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DYNAMICS OF KIRCHHOFF TYPE PLATE EQUATIONS WITH NONLINEAR DAMPING DRIVEN BY MULTIPLICATIVE NOISE*

Xiaobin Yao[1,†]

Abstract This paper investigates mainly the long term behavior of the kirchhoff type stochastic plate equations with multiplicative noise and nonlinear damping on unbounded domains. Due to the noncompactness of Sobolev embeddings on unbounded domains, pullback asymptotic compactness of random dynamical system associated with such random plate equation is proved by the tail-estimates method. This paper is an extension of [23].

Keywords Pullback attractors, plate equation, kirchhoff type, unbounded domains, multiplicative white noise.

MSC(2010) 35B40, 35B41.

1. Introduction

In this paper, we study the asymptotic behavior of solutions for the following non-autonomous kirchhoff type stochastic plate equation with multiplicative noise and nonlinear damping defined on the unbounded domain $\mathbb{R}^n$:

$$u_{tt} + \Delta^2 u + h(u_t) + \lambda u - M(\|\nabla u\|^2)\Delta u + f(x,u) = g(x,t) + \varepsilon u \circ \frac{dw}{dt} \tag{1.1}$$

with the initial value conditions

$$u(x,\tau) = u_0(x), \quad u_t(x,\tau) = u_1(x), \tag{1.2}$$

where $x \in \mathbb{R}^n$, $t > \tau$ with $\tau \in \mathbb{R}$, $\lambda > 0$ and ε are constants, $h(u_t)$ is a nonlinear damping term, f is a given interaction term, g is a given function satisfying $g \in L^2_{loc}(\mathbb{R}, H^1(\mathbb{R}^n))$, and w is a two-sided real-valued Wiener process on a probability space. The stochastic equation (1.1) is understood in the sense of Stratonovich's integration.

As for deterministic plate equations, there are a lot of works were researched to show the existence of global attractors, see [2, 6–9, 12, 14, 20, 21, 27, 28, 30]. The existence of random attractors for stochastic plate equations with additive noise or linear multiplicative noise has been investigated in [11, 15, 16] on bounded domains.

†The corresponding author.
[1]School of Mathematics and Statistics, Qinghai Minzu University, Xining 810007, Qinghai, China
*The authors were supported by National Natural Science Foundation of China (12161071).
Email: yaoxiaobin2008@163.com(X. Yao)

Moreover, there are results about the existence of random attractors and asymptotic compactness for stochastic plate equations on unbounded domains in [22–26, 29].

For the case $M(s) \equiv 0$ in (1.1), we have investigated the existence of a random attractor for the stochastic plate equations with multiplicative noise and nonlinear damping defined in the entire space, see [23]. However, when equation (1.1) is kirchhoff type, the problem is not yet considered by any predecessors.

Since equation (1.1) has not only random term, but also non-autonomous deterministic term, thus in order to study random attractor of (1.1), we need adapt the concept of random attractor which was introduced in [18]. On the other hand, the noncompactness of Sobolev embeddings on unbounded domains gives rise to difficulty in showing the pullback asymptotic compactness of solutions. To get through of it, we conquer the difficulty by using the uniform estimates on the tails of solutions outside a bounded ball in $\mathbb{R}^n$ and the splitting technique([19]), as well as the compactness methods introduced in [10].

The organizational structure of this paper is as follows. In the next section, we present some notations and proposition about random dynamical systems. In Section 3, we establish a continuous cocycle for Eq.(1.1) in $H^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$. In Section 4, we obtain all necessary uniform estimates of solutions. In Section 5, we show the existence and uniqueness of a random attractor for (1.1) defined $\mathbb{R}^n$.

Throughout the paper, we use $||\cdot||$ and $(\cdot,\cdot)$ to denote the norm and the inner product of $L^2(\mathbb{R}^n)$, respectively. The norms of $L^p(\mathbb{R}^n)$ and a Banach space X are generally written as $||\cdot||_p$ and $||\cdot||_X$, respectively. The letters c and c_i $(i = 1, 2, \ldots)$ are generic positive constants which may change their values from line to line or even in the same line and do not depend on ε.

2. Preliminaries

In this section, we first recall some notations and proposition on non-autonomous random dynamical systems which is can be found in [4,5,18,31], and then introduce some assumptions.

Let $(X, \|\cdot\|_X)$ be a complete separable metric space, and $(\Omega, \mathcal{F}, \mathcal{P}, \{\theta_t\}_{t\in\mathbb{R}})$ an ergodic metric dynamical system, where $\Omega = \{\omega \in C(\mathbb{R}, \mathbb{R}) : \omega(0) = 0\}$, $\mathcal{F}$ is the Borel σ-algebra induced by the compact open topology of Ω, and $\mathcal{P}$ is the Wiener measure on $(\Omega, \mathcal{F})$. There is a classical group $\{\theta_t\}_{t\in\mathbb{R}}$ acting on $(\Omega, \mathcal{F}, \mathcal{P})$ which is defined by

$$\theta_t\omega(\cdot) = \omega(\cdot + t) - \omega(t), \quad \text{for all } \omega \in \Omega,\ t \in \mathbb{R},$$

then $(\Omega, \mathcal{F}, \mathcal{P}, \{\theta_t\}_{t\in\mathbb{R}})$ is a parametric dynamical system, see [1].

Proposition 2.1. *Let $\mathcal{D}$ be an inclusion closed collection of some families of nonempty subsets of X, and Φ be a continuous cocycle on X over $(\Omega, \mathcal{F}, \mathcal{P}, \{\theta_t\}_{t\in\mathbb{R}})$. Then Φ has a unique $\mathcal{D}$-pullback random attractor $\mathcal{A}$ in $\mathcal{D}$ if Φ is $\mathcal{D}$-pullback asymptotically compact in X and Φ has a closed measurable $\mathcal{D}$-pullback absorbing set K in $\mathcal{D}$.*

We introduce the following hypotheses to complete the uniform estimates.

Assumption I. *Assume that the functions $h,\ f, M$ satisfy the following conditions:*
(1) *Let $F(x,u) = \int_0^u f(x,s)ds$ for $x \in \mathbb{R}^n$ and $u \in \mathbb{R}$, there exist positive*

constants $c_i (i = 1, 2, 3, 4)$, *such that*

$$|f(x,u)| \le c_1|u|^\gamma + \phi_1(x), \quad \phi_1 \in L^2(\mathbb{R}^n), \tag{2.1}$$

$$f(x,u)u - c_2F(x,u) \ge \phi_2(x), \quad \phi_2 \in L^1(\mathbb{R}^n), \tag{2.2}$$

$$F(x,u) \ge c_3|u|^{\gamma+1} - \phi_3(x), \quad \phi_3 \in L^1(\mathbb{R}^n), \tag{2.3}$$

$$|\frac{\partial f}{\partial u}(x,u)| \le \beta, \quad |\frac{\partial f}{\partial x}(x,u)| \le \phi_4(x), \quad \phi_4 \in L^2(\mathbb{R}^n), \tag{2.4}$$

where $\beta > 0,\ 1 \le \gamma \le \frac{n+4}{n-4}$. *Note that* (2.1) *and* (2.2) *imply*

$$F(x,u) \le c(|u|^2 + |u|^{\gamma+1} + \phi_1^2 + \phi_2). \tag{2.5}$$

(2) *There exist two constants* β_1, β_2 *such that*

$$h(0) = 0, \qquad 0 < \beta_1 \le h'(v) \le \beta_2 < \infty. \tag{2.6}$$

(3) *Assume* $M(\cdot) \in C^1(\mathbb{R})$, *denote* $\hat{M}(z) = \int_0^z M(r)dr$, *for* $\forall\ z \ge 0$,

$$M(z)z \ge \hat{M}(z) \ge 0, \tag{2.7}$$

$$M_1 \le M(s) \le M_2, \tag{2.8}$$

where M_1 *and* M_2 *are some positive real constants.*

Assumption II. *We assume that* $\sigma, \delta, \varepsilon$ *and* $g(x,t)$ *satisfy the following conditions:*

$$\sigma = \min\{\delta, \frac{\delta c_2}{2}\}, \tag{2.9}$$

$$\delta > 0 \text{ satisfies } \ \lambda + \delta^2 - \beta_2\delta > 1,\ \beta_1 > 5\delta + \frac{\beta^2}{\delta(\lambda + \delta^2 - \beta_2\delta)}, \tag{2.10}$$

$$|\varepsilon| < \min\left\{\frac{-2\sqrt{\delta}(\gamma_2\gamma_3 + \gamma_1) + \sqrt{4\delta(\gamma_2\gamma_3 + \gamma_1)^2 + \pi\delta\gamma_2\sigma}}{\gamma_2\sqrt{\pi}},\right.$$
$$\left.\frac{-2\sqrt{\delta}(\gamma_2\gamma_4 + 1) + \sqrt{4\delta(\gamma_2\gamma_4 + 1)^2 + \pi\delta\gamma_2\sigma}}{\gamma_2\sqrt{\pi}}\right\}, \tag{2.11}$$

where $\gamma_1 = \max\{2, \frac{c_1c_3^{-1}}{2}\},\ \gamma_2 = \frac{1}{\lambda + \delta^2 - \beta_2\delta}, \gamma_3 = \frac{M_2^2}{4} + 1 + \frac{3}{2}\delta + \frac{1}{2}\beta_2,\ \gamma_4 = \frac{3}{2}\delta + \frac{1}{2}\beta_2 + 2\beta_2\delta$.

Moreover,

$$\int_{-\infty}^0 e^{\sigma s}\|g(\cdot, \tau + s)\|_1^2 ds < \infty, \quad \forall\ \tau \in \mathbb{R}, \tag{2.12}$$

and

$$\lim_{k\to\infty}\int_{-\infty}^0 e^{\sigma s}\int_{|x|\ge k}|g(x, \tau + s)|^2 dxds = 0,\ \forall\ \tau \in \mathbb{R}, \tag{2.13}$$

where $|\cdot|$ denotes the absolute value of real number in $\mathbb{R}$.

3. Cocycles for stochastic plate equation

In this section, we show that (1.1)-(1.2) generates a continuous cocycle in $H^2(\mathbb{R}^n)\times L^2(\mathbb{R}^n)$.

Let $-\Delta$ denote the Laplace operator in $\mathbb{R}^n$, $A=\Delta^2$ with the domain $D(A)=H^4(\mathbb{R}^n)$. We can also define the powers A^ν of A for $\nu\in\mathbb{R}$. The space $V_\nu=D(A^{\frac{\nu}{4}})$ is a Hilbert space with the following inner product and norm

$$(u,v)_\nu=(A^{\frac{\nu}{4}}u,A^{\frac{\nu}{4}}v),\qquad \|\cdot\|_\nu=\|A^{\frac{\nu}{4}}\cdot\|.$$

For brevity, the notation $(\cdot,\cdot)$ for L^2-inner product will also be used for the notation of duality pairing between dual spaces, $\|\cdot\|$ denotes the L^2-norm.

Let $E=H^2\times L^2$, with the Sobolev norm

$$\|y\|_{H^2\times L^2}=(\|v\|^2+\|u\|^2+\|\Delta u\|^2)^{\frac{1}{2}},\quad \text{for}\quad y=(u,v)^\top\in E. \tag{3.1}$$

Let $\xi=u_t+\delta u$, where δ is a small positive constant whose value will be determined later. Substituting $u_t=\xi-\delta u$ into (1.1) we find

$$\frac{du}{dt}+\delta u=\xi, \tag{3.2}$$

$$\begin{aligned}&\frac{d\xi}{dt}-\delta\xi+(\delta^2+\lambda)u+\Delta^2u+h(\xi-\delta u)-M(\|\nabla u\|^2)\Delta u+f(x,u)\\ =&g(x,t)+\varepsilon u\circ\frac{dw}{dt}\end{aligned} \tag{3.3}$$

with the initial value conditions

$$u(x,\tau)=u_0(x),\qquad \xi(x,\tau)=z_0(x), \tag{3.4}$$

where $\xi_0(x)=u_1(x)+\delta u_0(x),\ x\in\mathbb{R}^n$.

Consider Ornstein-Uhlenbeck equation $dz+\delta zdt=d\omega,\ z(-\infty)=0$, and Ornstein-Uhlenbeck process

$$z(\theta_t\omega)=z(t,\omega)=-\delta\int_{-\infty}^0 e^{\delta s}(\theta_t\omega)(s)ds. \tag{3.5}$$

From [1], it is known that the random variable $|z(\omega)|$ is a stationary, ergodic and tempered stochastic process, and there is a θ_t-invariant set $\widetilde{\Omega}\subset\Omega$ of full $\mathcal{P}$ measure such that $z(\theta_t\omega)$ is continuous in t for every $\omega\in\widetilde{\Omega}$. For convenience, we shall simply write $\widetilde{\Omega}$ as Ω.

Now, let $v(x,t)=\xi(x,t)-\varepsilon u(x,t)z(\theta_t\omega)$, we obtain the equivalent system of (3.2)-(3.4),

$$\frac{du}{dt}+\delta u-v=\varepsilon uz(\theta_t\omega), \tag{3.6}$$

$$\begin{aligned}&\frac{dv}{dt}-\delta v+(\delta^2+\lambda+A)u-M(\|\nabla u\|^2)\Delta u+f(x,u)\\ =&g(x,t)-h(v+\varepsilon uz(\theta_t\omega)-\delta u)-\varepsilon(v-3\delta u+\varepsilon uz(\theta_t\omega))z(\theta_t\omega)\end{aligned} \tag{3.7}$$

with the initial value conditions

$$u(x,\tau)=u_0(x),\qquad v(x,\tau)=v_0(x), \tag{3.8}$$

where $v_0(x) = \xi_0(x) - \varepsilon z(\theta_t\omega)u_0(x),\ x \in \mathbb{R}^n$.

The well-posedness of local weak solutions for the problem of the random PDE (3.6)-(3.8) in $E = H^2(\mathbb{R}^n) \times H(\mathbb{R}^n)$ can be shown by Galerkin approximation and compactness method as in [3,13,17,29]. Under conditions (2.1)-(2.4) and (2.6)-(2.8), for every $\omega \in \Omega, \tau \in \mathbb{R}$ and $(u_0, v_0) \in E$, we can prove the problem (3.6)-(3.8) has a unique global solution $(u(\cdot,\tau,\omega,u_0), v(\cdot,\tau,\omega,v_0)) \in C([\tau,\infty), E)$. Moreover, for $t \geq \tau$, $(u(t,\tau,\omega,u_0), v(t,\tau,\omega,v_0))$ is $(\mathcal{F},\ \mathcal{B}(H^2(\mathbb{R}^n)) \times \mathcal{B}(L^2(\mathbb{R}^n)))$-measurable in ω and continuous in (u_0, v_0) with respect to the E-norm.

Thus the solution mapping can be used to define a continuous cocycle for (3.2)-(3.4). Let $\Phi : \mathbb{R}^+ \times \mathbb{R} \times \Omega \times E \to E$ be a mapping given by

$$\Phi(t,\tau,\omega,(u_0,v_0)) = (u(t+\tau,\tau,\theta_{-\tau}\omega,u_0), v(t+\tau,\tau,\theta_{-\tau}\omega,v_0)), \tag{3.9}$$

where $v(t+\tau,\tau,\theta_{-\tau}\omega,v_0) = \xi(t+\tau,\tau,\theta_{-\tau}\omega,\xi_0) - \varepsilon z(\theta_t\omega)u(t+\tau,\tau,\theta_{-\tau}\omega,u_0)$ with $v_0 = \xi_0 - \varepsilon z(\omega)u_0(x)$. Then Φ is a continuous cocycle over $\mathbb{R}$ and $(\Omega, \mathcal{F}, \mathbb{P}, \{\theta_t\}_{t\in\mathbb{R}})$ on E. For each $t \in \mathbb{R}^+, \tau \in \mathbb{R}, \omega \in \Omega$,

$$\begin{aligned}
&\Phi(t,\tau-t,\theta_{-t}\omega,(u_0,v_0))\\
=&(u(\tau,\tau-t,\theta_{-\tau}\omega,u_0), v(\tau,\tau-t,\theta_{-\tau}\omega,v_0))\\
=&(u(\tau,\tau-t,\theta_{-\tau}\omega,u_0), \xi(\tau,\tau-t,\theta_{-\tau}\omega,\xi_0) + \varepsilon z(\omega)u(\tau,\tau-t,\theta_{-\tau}\omega,u_0)).
\end{aligned} \tag{3.10}$$

Given a bounded nonempty subset B of E, we write $\|B\| = \sup\limits_{\phi\in B}\|\phi\|_E$. Let $D = \{D(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\}$ be a family of bounded nonempty subsets of E such that for every $\tau \in \mathbb{R}, \omega \in \Omega$,

$$\lim_{s\to-\infty} e^{\sigma s}\|D(\tau+s,\theta_s\omega)\|_E^2 = 0. \tag{3.11}$$

Let $\mathcal{D}$ be the collection of all such families, that is,

$$\mathcal{D} = \{D = \{D(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\} :\ D \text{ satisfies (3.11)}\}. \tag{3.12}$$

4. Uniform estimates of solutions

In this section, we impose the uniform estimates for solutions of (3.6)-(3.8) and get the absorbing set.

We define a new norm $\|\cdot\|_E$ by

$$\|Y\|_E = (\|v\|^2 + (\lambda + \delta^2 - \beta_2\delta)\|u\|^2 + \|\Delta u\|^2)^{\frac{1}{2}}, \quad \text{for } \ Y = (u,v) \in E. \tag{4.1}$$

It is easy to check that $\|\cdot\|_E$ is equivalent to the usual norm $\|\cdot\|_{H^2\times L^2}$ in (3.1). First we show that the cocycle Φ has a pullback $\mathcal{D}$-absorbing set in $\mathcal{D}$.

Lemma 4.1. *Under Assumptions* I *and* II, *for every* $\tau \in \mathbb{R}, \omega \in \Omega$, $D = \{D(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$, *there exists* $T = T(\tau,\omega,D) > 0$ *such that for all* $t \geq T$ *the solution of problem* (3.6)-(3.8) *satisfies*

$$\|Y(\tau,\tau-t,\theta_{-\tau}\omega,D(\tau-t,\theta_{-t}\omega))\|_E^2 \leq R_1(\tau,\omega),$$

and $R_1(\tau,\omega)$ *is given by*

$$R_1(\tau,\omega) = M + M\int_{-\infty}^0 e^{2\int_0^s \left[\sigma - \gamma_1|\varepsilon||z(\theta_r\omega)| - \gamma_2\left(\frac{1}{2}\varepsilon^2|z(\theta_r\omega)|^2 + \gamma_3|\varepsilon||z(\theta_t\omega)|\right)\right]dr}$$

$$\times \left(\|g(\cdot, s+\tau)\|^2 + |\varepsilon||z(\theta_s\omega)|\right)ds \tag{4.2}$$

where M is a positive constant independent of τ, ω, D and ε.

Proof. Taking the inner product of (3.7) with v in $L^2(\mathbb{R}^n)$, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|v\|^2 - (\delta - \varepsilon z(\theta_t\omega))\|v\|^2 + (\lambda+\delta^2)(u,v) + (Au,v)\\&-(M(\|\nabla u\|^2)\Delta u, v) + (f(x,u),v)\\=&\varepsilon z(\theta_t\omega)(3\delta - \varepsilon z(\theta_t\omega))(u,v) - (h(v+\varepsilon u z(\theta_t\omega) - \delta u), v) + (g(x,t),v).\end{aligned} \tag{4.3}$$

Next, we estimate some terms in (4.3).

By (3.6), we have

$$v = u_t - \varepsilon u z(\theta_t\omega) + \delta u. \tag{4.4}$$

By (2.6) and Lagrange's mean value theorem, we have

$$\begin{aligned}&-(h(v+\varepsilon u z(\theta_t\omega) - \delta u), v)\\=&-(h(v+\varepsilon u z(\theta_t\omega) - \delta u) - h(0), v)\\=&-(h'(\vartheta)(v+\varepsilon u z(\theta_t\omega) - \delta u), v)\\\le&-\beta_1\|v\|^2 - (h'(\vartheta)(\varepsilon u z(\theta_t\omega) - \delta u), v)\\\le&-\beta_1\|v\|^2 + \beta_2|\varepsilon||z(\theta_t\omega)|\|u\|\|v\| + h'(\vartheta)\delta(u,v),\end{aligned} \tag{4.5}$$

where ϑ is between 0 and $v+\varepsilon u z(\theta_t\omega) - \delta u$.

By (2.6) and (4.4), we get

$$\begin{aligned}&h'(\vartheta)\delta(u,v)\\=&h'(\vartheta)\delta(u, u_t - \varepsilon u z(\theta_t\omega) + \delta u)\\\le&\beta_2\delta\cdot\frac{1}{2}\frac{d}{dt}\|u\|^2 + \beta_2\delta^2\|u\|^2 + \beta_2\delta|\varepsilon||z(\theta_t\omega)|\|u\|^2.\end{aligned} \tag{4.6}$$

Substituting (4.4) into the third and fourth terms on the left-hand side of (4.3), we find that

$$\begin{aligned}(u,v) =&(u, u_t - \varepsilon u z(\theta_t\omega) + \delta u)\\\ge&\frac{1}{2}\frac{d}{dt}\|u\|^2 + \delta\|u\|^2 - |\varepsilon||z(\theta_t\omega)|\|u\|^2,\end{aligned} \tag{4.7}$$

and

$$\begin{aligned}(Au,v) &= (\Delta u, \Delta v)\\&= (\Delta u, \Delta u_t - \varepsilon z(\theta_t\omega)\Delta u + \delta\Delta u)\\&\ge \frac{1}{2}\frac{d}{dt}\|\Delta u\|^2 + \delta\|\Delta u\|^2 - |\varepsilon||z(\theta_t\omega)|\|\Delta u\|^2.\end{aligned} \tag{4.8}$$

For the first term on the right-hand side of (4.3), by (4.5), using the Cauchy-Schwarz inequality and Young's inequality, we have

$$\varepsilon z(\theta_t\omega)(3\delta - \varepsilon z(\theta_t\omega))(u,v) + \beta_2|\varepsilon||z(\theta_t\omega)|\|u\|\|v\|$$

$$
\begin{aligned}
&=(3\delta\varepsilon z(\theta_t\omega)-\varepsilon^2 z^2(\theta_t\omega))(u,v)+\beta_2|\varepsilon||z(\theta_t\omega)|\|u\|\|v\| \\
&\le(3\delta|\varepsilon||z(\theta_t\omega)|+\varepsilon^2|z(\theta_t\omega)|^2)\|u\|\|v\|+\beta_2|\varepsilon||z(\theta_t\omega)|\|u\|\|v\| \\
&=((3\delta+\beta_2)|\varepsilon||z(\theta_t\omega)|+\varepsilon^2|z(\theta_t\omega)|^2)\|u\|\|v\| \\
&\le(\frac{1}{2}(3\delta+\beta_2)|\varepsilon||z(\theta_t\omega)|+\frac{1}{2}\varepsilon^2|z(\theta_t\omega)|^2)(\|u\|^2+\|v\|^2), \qquad (4.9)
\end{aligned}
$$

and for the last term on the right-hand side of (4.3),

$$
(g,v)\le\|g\|\|v\|\le\frac{\|g\|^2}{2(\beta_1-\delta)}+\frac{\beta_1-\delta}{2}\|v\|^2. \qquad (4.10)
$$

Let $\widetilde{F}(x,u)=\int_{\mathbb{R}^n}F(x,u)dx$. Then for the last term on the left-hand side of (4.3) we have

$$
\begin{aligned}
(f(x,u),v)&=(f(x,u),u_t-\varepsilon z(\theta_t\omega)u+\delta u) \\
&=\frac{d}{dt}\widetilde{F}(x,u)+\delta(f(x,u),u)-\varepsilon z(\theta_t\omega)(f(x,u),u). \qquad (4.11)
\end{aligned}
$$

By condition (2.1) and (2.3), we have

$$
\begin{aligned}
&\varepsilon z(\theta_t\omega)\big(f(x,u),u\big) \\
\le& c_1|\varepsilon||z(\theta_t\omega)|\int_{\mathbb{R}^n}|u|^{\gamma+1}dx+\frac{1}{4}|\varepsilon||z(\theta_t\omega)|\|\phi_1\|^2+|\varepsilon||z(\theta_t\omega)|\|u\|^2 \\
\le& c_1c_3^{-1}|\varepsilon||z(\theta_t\omega)|\int_{\mathbb{R}^n}\big(F(x,u)+\phi_3\big)dx+\frac{1}{4}|\varepsilon||z(\theta_t\omega)|\|\phi_1\|^2+|\varepsilon||z(\theta_t\omega)|\|u\|^2 \\
\le& c_1c_3^{-1}|\varepsilon||z(\theta_t\omega)|\widetilde{F}(x,u)+c|\varepsilon||z(\theta_t\omega)|+|\varepsilon||z(\theta_t\omega)|\|u\|^2. \qquad (4.12)
\end{aligned}
$$

By (2.7) and (2.8) we have

$$
\begin{aligned}
&-(M(\|\nabla u\|^2)\Delta u,v) \\
=&-(M(\|\nabla u\|^2)\Delta u,u_t+\delta u-\varepsilon uz(\theta_t\omega)) \\
=&\frac{1}{2}\frac{d}{dt}\hat{M}(\|\nabla u\|^2)+\delta M(\|\nabla u\|^2)\|\nabla u\|^2-\varepsilon z(\theta_t\omega)(M(\|\nabla u\|^2)\Delta u,u) \\
\ge&\frac{1}{2}\frac{d}{dt}\hat{M}(\|\nabla u\|^2)+\delta M(\|\nabla u\|^2)\|\nabla u\|^2-M_2|\varepsilon|\cdot|z(\theta_t\omega)|\cdot|(\Delta u,u)| \\
\ge&\frac{1}{2}\frac{d}{dt}\hat{M}(\|\nabla u\|^2)+\delta\hat{M}(\|\nabla u\|^2)-|\varepsilon|\cdot|z(\theta_t\omega)|\|\Delta u\|^2-\frac{M_2^2}{4}|\varepsilon|\cdot|z(\theta_t\omega)|\|u\|^2. \qquad (4.13)
\end{aligned}
$$

Substitute (4.5)-(4.13) into (4.3), we get

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\big(\|v\|^2+(\lambda+\delta^2-\beta_2\delta)\|u\|^2+\|\Delta u\|^2+\hat{M}(\|\nabla u\|^2)+2\widetilde{F}(x,u)\big) \\
&+\delta\big(\|v\|^2+(\lambda+\delta^2-\beta_2\delta)\|u\|^2+\|\Delta u\|^2+\hat{M}(\|\nabla u\|^2)\big)+\delta c_2\widetilde{F}(x,u) \\
\le& c+\left(\frac{1}{2}(3\delta+\beta_2)|\varepsilon||z(\theta_t\omega)|+\frac{1}{2}\varepsilon^2|z(\theta_t\omega)|^2\right)(\|u\|^2+\|v\|^2) \\
&+2|\varepsilon||z(\theta_t\omega)|\big(\|v\|^2+(\lambda+\delta^2+\beta_2\delta)\|u\|^2+\|\Delta u\|^2\big)+(\frac{M_2^2}{4}+1)|\varepsilon||z(\theta_t\omega)|\|u\|^2
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{3\delta-\beta_1}{2}\|v\|^2+\frac{\|g\|^2}{2(\beta_1-\delta)}+c_1c_3^{-1}|\varepsilon||z(\theta_t\omega)|\widetilde{F}(x,u)+c|\varepsilon||z(\theta_t\omega)|\\
\leq&\bigg(\frac{1}{2}(3\delta+\beta_2)|\varepsilon||z(\theta_t\omega)|+\frac{1}{2}\varepsilon^2|z(\theta_t\omega)|^2\bigg)(\|u\|^2+\|v\|^2)\\
&+\gamma_1|\varepsilon||z(\theta_t\omega)|\big(\|v\|^2+(\lambda+\delta^2+\beta_2\delta)\|u\|^2+\|\Delta u\|^2+2\widetilde{F}(x,u)\big)\\
&+(\frac{M_2^2}{4}+1)|\varepsilon||z(\theta_t\omega)|\|u\|^2+c\big(1+\|g\|^2+|\varepsilon||z(\theta_t\omega)|\big).
\end{aligned}
$$

By (2.9), we get

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\big(\|v\|^2+(\lambda+\delta^2-\beta_2\delta)\|u\|^2+\|\Delta u\|^2+\hat{M}(\|\nabla u\|^2)+2\widetilde{F}(x,u)\big)\\
\leq&-\big[\sigma-\gamma_1|\varepsilon||z(\theta_t\omega)|-\gamma_2\big(\frac{1}{2}\varepsilon^2|z(\theta_t\omega)|^2+\gamma_3|\varepsilon||z(\theta_t\omega)|\big)\big]\\
&\times\big(\|v\|^2+(\lambda+\delta^2-\beta_2\delta)\|u\|^2+\|\Delta u\|^2+\hat{M}(\|\nabla u\|^2)+2\widetilde{F}(x,u)\big)\\
&+c\big(1+\|g\|^2+|\varepsilon||z(\theta_t\omega)|\big).
\end{aligned}
\tag{4.14}
$$

Let us denote

$$
\varrho(\tau,\omega)=\sigma-\gamma_1|\varepsilon||z(\theta_t\omega)|-\gamma_2\big(\frac{1}{2}\varepsilon^2|z(\theta_t\omega)|^2+\gamma_3|\varepsilon||z(\theta_t\omega)|\big).
\tag{4.15}
$$

Using the Gronwall's inequality to integrate (4.14) over $(\tau-t,\tau)$ with $t\geq 0$, we get

$$
\begin{aligned}
&\|v(\tau,\tau-t,\omega,v_0)\|^2+(\lambda+\delta^2-\beta_2\delta)\|u(\tau,\tau-t,\omega,u_0)\|^2\\
&+\|\Delta u(\tau,\tau-t,\omega,u_0)\|^2+\hat{M}(\|\nabla u(\tau,\tau-t,\omega,u_0)\|^2)+2\widetilde{F}\big(x,u(\tau,\tau-t,\omega,u_0)\big)\\
\leq&\big(\|v_0\|^2+(\lambda+\delta^2-\beta_2\delta)\|u_0\|^2+\|\Delta u_0\|^2+\hat{M}(\|\nabla u_0\|^2)\\
&+2\widetilde{F}(x,u_0)\big)e^{2\int_\tau^{\tau-t}\varrho(s,\omega)ds}+c\int_{\tau-t}^{\tau}e^{2\int_\tau^s\varrho(r,\omega)dr}\big(1+\|g(\cdot,s)\|^2+|\varepsilon||z(\theta_s\omega)|\big)ds.
\end{aligned}
\tag{4.16}
$$

Replacing ω by $\theta_{-\tau}\omega$ in the above we obtain, for every $t\in\mathbb{R}^+,\ \tau\in\mathbb{R},$ and $\omega\in\Omega$,

$$
\begin{aligned}
&\|v(\tau,\tau-t,\theta_{-\tau}\omega,v_0)\|^2+(\lambda+\delta^2-\beta_2\delta)\|u(\tau,\tau-t,\theta_{-\tau}\omega,u_0)\|^2\\
&+\|\Delta u(\tau,\tau-t,\theta_{-\tau}\omega,u_0)\|^2+\hat{M}(\|\nabla u(\tau,\tau-t,\theta_{-\tau}\omega,u_0)\|^2)\\
&+2\widetilde{F}\big(x,u(\tau,\tau-t,\theta_{-\tau}\omega,u_0)\big)\\
\leq&\big(\|v_0\|^2+(\lambda+\delta^2-\beta_2\delta)\|u_0\|^2+\|\Delta u_0\|^2+\hat{M}(\|\nabla u_0\|^2)\\
&+2\widetilde{F}(x,u_0)\big)e^{2\int_\tau^{\tau-t}\varrho(s-\tau,\omega)ds}\\
&+c\int_{\tau-t}^{\tau}e^{2\int_\tau^s\varrho(r-\tau,\omega)dr}\big(1+\|g(\cdot,s)\|^2+|\varepsilon||z(\theta_{s-\tau}\omega)|\big)ds,
\end{aligned}
\tag{4.17}
$$

then

$$
\begin{aligned}
&\|v(\tau,\tau-t,\theta_{-\tau}\omega,v_0)\|^2+(\lambda+\delta^2-\beta_2\delta)\|u(\tau,\tau-t,\theta_{-\tau}\omega,u_0)\|^2\\
&+\|\Delta u(\tau,\tau-t,\theta_{-\tau}\omega,u_0)\|^2+\hat{M}(\|\nabla u(\tau,\tau-t,\theta_{-\tau}\omega,u_0)\|^2)\\
&+2\widetilde{F}\big(x,u(\tau,\tau-t,\theta_{-\tau}\omega,u_0)\big)
\end{aligned}
$$

$$\leq \big(\|v_0\|^2 + (\lambda+\delta^2-\beta_2\delta)\|u_0\|^2 + \|\Delta u_0\|^2 + \hat{M}(\|\nabla u_0\|^2) + 2\widetilde{F}(x,u_0)\big) e^{2\int_0^{-t}\varrho(s,\omega)ds}$$
$$+ c\int_{-t}^{0} e^{2\int_0^s \varrho(r,\omega)dr}\big(1+\|g(\cdot,s+\tau)\|^2 + |\varepsilon||z(\theta_s\omega)|\big)ds. \tag{4.18}$$

Since $|z(\theta_t\omega)|$ is stationary and ergodic, from (3.5) and the ergodic theorem we can get

$$\lim_{t\to\infty}\frac{1}{t}\int_{-t}^{0}|z(\theta_r\omega)|dr = \mathbf{E}(|z(\theta_r\omega)|) = \frac{1}{\sqrt{\pi\delta}}, \tag{4.19}$$
$$\lim_{t\to\infty}\frac{1}{t}\int_{-t}^{0}|z(\theta_r\omega)|^2dr = \mathbf{E}(|z(\theta_r\omega)|^2) = \frac{1}{2\delta}. \tag{4.20}$$

By (4.19)-(4.20), there exists $T_1(\omega)>0$ such that for all $t\geq T_1(\omega)$,

$$\int_{-t}^{0}|z(\theta_r\omega)|dr < \frac{2}{\sqrt{\pi\delta}}\ t, \qquad \int_{-t}^{0}|z(\theta_r\omega)|^2dr < \frac{1}{\delta}\ t. \tag{4.21}$$

Next we show that for any $s\leq -T_1$

$$e^{2\int_0^s \varrho(r,\omega)dr} \leq e^{\sigma s}. \tag{4.22}$$

By using the two inequalities in (4.21), we have

$$\begin{aligned}&\int_0^s\Big[\sigma-\gamma_1|\varepsilon||z(\theta_r\omega)| - \gamma_2\big(\frac{1}{2}\varepsilon^2|z(\theta_r\omega)|^2+\gamma_3|\varepsilon||z(\theta_r\omega)|\big)\Big]dr\\ >&\sigma s - |\varepsilon|\frac{2\gamma_1}{\sqrt{\pi\delta}}s - \gamma_2\big[\frac{1}{2}\varepsilon^2\frac{1}{\delta}+\gamma_3|\varepsilon|\frac{2}{\sqrt{\pi\delta}}\big]s\\ =&-\frac{\gamma_2}{2\delta}\varepsilon^2 s - \frac{2}{\sqrt{\pi\delta}}\big[\gamma_3\gamma_2+\gamma_1\big]|\varepsilon|s+\sigma s.\end{aligned} \tag{4.23}$$

In order to have the inequality in (4.22) valid, we need

$$\int_0^s\Big[\sigma-\gamma_1|\varepsilon||z(\theta_r\omega)| - \gamma_2\big(\frac{1}{2}\varepsilon^2|z(\theta_r\omega)|^2+\gamma_3|\varepsilon||z(\theta_r\omega)|\big)\Big]dr \leq \frac{\sigma}{2}s.$$

Since $s\leq -T_1$, then it requires that

$$\frac{\gamma_2}{2\delta}\varepsilon^2 + \frac{2}{\sqrt{\pi\delta}}\big[\gamma_3\gamma_2+\gamma_1\big]|\varepsilon| - \frac{\sigma}{2} < 0.$$

Solving this quadratic inequality, ε needs to satisfy (4.21) as we have assumed in Assumption II.

Since $|z(\theta_t\omega)|$ is tempered, by (2.12) and (4.22), we see that the following integral is convergent,

$$R_2^2(\tau,\omega) = c\int_{-\infty}^{0} e^{2\int_0^s \varrho(r,\omega)dr}\big(1+\|g(\cdot,s+\tau)\|^2+|\varepsilon||z(\theta_s\omega)|\big)ds. \tag{4.24}$$

Note that (2.5) implies

$$\int_{\mathbb{R}^n} F(x,u_0)dx \leq c\big(1+\|u_0\|^2+\|u_0\|_{H^2}^{\gamma+1}\big). \tag{4.25}$$

Since $D \in \mathcal{D}$ and $(u_0, v_0) \in D(\tau - t, \theta_{-t}\omega)$, for all $t \geq T_1$, we get from (2.7) and (4.24)-(4.25) that

$$
\begin{aligned}
&\big(\|v_0\|^2 + (\lambda + \delta^2 - \beta_2\delta)\|u_0\|^2 + \|\Delta u_0\|^2 + \hat{M}(\|\nabla u_0\|^2) \\
&\quad + 2\widetilde{F}(x, u_0)\big)e^{2\int_0^{-t} \varrho(s,\omega)ds} \\
\leq& ce^{-\sigma t}\big(1 + \|v_0\|^2 + \|u_0\|_{H^2}^2 + \|u_0\|_{H^2}^{\gamma+1}\big) \\
\leq& ce^{-\sigma t}\big(1 + \|D(\tau - t, \theta_{-t}\omega)\|^2 + \|D(\tau - t, \theta_{-t}\omega)\|^{\gamma+1}\big) \to 0, \quad \text{as} \quad t \to +\infty. \quad (4.26)
\end{aligned}
$$

From (4.1), (4.18), (4.24) and (4.26), there exists $T_2 = T_2(\tau, \omega, D) \geq T_1$ such for all that $t \geq T_2$,

$$
\|Y(\tau, \tau - t, \theta_{-\tau}\omega, Y_0(\theta_{-\tau}\omega))\|_E^2 \leq c(1 + R_2^2(\tau, \omega)).
$$

□

Similar to the proof of Lemma 4.1, we have the following result:

Lemma 4.2. *Under Assumptions* I *and* II*, for every* $\tau \in \mathbb{R}, \omega \in \Omega$, $D = \{D(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$*, there exists* $T = T(\tau, \omega, D) > 0$ *such that for all* $t \geq T,\ s \in [-t, 0]$*, the solution of problem* (3.6)-(3.8) *satisfies*

$$
\|Y(\tau + s, \tau - t, \theta_{-\tau}\omega, D(\tau - t, \theta_{-t}\omega))\|_E^2 \leq M + R_3(\tau, \omega)e^{2\int_s^0 \varrho(r,\omega)dr},
$$

where $(u_0, v_0) \in D(\tau - t, \theta_{-t}\omega)$*,* M *is a positive constant independent of* τ, ω, D *and* ε*, and* $R_3(\tau, \omega)$ *is a specific random variable.*

Lemma 4.3. *Under Assumptions* I *and* II*, for every* $\tau \in \mathbb{R}, \omega \in \Omega$, $D = \{D(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$*, there exists* $T = T(\tau, \omega, D) > 0$ *such that for all* $t \geq T$ *the solution of problem* (3.6)-(3.8) *satisfies*

$$
\|A^{\frac{1}{4}}Y(\tau, \tau - t, \theta_{-\tau}\omega, D(\tau - t, \theta_{-t}\omega))\|_E^2 \leq R_4(\tau, \omega),
$$

and $R_4(\tau, \omega)$ *is given by*

$$
\begin{aligned}
&R_4(\tau, \omega) \\
=&R_5^2(\tau, \omega) + ce^{-\sigma t}(\|A^{\frac{1}{4}}v_0\|^2 + \|A^{\frac{1}{4}}u_0\|^2 + \|A^{\frac{3}{4}}u_0\|^2 + M(\|\nabla u_0\|^2)\|A^{\frac{1}{2}}u_0\|^2),
\end{aligned} \tag{4.27}
$$

where $(u_0, v_0) \in D(\tau - t, \theta_{-t}\omega)$*,* c *is a positive constant independent of* τ, ω, D *and* ε*, and* $R_5(\tau, \omega)$ *is a specific random variable.*

Proof. Taking the inner product of (3.7) with $A^{\frac{1}{2}}v$ in $L^2(\mathbb{R}^n)$, we find that

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|A^{\frac{1}{4}}v\|^2 - (\delta - \varepsilon z(\theta_t\omega))\|A^{\frac{1}{4}}v\|^2 + (\lambda + \delta^2)(u, A^{\frac{1}{2}}v) + (Au, A^{\frac{1}{2}}v) \\
&- (M(\|\nabla u\|^2)\Delta u, A^{\frac{1}{2}}v) + (f(x, u), A^{\frac{1}{2}}v) \\
=&\varepsilon z(\theta_t\omega)(3\delta - \varepsilon z(\theta_t\omega))(u, A^{\frac{1}{2}}v) - (h(v + \varepsilon uz(\theta_t\omega) - \delta u), A^{\frac{1}{2}}v) + (g(x, t), A^{\frac{1}{2}}v).
\end{aligned} \tag{4.28}
$$

Similar to the proof of 4.1, we have the following estimates:

$$
-\Big(h\big(v + \varepsilon uz(\theta_t\omega) - \delta u\big), A^{\frac{1}{2}}v\Big)
$$

$$
\begin{aligned}
&= -\left(h\big(v+\varepsilon uz(\theta_t\omega)-\delta u\big)-h(0), A^{\frac{1}{2}}v\right)\\
&= -\left(h'(\vartheta)\big(v+\varepsilon uz(\theta_t\omega)-\delta u\big), A^{\frac{1}{2}}v\right)\\
&\leq -\beta_1\|A^{\frac{1}{4}}v\|^2-\left(h'(\vartheta)\big(\varepsilon uz(\theta_t\omega)-\delta u\big), A^{\frac{1}{2}}v\right)\\
&\leq -\beta_1\|A^{\frac{1}{4}}v\|^2+\beta_2|\varepsilon||z(\theta_t\omega)|\|A^{\frac{1}{4}}u\|\|A^{\frac{1}{4}}v\|+h'(\vartheta)\delta(u, A^{\frac{1}{2}}v),
\end{aligned}
\tag{4.29}
$$

$$
\begin{aligned}
&h'(\vartheta)\delta(u, A^{\frac{1}{2}}v)\\
=&h'(\vartheta)\delta\big(u, A^{\frac{1}{2}}u_t-\varepsilon z(\theta_t\omega)A^{\frac{1}{2}}u\big)+\delta A^{\frac{1}{2}}u\big)\\
\leq&\beta_2\delta\cdot\frac{1}{2}\frac{d}{dt}\|A^{\frac{1}{4}}u\|^2+\beta_2\delta^2\|A^{\frac{1}{4}}u\|^2+\beta_2\delta|\varepsilon||z(\theta_t\omega)|\|A^{\frac{1}{4}}u\|^2,
\end{aligned}
\tag{4.30}
$$

$$
\begin{aligned}
&(u, A^{\frac{1}{2}}v)\\
=&\big(u, A^{\frac{1}{2}}u_t-\varepsilon z(\theta_t\omega)A^{\frac{1}{2}}u+\delta A^{\frac{1}{2}}u\big)\\
\geq&\frac{1}{2}\frac{d}{dt}\|A^{\frac{1}{4}}u\|^2+\delta\|A^{\frac{1}{4}}u\|^2-|\varepsilon||z(\theta_t\omega)|\|A^{\frac{1}{4}}u\|^2,
\end{aligned}
\tag{4.31}
$$

$$
\begin{aligned}
&(Au, A^{\frac{1}{2}}v)\\
=&\big(Au, A^{\frac{1}{2}}u_t-\varepsilon z(\theta_t\omega)A^{\frac{1}{2}}u+\delta A^{\frac{1}{2}}u\big)\\
\geq&\frac{1}{2}\frac{d}{dt}\|A^{\frac{3}{4}}u\|^2+\delta\|A^{\frac{3}{4}}u\|^2-|\varepsilon||z(\theta_t\omega)|\|A^{\frac{3}{4}}u\|^2,
\end{aligned}
\tag{4.32}
$$

$$
\begin{aligned}
&\varepsilon z(\theta_t\omega)\big(3\delta-\varepsilon z(\theta_t\omega)\big)(u, A^{\frac{1}{2}}v)+\beta_2|\varepsilon||z(\theta_t\omega)|\|A^{\frac{1}{4}}u\|\|A^{\frac{1}{4}}v\|\\
=&\big(3\delta\varepsilon z(\theta_t\omega)-\varepsilon^2z^2(\theta_t\omega)\big)(u, A^{\frac{1}{2}}v)+\beta_2|\varepsilon||z(\theta_t\omega)|\|A^{\frac{1}{4}}u\|\|A^{\frac{1}{4}}v\|\\
\leq&\big(3\delta|\varepsilon||z(\theta_t\omega)|+\varepsilon^2|z(\theta_t\omega)|^2\big)\|A^{\frac{1}{4}}u\|\|A^{\frac{1}{4}}v\|+\beta_2|\varepsilon||z(\theta_t\omega)|\|A^{\frac{1}{4}}u\|\|A^{\frac{1}{4}}v\|\\
=&\left((3\delta+\beta_2)|\varepsilon||z(\theta_t\omega)|+\varepsilon^2|z(\theta_t\omega)|^2\right)\|A^{\frac{1}{4}}u\|\|A^{\frac{1}{4}}v\|\\
\leq&\left(\frac{1}{2}(3\delta+\beta_2)|\varepsilon||z(\theta_t\omega)|+\frac{1}{2}\varepsilon^2|z(\theta_t\omega)|^2\right)(\|A^{\frac{1}{4}}u\|^2+\|A^{\frac{1}{4}}v\|^2),
\end{aligned}
\tag{4.33}
$$

$$
\big(g, A^{\frac{1}{2}}v\big)\leq\|g\|_1\|A^{\frac{1}{4}}v\|\leq\|g\|_1^2+\frac{\beta_1-\delta}{4}\|A^{\frac{1}{4}}v\|^2,
\tag{4.34}
$$

$$
\begin{aligned}
&-\big(f(x,u), A^{\frac{1}{2}}v\big)\\
=&-\int_{\mathbb{R}^n}\frac{\partial}{\partial x}f(x,u)\cdot A^{\frac{1}{4}}v\,dx-\int_{\mathbb{R}^n}\frac{\partial}{\partial u}f(x,u)\cdot A^{\frac{1}{4}}u\cdot A^{\frac{1}{4}}v\,dx\\
\leq&\int_{\mathbb{R}^n}|\frac{\partial}{\partial x}f(x,u)|\cdot|A^{\frac{1}{4}}v|dx+\beta\int_{\mathbb{R}^n}|A^{\frac{1}{4}}u|\cdot|A^{\frac{1}{4}}v|dx\\
\leq&\int_{\mathbb{R}^n}|\eta_4|\cdot|A^{\frac{1}{4}}v|dx+\beta\int_{\mathbb{R}^n}|A^{\frac{1}{4}}u|\cdot|A^{\frac{1}{4}}v|dx\\
\leq&\|\eta_4\|\|A^{\frac{1}{4}}v\|+\beta\|A^{\frac{1}{4}}u\|\|A^{\frac{1}{4}}v\|\\
\leq&c_{12}+\left(\delta+\frac{\beta^2}{2\delta(\lambda+\delta^2-\beta_2\delta)}\right)\|A^{\frac{1}{4}}v\|^2+\frac{1}{2}\delta(\lambda+\delta^2-\beta_2\delta)\|A^{\frac{1}{4}}u\|^2,
\end{aligned}
\tag{4.35}
$$

$$
\begin{aligned}
&-(M(\|\nabla u\|^2)\Delta u, A^{\frac{1}{2}}v)\\
=&-(M(\|\nabla u\|^2)\Delta u, A^{\frac{1}{2}}u_t-\varepsilon z(\theta_t\omega)A^{\frac{1}{2}}u+\delta A^{\frac{1}{2}}u)
\end{aligned}
$$

$$
\begin{aligned}
=&\frac{1}{2}\frac{d}{dt}(M(\|\nabla u\|^2)\|A^{\frac{1}{2}}u\|^2)+\delta M(\|\nabla u\|^2)\|A^{\frac{1}{2}}u\|^2-M'(\|\nabla u\|^2)(\nabla u,\nabla u_t)\cdot\|A^{\frac{1}{2}}u\|^2\\
&-(M(\|\nabla u\|^2)\Delta u,-\varepsilon z(\theta_t\omega)A^{\frac{1}{2}}u).
\end{aligned}\tag{4.36}
$$

From (2.7) and 4.1, there exists $C_0>0$, such that

$$
M'(\|\nabla u\|^2)\le C_0,\quad \forall\ t\ge 0.
$$

Consequently,

$$
\begin{aligned}
&M'(\|\nabla u\|^2)(\nabla u,\nabla u_t)\cdot\|A^{\frac{1}{2}}u\|^2\\
\le&C_0\|A^{\frac{1}{2}}u\|^2\|A^{\frac{1}{4}}u\|(\|A^{\frac{1}{4}}v\|+\delta\|A^{\frac{1}{4}}u\|+|\varepsilon|\cdot|z(\theta_t\omega)|\|A^{\frac{1}{4}}u\|)\\
\le&\frac{\beta_1-\delta}{4}\|A^{\frac{1}{4}}v\|^2+c|z(\theta_t\omega)|+c.
\end{aligned}\tag{4.37}
$$

Substituting (4.29)-(4.37) into (4.28) to obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\big(\|A^{\frac{1}{4}}v\|^2+(\lambda+\delta^2-\beta_2\delta)\|A^{\frac{1}{4}}u\|^2+\|A^{\frac{3}{4}}u\|^2+M(\|\nabla u\|^2)\|A^{\frac{1}{2}}u\|^2\big)\\
&+\sigma\big(\|A^{\frac{1}{4}}v\|^2+(\lambda+\delta^2-\beta_2\delta)\|A^{\frac{1}{4}}u\|^2+\|A^{\frac{3}{4}}u\|^2+M(\|\nabla u\|^2)\|A^{\frac{1}{2}}u\|^2\big)\\
\le&\left(\frac{1}{2}(3\delta+\beta_2)|\varepsilon||z(\theta_t\omega)|+\frac{1}{2}\varepsilon^2|z(\theta_t\omega)|^2\right)(\|A^{\frac{1}{4}}u\|^2+\|A^{\frac{1}{4}}v\|^2)\\
&+|\varepsilon||z(\theta_t\omega)|\big(\|A^{\frac{1}{4}}v\|^2+(\lambda+\delta^2+\beta_2\delta)\|A^{\frac{1}{4}}u\|^2+\|A^{\frac{3}{4}}u\|^2\big)\\
&+c(1+\|g\|_1^2+|z(\theta_t\omega)|).
\end{aligned}\tag{4.38}
$$

Then

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\big(\|A^{\frac{1}{4}}v\|^2+(\lambda+\delta^2-\beta_2\delta)\|A^{\frac{1}{4}}u\|^2+\|A^{\frac{3}{4}}u\|^2+M(\|\nabla u\|^2)\|A^{\frac{1}{2}}u\|^2\big)\\
\le&-\big[\sigma-|\varepsilon||z(\theta_t\omega)|-\gamma_2\big(\frac{1}{2}\varepsilon^2|z(\theta_t\omega)|^2+\gamma_4|\varepsilon||z(\theta_t\omega)|\big)\big]\\
&\times\big(\|A^{\frac{1}{4}}v\|^2+(\lambda+\delta^2-\beta_2\delta)\|A^{\frac{1}{4}}u\|^2+\|A^{\frac{3}{4}}u\|^2\\
&+M(\|\nabla u\|^2)\|A^{\frac{1}{2}}u\|^2\big)+c(1+\|g\|_1^2+|z(\theta_t\omega)|).
\end{aligned}\tag{4.39}
$$

Let us denote

$$
\varrho_1(\tau,\omega)=\sigma-|\varepsilon||z(\theta_t\omega)|-\gamma_2\big(\frac{1}{2}\varepsilon^2|z(\theta_t\omega)|^2+\gamma_4|\varepsilon||z(\theta_t\omega)|\big).\tag{4.40}
$$

Using the Gronwall's inequality to integrate (4.39) over $(\tau-t,\tau)$ with $t\ge 0$, we get

$$
\begin{aligned}
&\|A^{\frac{1}{4}}v(\tau,\tau-t,\omega,v_0)\|^2+(\lambda+\delta^2-\beta_2\delta)\|A^{\frac{1}{4}}u(\tau,\tau-t,\omega,u_0)\|^2\\
&+\|A^{\frac{3}{4}}u(\tau,\tau-t,\omega,u_0)\|^2+M(\|\nabla u\|^2)\|A^{\frac{1}{2}}u\|^2\\
\le&\big(\|A^{\frac{1}{4}}v_0\|^2+(\lambda+\delta^2-\beta_2\delta)\|A^{\frac{1}{4}}u_0\|^2+\|A^{\frac{3}{4}}u_0\|^2\\
&+M(\|\nabla u_0\|^2)\|A^{\frac{1}{2}}u_0\|^2\big)e^{2\int_\tau^{\tau-t}\varrho_1(s,\omega)ds}\\
&+c\int_{\tau-t}^{\tau}e^{2\int_\tau^s\varrho_1(r,\omega)dr}(1+\|g(\cdot,s)\|_1^2+|z(\theta_s\omega)|)ds.
\end{aligned}\tag{4.41}
$$

Replacing ω by $\theta_{-\tau}\omega$ in (4.41), for every $t\in\mathbb{R}^+,\ \tau\in\mathbb{R},$ and $\omega\in\Omega,$

$$\begin{aligned}
&\|A^{\frac14}v(\tau,\tau-t,\theta_{-\tau}\omega,v_0)\|^2+(\lambda+\delta^2-\beta_2\delta)\|A^{\frac14}u(\tau,\tau-t,\theta_{-\tau}\omega,u_0)\|^2\\
&+\|A^{\frac34}u(\tau,\tau-t,\theta_{-\tau}\omega,u_0)\|^2\\
&+M(\|\nabla u(\tau,\tau-t,\theta_{-\tau}\omega,u_0)\|^2)\|A^{\frac12}u(\tau,\tau-t,\theta_{-\tau}\omega,u_0)\|^2\\
\le&\big(\|A^{\frac14}v_0\|^2+(\lambda+\delta^2-\beta_2\delta)\|A^{\frac14}u_0\|^2\\
&+\|A^{\frac34}u_0\|^2+M(\|\nabla u_0\|^2)\|A^{\frac12}u_0\|^2\big)e^{2\int_\tau^{\tau-t}\varrho_1(s-\tau,\omega)ds}\\
&+c\int_{\tau-t}^{\tau}e^{2\int_\tau^s\varrho_1(r-\tau,\omega)dr}(1+\|g(\cdot,s)\|_1^2+|z(\theta_{s-\tau}\omega)|)ds,
\end{aligned}\tag{4.42}$$

then

$$\begin{aligned}
&\|A^{\frac14}v(\tau,\tau-t,\theta_{-\tau}\omega,v_0)\|^2+(\lambda+\delta^2-\beta_2\delta)\|A^{\frac14}u(\tau,\tau-t,\theta_{-\tau}\omega,u_0)\|^2\\
&+\|A^{\frac34}u(\tau,\tau-t,\theta_{-\tau}\omega,u_0)\|^2\\
&+M(\|\nabla u(\tau,\tau-t,\theta_{-\tau}\omega,u_0)\|^2)\|A^{\frac12}u(\tau,\tau-t,\theta_{-\tau}\omega,u_0)\|^2\\
\le&\big(\|A^{\frac14}v_0\|^2+(\lambda+\delta^2-\beta_2\delta)\|A^{\frac14}u_0\|^2+\|A^{\frac34}u_0\|^2\\
&+M(\|\nabla u_0\|^2)\|A^{\frac12}u_0\|^2\big)e^{2\int_0^{-t}\varrho_1(s,\omega)ds}\\
&+c\int_{-t}^{0}e^{2\int_0^s\varrho_1(r,\omega)dr}(1+\|g(\cdot,s+\tau)\|_1^2+|z(\theta_s\omega)|)ds.
\end{aligned}\tag{4.43}$$

Next we show that for any $s\le -T_1$

$$e^{2\int_0^s\varrho_1(r,\omega)dr}\le e^{\sigma s}.\tag{4.44}$$

In fact, using the two inequalities in (4.21), we have

$$\begin{aligned}
&\int_0^s\Big[\sigma-|\varepsilon||z(\theta_r\omega)|-\gamma_2\big(\frac12\varepsilon^2|z(\theta_r\omega)|^2+\gamma_4|\varepsilon||z(\theta_r\omega)|\big)\Big]dr\\
>&\sigma s-|\varepsilon|\frac{2}{\sqrt{\pi\delta}}s-\gamma_2\big[\frac12\varepsilon^2\frac1\delta+\gamma_4|\varepsilon|\frac{2}{\sqrt{\pi\delta}}\big]s\\
=&-\frac{\gamma_2}{2\delta}\varepsilon^2 s-\frac{2}{\sqrt{\pi\delta}}\big[\gamma_4\gamma_2+1\big]|\varepsilon|s+\delta s.
\end{aligned}$$

In order to have the inequality in (4.44) valid, we need

$$\int_0^s\Big[\sigma-|\varepsilon||z(\theta_r\omega)|-\gamma_2\big(\frac12\varepsilon^2|z(\theta_r\omega)|^2+\gamma_4|\varepsilon||z(\theta_r\omega)|\big)\Big]dr\le\frac{\sigma}{2}s.$$

Since $s\le -T_1$, then it requires that

$$\frac{\gamma_2}{2\delta}\varepsilon^2+\frac{2}{\sqrt{\pi\delta}}\big[\gamma_4\gamma_2+1\big]|\varepsilon|-\frac{\sigma}{2}<0.$$

Solving this quadratic inequality, ε needs to satisfy (2.11).

By (2.12) and (4.44), we see that the following integral is convergent,

$$R_5^2(\tau,\omega)=c\int_{-\infty}^0e^{2\int_0^s\varrho_1(r,\omega)dr}(1+\|g(\cdot,s+\tau)\|_1^2+|z(\theta_s\omega)|)ds.\tag{4.45}$$

For all $t \geq T_1$, we get from (4.44) that

$$\begin{aligned}
&(\|A^{\frac{1}{4}}v_0\|^2 + (\lambda + \delta^2 - \beta_2\delta)\|A^{\frac{1}{4}}u_0\|^2 + \|A^{\frac{3}{4}}u_0\|^2 \\
&+ M(\|\nabla u_0\|^2)\|A^{\frac{1}{2}}u_0\|^2)e^{2\int_0^{-t}\varrho_1(s,\omega)ds} \\
\leq & ce^{-\sigma t}(\|A^{\frac{1}{4}}v_0\|^2 + \|A^{\frac{1}{4}}u_0\|^2 + \|A^{\frac{3}{4}}u_0\|^2 + M(\|\nabla u_0\|^2)\|A^{\frac{1}{2}}u_0\|^2). \qquad (4.46)
\end{aligned}$$

From (4.1), (4.43), (4.45) and (4.46), there exists $T_4 = T_4(\tau, \omega, D) \geq T_1$ such for all that $t \geq T_4$,

$$\begin{aligned}
&\|A^{\frac{1}{4}}Y(\tau, \tau - t, \theta_{-\tau}\omega, Y_0(\theta_{-\tau}\omega))\|_E^2 \\
\leq & R_5^2(\tau, \omega) + ce^{-\sigma t}(\|A^{\frac{1}{4}}v_0\|^2 + \|A^{\frac{1}{4}}u_0\|^2 + \|A^{\frac{3}{4}}u_0\|^2 + M(\|\nabla u_0\|^2)\|A^{\frac{1}{2}}u_0\|^2).
\end{aligned}$$

□

Lemma 4.4. *Under Assumptions* I *and* II*, for every* $\eta > 0, \tau \in \mathbb{R}, \omega \in \Omega$*,* $D = \{D(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$*, there exists* $T = T(\tau, \omega, D, \eta) > 0, K = K(\tau, \omega, \eta) \geq 1$ *such that for all* $t \geq T,\ k \geq K$*, the solution of problem* (3.6)-(3.8) *satisfies*

$$\|Y(\tau, \tau - t, \theta_{-\tau}\omega, D(\tau - t, \theta_{-t}\omega))\|^2_{E(\mathbb{R}^n \setminus \mathbb{B}_k)} \leq \eta, \qquad (4.47)$$

where for $k \geq 1$*,* $\mathbb{B}_k = \{x \in \mathbb{R}^n : |x| \leq k\}$ *and* $\mathbb{R}^n \setminus \mathbb{B}_k$ *is the complement of* $\mathbb{B}_k$*.*

Proof. Choose a smooth function ρ, such that $0 \leq \rho \leq 1$ for $s \in \mathbb{R}$, and

$$\rho(s) = \begin{cases} 0, & \text{if } 0 \leq |s| \leq 1, \\ 1, & \text{if } |s| \geq 2, \end{cases} \qquad (4.48)$$

and there exist constants $\mu_1,\ \mu_2,\ \mu_3,\ \mu_4$ such that $|\rho'(s)| \leq \mu_1,\ |\rho''(s)| \leq \mu_2,\ |\rho'''(s)| \leq \mu_3,\ |\rho''''(s)| \leq \mu_4$ for $s \in \mathbb{R}$. Taking the inner product of (3.7) with $\rho(\frac{|x|^2}{k^2})v$ in $L^2(\mathbb{R}^n)$, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})|v|^2dx - (\delta - \varepsilon z(\theta_t\omega))\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})|v|^2dx \\
&+ (\lambda + \delta^2)\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})uvdx + \int_{\mathbb{R}^n}(Au)\rho(\frac{|x|^2}{k^2})vdx \\
&- \int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})M(\|\nabla u\|^2)\Delta uvdx + \int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})f(x, u)vdx \\
= & \varepsilon z(\theta_t\omega)(3\delta - \varepsilon z(\theta_t\omega))\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})uvdx \\
&- \int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})(h(v + \varepsilon uz(\theta_t\omega) - \delta u)vdx + \int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})g(x, t)vdx. \qquad (4.49)
\end{aligned}$$

Similar to (4.5), we have

$$\begin{aligned}
&- \int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})(h(v + \varepsilon uz(\theta_t\omega) - \delta u)vdx \\
= & - \int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})(h(v + \varepsilon uz(\theta_t\omega) - \delta u) - h(0))vdx
\end{aligned}$$

$$
\begin{aligned}
&\le -\beta_1\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})|v|^2dx + h'(\vartheta)\delta\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})uvdx\\
&\quad+\beta_2|\varepsilon||z(\theta_t\omega)|\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})|u||v|dx. \qquad (4.50)
\end{aligned}
$$

Taking (4.50) into (4.49), we have

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})|v|^2dx-(\delta-\varepsilon z(\theta_t\omega)-\beta_1)\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})|v|^2dx\\
&+(\lambda+\delta^2-h'(\vartheta)\delta)\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})uvdx+\int_{\mathbb{R}^n}(Au)\rho(\frac{|x|^2}{k^2})vdx\\
&-\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})M(\|\nabla u\|^2)\Delta uvdx+\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})f(x,u)vdx\\
\le&\varepsilon z(\theta_t\omega)(3\delta-\varepsilon z(\theta_t\omega))\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})uvdx+\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})g(x,t)vdx\\
&+\beta_2|\varepsilon||z(\theta_t\omega)|\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})|u||v|dx. \qquad (4.51)
\end{aligned}
$$

Next, we estimate some terms in (4.51).

$$
\begin{aligned}
&\big(\lambda+\delta^2-h'(\vartheta)\delta\big)\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})uvdx\\
=&\big(\lambda+\delta^2-h'(\vartheta)\delta\big)\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})u\big(\frac{du}{dt}+\delta u-\varepsilon uz(\theta_t\omega)\big)dx\\
=&\big(\lambda+\delta^2-h'(\vartheta)\delta\big)\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})\bigg(\frac{1}{2}\frac{d}{dt}u^2+\big(\delta-\varepsilon z(\theta_t\omega)\big)u^2\bigg)dx\\
\ge&(\lambda+\delta^2-\beta_2\delta)\bigg(\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})|u|^2dx+\delta\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})|u|^2dx\bigg)\\
&-(\lambda+\delta^2-\beta_2\delta)|\varepsilon||z(\theta_t\omega)|\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})|u|^2dx, \qquad (4.52)
\end{aligned}
$$

$$
\begin{aligned}
&\int_{\mathbb{R}^n}(Au)\rho(\frac{|x|^2}{k^2})vdx\\
=&\int_{\mathbb{R}^n}(Au)\rho(\frac{|x|^2}{k^2})\big(\frac{du}{dt}+\delta u-\varepsilon uz(\theta_t\omega)\big)dx\\
=&\int_{\mathbb{R}^n}(\Delta^2u)\rho(\frac{|x|^2}{k^2})\big(\frac{du}{dt}+\delta u-\varepsilon z(\theta_t\omega)u\big)dx\\
=&\int_{\mathbb{R}^n}(\Delta u)\Delta\bigg(\rho(\frac{|x|^2}{k^2})\big(\frac{du}{dt}+\delta u-\varepsilon z(\theta_t\omega)u\big)\bigg)dx\\
=&\int_{\mathbb{R}^n}(\Delta u)\bigg(\big(\frac{2}{k^2}\rho'(\frac{|x|^2}{k^2})+\frac{4x^2}{k^4}\rho''(\frac{|x|^2}{k^2})\big)\big(\frac{du}{dt}+\delta u-\varepsilon z(\theta_t\omega)u\big)\\
&+2\cdot\frac{2|x|}{k^2}\rho'(\frac{|x|^2}{k^2})\nabla\big(\frac{du}{dt}+\delta u-\varepsilon z(\theta_t\omega)u\big)+\rho(\frac{|x|^2}{k^2})\Delta\big(\frac{du}{dt}+\delta u-\varepsilon z(\theta_t\omega)u\big)\bigg)dx\\
\ge&-\int_{k<x<\sqrt{2}k}(\frac{2\mu_1}{k^2}+\frac{4\mu_2x^2}{k^4})|(\Delta u)v|dx-\int_{k<x<\sqrt{2}k}\frac{4\mu_1x}{k^2}|(\Delta u)(\nabla v)|dx\\
&+\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})|\Delta u|^2dx+\delta\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})|\Delta u|^2dx-\varepsilon z(\theta_t\omega)\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})|\Delta u|^2dx
\end{aligned}
$$

$$\geq -\int_{\mathbb{R}^n}(\frac{2\mu_1+8\mu_2}{k^2})|(\Delta u)v|dx - \int_{\mathbb{R}^n}\frac{4\sqrt{2}\mu_1}{k}|(\Delta u)(\nabla v)|dx$$
$$+\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})|\Delta u|^2dx + \delta\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})|\Delta u|^2dx - \varepsilon z(\theta_t\omega)\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})|\Delta u|^2dx$$
$$\geq -\frac{\mu_1+4\mu_2}{k^2}(\|\Delta u\|^2+\|v\|^2) - \frac{4\sqrt{2}\mu_1}{k}\|\Delta u\|\|\nabla v\| + \frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})|\Delta u|^2dx$$
$$+\delta\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})|\Delta u|^2dx - \varepsilon z(\theta_t\omega)\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})|\Delta u|^2dx$$
$$\geq -\frac{\mu_1+4\mu_2}{k^2}(\|\Delta u\|^2+\|v\|^2) - \frac{2\sqrt{2}\mu_1}{k}(\|\Delta u\|^2+\|\nabla v\|^2)$$
$$+\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})|\Delta u|^2dx - \big(|\varepsilon||z(\theta_t\omega)|-\delta\big)\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})|\Delta u|^2dx, \tag{4.53}$$

$$\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})M(\|\nabla u\|^2)\Delta uvdx$$
$$= -\int_{\mathbb{R}^n}M(\|\nabla u\|^2)\rho'(\frac{|x|^2}{k^2})\frac{2x}{k^2}v\nabla udx - \int_{\mathbb{R}^n}M(\|\nabla u\|^2)\rho(\frac{|x|^2}{k^2})\nabla v\nabla udx$$
$$\leq\frac{c}{k}(\|v\|^2+\|\nabla v\|^2+\|\nabla u\|^2), \tag{4.54}$$

$$\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})f(x,u)vdx$$
$$=\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})f(x,u)\big(\frac{du}{dt}+\delta u-\varepsilon z(\theta_t\omega)u\big)dx$$
$$=\frac{d}{dt}\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})F(x,u)dx + \delta\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})f(x,u)udx$$
$$-\varepsilon z(\theta_t\omega)\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})f(x,u)udx. \tag{4.55}$$

By (2.2), we get

$$\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})f(x,u)udx \geq c_2\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})F(x,u)dx + \int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})\phi_2(x)dx. \tag{4.56}$$

On the other hand, by (2.1) and (2.3),

$$\varepsilon z(\theta_t\omega)\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})f(x,u)udx$$
$$\leq c|\varepsilon||z(\theta_t\omega)|\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})F(x,u)dx + c|\varepsilon||z(\theta_t\omega)|\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})|u|^2dx$$
$$+c|\varepsilon||z(\theta_t\omega)|\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})(|\phi_1|^2+|\phi_3|)dx. \tag{4.57}$$

Similar to (4.9) and (4.10) in Lemma 4.1, we get

$$\varepsilon z(\theta_t\omega)(3\delta-\varepsilon z(\theta_t\omega))\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})uvdx + \beta_2|\varepsilon||z(\theta_t\omega)\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})||u||v|dx$$
$$\leq(\frac{1}{2}(3\delta+\beta_2)|\varepsilon||z(\theta_t\omega)| + \frac{1}{2}\varepsilon^2|z(\theta_t\omega)|^2)\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})(|u|^2+|v|^2)dx. \tag{4.58}$$

$$\int_{\mathbb{R}^n} \rho(\frac{|x|^2}{k^2}) g(x,t) v dx \leq c \int_{\mathbb{R}^n} \rho(\frac{|x|^2}{k^2}) |g(x,t)|^2 dx + \frac{\beta_1 - \delta}{2} \int_{\mathbb{R}^n} \rho(\frac{|x|^2}{k^2}) |v|^2 dx. \tag{4.59}$$

Assemble together (4.51)-(4.59) to obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^n} \rho(\frac{|x|^2}{k^2})\big(|v|^2 + (\lambda + \delta^2 - \beta_2\delta)|u|^2 + |\Delta u|^2) + 2F(x,u)\big)dx \\
&+ \delta \int_{\mathbb{R}^n} \rho(\frac{|x|^2}{k^2})\big(|v|^2 + (\lambda + \delta^2 - \beta_2\delta)|u|^2 + |\Delta u|^2)\big)dx + \delta c_2 \int_{\mathbb{R}^n} \rho(\frac{|x|^2}{k^2}) F(x,u) dx \\
\leq& \frac{c}{k}(|\Delta u|^2 + |v|^2 + |\nabla v|^2) \\
&+ \left(\frac{1}{2}(3\delta + \beta_2)|\varepsilon||z(\theta_t\omega)| + \frac{1}{2}\varepsilon^2 |z(\theta_t\omega)|^2\right) \int_{\mathbb{R}^n} \rho(\frac{|x|^2}{k^2})(|u|^2 + |v|^2) dx \\
&+ c\int_{\mathbb{R}^n} \rho(\frac{|x|^2}{k^2})|g(x,t)|^2 dx + c|\varepsilon||z(\theta_t\omega)| \int_{\mathbb{R}^n} \rho(\frac{|x|^2}{k^2}) F(x,u) dx \\
&+ c|\varepsilon||z(\theta_t\omega)| \int_{\mathbb{R}^n} \rho(\frac{|x|^2}{k^2})|u|^2 dx \\
&+ c|\varepsilon||z(\theta_t\omega)| \int_{\mathbb{R}^n} \rho(\frac{|x|^2}{k^2})(|\phi_1|^2 + |\phi_3|) dx + c\int_{\mathbb{R}^n} \rho(\frac{|x|^2}{k^2})\phi_2(x) dx \\
&+ |\varepsilon||z(\theta_t\omega)| \int_{\mathbb{R}^n} \rho(\frac{|x|^2}{k^2})\big(|v|^2 + (\lambda + \delta^2 + \beta_2\delta)|u|^2 + |\Delta u|^2\big)dx + c|\varepsilon| \cdot |z(\theta_t\omega)|.
\end{aligned} \tag{4.60}$$

Since that $\phi_1 \in L^2(\mathbb{R}^n)$, ϕ_2, $\phi_3 \in L^1(\mathbb{R}^n)$, for given $\eta > 0$, there exists $K_0 = K_0(\eta) \geq 1$ such that for all $k \geq K_0$,

$$\begin{aligned}
&c\int_{\mathbb{R}^n} \rho(\frac{|x|^2}{k^2})(|\phi_1|^2 + |\phi_2| + |\phi_3|) dx \\
=& c\int_{|x|\geq k} \rho(\frac{|x|^2}{k^2})(|\phi_1|^2 + |\phi_2| + |\phi_3|) dx \\
\leq& c\int_{|x|\geq k} (|\phi_1|^2 + |\phi_2| + |\phi_3|) dx \\
\leq& \eta.
\end{aligned} \tag{4.61}$$

We conclude from (4.15) and (4.60) that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^n} \rho(\frac{|x|^2}{k^2})\big(|v|^2 + (\lambda + \delta^2 - \beta_2\delta)|u|^2 + |\Delta u|^2 + 2F(x,u)\big)dx \\
\leq& -\varrho(t,\omega)\int_{\mathbb{R}^n} \rho(\frac{|x|^2}{k^2})\big(|v|^2 + (\lambda + \delta^2 - \beta_2\delta)|u|^2 + |\Delta u|^2 + 2F(x,u)\big)dx \\
&+ \frac{c}{k}(|\Delta u|^2 + |v|^2 + |\nabla v|^2) + c\int_{\mathbb{R}^n} \rho(\frac{|x|^2}{k^2})|g(x,t)|^2 dx + \eta\big(1 + |\varepsilon||z(\theta_t\omega)|\big).
\end{aligned} \tag{4.62}$$

Integrating (4.62) over $(\tau - t, \tau)$ for $t \in \mathbb{R}^+$ and $\tau \in \mathbb{R}$, we get

$$\int_{\mathbb{R}^n} \rho(\frac{|x|^2}{k^2})\big(|v(\tau, \tau - t, \omega, v_0)|^2 + (\lambda + \delta^2 - \beta_2\delta)|u(\tau, \tau - t, \omega, u_0)|^2\big)dx$$

$$
\begin{aligned}
&+\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})\Big(|\Delta u(\tau,\tau-t,\omega,u_0)|^2+\hat{M}(\|\nabla u(\tau,\tau-t,\omega,u_0)\|^2)\\
&+2F\big(x,u(\tau,\tau-t,\omega,u_0)\big)\Big)dx\\
\le& e^{2\int_\tau^{\tau-t}\varrho(\mu,\omega)d\mu}\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})\big(|v_0|^2+(\lambda+\delta^2-\beta_2\delta)|u_0|^2\big)dx\\
&+e^{2\int_\tau^{\tau-t}\varrho(\mu,\omega)d\mu}\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})\Big(|\Delta u_0|^2+\hat{M}(\|\nabla u_0\|^2)+2F\big(x,u_0\big)\Big)dx\\
&+c\int_{\tau-t}^{\tau}e^{2\int_\tau^s\varrho(\mu,\omega)d\mu}\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})|g(x,s)|^2dsdx\\
&+\eta\int_{\tau-t}^{\tau}e^{2\int_\tau^s\varrho(\mu,\omega)d\mu}\big(1+|\varepsilon||z(\theta_s\omega)|\big)ds\\
&+\frac{c}{k}\int_{\tau-t}^{\tau}e^{2\int_\tau^s\varrho(\mu,\omega)d\mu}\big(|\Delta u(s,\tau-t,\omega,u_0)|^2+|v(s,\tau-t,\omega,v_0)|^2\\
&+|\nabla v(s,\tau-t,\omega,v_0)|^2\big)ds.
\end{aligned}
\tag{4.63}
$$

Replacing ω by $\theta_{-\tau}\omega$ in (4.63), we obtain, for every $t\in\mathbb{R}^+,\ \tau\in\mathbb{R}$, and $\omega\in\Omega$,

$$
\begin{aligned}
&\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})\big(|v(\tau,\tau-t,\theta_{-\tau}\omega,v_0)|^2+(\lambda+\delta^2-\beta_2\delta)|u(\tau,\tau-t,\theta_{-\tau}\omega,u_0)|^2\big)dx\\
&+\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})\Big(|\Delta u(\tau,\tau-t,\theta_{-\tau}\omega,u_0)|^2)+2F\big(x,u(\tau,\tau-t,\theta_{-\tau}\omega,u_0)\big)\Big)dx\\
\le& e^{2\int_\tau^{\tau-t}\varrho(\mu-\tau,\omega)d\mu}\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})\big(|v_0|^2+(\lambda+\delta^2-\beta_2\delta)|u_0|^2\big)dx\\
&+e^{2\int_\tau^{\tau-t}\varrho(\mu-\tau,\omega)d\mu}\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})\Big(|\Delta u_0|^2+2F\big(x,u_0\big)\Big)dx\\
&+c\int_{\tau-t}^{\tau}e^{2\int_\tau^s\varrho(\mu-\tau,\omega)d\mu}\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})|g(x,s)|^2dsdx\\
&+\eta\int_{\tau-t}^{\tau}e^{2\int_\tau^s\varrho(\mu-\tau,\omega)d\mu}\big(1+|\varepsilon||z(\theta_{s-\tau}\omega)|\big)ds\\
&+\frac{\mu_1+4\mu_2}{k^2}\int_{\tau-t}^{\tau}e^{2\int_\tau^s\varrho(\mu-\tau,\omega)d\mu}\big(\|\Delta u(s,\tau-t,\theta_{-\tau}\omega,u_0)\|^2\\
&+\|v(s,\tau-t,\theta_{-\tau}\omega,v_0)\|^2\big)ds\\
&+\frac{2\sqrt{2}\mu_1}{k}\int_{\tau-t}^{\tau}e^{2\int_\tau^s\varrho(\mu-\tau,\omega)d\mu}\big(\|\Delta u(s,\tau-t,\theta_{-\tau}\omega,u_0)\|^2\\
&+\|\nabla v(s,\tau-t,\theta_{-\tau}\omega,v_0)\|^2\big)ds\\
\le& e^{2\int_0^{-t}\varrho(\mu,\omega)d\mu}\Big(\|v_0\|^2+(\lambda+\delta^2-\beta_2\delta)\|u_0\|^2+\|\Delta u_0\|^2+2\widetilde{F}\big(x,u_0\big)\Big)dx\\
&+c\int_{-t}^{0}e^{2\int_0^s\varrho(\mu,\omega)d\mu}\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})|g(x,s+\tau)|^2dsdx\\
&+\eta\int_{-t}^{0}e^{2\int_0^s\varrho(\mu,\omega)d\mu}\big(1+|\varepsilon||z(\theta_s\omega)|\big)ds
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{c}{k}\int_{\tau-t}^{\tau} e^{2\int_\tau^s \varrho(\mu-\tau,\omega)d\mu}\big(\|\Delta u(s,\tau-t,\theta_{-\tau}\omega,u_0)\|^2+\|v(s,\tau-t,\theta_{-\tau}\omega,v_0)\|^2\\
&+\|\nabla v(s,\tau-t,\theta_{-\tau}\omega,v_0)\|^2\big)ds. \qquad (4.64)
\end{aligned}
$$

Similar to (4.26), for an arbitrarily given $\eta>0$, there exists $T=T(\tau,\omega,D,\eta)$ such that for all $t\geq T$,

$$
e^{2\int_0^{-t}\varrho(\mu,\omega)d\mu}\bigg(\|v_0\|^2+(\lambda+\delta^2-\beta_2\delta)\|u_0\|^2+\|\Delta u_0\|^2+2\widetilde{F}\big(x,u_0\big)\bigg)dx\leq\eta. \qquad (4.65)
$$

By Lemma 4.1 and Lemma 4.3, for all $t\geq\max\{T_2,\ T_4\}$,

$$
\begin{aligned}
&\frac{c}{k}\int_{\tau-t}^{\tau} e^{2\int_\tau^s \varrho(\mu-\tau,\omega)d\mu}\big(\|\Delta u(s,\tau-t,\theta_{-\tau}\omega,u_0)\|^2+\|v(s,\tau-t,\theta_{-\tau}\omega,v_0)\|^2\big)ds\\
&+\|\nabla v(s,\tau-t,\theta_{-\tau}\omega,v_0)\|^2\big)ds\\
\leq&\eta\big(R_1^2(\tau,\omega)+R_5^2(\tau,\omega)\big). \qquad (4.66)
\end{aligned}
$$

By (4.22), we get

$$
\begin{aligned}
&\int_{-\infty}^{0} e^{2\int_0^s \varrho(\mu,\omega)d\mu}\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})|g(x,s+\tau)|^2dsdx\\
\leq&\int_{-\infty}^{-T_1} e^{2\int_0^s \varrho(\mu,\omega)d\mu}\int_{|x|\geq k}|g(x,s+\tau)|^2dsdx\\
&+\int_{-T_1}^{0} e^{2\int_0^s \varrho(\mu,\omega)d\mu}\int_{|x|\geq k}|g(x,s+\tau)|^2dsdx\\
\leq&\int_{-\infty}^{-T_1} e^{\sigma s}\int_{|x|\geq k}|g(x,s+\tau)|^2dsdx+e^{c^*}\int_{-T_1}^{0} e^{\sigma s}\int_{|x|\geq k}|g(x,s+\tau)|^2dsdx, \qquad (4.67)
\end{aligned}
$$

where $c^*>0$ is a random variable independent of $\tau\in\mathbb{R}$ and $D\in\mathcal{D}$, i.e.

$$
\begin{aligned}
c^*=&\bigg(\frac{\sigma}{2}+|\varepsilon|\max_{-T_1\leq\mu\leq0}|z(\theta_\mu\omega)|\\
&+\gamma_2\big(\frac{1}{2}\varepsilon^2\max_{-T_1\leq\mu\leq0}z^2(\theta_\mu\omega)+\gamma_3|\varepsilon|\max_{-T_1\leq\mu\leq0}|z(\theta_\mu\omega)|\big)\bigg)T_1.
\end{aligned}
$$

Therefore, by (2.13) there exists $K_2(\tau,\omega)\geq K_1$ such that for all $k\geq K_2$, we obtain

$$
\begin{aligned}
&c\int_{-\infty}^{0} e^{2\int_0^s \varrho(\mu,\omega)d\mu}\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})|g(x,s+\tau)|^2dsdx\\
\leq&e^c\int_{-\infty}^{0} e^{\sigma s}\int_{|x|\geq k}|g(x,s+\tau)|^2dsdx\\
\leq&\eta. \qquad (4.68)
\end{aligned}
$$

Let

$$
R_6(\tau,\omega)=\int_{-\infty}^{0} e^{2\int_0^s \varrho(\mu,\omega)d\mu}\big(1+|\varepsilon||z(\theta_s\omega)|\big)ds, \qquad (4.69)
$$

by (4.22), we know that the integral of (4.69) is convergent.

Together with (4.64)-(4.69), we have

$$\begin{aligned}
&\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})(|v(\tau,\tau-t,\theta_{-\tau}\omega,v_0)|^2+(\lambda+\delta^2-\beta_2\delta)|u(\tau,\tau-t,\theta_{-\tau}\omega,u_0)|^2)dx\\
&+\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})(|\Delta u(\tau,\tau-t,\theta_{-\tau}\omega,u_0)|^2)+2F(x,u(\tau,\tau-t,\theta_{-\tau}\omega,u_0)))dx\\
\leq&2\eta(1+R_1^2(\tau,\omega)+R_5^2(\tau,\omega)+R_6(\tau,\omega)). \qquad (4.70)
\end{aligned}$$

It follows from (2.8), (4.25) and (4.70) that there exists $K_3=K_3(\tau,\omega)\geq K_2$, such for all $k\geq K_3$, $t\geq\max\{T_2,\ T_4\}$,

$$\begin{aligned}
&\int_{|x|\geq\sqrt{2}k}\rho(\frac{|x|^2}{k^2})(|v(\tau,\tau-t,\theta_{-\tau}\omega,v_0)|^2\\
&+(\lambda+\delta^2-\beta_2\delta)|u(\tau,\tau-t,\theta_{-\tau}\omega,u_0)|^2)dx\\
&+\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})(|\Delta u(\tau,\tau-t,\theta_{-\tau}\omega,u_0)|^2)dx\\
\leq&3\eta(1+R_1^2(\tau,\omega)+R_5^2(\tau,\omega)+R_6(\tau,\omega)),
\end{aligned}$$

which implies (4.47). □

We now derive uniform estimates of solutions in bounded domains. Let $\widehat{\rho}=1-\rho$ with ρ given by (4.48). Fix $k\geq1$, and set

$$\begin{cases}\widehat{u}(t,\tau,\omega,\widehat{u_0})=\widehat{\rho}(\frac{|x|^2}{k^2})u(t,\tau,\omega,u_0),\\ \widehat{v}(t,\tau,\omega,\widehat{v}_0)=\widehat{\rho}(\frac{|x|^2}{k^2})v(t,\tau,\omega,v_0).\end{cases} \qquad (4.71)$$

By (3.6)-(3.8) we find that $\widehat{u}$ and $\widehat{v}$ satisfy the following system in $\mathbb{B}_{2k}=\{x\in\mathbb{R}^n:|x|\leq 2k\}$:

$$\frac{d\widehat{u}}{dt}=\widehat{v}+\varepsilon\widehat{u}z(\theta_t\omega)-\delta\widehat{u}, \qquad (4.72)$$

$$\begin{aligned}
&\frac{d\widehat{v}}{dt}-\delta\widehat{v}+(\delta^2+\lambda+A)\widehat{u}-M(\|\nabla u\|^2)\Delta\widehat{u}+\widehat{\rho}(\frac{|x|^2}{k^2})f(x,u)\\
=&\widehat{\rho}(\frac{|x|^2}{k^2})g(x,t)-\widehat{\rho}(\frac{|x|^2}{k^2})h(v+\varepsilon uz(\theta_t\omega)-\delta u)-\varepsilon(\widehat{v}-3\delta\widehat{u}+\varepsilon\widehat{u}z(\theta_t\omega))z(\theta_t\omega)\\
&+4\Delta\nabla\widehat{\rho}(\frac{|x|^2}{k^2})\nabla u+6\Delta\widehat{\rho}(\frac{|x|^2}{k^2})\Delta u+4\nabla\widehat{\rho}(\frac{|x|^2}{k^2})\Delta\nabla u+u\Delta^2\widehat{\rho}(\frac{|x|^2}{k^2}), \qquad (4.73)
\end{aligned}$$

with boundary conditions

$$\widehat{u}=\widehat{v}=0 \quad \text{for} \quad x\in\partial\mathbb{B}_{2k}. \qquad (4.74)$$

Let $\{e_n\}_{n=1}^{\infty}$ be an orthonormal basis of $L^2(\mathbb{B}_{2k})$ such that $Ae_n=\lambda_n e_n$ with zero boundary condition in $\mathbb{B}_{2k}$. Given n, let $X_n=\text{span}\{e_1,\cdots,e_n\}$ and $P_n:L^2(\mathbb{B}_{2k})\to X_n$ be the projection operator.

Lemma 4.5. *Under Assumptions* I *and* II, *for every* $\eta>0,\tau\in\mathbb{R},\omega\in\Omega$, $D=\{D(\tau,\omega):\tau\in\mathbb{R},\omega\in\Omega\}\in\mathcal{D}$, *there exists* $T=T(\tau,\omega,D,\eta)>0,K=K(\tau,\omega,\eta)\geq$

1 and $N = N(\tau, \omega, \eta) \geq 1$ such that for all $t \geq T,\ k \geq K$ and $n \geq N$, the solution of problem (4.72)-(4.74) *satisfies*

$$\|(I - P_n)\widehat{Y}(\tau, \tau - t, \theta_{-\tau}\omega, D(\tau - t, \theta_{-\tau}\omega))\|^2_{E(\mathbb{B}_{2k})} \leq \eta.$$

Proof. Let $\widehat{u}_{n,1} = P_n\widehat{u},\ \widehat{u}_{n,2} = (I-P_n)\widehat{u},\ \widehat{v}_{n,1} = P_n\widehat{v},\ \widehat{v}_{n,2} = (I-P_n)\widehat{v}$. Applying $I - P_n$ to (4.74), we obtain

$$\widehat{v}_{n,2} = \frac{d\widehat{u}_{n,2}}{dt} + \delta\widehat{u}_{n,2} - \varepsilon z(\theta_t\omega)\widehat{u}_{n,2}. \tag{4.75}$$

Then applying $I - P_n$ to (4.73) and taking the inner product with $\widehat{v}_{n,2}$ in $L^2(\mathbb{B}_{2k})$, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\widehat{v}_{n,2}\|^2 - (\delta - \varepsilon z(\theta_t\omega))\|\widehat{v}_{n,2}\|^2 + (\lambda + \delta^2 + A)(\widehat{u}_{n,2}, \widehat{v}_{n,2}) \\
&- (M(\|\nabla u\|^2)\Delta\widehat{u}_{n,2}, \widehat{v}_{n,2}) + ((I - P_n)\widehat{\rho}(\frac{|x|^2}{k^2})f(x, u), \widehat{v}_{n,2}) \\
=&((I - P_n)\widehat{\rho}(\frac{|x|^2}{k^2})g(x, t), \widehat{v}_{n,2}) + \varepsilon z(\theta_t\omega)(3\delta - \varepsilon z(\theta_t\omega))(\widehat{u}_{n,2}, \widehat{v}_{n,2}) \\
&- (I - P_n)\widehat{\rho}(\frac{|x|^2}{k^2})(h(v + \varepsilon z(\theta_t\omega) - \delta u), \widehat{v}_{n,2}) \\
&+ (4\Delta\nabla\widehat{\rho}(\frac{|x|^2}{k^2})\nabla u + 6\Delta\widehat{\rho}(\frac{|x|^2}{k^2})\Delta u + 4\nabla\widehat{\rho}(\frac{|x|^2}{k^2})\Delta\nabla u + u\Delta^2\widehat{\rho}(\frac{|x|^2}{k^2}), \widehat{v}_{n,2}).
\end{aligned} \tag{4.76}$$

Next, we estimate some terms of (4.76).

$$\begin{aligned}
&(\widehat{u}_{n,2}, \widehat{v}_{n,2}) \\
=&(\widehat{u}_{n,2}, \frac{d\widehat{u}_{n,2}}{dt} + \delta\widehat{u}_{n,2} - \varepsilon z(\theta_t\omega)\widehat{u}_{n,2}) \\
\geq&\frac{1}{2}\frac{d}{dt}\|\widehat{u}_{n,2}\|^2 + \delta\|\widehat{u}_{n,2}\|^2 - |\varepsilon||z(\theta_t\omega)|\|\widehat{u}_{n,2}\|^2,
\end{aligned} \tag{4.77}$$

$$\begin{aligned}
&(A\widehat{u}_{n,2}, \widehat{v}_{n,2}) \\
=&(\Delta\widehat{u}_{n,2}, \Delta(\frac{d\widehat{u}_{n,2}}{dt} + \delta\widehat{u}_{n,2} - \varepsilon z(\theta_t\omega)\widehat{u}_{n,2})) \\
\geq&\frac{1}{2}\frac{d}{dt}\|\Delta\widehat{u}_{n,2}\|^2 + \delta\|\Delta\widehat{u}_{n,2}\|^2 - |\varepsilon||z(\theta_t\omega)|\|\Delta\widehat{u}_{n,2}\|^2,
\end{aligned} \tag{4.78}$$

$$\begin{aligned}
&((I - P_n)\widehat{\rho}(\frac{|x|^2}{k^2})f(x, u), \widehat{v}_{n,2}) \\
=&((I - P_n)\widehat{\rho}(\frac{|x|^2}{k^2})f(x, u), \frac{d\widehat{u}_{n,2}}{dt} + \delta\widehat{u}_{n,2} - \varepsilon z(\theta_t\omega)\widehat{u}_{n,2}) \\
=&\frac{d}{dt}((I - P_n)\widehat{\rho}(\frac{|x|^2}{k^2})f(x, u), \widehat{u}_{n,2}) - ((I - P_n)\widehat{\rho}(\frac{|x|^2}{k^2})f'_u(x, u)u_t, \widehat{u}_{n,2}) \\
&+ (\delta - \varepsilon z(\theta_t\omega))((I - P_n)\widehat{\rho}(\frac{|x|^2}{k^2})f(x, u), \widehat{u}_{n,2}),
\end{aligned} \tag{4.79}$$

$$\begin{aligned}
&- (I - P_n)\widehat{\rho}(\frac{|x|^2}{k^2})(h(v + \varepsilon z(\theta_t\omega) - \delta u), \widehat{v}_{n,2}) \\
=&- (I - P_n)\widehat{\rho}(\frac{|x|^2}{k^2})(h(v + \varepsilon z(\theta_t\omega) - \delta u) - h(0), \widehat{v}_{n,2})
\end{aligned}$$

$$\begin{aligned}
&\le -\beta_1\|\widehat{v}_{n,2}\|^2+h'(\vartheta)\delta(\widehat{u}_{n,2},\widehat{v}_{n,2})+\beta_2|\varepsilon||z(\theta_t\omega)|\|\widehat{u}_{n,2}\|\|\widehat{v}_{n,2}\|, \quad (4.80)\\
&\varepsilon z(\theta_t\omega)(3\delta-\varepsilon z(\theta_t\omega))(\widehat{u}_{n,2},\widehat{v}_{n,2})+\beta_2|\varepsilon||z(\theta_t\omega)|\|\widehat{u}_{n,2}\|\|\widehat{v}_{n,2}\|\\
=&(3\delta\varepsilon z(\theta_t\omega)-\varepsilon^2z^2(\theta_t\omega))(\widehat{u}_{n,2},\widehat{v}_{n,2})+\beta_2|\varepsilon||z(\theta_t\omega)|\|\widehat{u}_{n,2}\|\|\widehat{v}_{n,2}\|\\
\le&(3\delta|\varepsilon||z(\theta_t\omega)|+\varepsilon^2|z(\theta_t\omega)|^2)\|\widehat{u}_{n,2}\|\|\widehat{v}_{n,2}\|+\beta_2|\varepsilon||z(\theta_t\omega)|\|\widehat{u}_{n,2}\|\|\widehat{v}_{n,2}\|\\
=&((3\delta+\beta_2)|\varepsilon||z(\theta_t\omega)|+\varepsilon^2|z(\theta_t\omega)|^2)\|\widehat{u}_{n,2}\|\|\widehat{v}_{n,2}\|\\
\le&(\frac{1}{2}(3\delta+\beta_2)|\varepsilon||z(\theta_t\omega)|+\frac{1}{2}\varepsilon^2|z(\theta_t\omega)|^2)(\|\widehat{u}_{n,2}\|^2+\|\widehat{v}_{n,2}\|^2), \quad (4.81)\\
&((I-P_n)\widehat{\rho}(\frac{|x|^2}{k^2})g(x,t),\widehat{v}_{n,2})\\
\le&\frac{\beta_1-\delta}{4}\|\widehat{v}_{n,2}\|^2+\frac{1}{\beta_1-\delta}\|(I-P_n)(\widehat{\rho}(\frac{|x|^2}{k^2})g(x,t))\|^2, \quad (4.82)\\
&(4\Delta\nabla\widehat{\rho}(\frac{|x|^2}{k^2})\cdot\nabla u+6\Delta\widehat{\rho}(\frac{|x|^2}{k^2})\cdot\Delta u+4\nabla\widehat{\rho}(\frac{|x|^2}{k^2})\cdot\Delta\nabla u+u\Delta^2\widehat{\rho}(\frac{|x|^2}{k^2}),\widehat{v}_{n,2})\\
=&(4\nabla u\cdot(\frac{12|x|}{k^4}\widehat{\rho}\,''(\frac{|x|^2}{k^2})+\frac{8|x|^3}{k^6}\widehat{\rho}\,'''(\frac{|x|^2}{k^2}))+6\Delta u\cdot(\frac{2}{k^2}\widehat{\rho}\,'(\frac{|x|^2}{r^2})\\
&+\frac{4x^2}{k^4}\widehat{\rho}\,''(\frac{|x|^2}{k^2}))+\frac{8|x|}{k^2}\Delta\nabla u\cdot\widehat{\rho}\,'(\frac{|x|^2}{k^2})+u(\frac{12}{k^4}\widehat{\rho}\,''(\frac{|x|^2}{k^2})+\frac{48x^2}{k^6}\widehat{\rho}\,'''(\frac{|x|^2}{k^2})\\
&+\frac{16x^4}{k^8}\widehat{\rho}\,''''(\frac{|x|^2}{k^2})),\widehat{v}_{n,2})\\
\le&\frac{16\sqrt{2}(3\mu_2+4\mu_3)}{k^3}\|\nabla u\|\cdot\|\widehat{v}_{n,2}\|+\frac{12(\mu_1+4\mu_2)}{k^2}\|\Delta u\|\cdot\|\widehat{v}_{n,2}\|\\
&+\frac{8\sqrt{2}\mu_1}{k}\|A^{\frac{3}{4}}u\|\cdot\|\widehat{v}_{n,2}\|+\frac{4(3\mu_2+24\mu_3+16\mu_4)}{k^4}\|u\|\cdot\|\widehat{v}_{n,2}\|\\
\le&\frac{8(48\mu_2+64\mu_3)^2}{(\beta_1-\delta)k^6}\|\nabla u\|^2+\frac{4(12\mu_1+48\mu_2)^2}{(\beta_1-\delta)k^4}\|\Delta u\|^2+\frac{512\mu_1^2}{(\beta_1-\delta)k^2}\|A^{\frac{3}{4}}u\|^2\\
&+\frac{4(12\mu_2+96\mu_3+64\mu_4)^2}{(\beta_1-\delta)k^8}\|u\|^2+\frac{\beta_1-\delta}{8}\|\widehat{v}_{n,2}\|^2, \quad (4.83)\\
&-(M(\|\nabla u\|^2)\Delta\widehat{u}_{n,2},\widehat{v}_{n,2})\ge-\frac{2}{\beta_1-\delta}\|\Delta\widehat{u}_{n,2}\|^2-\frac{\beta_1-\delta}{8}\|\widehat{v}_{n,2}\|^2. \quad (4.84)
\end{aligned}$$

Assemble together (4.76)-(4.84) to obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}[\|\widehat{v}_{n,2}\|^2+(\lambda+\delta^2-\beta_2\delta)\|\widehat{u}_{n,2}\|^2+\|\Delta\widehat{u}_{n,2}\|^2\\
&+2((I-P_n)\widehat{\rho}(\frac{|x|^2}{k^2})f(x,u),\widehat{u}_{n,2})]+(\delta-|\varepsilon||z(\theta_t\omega)|)[\|\widehat{v}_{n,2}\|^2\\
&+(\lambda+\delta^2-\beta_2\delta)\|\widehat{u}_{n,2}\|^2+\|\Delta\widehat{u}_{n,2}\|^2+((I-P_n)\widehat{\rho}(\frac{|x|^2}{k^2})f(u),\widehat{u}_{n,2})]\\
\le&(\frac{1}{2}(3\delta+\beta_2+4\beta_2\delta)|\varepsilon||z(\theta_t\omega)|+\frac{1}{2}\varepsilon^2|z(\theta_t\omega)|^2)(\|\widehat{v}_{n,2}\|^2+\|\widehat{u}_{n,2}\|^2)\\
&+\frac{2}{\beta_1-\delta}(\frac{4(48\mu_2+64\mu_3)^2}{k^6}\|\nabla u\|^2+\frac{2(12\mu_1+48\mu_2)^2}{k^4}\|\Delta u\|^2+\frac{256\mu_1^2}{k^2}\|A^{\frac{3}{4}}u\|^2\\
&+\frac{2(12\mu_2+96\mu_3+64\mu_4)^2}{k^8}\|u\|^2+\frac{1}{2}\|(I-P_n)(\widehat{\rho}(\frac{|x|^2}{k^2})g(x,t))\|^2)\\
&+\frac{3\delta-\beta_1}{2}\|\widehat{v}_{n,2}\|^2+((I-P_n)\widehat{\rho}(\frac{|x|^2}{k^2})f'_u(x,u)u_t,\widehat{u}_{n,2})+\frac{2}{\beta_1-\delta}\|\Delta\widehat{u}_{n,2}\|^2.
\end{aligned}$$

It follows that

$$
\begin{aligned}
&\frac{d}{dt}\Big[\|\widehat{v}_{n,2}\|^2+(\lambda+\delta^2-\beta_2\delta)\|\widehat{u}_{n,2}\|^2\\
&+\|\Delta\widehat{u}_{n,2}\|^2+2\big((I-P_n)\widehat{\rho}(\frac{|x|^2}{k^2})f(x,u),\widehat{u}_{n,2}\big)\Big]\\
\le&2\Bigg(-\delta+|\varepsilon||z(\theta_t\omega)|+\gamma_2\big(\frac{1}{2}(3\delta+\beta_2+4\beta_2\delta)|\varepsilon||z(\theta_t\omega)|+\frac{1}{2}\varepsilon^2|z(\theta_t\omega)|^2\big)\Bigg)\\
&\times\Big[\|\widehat{v}_{n,2}\|^2+(\lambda+\delta^2-\beta_2\delta)\|\widehat{u}_{n,2}\|^2\\
&+\|\Delta\widehat{u}_{n,2}\|^2+2\big((I-P_n)\widehat{\rho}(\frac{|x|^2}{k^2})f(x,u),\widehat{u}_{n,2}\big)\Big]\\
&+\frac{4}{\beta_1-\delta}\Bigg(\frac{4(48\mu_2+64\mu_3)^2}{k^6}\|\nabla u\|^2+\frac{2(12\mu_1+48\mu_2)^2}{k^4}\|\Delta u\|^2\\
&+\frac{256\mu_1^2}{k^2}\|A^{\frac{3}{4}}u\|^2+\frac{2(12\mu_2+96\mu_3+64\mu_4)^2}{k^8}\|u\|^2\\
&+\frac{1}{2}\|(I-P_n)(\widehat{\rho}(\frac{|x|^2}{k^2})g(x,t))\|^2\Bigg)+2\Bigg((I-P_n)\widehat{\rho}(\frac{|x|^2}{k^2})f'_u(x,u)u_t,\widehat{u}_{n,2}\Bigg)\\
&+2\Bigg(\delta-|\varepsilon||z(\theta_t\omega)|-\gamma_2\big(\frac{1}{2}(3\delta+\beta_2+4\beta_2\delta)|\varepsilon||z(\theta_t\omega)|+\frac{1}{2}\varepsilon^2|z(\theta_t\omega)|^2\big)\Bigg)\\
&\times\big((I-P_n)\widehat{\rho}(\frac{|x|^2}{k^2})f(x,u),\widehat{u}_{n,2}\big)+\frac{2}{\beta_1-\delta}\|\Delta\widehat{u}_{n,2}\|^2\\
\le&-2\varrho(\tau,\omega)\Big[\|\widehat{v}_{n,2}\|^2+(\lambda+\delta^2-\beta_2\delta)\|\widehat{u}_{n,2}\|^2+\|\Delta\widehat{u}_{n,2}\|^2\\
&+2\big((I-P_n)\widehat{\rho}(\frac{|x|^2}{k^2})f(x,u),\widehat{u}_{n,2}\big)\Big]+2\Big[-\delta+\sigma-(\gamma_1+\gamma_2-1)|\varepsilon||z(\theta_t\omega)|\Big]\\
&\times\Big[\|\widehat{v}_{n,2}\|^2+(\lambda+\delta^2-\beta_2\delta)\|\widehat{u}_{n,2}\|^2\\
&+\|\Delta\widehat{u}_{n,2}\|^2\Big]+2\big((I-P_n)\widehat{\rho}(\frac{|x|^2}{k^2})f'_u(x,u)u_t,\widehat{u}_{n,2}\big)\\
&+\frac{4}{\beta_1-\delta}\Bigg(\frac{4(48\mu_2+64\mu_3)^2}{k^6}\|\nabla u\|^2+\frac{2(12\mu_1+48\mu_2)^2}{k^4}\|\Delta u\|^2+\frac{256\mu_1^2}{k^2}\|A^{\frac{3}{4}}u\|^2\\
&+\frac{2(12\mu_2+96\mu_3+64\mu_4)^2}{k^8}\|u\|^2+\frac{1}{2}\|(I-P_n)\big(\widehat{\rho}(\frac{|x|^2}{k^2})g(x,t)\big)\|^2\Bigg)\\
&+4(\sigma-\frac{1}{2}\delta)\cdot\Bigg((I-P_n)\widehat{\rho}(\frac{|x|^2}{k^2})f(x,u),\widehat{u}_{n,2}\Bigg)+4\Bigg[-\frac{\gamma_1}{2}|\varepsilon||z(\theta_t\omega)|\\
&-\frac{\gamma_2}{2}\Bigg(\frac{1}{2}\varepsilon^2|z(\theta_t\omega)|^2+\big(\frac{1}{2}(3\delta+\beta_2+4\beta_2\delta+4)|\varepsilon||z(\theta_t\omega)|\big)\Bigg)\Bigg]\\
&\times\Bigg((I-P_n)\widehat{\rho}(\frac{|x|^2}{k^2})f(x,u),\widehat{u}_{n,2}\Bigg).
\end{aligned}
\tag{4.85}
$$

Let $\theta=\frac{n(\gamma-1)}{4(\gamma+1)}$. Since $1\le\gamma\le\frac{n+4}{n-4}$, we find that $0\le\theta\le 1$. Then by (2.1) and interpolation inequalities, the last term on the right hand of (4.85) is bounded by

$$\begin{aligned}
&4\Big[-\frac{\gamma_1}{2}|\varepsilon||z(\theta_t\omega)|-\frac{\gamma_2}{2}\Big(\frac{1}{2}\varepsilon^2|z(\theta_t\omega)|^2+\big(\frac{1}{2}(3\delta+\beta_2+4\beta_2\delta+4)|\varepsilon||z(\theta_t\omega)|\big)\Big)\Big]\\
&\times\Big((I-P_n)\widehat{\rho}(\frac{|x|^2}{k^2})f(x,u),\widehat{u}_{n,2}\Big)\\
\le& c\big(1+|z(\theta_t\omega)|^2\big)\Big[c_1\int_{\mathbb{R}^n}\widehat{\rho}(\frac{|x|^2}{k^2})|u|^\gamma|\widehat{u}_{n,2}|dx+\int_{\mathbb{R}^n}\widehat{\rho}(\frac{|x|^2}{k^2})|\phi_1(x)||\widehat{u}_{n,2}|dx\Big]\\
\le& c\big(1+|z(\theta_t\omega)|^2\big)\big(c_1\|u\|^\gamma_{\gamma+1}\|\widehat{u}_{n,2}\|_{\gamma+1}+\|\phi_1\|\|\widehat{u}_{n,2}\|\big)\\
\le& c\big(1+|z(\theta_t\omega)|^2\big)\big(c_1\|u\|^\gamma_{\gamma+1}\|\Delta\widehat{u}_{n,2}\|^\theta\|\widehat{u}_{n,2}\|^{1-\theta}+\lambda_{n+1}^{-\frac{1}{2}}\|\phi_1\|\|\Delta\widehat{u}_{n,2}\|\big)\\
\le& c\big(1+|z(\theta_t\omega)|^2\big)\Big[\lambda_{n+1}^{-\frac{1}{2}}\|\Delta\widehat{u}_{n,2}\|\big(c_1\lambda_{n+1}^{\frac{\theta}{2}}\|u\|^\gamma_{H^2}+\|\phi_1\|\big)\Big]\\
\le&\frac{1}{6}(\delta-\sigma)\|\Delta\widehat{u}_{n,2}\|^2+c\lambda_{n+1}^{-1}\big(1+|z(\theta_t\omega)|^{18}+\lambda_{n+1}^{\theta}\|u\|^{18}_{H^2(\mathbb{R}^n)}\big).
\end{aligned}\tag{4.86}$$

Similarly we have

$$\begin{aligned}
&4(\sigma-\frac{1}{2}\delta)\Big((I-P_n)\widehat{\rho}(\frac{|x|^2}{k^2})f(x,u),\widehat{u}_{n,2}\Big)\\
\le&\frac{1}{6}(\delta-\sigma)\|\Delta\widehat{u}_{n,2}\|^2+c\lambda_{n+1}^{-1}\big(1+\lambda_{n+1}^{\theta}\|u\|^{18}_{H^2(\mathbb{R}^n)}\big).
\end{aligned}\tag{4.87}$$

On the other hand, by (2.4), using Hölder inequality and Young's inequality, we obtain

$$\begin{aligned}
&2\Big((I-P_n)\widehat{\rho}(\frac{|x|^2}{k^2})f'_u(x,u)u_t,\widehat{u}_{n,2}\Big)\\
\le&\frac{1}{6}(\delta-\sigma)\|\Delta\widehat{u}_{n,2}\|^2+c\lambda_{n+1}^{-1}\|u_t\|^2\\
\le&\frac{1}{6}(\delta-\sigma)\|\Delta\widehat{u}_{n,2}\|^2+c\lambda_{n+1}^{-1}\big(\|u\|^2+\|v\|^2+\|u\|^4+|z(\theta_t\omega)|^4\big).
\end{aligned}\tag{4.88}$$

Then by (4.85)-(4.88), we obtain

$$\begin{aligned}
&\frac{d}{dt}\Big[\|\widehat{v}_{n,2}\|^2+(\lambda+\delta^2-\beta_2\delta)\|\widehat{u}_{n,2}\|^2+\|\Delta\widehat{u}_{n,2}\|^2\\
&+2\big((I-P_n)\widehat{\rho}(\frac{|x|^2}{k^2})f(x,u),\widehat{u}_{n,2}\big)\Big]\\
\le&-2\varrho(\tau,\omega)[\|\widehat{v}_{n,2}\|^2+(\lambda+\delta^2-\beta_2\delta)\|\widehat{u}_{n,2}\|^2+\|\Delta\widehat{u}_{n,2}\|^2\\
&+2((I-P_n)\widehat{\rho}(\frac{|x|^2}{k^2})f(x,u),\widehat{u}_{n,2})]\\
&+c\lambda_{n+1}^{-1}\big[1+\|v\|^{18}+(1+\lambda_{n+1}^{\theta})\|u\|^{18}_{H^2(\mathbb{R}^n)}+|z(\theta_t\omega)|^{18}\big]+\frac{c}{k^6}\|\nabla u\|^2+\frac{c}{k^4}\|\Delta u\|^2\\
&+\frac{c}{k^2}\|A^{\frac{3}{4}}u\|^2+\frac{c}{k^8}\|u\|^2+c\|(I-P_n)\big(\widehat{\rho}(\frac{|x|^2}{k^2})g(x,t)\big)\|^2.
\end{aligned}\tag{4.89}$$

Note that $\lambda_n\to\infty$ when $n\to\infty$. Therefore, given $\eta>0$, by Lemma 4.1 and 4.3, we know there exist $N_1=N_1(\eta)\ge 1$ and $K_4=K_4(\eta)\ge 1$ such for all $n\ge N_1$ and

$k \geq K_4$,

$$\begin{aligned}
&\frac{d}{dt}\Big[\|\widehat{v}_{n,2}\|^2 + (\lambda+\delta^2-\beta_2\delta)\|\widehat{u}_{n,2}\|^2 \\
&\quad + \|\Delta\widehat{u}_{n,2}\|^2 + 2\big((I-P_n)\widehat{\rho}(\frac{|x|^2}{k^2})f(x,u), \widehat{u}_{n,2}\big)\Big] \\
&\leq -2\varrho(\tau,\omega)[\|\widehat{v}_{n,2}\|^2 + (\lambda+\delta^2-\beta_2\delta)\|\widehat{u}_{n,2}\|^2 + \|\Delta\widehat{u}_{n,2}\|^2 \\
&\quad + 2((I-P_n)\widehat{\rho}(\frac{|x|^2}{k^2})f(x,u), \widehat{u}_{n,2})] \\
&\quad + \eta\big(1+\|v\|^{18} + \|u\|^{18}_{H^2(\mathbb{R}^n)} + |z(\theta_t\omega)|^{18}\big) + c\|(I-P_n)\big(\widehat{\rho}(\frac{|x|^2}{k^2})g(x,t)\big)\|^2. \quad (4.90)
\end{aligned}$$

Integrating (4.90) over $(\tau-t,\tau)$ with $t\geq 0$, we get for all $n\geq N_1$ and $k\geq K_4$,

$$\begin{aligned}
&\|\widehat{v}_{n,2}(\tau,\tau-t,\omega)\|^2 + (\lambda+\delta^2-\beta_2\delta)\|\widehat{u}_{n,2}(\tau,\tau-t,\omega)\|^2 + \|\Delta\widehat{u}_{n,2}(\tau,\tau-t,\omega)\|^2 \\
&\quad + 2\left((I-P_n)\widehat{\rho}(\frac{|x|^2}{k^2})f(x,u), \widehat{u}_{n,2}(\tau,\tau-t,\omega)\right) \\
&\leq ce^{2\int_\tau^{\tau-t}\varrho(\mu,\omega)d\mu}\big(1+\|v_0\|^2 + \|u_0\|^2_{H^2(\mathbb{R}^n)}\big) \\
&\quad + \eta\int_{\tau-t}^{\tau} e^{2\int_\tau^s \varrho(\mu,\omega)d\mu}\big(\|u(s,\tau-t,\omega,u_0)\|^{18}_{H^2(\mathbb{R}^n)} + \|v(s,\tau-t,\omega,v_0)\|^{18}\big)ds \\
&\quad + \eta\int_{\tau-t}^{\tau} e^{2\int_\tau^s \varrho(\mu,\omega)d\mu}\big(1+|z(\theta_s\omega)|^{18}\big)ds \\
&\quad + c\int_{\tau-t}^{\tau} e^{2\int_\tau^s \varrho(\mu,\omega)d\mu}\|(I-P_n)\big(\widehat{\rho}(\frac{|x|^2}{k^2})g(x,s)\big)\|^2 ds.
\end{aligned}$$

Replacing ω by $\theta_{-\tau}\omega$ in the above we obtain, for every $t\in\mathbb{R}^+$, $\tau\in\mathbb{R}$, $\omega\in\Omega$, $n\geq N_1$ and $k\geq K_4$,

$$\begin{aligned}
&\|\widehat{v}_{n,2}(\tau,\tau-t,\theta_{-\tau}\omega)\|^2 + (\lambda+\delta^2-\beta_2\delta)\|\widehat{u}_{n,2}(\tau,\tau-t,\theta_{-\tau}\omega)\|^2 \\
&\quad + \|\Delta\widehat{u}_{n,2}(\tau,\tau-t,\theta_{-\tau}\omega)\|^2 + 2\left((I-P_n)\widehat{\rho}(\frac{|x|^2}{k^2})f(x,u), \widehat{u}_{n,2}(\tau,\tau-t,\theta_{-\tau}\omega)\right) \\
&\leq ce^{2\int_\tau^{\tau-t}\varrho(\mu-\tau,\omega)d\mu}\big(1+\|v_0\|^2 + \|u_0\|^{\gamma+1}_{H^2(\mathbb{R}^n)}\big) \\
&\quad + \eta\int_{\tau-t}^{\tau} e^{2\int_\tau^s \varrho(\mu-\tau,\omega)d\mu}\big(\|u(s,\tau-t,\theta_{-\tau}\omega,u_0)\|^{18}_{H^2(\mathbb{R}^n)} \\
&\quad + \|v(s,\tau-t,\theta_{-\tau}\omega,v_0)\|^{18}\big)ds \\
&\quad + \eta\int_{\tau-t}^{\tau} e^{2\int_\tau^s \varrho(\mu-\tau,\omega)d\mu}\big(1+|z(\theta_{s-\tau}\omega)|^{18}\big)ds \\
&\quad + c\int_{\tau-t}^{\tau} e^{2\int_\tau^s \varrho(\mu-\tau,\omega)d\mu}\|(I-P_n)\big(\widehat{\rho}(\frac{|x|^2}{k^2})g(x,s)\big)\|^2 ds \\
&\leq ce^{2\int_0^{-t}\varrho(\mu,\omega)d\mu}\big(1+\|v_0\|^2 + \|u_0\|^{\gamma+1}_{H^2(\mathbb{R}^n)}\big) \\
&\quad + \eta\int_{-t}^{0} e^{2\int_0^s \varrho(\mu,\omega)d\mu}(\|u(s+\tau,\tau-t,\theta_{-\tau}\omega,u_0)\|^{18}_{H^2(\mathbb{R}^n)} \\
&\quad + \|v(s+\tau,\tau-t,\theta_{-\tau}\omega,v_0)\|^{18})ds + \eta\int_{-t}^{0} e^{2\int_0^s \varrho(\mu,\omega)d\mu}(1+|z(\theta_s\omega)|^{18})ds
\end{aligned}$$

$$+ c\int_{-t}^{0} e^{2\int_0^s \varrho(\mu,\omega)d\mu}\|(I-P_n)\big(\widehat{\rho}(\frac{|x|^2}{k^2})g(x,s+\tau)\big)\|^2 ds. \tag{4.91}$$

We now estimate every term on the right-hand side of (4.91), we find that there exists $\widetilde{T}=\widetilde{T}(\tau,\omega,D,\eta)>0$ such for all $t\geq\widetilde{T}$,

$$ce^{2\int_0^{-t}\varrho(\mu,\omega)d\mu}\big(1+\|v_0\|^2+\|u_0\|^{\gamma+1}_{H^2(\mathbb{R}^n)}\big)\leq\eta. \tag{4.92}$$

For the second term on the right-hand side of (4.91), by Lemma 4.2 we have

$$\begin{aligned}
&\eta\int_{-t}^{0} e^{2\int_0^s \varrho(\mu,\omega)d\mu}\big(\|u(s+\tau,\tau-t,\theta_{-\tau}\omega,u_0)\|^{18}_{H^2(\mathbb{R}^n)}\\
&\quad+\|v(s+\tau,\tau-t,\theta_{-\tau}\omega,v_0)\|^{18}\big)ds\\
\leq&\eta c\int_{-t}^{0} e^{2\int_0^s \varrho(\mu,\omega)d\mu}ds+\eta R_3^9(\tau,\omega)\int_{-t}^{0} e^{-16\int_0^s \varrho(\mu,\omega)d\mu}ds\\
\leq&\eta c\int_{-\infty}^{0} e^{2\int_0^s \varrho(\mu,\omega)d\mu}ds+\frac{\eta}{8\sigma}R_3^9(\tau,\omega),
\end{aligned} \tag{4.93}$$

where $R_3(\tau,\omega)$ is the random variable given in Lemma 4.2. Note that by (4.22) the above integral is well defined, and so is the following one

$$\int_{-\infty}^{0} e^{2\int_0^s \varrho(\mu,\omega)d\mu}\big(1+|z(\theta_s\omega)|^{18}\big)ds<\infty. \tag{4.94}$$

For the last term on the right-hand side of (4.91), by (2.13) and (4.22), since $g\in L^2(\mathbb{R}^n)$, there exists $N_2=N_2(\tau,\omega,\eta)\geq N_1$, such that for all $n\geq N_2$,

$$\int_{-\infty}^{0} e^{2\int_0^s \varrho(\mu,\omega)d\mu}\|(I-P_n)\big(\widehat{\rho}(\frac{|x|^2}{k^2})g(x,s+\tau)\big)\|^2 ds<\eta. \tag{4.95}$$

According to (4.91)-(4.95) we find that, for every $\tau\in\mathbb{R},\ \omega\in\Omega,\ t\geq\widetilde{T},\ n\geq N_2$ and $k\geq K_4$,

$$\begin{aligned}
&\|\widehat{v}_{n,2}(\tau,\tau-t,\theta_{-\tau}\omega)\|^2+(\lambda+\delta^2-\beta_2\delta)\|\widehat{u}_{n,2}(\tau,\tau-t,\theta_{-\tau}\omega)\|^2\\
&\quad+\|\Delta\widehat{u}_{n,2}(\tau,\tau-t,\theta_{-\tau}\omega)\|^2+2((I-P_n)\widehat{\rho}(\frac{|x|^2}{k^2})f(x,u),\widehat{u}_{n,2}(\tau,\tau-t,\theta_{-\tau}\omega))\\
\leq&\eta R_7(\tau,\omega),
\end{aligned} \tag{4.96}$$

where $R_7(\tau,\omega)$ is a positive random variable. The proof is completed by (2.1) and (4.96). □

5. Random attractors

In this section, we prove existence and uniqueness of $\mathcal{D}$- pullback attractors for the stochastic system (3.6)-(3.8). First we apply the Lemmas shown in Section 4 to prove the asymptotic compactness of solutions of (3.6)-(3.8) in E.

Lemma 5.1. *Under Assumptions* I *and* II*, for every $\tau\in\mathbb{R},\ \omega\in\Omega$, the sequence of weak solutions of* (3.6)-(3.8)*, $\{Y(\tau,\tau-t_m,\theta_{-\tau}\omega,Y_0(\theta_{-t_m}\omega))\}_{m=1}^{\infty}$ has a convergent subsequence in E whenever $t_m\to\infty$ and $Y_0(\theta_{-t_m}\omega)\in D(\tau-t_m,\theta_{-t_m}\omega)$ with $D\in\mathcal{D}$.*

Proof. Let $t_m \to \infty$ and $Y_0(\theta_{-t_m}\omega) \in D(\tau - t_m, \theta_{-t_m}\omega)$ with $D \in \mathcal{D}$. By Lemma 4.1, there exists $m_1 = m_1(\tau, \omega, D) > 0$ such for all $m \geq m_1$, we have

$$\|Y(\tau, \tau - t_m, \theta_{-\tau}\omega, Y_0(\theta_{-t_m}\omega))\|_E^2 \leq R_1(\tau, \omega). \tag{5.1}$$

By Lemma 4.4, for every $\eta > 0$, there exist $k_0 = k_0(\tau, \omega, \eta) \geq 1$ and $m_2 = m_2(\tau, \omega, D, \eta) \geq m_1$ such for all $m \geq m_2$,

$$\|Y(\tau, \tau - t, \theta_{-\tau}\omega, D(\tau - t, \theta_{-t}\omega))\|^2_{E(\mathbb{R}^n \setminus \mathbb{B}_{k_0})} \leq \eta. \tag{5.2}$$

By Lemma 4.5, there exist $k_1 = k_1(\tau, \omega, \eta) \geq k_0$ and $m_3 = m_3(\tau, \omega, D, \eta) \geq m_2$ and $n_1 = n_1(\tau, \omega, \eta) \geq 0$ such for all $m \geq m_3$,

$$\|(I - P_n)\widehat{Y}(\tau, \tau - t, \theta_{-\tau}\omega, D(\tau - t, \theta_{-\tau}\omega))\|^2_{E(\mathbb{B}_{2k_1})} \leq \eta. \tag{5.3}$$

Using (4.71) and (5.1), we get

$$\|P_n\widehat{Y}(\tau, \tau - t, \theta_{-\tau}\omega, D(\tau - t, \theta_{-\tau}\omega))\|^2_{P_nE(\mathbb{B}_{2k_1})} \leq c_{19}R_1(\tau, \omega), \tag{5.4}$$

which together with (5.3) implies that $\{Y(\tau, \tau - t_m, \theta_{-\tau}\omega, Y_0(\theta_{-t_m}\omega))\}$ is precompact in $E(\mathbb{B}_{2k_1})$. Note that $\widehat{\rho}(\frac{|x|^2}{k_1^2}) = 1$ for $|x| \leq k_1$. Therefore, $\{Y(\tau, \tau - t_m, \theta_{-\tau}\omega, Y_0(\theta_{-t_m}\omega))\}$ is precompact in $E(\mathbb{B}_{k_1})$, which along with (5.2) shows the precompactness of this sequence in E. □

Theorem 5.1. *Under Assumptions* I *and* II*, the random dynamical system* Φ *generated by the stochastic plate equation* (3.6)-(3.8) *has a unique pullback* $\mathcal{D}$*-attractor* $\mathcal{A} = \{\mathcal{A}(\tau, \omega) : \tau \in \mathbb{R},\ \omega \in \Omega\} \in \mathcal{D}$ *in the space* E*.*

Proof. This is an immediate consequence of Proposition 2.1, Lemma 4.1 and Lemma 5.1. □

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