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SOLVABILITY OF SOME RIEMANN-HILBERT PROBLEMS RELATED TO DIRAC OPERATOR WITH GRADIENT POTENTIAL IN $\mathbb{R}^3$*

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Abstract Using quaternionic analysis and Schauder's fixed point theorem, we establish sufficient conditions for the existence of solutions for some nonlinear Riemann-Hilbert boundary value problems for Dirac operator with gradient potential.

Keywords Quaternionic analysis, nonlinear Riemann-Hilbert problem, Dirac operator.

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1. Introduction

In [4, 5, 8, 13–15, 17, 28, 32] and [29, 30] several results on Riemann-Hilbert boundary value problems in the complex plane and its applications are presented. The Riemann-Hilbert boundary value problems have also become a very important and useful area of mathematics over the last few decades and are used by Deift, Its and Zhou solve some problems in random matrix theory. This method is called Riemann-Hilbert approach which plays important role in the long-time behavior of solutions of KdV equations, mathematical physics, inverse problems for partial differential equations and engineering, etc. We refer to [10, 11]. A natural and interesting question is nonlinear Riemann-Hilbert problem in higher dimension ($\mathbb{R}^n, n \geq 3$). The quaternionic and Clifford analysis approaches that may be considered as a generalization to higher dimension for the theory of holomorphic functions in the complex plane are powerful mathematical tools for the treatment of linear boundary value problems in higher dimensions, see [1–3, 6, 7, 9, 18, 19, 21–24, 33, 35]. In this paper we consider the Riemann-Hilbert problem for Dirac operator with gradient potential

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about boundary value condition of nonlinearity:

$$\begin{cases} D_{b(\mathbf{x})}[f] = 0, & \text{in } \mathbb{R}^3 \setminus \partial\Omega, \\ f^+(\mathbf{x}) = f^-(\mathbf{x})A(\mathbf{x}) + \lambda G(\mathbf{x}, f^+,\ f^-), & \mathbf{x} \in \partial\Omega, \end{cases}$$

where $f(\mathbf{x}) : \mathbb{R}^3 \to \mathbb{H}$ is the unknown quaternion value function with vanishing at infinity.

This article is organized as follows: In Sect.2, we recall some basic facts about quaternionic analysis which will be needed in the article and Dirac operators with gradient potentials. In Sect. 3, we prove a compact embedding theorem related to Hölder space and continuous function space in quaternion analysis, which is necessary to study nonlinear boundary value problems. Finally, we mainly consider that a kind of nonlinear Riemann-Hilbert boundary value problem.

2. Preliminaries

Let $\mathbb{H}$ denote the set of real quaternions. Then, each $\mathbf{x} \in \mathbb{H}$ can be represented as $\mathbf{x} = \sum_{k=0}^{3} e_k x_k$, where $e_0 e_k = e_k e_0$, $e_k^2 = -e_0$, $e_1 e_2 = -e_2 e_1 = e_3$, $e_2 e_3 = -e_3 e_2 = e_1$, $e_3 e_1 = e_1 e_3 = e_2$, $\{x_0, x_1, x_2, x_3\} \subset \mathbb{R}$. With the natural addition from $\mathbb{R}^4$ and the multiplication based on the above mentioned rules $\mathbb{H}$ is non-commutative associative skew field. The quaternionic conjugation is defined by

$$\overline{e_0} = e_0 := 1,\ \overline{e_k} = -e_k,\ k \in \{1, 2, 3\}$$

and it extends onto $\mathbb{H}$ as an $\mathbb{R}$-linear mapping: if $\mathbf{x} \in \mathbb{H}$ then

$$\mathbf{x} := \sum_{k=0}^{3} \overline{x_k e_k} = \sum_{k=0}^{3} x_k \bar{e}_k = x_0 - \sum_{k=1}^{3} x_k e_k := x_0 - \underline{\mathbf{x}}.$$

$Sc(\mathbf{x}) := \mathbf{x}_0$ is called scalar part and $Vec(\mathbf{x}) := \underline{\mathbf{x}}$ is the vector part of the quaternion $\mathbf{x}$. Furthermore $\overline{\mathbf{x}} = \mathbf{x}_0 - \underline{\mathbf{x}}$ is called conjugated quaternion. For arbitrary quaternions $\mathbf{x}$ and $\mathbf{y}$ we have $\overline{\mathbf{x}\mathbf{y}} = \overline{\mathbf{y}}\ \overline{\mathbf{x}}$ and

$$\mathbf{x}\overline{\mathbf{x}} = \overline{\mathbf{x}}\mathbf{x} = \sum_{k=0}^{3} x_k^2 =: \|\mathbf{x}\|^2 \in \mathbb{R}.$$

This norm of a quaternion coincides with the usual Euclidean norm in $\mathbb{R}^4$. It follows that for any $\mathbf{x} \in \mathbb{H} \setminus \{0\}$ has a multiplicative inverse

$$\mathbf{x}^{-1} := \frac{\overline{\mathbf{x}}}{\|\mathbf{x}\|^2}.$$

Suppose Ω is an open bounded non-empty subset of $\mathbb{R}^3$. We introduce the Dirac operator $D = \sum_{k=1}^{3} e_k \frac{\partial}{\partial x_k}$. In particular, we obtain that $DD = -\Delta$ where Δ is the Laplacian over $\mathbb{R}^3$.

In this article, assume that $\mathcal{B}:\mathbb{R}^3\to\mathbb{R}$ is a bounded continuous differentiable function, i.e., C^1-function and $b:=\sum_{k=1}^{3} e_k\frac{\partial\mathcal{B}}{\partial x_k}$. We now consider the following inhomogeneous Dirac operator:

$$D_{b(\mathbf{x})}[f](\mathbf{x}) = D[f](\mathbf{x}) + b(\mathbf{x})f(\mathbf{x}).$$

Definition 2.1. A C^1-function $f:\Omega\to\mathbb{H}$ is said to be (left) regular with respect to the potential $\mathcal{B}$ if $D_{b(\mathbf{x})}[f](\mathbf{x})=0$ for all $\mathbf{x}\in\Omega$.

Definition 2.2. A compact surface $\partial\Omega$ in $\mathbb{R}^3$ is called Lyapunov surface, if the following conditions are satisfied:

1. At each point $\mathbf{x}\in\partial\Omega$ there is the tangential space.
2. There exists a number r, that for any point $\mathbf{x}\in\partial\Omega$ the set $\partial\Omega\bigcap B_r(\mathbf{x})$ (Lyapunov ball) is connected and parallel lines to the outer normal $n(\mathbf{x})$ intersect with $\partial\Omega$ at not more than one point.
3. The normal $n(\mathbf{x})$ is Hölder continuous on $\partial\Omega$ i.e., there are constants $C>0$ and $0<\delta\le 1$ such that for $\mathbf{x},\mathbf{y}\in\partial\Omega$

$$|n(\mathbf{x})-n(\mathbf{y})|\le C\|\mathbf{x}-\mathbf{y}\|^{\delta}.$$

Remark 2.1. A Lyapunov surface is necessarily C^1 surface, and on the other hand a compact surface of class C^2 is a Lyapunov surface. Throughout this work, suppose Ω is a bounded, open set of $\mathbb{R}^3$ with a Lyapunov boundary $\partial\Omega$. For more information about the theory of surface, we refer to [23]. For reasons of simplicity, we will only consider $\delta=1$ in this article.

Denote the fundamental solution of $D_{b(\mathbf{x})}$ by

$$K_{\mathcal{B}}(\mathbf{y},\mathbf{x}) = \frac{1}{4\pi}\frac{\overline{\mathbf{y}-\mathbf{x}}}{\|\mathbf{y}-\mathbf{x}\|^3}e^{\mathcal{B}(\mathbf{y})-\mathcal{B}(\mathbf{x})}, \tag{2.1}$$

where $\mathbf{y}=\sum_{k=1}^{3} y_k e_k$, $\mathbf{x}=\sum_{k=1}^{3} x_k e_k$.

Lemma 2.1. [20] *If $f(\mathbf{x})\in C^1(\Omega,\mathbb{H})\bigcap C(\overline{\Omega},\mathbb{H})$ and $D_{b(\mathbf{x})}[f](\mathbf{x})=0$ in Ω, then*

$$\int\limits_{\partial\Omega} K_{\mathcal{B}}(\mathbf{y},\mathbf{x})d\sigma_{\mathbf{y}}f(\mathbf{y}) = \begin{cases} f(\mathbf{x}), & \mathbf{x}\in\Omega,\\ 0, & \mathbf{x}\in\mathbb{R}^3\setminus\overline{\Omega},\end{cases}$$

where $K_{\mathcal{B}}(\mathbf{y},\mathbf{x})$ is as in (2.1), $d\sigma_{\mathbf{y}}=\sum_{i=1}^{3}(-1)^{i-1}e_i d\widehat{y}_i$ and $d\widehat{y}_i = dy_1\wedge\cdots\wedge dy_{i-1}\wedge dy_{i+1}\wedge\cdots\wedge dy_3$.

Let Ω be an open nonempty subset of $\mathbb{R}^3$ with a Lyapunov boundary, $f(\mathbf{x})=\sum_{k=0}^{3} e_k f_k(\mathbf{x})$, where $f_k(\mathbf{x})$ are real functions. $f(\mathbf{x})$ is called a Hölder continuous functions on $\overline{\Omega}$ if the following condition is satisfied,

$$\|f(\mathbf{x}_1)-f(\mathbf{x}_2)\| = [\sum_{k=0}^{3}\|f_k(\mathbf{x}_1)-f_k(\mathbf{x}_2)\|]^{\frac{1}{2}} \le C\|\mathbf{x}_1-\mathbf{x}_2\|^{\alpha},$$

where for any $\mathbf{x}_1$, $\mathbf{x}_2 \in \overline{\Omega}$, $\mathbf{x}_1 \neq \mathbf{x}_2$, $0 < \alpha \leq 1$, C is a positive constant independent of $\mathbf{x}_1$, $\mathbf{x}_2$.

The space $H^\alpha(\partial\Omega, \mathbb{H})$ consists of all Hölder continuous functions with values in $\mathbb{H}$ on $\partial\Omega$ (the Hölder exponent is α, $0 < \alpha \leq 1$), which the norm

$$\|f\|_{(\alpha,\partial\Omega)} = \|f\|_\infty + \|f\|_\alpha \tag{2.2}$$

is finite, where $\|f\|_\infty := \sup\limits_{x\in\partial\Omega} \|f(\mathbf{x})\|$, $\|f\|_\alpha := \sup\limits_{\substack{\mathbf{x}_1,\mathbf{x}_2\in\partial\Omega \\ \mathbf{x}_1\neq\mathbf{x}_2}} \frac{\|f(\mathbf{x}_1)-f(\mathbf{x}_2)\|}{\|\mathbf{x}_1-\mathbf{x}_2\|^\alpha}$.

Lemma 2.2. [21] *The Hölder space $H^\alpha(\partial\Omega, \mathbb{H})$ is a Banach space with norm (2.2).*

Remark 2.2. It is clear that the space $C(\partial\Omega, \mathbb{H})$ of quaternion valued continuous functions defined on the $\partial\Omega$ equipped with the norm $\|u\|_\infty := \sup\limits_{\mathbf{x}\in\partial\Omega} \|u(\mathbf{x})\|$ is a Banach space.

Next, we introduce the following integral operators:

$$\mathcal{F}_{\partial\Omega}[f](\mathbf{x}) := \int\limits_{\partial\Omega} K_{\mathcal{B}}(\mathbf{y}, \mathbf{x}) d\sigma_{\mathbf{y}} f(\mathbf{y}), \mathbf{x} \in \mathbb{R}^3 \setminus \partial\Omega, \tag{2.3}$$

$$\mathcal{S}_{\partial\Omega}[f](\mathbf{x}) := \int\limits_{\partial\Omega} K_{\mathcal{B}}(\mathbf{y}, \mathbf{x}) d\sigma_{\mathbf{y}} f(\mathbf{y}), \mathbf{x} \in \partial\Omega, \tag{2.4}$$

where $K_{\mathcal{B}}(\mathbf{y}, \mathbf{x})$ is as in (2.1).

Due to the universal quaternion generalized the complex number to higher dimension, we naturally have the following results by the weak singularity of kernel function $K_{\mathcal{B}}(\mathbf{y}, \mathbf{x})$ in quaternion analysis and the same proof technique in [20, 34].

Lemma 2.3. *Let Ω be a bounded, convex domain in $\mathbb{R}^3$ with a Lyapunov boundary $\partial\Omega$. Then the integral transform $\mathcal{S}_{\partial\Omega} : H^\alpha(\partial\Omega, \mathbb{H}) \to H^\alpha(\partial\Omega, \mathbb{H})$ defined by* (2.4) *is bounded, i.e.,*

$$\|\mathcal{S}_{\partial\Omega}[f]\|_{(\alpha,\partial\Omega)} \leq \widehat{C}\|f\|_{(\alpha,\partial\Omega)},$$

where $\widehat{C}$ is a nonnegative constant.

Lemma 2.4. [Plemelj-Sokhotski] *Suppose that Ω is an open nonempty bounded subset of $\mathbb{R}^3$ with oriented Lyapunov boundary $\partial\Omega$, and $f \in H^\alpha(\partial\Omega, \mathbb{H})$, $0 < \alpha \leq 1$. Then*

$$\lim_{\substack{\mathbf{x}\to\mathbf{x}_0\in\partial\Omega \\ \mathbf{x}\in\Omega}} \mathcal{F}_{\partial\Omega}[f](\mathbf{x}) = \frac{f(\mathbf{x}_0)}{2} + \mathcal{S}_{\partial\Omega}[f](\mathbf{x}_0),$$

$$\lim_{\substack{\mathbf{x}\to\mathbf{x}_0\in\partial\Omega \\ \mathbf{x}\in\mathbb{R}^n\setminus\overline{\Omega}}} \mathcal{F}_{\partial\Omega}[f](\mathbf{x}) = -\frac{f(\mathbf{x}_0)}{2} + \mathcal{S}_{\partial\Omega}[f](\mathbf{x}_0).$$

For simplicity, we denote

$$f^\pm(\mathbf{x}) = \lim_{\substack{\mathbf{y}\to\mathbf{x}\in\partial\Omega \\ \mathbf{y}\in\Omega^\pm}} f(\mathbf{y}),$$

where $\Omega^+ = \Omega$ and $\Omega^- = \mathbb{R}^3 \setminus \overline{\Omega}$.

Elementary properties of the Dirac operators, regular functions in quaternion analysis and Dirac operators with gradient potentials in Clifford analysis can be found in References [20, 31].

3. A nonlinear Riemann-Hilbert problem in quaternion analysis

Theorem 3.1. *If $\partial\Omega$ is a Lyapunov boundary in $\mathbb{R}^3$, then the imbedding operator*

$$\mathbb{I}^{\alpha}: H^{\alpha}(\partial\Omega, \mathbb{H}) \to C(\partial\Omega, \mathbb{H}) \tag{3.1}$$

is compact.

Proof. Let $\mathcal{M}$ be a bounded set in $H^{\alpha}(\partial\Omega, \mathbb{H})$ i.e., $\|u\|_{(\alpha,\partial\Omega)} \leq C$ for all $u \in \mathcal{M}$. It is obvious that

$$\|u(\mathbf{x})\| \leq C$$

for all $\mathbf{x} \in \partial\Omega$ and

$$\|u(\mathbf{x}) - u(\mathbf{y})\| \leq C\|\mathbf{x} - \mathbf{y}\|^{\alpha}$$

for all $\mathbf{x}, \mathbf{y} \in \Omega$ and $u \in \mathcal{M}$, whence $\mathcal{M}$ is relatively compact in $C(\partial\Omega, \mathbb{H})$ by the Arzela-Ascoli Theorem, which means that the imbedding operators

$$\mathbb{I}^{\alpha}: H^{\alpha}(\partial\Omega, \mathbb{H}) \to C(\partial\Omega, \mathbb{H})$$

is compact. The proof is done. □

Corollary 3.2. *Let $\mathcal{M}$ be a bounded and closed set in $C(\partial\Omega, \mathbb{H})$ which satisfies the following condition:*

$$\|u(\mathbf{x}_1) - u(\mathbf{x}_2)\| \leq C\|\mathbf{x}_1 - \mathbf{x}_2\|^{\alpha},\ \mathbf{x}_1,\ \mathbf{x}_2 \in \partial\Omega,\ \forall u \in \mathcal{M}.$$

Then the set $\mathcal{M}$ is compact in $C(\partial\Omega, \mathbb{H})$.

Remark 3.1. The above results demonstrate that $H^{\alpha}(\partial\Omega, \mathbb{H})$ is in fact compactly embedded in $C(\partial\Omega, \mathbb{H})$. The compactness is fundamental for our research of the nonlinear Riemann-Hilbert problem in this article.

The following lemmas are fundamental in the proof of existence of solutions for the nonlinear problem.

Lemma 3.1. [16](Schauder's Fixed Point Theorem) *Suppose X is a Banach space and K is a compact and convex subset of X, and assume also*

$$\mathcal{T}: K \to K$$

is continuous. Then $\mathcal{T}$ has a fixed point in K.

Lemma 3.2. *Let Ω be an open bounded non-empty subset of $\mathbb{R}^3$ with a Lyapunov boundary $\partial\Omega$, $f \in C^1(\Omega, \mathbb{H}) \bigcap C(\overline{\Omega}, \mathbb{H})$. Then*

$$D_{b(\mathbf{x})}[\mathcal{F}_{\partial\Omega}[f](\mathbf{x})] = 0.$$

Proof. Using Lemma 3.2 in [12], the results can be directly proved. □

Let Ω be a bounded convex domain of $\mathbb{R}^3$ with Lyapunov boundary $\partial\Omega$, we consider the following nonlinear Riemann-Hilbert problem:

$$\begin{cases} D_{b(\mathbf{x})}[f] = 0, & \text{in } \mathbb{R}^3 \setminus \partial\Omega, \\ f^+(\mathbf{x}) = f^-(\mathbf{x})A(\mathbf{x}) + \lambda G(\mathbf{x}, f^+, f^-), \ \mathbf{x} \in \partial\Omega, & \\ \|f(\infty)\| = 0, & \end{cases} \tag{3.2}$$

where $A(\mathbf{x}) \in H^\alpha(\partial\Omega, \mathbb{H})$, $0 < \alpha < 1$ and λ is a real parameter with a certain condition, see (3.5). We assume the following conditions to be fulfilled:

1) For each f_1, f_2 in $H^\alpha(\partial\Omega, \mathbb{H})$ the function

$$G(\mathbf{x}, f_1, f_2) = G(\mathbf{x}, f_1(\mathbf{x}), f_2(\mathbf{x}))$$

is in $H^\alpha(\partial\Omega, \mathbb{H})$ as a function of $\mathbf{x}$ and $G(\mathbf{x}, 0, 0) = 0$. Moreover there exists a nonnegative constant N such that for all f_1, $\widetilde{f}_1$, f_2, $\widetilde{f}_2$ in $H^\alpha(\partial\Omega, \mathbb{H})$ we have

$$\begin{aligned} &\|G(\mathbf{x}, f_1, f_2) - G(\mathbf{x}, \widetilde{f}_1, \widetilde{f}_2)\|_{(\alpha,\partial\Omega)} \\ \leq &N[\|f_1 - \widetilde{f}_1\| + \|f_2 - \widetilde{f}_2\|], \ 0 < \alpha < 1. \end{aligned} \tag{3.3}$$

2) The quaternion value function $A(\mathbf{x})$ is in $H^\alpha(\partial\Omega, \mathbb{H})$ and $A(\mathbf{x}) \neq 0$ on $\partial\Omega$.

We establish solvability conditions of the Riemann-Hilbert problem (3.2).

Theorem 3.3. *Suppose the function $G(\mathbf{x}, f_1, f_2)$ satisfies the condition (3.3), $A(\mathbf{x})$ satisfies the following condition*

$$\mu{:=}\|A - 1\|_{(\alpha,\partial\Omega)}(\widehat{C} + 1) < 1, \tag{3.4}$$

where $\widehat{C}$ is the positive constant mentioned in Lemma 2.3 and λ satisfies

$$|\lambda| \leq \frac{2(1-\mu)}{N(1+2\widehat{C})}. \tag{3.5}$$

Then (3.2) there admits at least one solution.

Proof. In view of Lemma 3.2, the solution to this Riemann-Hilbert problem (3.2) can be written in the form

$$f(\mathbf{x}) = \int_{\partial\Omega} K_{\mathcal{B}}(\mathbf{x}, \mathbf{y}) d\sigma_{\mathbf{y}} u(\mathbf{y}), \ \mathbf{x} \in \mathbb{R}^3 \setminus \partial\Omega, \tag{3.6}$$

where $u(\mathbf{y}) \in H^\alpha(\partial\Omega, \mathbb{H})$. By Lemma 2.4, it follows that

$$f^+(\mathbf{x}) = \frac{u(\mathbf{x})}{2} + \mathcal{S}_{\partial\Omega}[u](\mathbf{x}), \tag{3.7}$$

and

$$f^-(\mathbf{x}) = -\frac{u(\mathbf{x})}{2} + \mathcal{S}_{\partial\Omega}[u](\mathbf{x}). \tag{3.8}$$

We then combine (3.7), (3.8) and the boundary value condition in (3.2) to reduce the Riemann-Hilbert problem (3.2) to an equivalent nonlinear singular integral equation

$$\begin{aligned} u(\mathbf{x}) =& [\frac{u(\mathbf{x})}{2} - \mathcal{S}_{\partial\Omega}[u](\mathbf{x})](A(\mathbf{x}) - 1) \\ &+ \lambda G(\mathbf{x}, f^+(\mathbf{x}), f^-(\mathbf{x})), \qquad \mathbf{x} \in \partial\Omega. \end{aligned} \tag{3.9}$$

Let $\mathfrak{F}$ denote an integral operator defined by the right side of (3.9), i.e.

$$\begin{aligned} (\mathfrak{F}u)(\mathbf{x}) =& [\frac{u(\mathbf{x})}{2} - \mathcal{S}_{\partial\Omega}[u](\mathbf{x})](A(\mathbf{x}) - 1) \\ &+ \lambda G(\mathbf{x}, f^+(\mathbf{x}), f^-(\mathbf{x})), \qquad \mathbf{x} \in \partial\Omega. \end{aligned} \tag{3.10}$$

The quaternion valued continuous function space is denoted by $C(\partial\Omega, \mathbb{H})$ which the norm is defined as

$$\|u\|_\infty = \max_{\mathbf{x}\in\partial\Omega} \|u(\mathbf{x})\|$$

and $C(\partial\Omega, \mathbb{H})$ is a Banach space. Denote

$$\mathcal{M} = \{u(\mathbf{x}) | u(\mathbf{x}) \in H^\alpha(\partial\Omega, \mathbb{H}), \|u\|_{(\alpha,\partial\Omega)} \leq r\},$$

here r is a positive constant and $0 < r < 1$, by using Theorem 3.1, then $\mathcal{M}$ is a compact and convex closed set in $C(\partial\Omega, \mathbb{H})$. Thus, for any $u \in \mathcal{M}$, in view of (3.4) and (3.5), we have

$$\begin{aligned}
&\|\mathfrak{F}u\|_{(\alpha,\partial\Omega)} \\
\leq&(\frac{1}{2} + \widehat{C})\|A-1\|_{(\alpha,\partial\Omega)}\|u\|_{(\alpha,\partial\Omega)} + |\lambda|\|G(\mathbf{x}, f^+(\mathbf{x}), f^-(\mathbf{x}))\|_{(\alpha,\partial\Omega)} \\
=&(\frac{1}{2} + \widehat{C})\|A-1\|_{(\alpha,\partial\Omega)}\|u\|_{(\alpha,\partial\Omega)} + |\lambda| \left[\|G(\mathbf{x}, f^+(\mathbf{x}), f^-(\mathbf{x})) - G(\mathbf{x},0,0)\|_{(\alpha,\partial\Omega)}\right] \\
\leq&(\frac{1}{2} + \widehat{C})\|A-1\|_{(\alpha,\partial\Omega)}\|u\|_{(\alpha,\partial\Omega)} + |\lambda| N \left[\|f^+(\mathbf{x})\| + \|f^-(\mathbf{x})\|\right] \\
\leq&(\frac{1}{2} + \widehat{C})\|A-1\|_{(\alpha,\partial\Omega)}\|u\|_{(\alpha,\partial\Omega)} + |\lambda| N \left[\frac{1}{2}\|u\|_{(\alpha,\partial\Omega)} + \widehat{C}\|u\|_{(\alpha,\partial\Omega)}\right] \\
\leq&\|u\|_{(\alpha,\partial\Omega)} \\
\leq& r,
\end{aligned} \tag{3.11}$$

hence the operator $\mathfrak{F}$ maps $\mathcal{M}$ into itself.

Now we assert that $\mathfrak{F} : \mathcal{M} \to \mathcal{M}$ is continuous. Indeed, let u_n be a sequence in $\mathcal{M}$ such that $\{u_n(\mathbf{x})\}_{n=1}^{+\infty} \subset \mathcal{M}$ is uniformly convergent to a function $u(\mathbf{x})$ i.e., $u_n \to u$ in $C(\partial\Omega, \mathbb{H})$. By (3.10), we then have

$$\begin{aligned}
&\|(\mathfrak{F}u_n)(\mathbf{x}) - (\mathfrak{F}u)(\mathbf{x})\| \\
\leq&\|\mathcal{S}_{\partial\Omega}[u_n - u](\mathbf{x})(A(\mathbf{x}) - 1)\| \\
&+ \frac{1}{2}\|(u_n(\mathbf{x}) - u(\mathbf{x}))(A(\mathbf{x}) - 1)\| \\
&+ |\lambda|\|G(\mathbf{x}, \frac{u_n(\mathbf{x})}{2} + \mathcal{S}_{\partial\Omega}[u_n](\mathbf{x}), -\frac{u_n(\mathbf{x})}{2} + \mathcal{S}_{\partial\Omega}[u_n](\mathbf{x})) \\
&- G(\mathbf{x}, \frac{u(\mathbf{x})}{2} + \mathcal{S}_{\partial\Omega}[u](\mathbf{x}), -\frac{u(\mathbf{x})}{2} + \mathcal{S}_{\partial\Omega}[u](\mathbf{x}))\| \\
\leq&(\|\mathcal{S}_{\partial\Omega}[u_n](\mathbf{x}) - \mathcal{S}_{\partial\Omega}[u](\mathbf{x})\| + \frac{1}{2}\|u_n(\mathbf{x}) - u(\mathbf{x})\|)\|A-1\|_\infty \\
&+ |\lambda| N(\|u_n(\mathbf{x}) - u(\mathbf{x})\| + \|\mathcal{S}_{\partial\Omega}[u_n](\mathbf{x}) - \mathcal{S}_{\partial\Omega}[u](\mathbf{x})\|).
\end{aligned} \tag{3.12}$$

In the following, we only need to prove that $\mathcal{S}[u_n](\mathbf{x})$ converges uniformly to $\mathcal{S}[u](\mathbf{x})$. Since $\partial\Omega$ is Lyapunov boundary, the normal vector $\mathbf{n}$ is continuous on $\partial\Omega$, we can choose d $(0 < d \leq 1)$ small enough such that for the scalar product $(\mathbf{n}(\mathbf{x}), \mathbf{n}(\mathbf{y})) > \frac{1}{2}$ for all $\mathbf{x}, \mathbf{y} \in \partial\Omega$ with $\|\mathbf{x} - \mathbf{y}\| < d$. Furthermore, we can assume that d is small enough such that the set $\partial L \triangleq \{\mathbf{y} \in \partial\Omega : \|\mathbf{y}-\mathbf{x}\| < d\}$ is connected for each $\mathbf{x} \in \partial\Omega$, we have

$$\begin{aligned}\|\mathcal{S}_{\partial\Omega}[u_n](\mathbf{x}) - \mathcal{S}_{\partial\Omega}[u](\mathbf{x})\| \leq & \| \int_{\partial\Omega\setminus\partial L} K_{\mathcal{B}}(\mathbf{x},\mathbf{y}) d\sigma_y (u_n(\mathbf{y}) - u(\mathbf{y}))\| \\ & + \| \int_{\partial L} K_{\mathcal{B}}(\mathbf{x},\mathbf{y}) d\sigma_y (u_n(\mathbf{y}) - u(\mathbf{y}))\|.\end{aligned} \tag{3.13}$$

Denote

$$I_1 := \| \int_{\partial\Omega\setminus\partial L} K_{\mathcal{B}}(\mathbf{x},\mathbf{y}) d\sigma_y (u_n(\mathbf{y}) - u(\mathbf{y}))\|$$

and

$$I_2 := \| \int_{\partial L} K_{\mathcal{B}}(\mathbf{x},\mathbf{y}) d\sigma_y (u_n(\mathbf{y}) - u(\mathbf{y}))\|.$$

Now we have

$$I_1 \leq \frac{\widetilde{b}}{4\pi} \int_{\partial\Omega\setminus\partial L} \frac{1}{d} dS \|u_n - u\|_\infty \leq \frac{\widetilde{b}|\partial\Omega|}{4\pi d} \|u_n - u\|_\infty.$$

We continue to estimate I_2. The condition $(\mathbf{n}(\mathbf{x}), \mathbf{n}(\mathbf{y})) > \frac{1}{2}$ implies that ∂L can be projected bijectively onto the tangent plane to $\partial\Omega$ at the point $\mathbf{x}$. With the help of polar coordinates (r, θ) in the tangent plane with origin $\mathbf{x}$ and the surface element $dS_{\mathbf{y}} = \frac{r dr d\theta}{(\mathbf{n}(\mathbf{x}), \mathbf{n}(\mathbf{y}))}$, we have

$$I_2 \leq \widetilde{b} e^{\widetilde{M}} d \|u_n - u\|_\infty,$$

where $\widetilde{M} = \max_{\mathbf{x}\in\overline{\Omega}} |\mathcal{B}(\mathbf{x})|$ and $\widetilde{b} = \max_{\mathbf{x}\in\overline{\Omega}} \|b(\mathbf{x})\|$.

In view of I_1 and I_2, it follows that

$$\|\mathcal{S}_{\partial\Omega}[u_n](\mathbf{x}) - \mathcal{S}_{\partial\Omega}[u](\mathbf{x})\| \leq \widetilde{b}(\frac{|\partial\Omega|}{4\pi d} + e^{\widetilde{M}} d)\|u_n - u\|_\infty. \tag{3.14}$$

By (3.14) and (3.12), we obtain that

$$\begin{aligned}\|(\mathfrak{F}u_n)(\mathbf{x}) - (\mathfrak{F}u)(\mathbf{x})\| & \leq \widetilde{C}\|u_n - u\|_\infty \\ & \to 0 \quad (n \to +\infty),\end{aligned}$$

where $\widetilde{C} = \widetilde{b}(\frac{|\partial\Omega|}{4\pi d} + e^{\widetilde{M}} d)(\|A-1\|_\infty + 1) + |\lambda| N + \frac{\|A-1\|_\infty}{2}$.

Then $\mathfrak{F}$ is a continuous mapping from $\mathcal{M}$ into itself. Based on Lemma 3.1, this problem is solvable. The proof is finished. □

Example 3.1.

Consider the following boundary value problem of Riemann-Hilbert type:

$$\begin{cases} D_{b(\mathbf{x})}[f] = 0, & \text{in } \mathbb{R}^3 \setminus \partial\Omega, \\ f^+(\mathbf{x}) = f^-(\mathbf{x}) + \lambda G(\mathbf{x}, f^+, f^-), \ \mathbf{x} \in \partial\Omega, \\ \|f(\infty)\| = 0, \end{cases} \tag{3.15}$$

here $G(\mathbf{x}, f_1, f_2)$ satisfies two conditions in the above nonlinear Riemann-Hilbert problem (3.2) and $|\lambda| \leq \frac{2}{N(1+2\widehat{C})}$.

Since $A(x) = 1$, it is clear that the problem (3.15) has at least one solution by Theorem 3.3.

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