{us}(r) = 0$ in $(0,1)$, where

$$Q_{us}(r) = 1 - 20r + 16r^2 - 12r^3 + 3r^4. \tag{4.4}$$

Corollary 4.3. *Let $f \in \mathcal{K}_H^0$. Then $D[f] \in \mathcal{US}_H^{*0}$ in at least $|z| < r_{us}$, where r_{us} is the real root of* (4.4) *in* $(0,1)$.

Theorem 4.4. *Let $f = h + \overline{g} \in \mathcal{S}_H^0$ be given by* (1.1) *with $b_1 = g^{'}(0) = 0$, and the coefficients of the series satisfy the conditions*

$$|a_n| \leq \frac{n+1}{2} \quad \text{and} \quad |b_n| \leq \frac{n-1}{2}.$$

Then $D[f] \in \mathcal{UK}_H^0$ on the disk $|z| < r_{uc}$, where $r_{uc} \approx 0.034249$ is the unique positive real root of $Q_{uc}(r) = 0$ in $(0,1)$, where

$$Q_{uc}(r) = 1 - 29r - 5r^2 - 23r^3 + 10r^4 - 2r^5. \tag{4.5}$$

Corollary 4.4. *Let $f \in \mathcal{K}_H^0$. Then $D[f] \in \mathcal{UK}_H^0$ at least $|z| < r_{uc}$, where r_{uc} is the real root of* (4.5) *in* $(0,1)$.

5. Radius of harmonic convolution

In this section, we investigate the radius of uniformly starlikeness and uniformly convexity for differential operator after harmonic convolutions.

Theorem 5.1. *Let $f = h + \overline{g} \in \mathcal{S}_H^0$ be given by* (1.1)*, and the coefficients of the series satisfy the conditions*

$$|a_n| \le \frac{(n+1)^2}{4} \quad \text{and} \quad |b_n| \le \frac{(n-1)^2}{4}. \tag{5.1}$$

Then $D[f] \in \mathcal{US}_H^{*0}$ on the disk $|z| < r_{us}$, where $r_{us} \approx 0.0412746$ is the unique positive real root of $P_{us}(r) = 0$ in $(0,1)$, where

$$P_{us}(r) = 1 - 25r + 20r^2 - 32r^3 + 15r^4 - 3r^5. \tag{5.2}$$

Proof. For $r \in (0,1)$, it is only necessary to prove that $D[f]_r(z) \in \mathcal{US}_H^{*0}$, where

$$D[f]_r(z) = \frac{D[f](rz)}{r} = z + \sum_{n=2}^{\infty} n a_n r^{n-1} z^n - \overline{\sum_{n=2}^{\infty} n b_n r^{n-1} z^n}.$$

Consider the sum

$$S = \sum_{n=2}^{\infty} n^2(|a_n| + |b_n|) r^{n-1}.$$

By using (5.1), we obtain

$$S \le \sum_{n=2}^{\infty} \frac{(n^2+1)n^2}{2} r^{n-1} = \frac{r(10 - 5r + 11r^2 - 5r^3 + r^4)}{(1-r)^5} := C_1.$$

From Lemma 1.1, we note that $D[f]_r(z) \in \mathcal{US}_H^{*0}$ on $\mathbb{D}$ if $C_1 \le 1/2$, which implies that

$$1 - 25r + 20r^2 - 32r^3 + 15r^4 - 3r^5 \ge 0.$$

Therefore $C_1 \le \frac{1}{2}$, when $P_{us}(r) \ge 0$. A simple calculation shows that $P_{us}(r)$ has exactly one real root (say $r_{us} \approx 0.0412746$) in $(0,1)$. □

Corollary 5.1. *Let $f_1,\ f_2 \in \mathcal{K}_H^0$. Then $D[f_1 * f_2] \in \mathcal{US}_H^{*0}$ in at least $|z| < r_{us}$, where r_{us} is the real root of* (5.2) *in* $(0,1)$.

Using Lemma 1.2 instead of Lemma 1.1 and using the same proof method as Theorem 5.1, we obtain the following results. So the proof is omitted here.

Theorem 5.2. *Let h and g have the form* (1.1)*, and the coefficients of the series satisfy the conditions*

$$|a_n| \le \frac{(n+1)^2}{4}, \quad |b_n| \le \frac{(n-1)^2}{4}.$$

Then $D[f] \in \mathcal{UK}_H^0$ on the disk $|z| < r_{uc}$, where $r_{uc} \approx 0.02712512$ is the unique positive real root of $P_{uc}(r) = 0$ in $(0,1)$, where

$$P_{uc}(r) = 1 - 36r - 30r^2 - 72r^3 + 27r^4 - 12r^5 + 2r^6. \tag{5.3}$$

Corollary 5.2. *Let $f_1, f_2 \in \mathcal{K}_H^0$. Then $D[f_1 * f_2] \in \mathcal{UK}_H^0$ in at least $|z| < r_{uc}$, where r_{uc} is the real root of* (5.3) *in* $(0,1)$.

The next theorem comes from a simple modification to the proof of Theorem 5.1.

Theorem 5.3. *Let $f = h + \overline{g}$ have the form* (1.1), *and the coefficients of the series satisfy the conditions*

$$|a_n| \le \frac{(n+1)^2(2n+1)}{12} \quad \text{and} \quad |b_n| \le \frac{(n-1)^2(2n-1)}{12}.$$

Then $D[f] \in \mathcal{US}_H^{*0}$ on the disk $|z| < r_{us}$, where $r_{us} \approx 0.026443019$ is the unique positive real root of $q_{us}(r) = 0$ in $(0,1)$, where

$$q_{us}(r) = 1 - 38r + 9r^2 - 80r^3 + 43r^4 - 18r^5 + 3r^6. \tag{5.4}$$

Corollary 5.3. *Let $f_1 \in \mathcal{S}_H^{*0}$, $f_2 \in \mathcal{K}_H^0$, then $D[f_1 * f_2] \in \mathcal{US}_H^{*0}$ in at least $|z| < r_{us}$, where r_{us} is the real root of* (5.4) *in* $(0,1)$.

Theorem 5.4. *Let $f = h + \overline{g}$ have the form* (1.1), *and the coefficients of the series satisfy the conditions*

$$|a_n| \le \frac{(n+1)^2(2n+1)}{12} \quad \text{and} \quad |b_n| \le \frac{(n-1)^2(2n-1)}{12}.$$

Then $D[f] \in \mathcal{UK}_H^0$ on the disk $|z| < r_{uc}$, where $r_{uc} \approx 0.017396982$ is the unique positive real root of $Q_{uc}(r) = 0$ in $(0,1)$, where

$$Q_{uc}(r) = -1 + 55r + 138r^2 + 266r^3 - 11r^4 + 45r^5 - 14r^6 + 2r^7. \tag{5.5}$$

Corollary 5.4. *Let $f_1 \in \mathcal{S}_H^{*0}$, $f_2 \in \mathcal{K}_H^0$, then $D[f_1 * f_2] \in \mathcal{UK}_H^0$ in at least $|z| < r_{uc}$, where r_{uc} is the real root of* (5.5) *in* $(0,1)$.

Data availability

No data were used to support this study.

Conflict of interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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