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SOLVABILITY AND STABILITY OF MULTI-TERM FRACTIONAL DELAY Q-DIFFERENCE EQUATION

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Abstract The research of multi-term fractional differential equations has attracted the attention of scholars and obtained abundant results in recent years. However, there are few studies on the multi-term fractional q-difference equations. In this paper, we investigate boundary value problems for multi-term fractional delay q-difference equation. By virtue of Banach contraction mapping principle and Leray-Schauder nonlinear alternative theorem, we obtain the uniqueness and existence of the solution. In addition, we get four different results for functional stability, including Ulam-Hyres stability, generalized Ulam-Hyres stability, Ulam-Hyres Rassias stability and generalized Ulam-Hyres Rassias stability. Finally, give relevant examples to demonstrate the main results.

Keywords Fractional q-difference equations, boundary value problems, multi-term operators, existence and uniqueness, functional stability.

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1. Introduction

Fractional calculus has more advantages than integer calculus in describing and modeling aspects of natural science and engineering technology. Especially with the development of science and technology, the demand for complex engineering modeling is increasing gradually. Fractional model has been used to model different systems. For example, Dabiri [8] simulated several fractional viscoelastic impact models by using fractional Chebyshev allocation method and instantaneous memory principle. Abdullah [1] established a simulation model of cell invasion using fractional differential equations of two basic cell functions. Fractional derivatives are widely used in dynamic systems and the theory of fractional calculus has been further developed.

The q-difference is an important branch of discrete mathematics, which was first proposed by Jackson [14] in 1910 and is a bridge between mathematics and physics. Later, Al-Salam [3] and Agarwal [2] proposed the basic concept and properties of fractional q-difference. Since fractional q-difference theory combined the advantages of discrete mathematics and fractional calculus, scholars introduced equation theory

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into fractional q-difference and obtained abundant theoretical results [12,15–18]. In recent years, q-difference has been applied more and more, especially in quantum physics, economics and dynamical systems [5, 20]. For example, in financial markets, economists use the theoretical knowledge of q-difference to derive the density function of the corresponding q-statistical distribution

$$f_q(x) = (1-q)\sum_{i=-\infty}^{+\infty} q^i \exp_q(-\lambda q^i), \quad \lambda > 0, \quad 0 < q < 1,$$

and mathematical expectations, variances, k-order matrices, etc., and apply them to problems like stock returns [10].

The delay appears because certain processes (periods of infection) are not only related to the current state, but also affected by past state [22]. The delay has applications in various fields of life sciences, such as epidemiology, population dynamics and immunology [13, 19, 27]. Yan et al. [29] proposed a fractional delay differential model for HIV transmission and discussed its stability:

$$\begin{cases} D^\alpha T(t) = s - \mu_T T(t) - rT(t)\left(1 - \frac{T(t)+I(t)}{T_{\max}}\right) - k_1 T(t)V(t), \\ D^\alpha I(t) = k_1' T(t-\tau)V(t-\tau) - \mu_1 I(t), \\ D^\alpha V(t) = N\mu_b I(t) - k_1 T(t)V(t) - \mu v V(t). \end{cases}$$

Therefore, fractional models with delay can better describe practical problems in some cases. According to the different delay parameters involved, it can be divided into continuous delay, discrete delay and proportional delay. Proportional delay can be applied to automatic control systems, infectious diseases and transmission lines [11, 28]. In this paper, we can consider the effect of proportional delay on fractional q-difference equation.

When establishing the mathematical model, we notice that not only a single fractional operator is involved, but in some cases, multi-term fractional operators are required. For example, the famous Basset equation [7] and Bagley-Torvik equation [25]. Rahman et al. [21] investigated the following multi-term fractional delay differential equation

$$\begin{cases} \sum\limits_{i=1}^{n} \lambda_i {}^C D^{\alpha_i} u(t) = f(t, u(t), u(\tau t)), \quad t \in [0,1], \\ u(0) = 0, \frac{d^j u(0)}{du^j} = 0, u(1) = \sum\limits_{j=1}^{n-2} \eta_j u(\xi_j). \end{cases}$$

Boutiara [6] obtained the existence and uniqueness of solution of the multi-term fractional q-difference equation by using coincidence degree theory and Banach contraction principle,

$$\begin{cases} {}^C D_q^\alpha [{}^C D_q^\beta ({}^C D_q^\gamma x(t) - g(t, x(t))) - h(t, x(t))] = f(t, x(t)), \quad t \in [0, T], \\ {}^C D_q^\gamma x(0) = x(0) = 0, ax(\eta) + bx(T) = \sum\limits_{i=1}^{n} \lambda_i I_q^{\omega_i} x(\xi_i), \quad 0 < \eta, \xi_i < T, \omega_i > 0. \end{cases}$$

Inspired by the work above, we consider the following multi-term fractional delay q-difference equation:

$$\begin{cases} \sum\limits_{i=1}^{n} \lambda_i {}^C D_q^{\alpha_i} u(t) = f(t, u(t), u(\tau t)), \quad t \in [0,1], \\ u(0) = 0, \frac{d_q^j u(0)}{d_q u^j} = 0, u(1) = \sum\limits_{j=1}^{n-2} \eta_j u(\xi_j), \end{cases} \tag{1.1}$$

where $n-1<\alpha_1\leq n$, $n\geq 3$, $0<\alpha_i\leq 1$, $i=2,\cdots,n$, $\lambda_i\in\mathbb{R}$, $0<\tau<1$, $\eta_j\in\mathbb{R}$, $0<\xi_j<1$, $j=1,\cdots,n-2$, ${}^CD_q^\alpha$ is fractional q-derivative of Caputo type, $f:[0,1]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is continuous.

For the study of boundary value problem (1.1), besides the existence and uniqueness of solution, we also investigate the stability of solution, stability is an important performance index for the safe operation of the system. Under the premise of existence of the solution, using mathematical methods to study the conditions to ensure the stable operation of the system can provide theoretical guarantee for the safe and stable operation of the actual system. Rizwan et al. [23] by using generalized Diaz-Margolis's fixed point obtained the Ulam-Hyers and Ulam-Hyers-Rassias stability of the following equation

$$ {}^CD_s^\nu({}^CD_d^\mu+\lambda)w(s)=F(s,w(s)),\quad 0<\nu,\mu<1. $$

Waheed et al. [26] investigated different types of Hyers-Ulam stability for a coupled system of the following equation

$$ \begin{cases} {}^CD_{0^+}^{\beta_1}(L_p({}^CD_{0^+}^{\alpha_1}\omega(\rho)))=A_1(\rho)\omega(\rho)+\Phi(\rho,\omega(\rho),{}^CD_{0^+}^{\beta_1}(L_p({}^CD_{0^+}^{\alpha_1}\omega(\rho)))),\\ {}^CD_{0^+}^{\beta_2}(L_p({}^CD_{0^+}^{\alpha_2}\upsilon(\rho)))=A_2(\rho)\upsilon(\rho)+\Psi(\rho,{}^CD_{0^+}^{\beta_2}(L_p({}^CD_{0^+}^{\alpha_2}\omega(\rho))),\upsilon(\rho)),\\ {}^CD_{0^+}^{\alpha_1}\omega(0))=\omega(0)=\omega''(0)=0,\omega'(T)=c_1\int_0^T g_1(s,\omega(s),{}^CD_{0^+}^{\beta_1}(L_p({}^CD_{0^+}^{\alpha_1}\omega(s))))ds,\\ {}^CD_{0^+}^{\alpha_2}\upsilon(0))=\upsilon(0)=\upsilon''(0)=0,\upsilon'(T)=c_2\int_0^T g_2(s,{}^CD_{0^+}^{\beta_2}(L_p({}^CD_{0^+}^{\alpha_2}\upsilon(s))),\upsilon(s))ds. \end{cases} $$

Therefore, in this paper, we consider four different functional stability results, including Ulam-Hyres stability, generalized Ulam-Hyres stability, Ulam-Hyres Rassias stability and generalized Ulam-Hyres Rassias stability.

The structure of this paper is as follows. In section 2, we give the definitions and lemmas of fractional q-derivative and q-integral and some basic theorems. In section 3, we investigate the uniqueness and existence of solution by using Banach contraction mapping principle and Leray-Schauder nonlinear alternative theorem. In Section 4, we obtain four different results for functional stability. In Section 5, relevant examples are used to prove the main results.

2. Preliminaries

Let $q\in(0,1)$ and define

$$ [a]_q=\frac{1-q^a}{1-q},\quad a\in\mathbb{R}. $$

The q-analogue of the power function is given by

$$ (a-b)^{(0)}=1,\quad (a-b)^{(n)}=\prod_{k=0}^{n-1}(a-bq^k),\quad n\in\mathbb{N}^+, $$

$$ (a-b)^{(\alpha)}=a^\alpha\prod_{k=0}^{+\infty}\frac{a-bq^k}{a-bq^{\alpha+k}},\quad \alpha\in\mathbb{R}. $$

If $b=0$, then $a^{(\alpha)}=a^\alpha$. We can also get the following properties

$$ [a(t-s)]^{(\alpha)}=a^\alpha(t-s)^{(\alpha)}, $$

$$
\begin{aligned}
&{}_tD_q(t-s)^{(\alpha)} = [\alpha]_q(t-s)^{(\alpha-1)},\\
&{}_sD_q(t-s)^{(\alpha)} = -[\alpha]_q(t-qs)^{(\alpha-1)},\\
&\int_a^b {}^CD_q f(t)d_qt = f(b)-f(a).
\end{aligned}
$$

The q-Gamma function is given by

$$\Gamma_q(x) = \frac{(1-q)^{(x-1)}}{(1-q)^{x-1}}, \quad x \in \mathbb{R} \setminus \{0,-1,-2,\cdots\},$$

and satisfies $\Gamma_q(x+1) = [x]_q\Gamma_q(x)$.

The q-Beta function is given by

$$B_q(x,y) = \int_0^1 t^{x-1}(1-qt)^{(y-1)}d_qt, \quad x,y \in \mathbb{R}^+,$$

and satisfies $B_q(x,y) = \frac{\Gamma_q(x)\Gamma_q(y)}{\Gamma_q(x+y)}$.

Definition 2.1. [4] The fractional q-integral of Riemann-Liouville type of order $\alpha > 0$ of a function $u : (0,+\infty) \to \mathbb{R}$ is given by

$$I_q^\alpha u(t) = \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} u(s)d_qs.$$

Definition 2.2. [4] The fractional q-derivative of Caputo type of order $\alpha > 0$ of a function $u : (0,+\infty) \to \mathbb{R}$ is given by

$${}^CD_q^\alpha u(t) = I_q^{n-\alpha}({}^CD_q^n u)(t),$$

where n is the smallest integer greater than or equal to α.

Lemma 2.1. [4] *Let $\alpha > 0$. If $u \in L_q^1[0,1]$ such that $I_q^{n-\alpha}u \in AC_q^n[0,1]$. Then*

$$I_q^{\alpha C}D_q^\alpha u(t) = u(t) - c_1 - c_2t - \cdots - c_nt^{n-1},$$

where $c_i \in \mathbb{R}, i = 1,\cdots,n$ and n is the smallest integer greater than or equal to α.

Lemma 2.2. [4] *Let $\alpha > 0$. If $u \in L_q^1[0,1]$. Then*

$${}^CD_q^\alpha I_q^\alpha u(t) = u(t).$$

Theorem 2.1. [24] *Let X be a Banach space and $W \subset X$ be a nonempty closed subset, the mapping $T : W \to W$ is a contraction,*

$$\forall u,v \in W, \quad \|Tu - Tv\| \le k\|u-v\|, \quad 0 < k < 1.$$

Then there is a unique point $u^* \in W$, such that $Tu^* = u^*$, that is, T has a unique fixed point on W.

Theorem 2.2. [9] *Let X be a Banach space and $\Omega \subset X$ be a nonempty convex subset. Let $W \subset \Omega$ be a nonempty open subset with $0 \in W$ and $T : \overline{W} \to \Omega$ be a completely continuous operator. Then either there exists $u \in \partial W$ and $\lambda \in (0,1)$, such that $u = \lambda T(u)$, or T has at least a fixed point on W.*

Definition 2.3. The solution is Ulam-Hyers (UH) stable, if there exist $M_1 \geq 0$ and $\varepsilon > 0$, for each solution $u \in C([0,1],\mathbb{R})$,

$$\Big|\sum_{i=1}^{n} \lambda_i {}^C D_q^{\alpha_i} u(t) - f(t,u(t),u(\tau t))\Big| \leq \varepsilon, \quad t \in [0,1], \tag{2.1}$$

and a unique solution $u^* \in C([0,1],\mathbb{R})$, such that $|u-u^*| \leq M_1\varepsilon$. The solution is generalised Ulam-Hyers (GUH) stable, if there exists a positive function ψ : $(0,+\infty) \to (0,+\infty)$ with $\psi(0) = 0$, such that $|u-u^*| \leq M_1\psi(\varepsilon)$.

Definition 2.4. The solution is Ulam-Hyers Rassias (UHR) stable, function $\varphi \in X$ is continuous and constant $\rho > 0$, if there exist $M_2 \geq 0$ and $\varepsilon > 0$, for each solution $u \in C([0,1],\mathbb{R})$,

$$\Big|\sum_{i=1}^{n} \lambda_i {}^C D_q^{\alpha_i} u(t) - f(t,u(t),u(\tau t))\Big| \leq (\varphi(t)+\rho)\varepsilon, \quad t \in [0,1], \tag{2.2}$$

and a unique solution $u^* \in C([0,1],\mathbb{R})$, such that $|u-u^*| \leq M_2(\varphi(t)+\rho)\varepsilon$.

Definition 2.5. The solution is generalised Ulam-Hyers Rassias ($GUHR$) stable, function $\varphi \in X$ is continuous and constant $\rho > 0$, if there exists $M_2 \geq 0$, for each solution $u \in C([0,1],\mathbb{R})$,

$$\Big|\sum_{i=1}^{n} \lambda_i {}^C D_q^{\alpha_i} u(t) - f(t,u(t),u(\tau t))\Big| \leq \varphi(t)+\rho, \quad t \in [0,1], \tag{2.3}$$

and a unique solution $u^* \in C([0,1],\mathbb{R})$, such that $|u-u^*| \leq M_2(\varphi(t)+\rho)$.

Lemma 2.3. *Function $u \in C([0,1],\mathbb{R})$ is the solution of* (2.1)*, iff there exists a function $\zeta \in C([0,1],\mathbb{R})$ depends on u, such that*

(i) $\zeta(t) \leq \varepsilon$, $t \in [0,1]$.

(ii) $\sum_{i=1}^{n} \lambda_i {}^C D_q^{\alpha_i} u(t) - f(t,u(t),u(\tau t)) - \zeta(t) = 0$.

Proof. If (i) and (ii) hold, we obtain,

$$\Big|\sum_{i=1}^{n} \lambda_i {}^C D_q^{\alpha_i} u(t) - f(t,u(t),u(\tau t))\Big| = \zeta(t) \leq \varepsilon.$$

If $u \in C([0,1],\mathbb{R})$ is the solution of (2.1), we get,

$$-\varepsilon \leq \sum_{i=1}^{n} \lambda_i {}^C D_q^{\alpha_i} u(t) - f(t,u(t),u(\tau t)) \leq \varepsilon.$$

Hence, there exists $\zeta(t) \in [-\varepsilon,\varepsilon]$, such that

$$\sum_{i=1}^{n} \lambda_i {}^C D_q^{\alpha_i} u(t) - f(t,u(t),u(\tau t)) = \zeta(t).$$

This completes the proof. □

Lemma 2.4. *Function $u \in C([0,1],\mathbb{R})$ is the solution of (2.2), iff there exists a function $\zeta \in C([0,1],\mathbb{R})$ depends on u, such that*

(*i*) $\zeta(t) \le \varphi(t)\varepsilon$ *and* $\varphi(t)\varepsilon \le \rho\varepsilon$, $t \in [0,1]$.

(*ii*) $\sum\limits_{i=1}^{n} \lambda_i {}^C D_q^{\alpha_i} u(t) - f(t,u(t),u(\tau t)) - \zeta(t) = 0.$

Proof. If (*i*) and (*ii*) hold, we obtain,

$$\Big|\sum_{i=1}^{n} \lambda_i {}^C D_q^{\alpha_i} u(t) - f(t,u(t),u(\tau t))\Big| = \zeta(t) \le \varphi(t)\varepsilon \le (\varphi(t)+\rho)\varepsilon.$$

If $u \in C([0,1],\mathbb{R})$ is the solution of (2.2), we get,

$$-(\varphi(t)+\rho)\varepsilon \le \sum_{i=1}^{n} \lambda_i {}^C D_q^{\alpha_i} u(t) - f(t,u(t),u(\tau t)) \le (\varphi(t)+\rho)\varepsilon.$$

Hence, there exists $\zeta(t) \in [-(\varphi(t)+\rho)\varepsilon, (\varphi(t)+\rho)\varepsilon]$, such that

$$\sum_{i=1}^{n} \lambda_i {}^C D_q^{\alpha_i} u(t) - f(t,u(t),u(\tau t)) = \zeta(t).$$

This completes the proof. □

Lemma 2.5. *Function $u \in C([0,1],\mathbb{R})$ is the solution of (2.3), iff there exists a function $\zeta \in C([0,1],\mathbb{R})$ depends on u, such that*

(*i*) $\zeta(t) \le \varphi(t)$ *and* $\varphi(t) \le \rho$, $t \in [0,1]$.

(*ii*) $\sum\limits_{i=1}^{n} \lambda_i {}^C D_q^{\alpha_i} u(t) - f(t,u(t),u(\tau t)) - \zeta(t) = 0.$

Proof. If (*i*) and (*ii*) hold, we obtain,

$$\Big|\sum_{i=1}^{n} \lambda_i {}^C D_q^{\alpha_i} u(t) - f(t,u(t),u(\tau t))\Big| = \zeta(t) \le \varphi(t) \le \varphi(t)+\rho.$$

If $u \in C([0,1],\mathbb{R})$ is the solution of (2.3), we get,

$$-(\varphi(t)+\rho) \le \sum_{i=1}^{n} \lambda_i {}^C D_q^{\alpha_i} u(t) - f(t,u(t),u(\tau t)) \le \varphi(t)+\rho.$$

Hence, there exists $\zeta(t) \in [-(\varphi(t)+\rho), \varphi(t)+\rho]$, such that

$$\sum_{i=1}^{n} \lambda_i {}^C D_q^{\alpha_i} u(t) - f(t,u(t),u(\tau t)) = \zeta(t).$$

This completes the proof. □

3. The solvability of fractional q-difference equation

Lemma 3.1. *If $y \in C([0,1],\mathbb{R})$. Then the unique solution of problem*

$$\begin{cases} \sum\limits_{i=1}^{n} \lambda_i {}^C D_q^{\alpha_i} u(t) = y(t), \quad t \in [0,1], \\ u(0) = 0, \frac{d_q^j u(0)}{d_q u^j} = 0, u(1) = \sum\limits_{j=1}^{n-2} \eta_j u(\xi_j), \end{cases} \tag{3.1}$$

is

$$u(t) =\frac{t^{n-1}}{\beta}\Big[-\sum_{j=1}^{n-2}\sum_{i=2}^{n}\frac{\lambda_i\eta_j}{\lambda_1}\int_0^{\xi_j}\frac{(\xi_j-qs)^{(\alpha_1-\alpha_i-1)}}{\Gamma_q(\alpha_1-\alpha_i)}u(s)d_qs + \sum_{j=1}^{n-2}\frac{\eta_j}{\lambda_1}\int_0^{\xi_j}\frac{(\xi_j-qs)^{(\alpha_1-1)}}{\Gamma_q(\alpha_1)}y(s)d_qs + \sum_{i=2}^{n}\frac{\lambda_i}{\lambda_1}\int_0^{1}\frac{(1-qs)^{(\alpha_1-\alpha_i-1)}}{\Gamma_q(\alpha_1-\alpha_i)}u(s)d_qs - \frac{1}{\lambda_1}\int_0^{1}\frac{(1-qs)^{(\alpha_1-1)}}{\Gamma_q(\alpha_1)}y(s)d_qs\Big] - \sum_{i=2}^{n}\frac{\lambda_i}{\lambda_1}\int_0^{t}\frac{(t-qs)^{(\alpha_1-\alpha_i-1)}}{\Gamma_q(\alpha_1-\alpha_i)}u(s)d_qs + \frac{1}{\lambda_1}\int_0^{t}\frac{(t-qs)^{(\alpha_1-1)}}{\Gamma_q(\alpha_1)}y(s)d_qs, \tag{3.2}$$

where $\beta = 1-\sum\limits_{j=1}^{n-2}\eta_j\xi_j^{n-1}\neq 0.$

Proof. (i) We show that if $u(t)$ is the solution of (3.1), then it can be expressed as (3.2). By Lemma 2.1, we obtain,

$$u(t) =c_1+c_2t+\cdots+c_nt^{n-1}-\sum_{i=2}^{n}\frac{\lambda_i}{\lambda_1}\int_0^{t}\frac{(t-qs)^{(\alpha_1-\alpha_i-1)}}{\Gamma_q(\alpha_1-\alpha_i)}u(s)d_qs + \frac{1}{\lambda_1}\int_0^{t}\frac{(t-qs)^{(\alpha_1-1)}}{\Gamma_q(\alpha_1)}y(s)d_qs.$$

From the boundary condition $u(0)=0$, we get $c_1=0$. Therefore,

$$\frac{d_qu(t)}{d_qu} =c_2+\cdots+c_n[n-1]_qt^{n-2}-\sum_{i=2}^{n}\frac{\lambda_i}{\lambda_1}\int_0^{t}\frac{(t-qs)^{(\alpha_1-\alpha_i-2)}}{\Gamma_q(\alpha_1-\alpha_i-1)}u(s)d_qs + \frac{1}{\lambda_1}\int_0^{t}\frac{(t-qs)^{(\alpha_1-2)}}{\Gamma_q(\alpha_1-1)}y(s)d_qs.$$

By $\frac{d_qu(0)}{d_qu}=0$, we have $c_2=0$. Hence,

$$\frac{d_q^2u(t)}{d_qu^2} =c_3[2]_q+\cdots+c_n[n-1]_q[n-2]_qt^{n-3}-\sum_{i=2}^{n}\frac{\lambda_i}{\lambda_1}\int_0^{t}\frac{(t-qs)^{(\alpha_1-\alpha_i-3)}}{\Gamma_q(\alpha_1-\alpha_i-2)}u(s)d_qs + \frac{1}{\lambda_1}\int_0^{t}\frac{(t-qs)^{(\alpha_1-3)}}{\Gamma_q(\alpha_1-2)}y(s)d_qs.$$

From $\frac{d_q^2u(0)}{d_qu^2}=0$, we obtain $c_3=0$. By $\frac{d_q^ju(0)}{d_qu^j}=0$, $j=3,\ldots,n-2$, we get, $c_4=\cdots=c_{n-1}=0$. Therefore,

$$u(t) = c_nt^{n-1}-\sum_{i=2}^{n}\frac{\lambda_i}{\lambda_1}\int_0^{t}\frac{(t-qs)^{(\alpha_1-\alpha_i-1)}}{\Gamma_q(\alpha_1-\alpha_i)}u(s)d_qs+\frac{1}{\lambda_1}\int_0^{t}\frac{(t-qs)^{(\alpha_1-1)}}{\Gamma_q(\alpha_1)}y(s)d_qs.$$

From $u(1)=\sum\limits_{j=1}^{n-2}\eta_ju(\xi_j)$, we have,

$$c_n-\sum_{i=2}^{n}\frac{\lambda_i}{\lambda_1}\int_0^{1}\frac{(1-qs)^{(\alpha_1-\alpha_i-1)}}{\Gamma_q(\alpha_1-\alpha_i)}u(s)d_qs+\frac{1}{\lambda_1}\int_0^{1}\frac{(1-qs)^{(\alpha_1-1)}}{\Gamma_q(\alpha_1)}y(s)d_qs$$

$$
\begin{aligned}
&=\sum_{j=1}^{n-2}\eta_j\Big[c_n\xi_j^{n-1}-\sum_{i=2}^{n}\frac{\lambda_i}{\lambda_1}\int_0^{\xi_j}\frac{(\xi_j-qs)^{(\alpha_1-\alpha_i-1)}}{\Gamma_q(\alpha_1-\alpha_i)}u(s)d_qs\\
&\quad+\frac{1}{\lambda_1}\int_0^{\xi_j}\frac{(\xi_j-qs)^{(\alpha_1-1)}}{\Gamma_q(\alpha_1)}y(s)d_qs\Big].
\end{aligned}
$$

By simple calculation, we obtain,

$$
\begin{aligned}
c_n=&\frac{1}{\beta}\Big[-\sum_{j=1}^{n-2}\sum_{i=2}^{n}\frac{\lambda_i\eta_j}{\lambda_1}\int_0^{\xi_j}\frac{(\xi_j-qs)^{(\alpha_1-\alpha_i-1)}}{\Gamma_q(\alpha_1-\alpha_i)}u(s)d_qs\\
&+\sum_{j=1}^{n-2}\frac{\eta_j}{\lambda_1}\int_0^{\xi_j}\frac{(\xi_j-qs)^{(\alpha_1-1)}}{\Gamma_q(\alpha_1)}y(s)d_qs\\
&+\sum_{i=2}^{n}\frac{\lambda_i}{\lambda_1}\int_0^{1}\frac{(1-qs)^{(\alpha_1-\alpha_i-1)}}{\Gamma_q(\alpha_1-\alpha_i)}u(s)d_qs-\frac{1}{\lambda_1}\int_0^{1}\frac{(1-qs)^{(\alpha_1-1)}}{\Gamma_q(\alpha_1)}y(s)d_qs\Big],
\end{aligned}
$$

where $\beta=1-\sum\limits_{j=1}^{n-2}\eta_j\xi_j^{n-1}\neq 0$. Then,

$$
\begin{aligned}
u(t)=&\frac{t^{n-1}}{\beta}\Big[-\sum_{j=1}^{n-2}\sum_{i=2}^{n}\frac{\lambda_i\eta_j}{\lambda_1}\int_0^{\xi_j}\frac{(\xi_j-qs)^{(\alpha_1-\alpha_i-1)}}{\Gamma_q(\alpha_1-\alpha_i)}u(s)d_qs\\
&+\sum_{j=1}^{n-2}\frac{\eta_j}{\lambda_1}\int_0^{\xi_j}\frac{(\xi_j-qs)^{(\alpha_1-1)}}{\Gamma_q(\alpha_1)}y(s)d_qs+\sum_{i=2}^{n}\frac{\lambda_i}{\lambda_1}\int_0^{1}\frac{(1-qs)^{(\alpha_1-\alpha_i-1)}}{\Gamma_q(\alpha_1-\alpha_i)}u(s)d_qs\\
&-\frac{1}{\lambda_1}\int_0^{1}\frac{(1-qs)^{(\alpha_1-1)}}{\Gamma_q(\alpha_1)}y(s)d_qs\Big]-\sum_{i=2}^{n}\frac{\lambda_i}{\lambda_1}\int_0^{t}\frac{(t-qs)^{(\alpha_1-\alpha_i-1)}}{\Gamma_q(\alpha_1-\alpha_i)}u(s)d_qs\\
&+\frac{1}{\lambda_1}\int_0^{t}\frac{(t-qs)^{(\alpha_1-1)}}{\Gamma_q(\alpha_1)}y(s)d_qs.
\end{aligned}
$$

(ii) We present that if $u(t)$ can be expressed as (3.2), then it is the solution of (3.1). In fact,

$$
\begin{aligned}
u(t)&=c_nt^{n-1}-\sum_{i=2}^{n}\frac{\lambda_i}{\lambda_1}\int_0^{t}\frac{(t-qs)^{(\alpha_1-\alpha_i-1)}}{\Gamma_q(\alpha_1-\alpha_i)}u(s)d_qs+\frac{1}{\lambda_1}\int_0^{t}\frac{(t-qs)^{(\alpha_1-1)}}{\Gamma_q(\alpha_1)}y(s)d_qs\\
&=c_nt^{n-1}-\sum_{i=2}^{n}\frac{\lambda_i}{\lambda_1}I_q^{\alpha_1-\alpha_i}u(t)+\frac{1}{\lambda_1}I_q^{\alpha_1}y(t),
\end{aligned}
$$

where

$$
\begin{aligned}
c_n=&\frac{1}{\beta}\Big[-\sum_{j=1}^{n-2}\sum_{i=2}^{n}\frac{\lambda_i\eta_j}{\lambda_1}\int_0^{\xi_j}\frac{(\xi_j-qs)^{(\alpha_1-\alpha_i-1)}}{\Gamma_q(\alpha_1-\alpha_i)}u(s)d_qs\\
&+\sum_{j=1}^{n-2}\frac{\eta_j}{\lambda_1}\int_0^{\xi_j}\frac{(\xi_j-qs)^{(\alpha_1-1)}}{\Gamma_q(\alpha_1)}y(s)d_qs\\
&+\sum_{i=2}^{n}\frac{\lambda_i}{\lambda_1}\int_0^{1}\frac{(1-qs)^{(\alpha_1-\alpha_i-1)}}{\Gamma_q(\alpha_1-\alpha_i)}u(s)d_qs-\frac{1}{\lambda_1}\int_0^{1}\frac{(1-qs)^{(\alpha_1-1)}}{\Gamma_q(\alpha_1)}y(s)d_qs\Big].
\end{aligned}
$$

From Lemma 2.2, we get,

$$ {}^CD_q^{\alpha_1}u(t)=-\sum_{i=2}^{n}\frac{\lambda_i}{\lambda_1}{}^CD_q^{\alpha_i}u(t)+\frac{1}{\lambda_1}y(t). $$

Therefore,

$$ \sum_{i=1}^{n}\lambda_i{}^CD_q^{\alpha_i}u(t)=y(t). $$

This completes the proof. □

Let $X=C([0,1],\mathbb{R})$ with $\|x\|=\sup_{t\in[0,1]}|x(t)|$. Then X is a Banach space. From Lemma 3.1, the problem (1.1) can be converted into the following fixed point problem. Define an operator $T:C([0,1],\mathbb{R})\to C([0,1],\mathbb{R})$,

$$ \begin{aligned} &Tu(t)\\ =&\frac{t^{n-1}}{\beta}\Big[-\sum_{j=1}^{n-2}\sum_{i=2}^{n}\frac{\lambda_i\eta_j}{\lambda_1}\int_0^{\xi_j}\frac{(\xi_j-qs)^{(\alpha_1-\alpha_i-1)}}{\Gamma_q(\alpha_1-\alpha_i)}u(s)d_qs\\ &+\sum_{j=1}^{n-2}\frac{\eta_j}{\lambda_1}\int_0^{\xi_j}\frac{(\xi_j-qs)^{(\alpha_1-1)}}{\Gamma_q(\alpha_1)}f(s,u(s),u(\tau s))d_qs\\ &+\sum_{i=2}^{n}\frac{\lambda_i}{\lambda_1}\int_0^{1}\frac{(1-qs)^{(\alpha_1-\alpha_i-1)}}{\Gamma_q(\alpha_1-\alpha_i)}u(s)d_qs-\frac{1}{\lambda_1}\int_0^{1}\frac{(1-qs)^{(\alpha_1-1)}}{\Gamma_q(\alpha_1)}f(s,u(s),u(\tau s))d_qs\Big]\\ &-\sum_{i=2}^{n}\frac{\lambda_i}{\lambda_1}\int_0^{t}\frac{(t-qs)^{(\alpha_1-\alpha_i-1)}}{\Gamma_q(\alpha_1-\alpha_i)}u(s)d_qs+\frac{1}{\lambda_1}\int_0^{t}\frac{(t-qs)^{(\alpha_1-1)}}{\Gamma_q(\alpha_1)}f(s,u(s),u(\tau s))d_qs. \end{aligned} $$

In this paper, we give the following assumptions:

(A_1) $f:[0,1]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is continuous.

(A_2) $\big|f(t,u_1,u_2)-f(t,v_1,v_2)\big|\le k_1\big|u_1-v_1\big|+k_2\big|u_2-v_2\big|$, $t\in[0,1]$, $u_1,u_2,v_1,v_2\in\mathbb{R}$, $k_1,k_2>0$.

(A_3) $\big|f(t,u_1,u_2)\big|\le\big|\phi_f(t)\big|$, $t\in[0,1]$, $u_1,u_2\in\mathbb{R}$, $\phi_f(t)\in C([0,1],\mathbb{R})$.

(A_4) For any non-decreasing function $\zeta\in C([0,1],\mathbb{R})$, there exists a constant $M>0$, such that $I_q^{\alpha_1}\zeta(t)=M\zeta(t)$, $t\in[0,1]$.

Theorem 3.1. *If (A_1) and (A_2) hold and*

$$ \begin{aligned} k=&\sum_{j=1}^{n-2}\sum_{i=2}^{n}\frac{|\lambda_i|}{|\beta||\lambda_1|\Gamma_q(\alpha_1-\alpha_i+1)}\big(|\eta_j|\xi_j^{\alpha_1-\alpha_i}+1+|\beta|\big)\\ &+\sum_{j=1}^{n-2}\frac{k_1+k_2}{|\beta||\lambda_1|\Gamma_q(\alpha_1+1)}\big(|\eta_j|\xi_j^{\alpha_1}+1+|\beta|\big)<1. \end{aligned} $$

Then boundary value problem (1.1) *has a unique solution.*

Proof. For any $u_1,u_2\in C([0,1],\mathbb{R})$, in virtue of (A_2), we obtain,

$$ \begin{aligned} &\big\|Tu_1-Tu_2\big\|\\ \le&\sup_{t\in[0,1]}\Big\{\frac{t^{n-1}}{|\beta|}\Big[\sum_{j=1}^{n-2}\sum_{i=2}^{n}\frac{|\lambda_i||\eta_j|}{|\lambda_1|}\int_0^{\xi_j}\frac{(\xi_j-qs)^{(\alpha_1-\alpha_i-1)}}{\Gamma_q(\alpha_1-\alpha_i)}\big|u_1(s)-u_2(s)\big|d_qs \end{aligned} $$

$$+\sum_{j=1}^{n-2}\frac{|\eta_j|}{|\lambda_1|}\int_0^{\xi_j}\frac{(\xi_j-qs)^{(\alpha_1-1)}}{\Gamma_q(\alpha_1)}\big|f(s,u_1(s),u_1(\tau s))-f(s,u_2(s),u_2(\tau s))\big|d_qs$$
$$+\sum_{i=2}^{n}\frac{|\lambda_i|}{|\lambda_1|}\int_0^1\frac{(1-qs)^{(\alpha_1-\alpha_i-1)}}{\Gamma_q(\alpha_1-\alpha_i)}\big|u_1(s)-u_2(s)\big|d_qs$$
$$+\frac{1}{|\lambda_1|}\int_0^1\frac{(1-qs)^{(\alpha_1-1)}}{\Gamma_q(\alpha_1)}\big|f(s,u_1(s),u_1(\tau s))-f(s,u_2(s),u_2(\tau s))\big|d_qs\Big]$$
$$+\sum_{i=2}^{n}\frac{|\lambda_i|}{|\lambda_1|}\int_0^t\frac{(t-qs)^{(\alpha_1-\alpha_i-1)}}{\Gamma_q(\alpha_1-\alpha_i)}\big|u_1(s)-u_2(s)\big|d_qs$$
$$+\frac{1}{|\lambda_1|}\int_0^t\frac{(t-qs)^{(\alpha_1-1)}}{\Gamma_q(\alpha_1)}\big|f(s,u_1(s),u_1(\tau s))-f(s,u_2(s),u_2(\tau s))\big|d_qs\Big\}$$
$$\le\|u_1-u_2\|\sup_{t\in[0,1]}\Big\{\frac{t^{n-1}}{|\beta|}\Big[\sum_{j=1}^{n-2}\sum_{i=2}^{n}\frac{|\lambda_i||\eta_j|}{|\lambda_1|}\int_0^{\xi_j}\frac{(\xi_j-qs)^{(\alpha_1-\alpha_i-1)}}{\Gamma_q(\alpha_1-\alpha_i)}d_qs$$
$$+\sum_{j=1}^{n-2}\frac{|\eta_j|}{|\lambda_1|}\int_0^{\xi_j}\frac{(\xi_j-qs)^{(\alpha_1-1)}}{\Gamma_q(\alpha_1)}(k_1+k_2)d_qs$$
$$+\sum_{i=2}^{n}\frac{|\lambda_i|}{|\lambda_1|}\int_0^1\frac{(1-qs)^{(\alpha_1-\alpha_i-1)}}{\Gamma_q(\alpha_1-\alpha_i)}d_qs+\frac{1}{|\lambda_1|}\int_0^1\frac{(1-qs)^{(\alpha_1-1)}}{\Gamma_q(\alpha_1)}(k_1+k_2)d_qs\Big]$$
$$+\sum_{i=2}^{n}\frac{|\lambda_i|}{|\lambda_1|}\int_0^t\frac{(t-qs)^{(\alpha_1-\alpha_i-1)}}{\Gamma_q(\alpha_1-\alpha_i)}d_qs+\frac{1}{|\lambda_1|}\int_0^t\frac{(t-qs)^{(\alpha_1-1)}}{\Gamma_q(\alpha_1)}(k_1+k_2)d_qs\Big\}$$
$$\le\|u_1-u_2\|\Big\{\frac{1}{|\beta|}\Big[\sum_{j=1}^{n-2}\sum_{i=2}^{n}\frac{|\lambda_i||\eta_j|}{|\lambda_1|}\frac{\xi_j^{\alpha_1-\alpha_i}}{\Gamma_q(\alpha_1-\alpha_i+1)}+\sum_{j=1}^{n-2}\frac{|\eta_j|}{|\lambda_1|}\frac{(k_1+k_2)\xi_j^{\alpha_1}}{\Gamma_q(\alpha_1+1)}$$
$$+\sum_{i=2}^{n}\frac{|\lambda_i|}{|\lambda_1|}\frac{1}{\Gamma_q(\alpha_1-\alpha_i+1)}+\frac{1}{|\lambda_1|}\frac{k_1+k_2}{\Gamma_q(\alpha_1+1)}\Big]$$
$$+\sum_{i=2}^{n}\frac{|\lambda_i|}{|\lambda_1|}\frac{1}{\Gamma_q(\alpha_1-\alpha_i+1)}+\frac{1}{|\lambda_1|}\frac{k_1+k_2}{\Gamma_q(\alpha_1+1)}\Big\}$$
$$=\|u_1-u_2\|\Big[\sum_{j=1}^{n-2}\sum_{i=2}^{n}\frac{|\lambda_i|}{|\beta||\lambda_1|\Gamma_q(\alpha_1-\alpha_i+1)}\big(|\eta_j|\xi_j^{\alpha_1-\alpha_i}+1+|\beta|\big)$$
$$+\sum_{j=1}^{n-2}\frac{k_1+k_2}{|\beta||\lambda_1|\Gamma_q(\alpha_1+1)}\big(|\eta_j|\xi_j^{\alpha_1}+1+|\beta|\big)\Big].$$

Hence,

$$\|Tu_1-Tu_2\|\le k\|u_1-u_2\|.$$

From Theorem 2.1, boundary value problem (1.1) has a unique solution. This completes the proof. □

Theorem 3.2. *If* (A_1) *and* (A_3) *hold. Then boundary value problem* (1.1) *has at least a solution.*

Proof. Denote $\Omega = \{u \in C([0,1],\mathbb{R}) : \|u\| \le r\}$ where

$$r \ge \frac{\sum\limits_{j=1}^{n-2} \frac{\|\phi_f\|}{|\beta||\lambda_1|\Gamma_q(\alpha_1+1)}\big(|\eta_j|\xi_j^{\alpha_1}+1+|\beta|\big)}{1-\sum\limits_{j=1}^{n-2}\sum\limits_{i=2}^{n} \frac{|\lambda_i|}{|\beta||\lambda_1|\Gamma_q(\alpha_1-\alpha_i+1)}\big(|\eta_j|\xi_j^{\alpha_1-\alpha_i}+1+|\beta|\big)}.$$

(i) We present that T is uniformly bounded. For any $u \in \Omega$, by (A_3), we have

$$\begin{aligned}
\|Tu\| \le \sup_{t\in[0,1]} \Big\{ & \frac{t^{n-1}}{|\beta|}\Big[\sum_{j=1}^{n-2}\sum_{i=2}^{n} \frac{|\lambda_i||\eta_j|}{|\lambda_1|}\int_0^{\xi_j} \frac{(\xi_j-qs)^{(\alpha_1-\alpha_i-1)}}{\Gamma_q(\alpha_1-\alpha_i)}\big|u(s)\big|d_qs \\
&+\sum_{j=1}^{n-2}\frac{|\eta_j|}{|\lambda_1|}\int_0^{\xi_j}\frac{(\xi_j-qs)^{(\alpha_1-1)}}{\Gamma_q(\alpha_1)}\big|f(s,u_(s),u_(\tau s))\big|d_qs \\
&+\sum_{i=2}^{n}\frac{|\lambda_i|}{|\lambda_1|}\int_0^1\frac{(1-qs)^{(\alpha_1-\alpha_i-1)}}{\Gamma_q(\alpha_1-\alpha_i)}\big|u(s)\big|d_qs \\
&+\frac{1}{|\lambda_1|}\int_0^1\frac{(1-qs)^{(\alpha_1-1)}}{\Gamma_q(\alpha_1)}\big|f(s,u(s),u(\tau s))\big|d_qs\Big] \\
&+\sum_{i=2}^{n}\frac{|\lambda_i|}{|\lambda_1|}\int_0^t\frac{(t-qs)^{(\alpha_1-\alpha_i-1)}}{\Gamma_q(\alpha_1-\alpha_i)}\big|u(s)\big|d_qs \\
&+\frac{1}{|\lambda_1|}\int_0^t\frac{(t-qs)^{(\alpha_1-1)}}{\Gamma_q(\alpha_1)}\big|f(s,u(s),u(\tau s))\big|d_qs\Big\} \\
\le \sup_{t\in[0,1]}\Big\{ & \frac{t^{n-1}}{|\beta|}\Big[\sum_{j=1}^{n-2}\sum_{i=2}^{n}\frac{|\lambda_i||\eta_j|}{|\lambda_1|}\int_0^{\xi_j}\frac{(\xi_j-qs)^{(\alpha_1-\alpha_i-1)}}{\Gamma_q(\alpha_1-\alpha_i)}r d_qs \\
&+\sum_{j=1}^{n-2}\frac{|\eta_j|}{|\lambda_1|}\int_0^{\xi_j}\frac{(\xi_j-qs)^{(\alpha_1-1)}}{\Gamma_q(\alpha_1)}\|\phi_f\|d_qs \\
&+\sum_{i=2}^{n}\frac{|\lambda_i|}{|\lambda_1|}\int_0^1\frac{(1-qs)^{(\alpha_1-\alpha_i-1)}}{\Gamma_q(\alpha_1-\alpha_i)}r d_qs+\frac{1}{|\lambda_1|}\int_0^1\frac{(1-qs)^{(\alpha_1-1)}}{\Gamma_q(\alpha_1)}\|\phi_f\|d_qs\Big] \\
&+\sum_{i=2}^{n}\frac{|\lambda_i|}{|\lambda_1|}\int_0^t\frac{(t-qs)^{(\alpha_1-\alpha_i-1)}}{\Gamma_q(\alpha_1-\alpha_i)}r d_qs+\frac{1}{|\lambda_1|}\int_0^t\frac{(t-qs)^{(\alpha_1-1)}}{\Gamma_q(\alpha_1)}\|\phi_f\|d_qs\Big\} \\
\le & \frac{1}{|\beta|}\Big[\sum_{j=1}^{n-2}\sum_{i=2}^{n}\frac{|\lambda_i||\eta_j|}{|\lambda_1|}\frac{\xi_j^{\alpha_1-\alpha_i}}{\Gamma_q(\alpha_1-\alpha_i+1)}r+\sum_{j=1}^{n-2}\frac{|\eta_j|}{|\lambda_1|}\frac{\xi_j^{\alpha_1}}{\Gamma_q(\alpha_1+1)}\|\phi_f\| \\
&+\sum_{i=2}^{n}\frac{|\lambda_i|}{|\lambda_1|}\frac{1}{\Gamma_q(\alpha_1-\alpha_i+1)}r+\frac{1}{|\lambda_1|}\frac{1}{\Gamma_q(\alpha_1+1)}\|\phi_f\|\Big] \\
&+\sum_{i=2}^{n}\frac{|\lambda_i|}{|\lambda_1|}\frac{1}{\Gamma_q(\alpha_1-\alpha_i+1)}r+\frac{1}{|\lambda_1|}\frac{1}{\Gamma_q(\alpha_1+1)}\|\phi_f\| \\
= & \sum_{j=1}^{n-2}\sum_{i=2}^{n}\frac{|\lambda_i|r}{|\beta||\lambda_1|\Gamma_q(\alpha_1-\alpha_i+1)}\big(|\eta_j|\xi_j^{\alpha_1-\alpha_i}+1+|\beta|\big)
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{j=1}^{n-2}\frac{\|\phi_f\|}{|\beta||\lambda_1|\Gamma_q(\alpha_1+1)}\big(|\eta_j|\xi_j^{\alpha_1}+1+|\beta|\big)\\
\leq& r.
\end{aligned}
$$

(ii) We show that T is equicontinuous. For any $u\in\Omega$, $t_1,t_2\in[0,1]$, $t_1\leq t_2$, from (A_3), we get

$$
\begin{aligned}
&\big|Tu(t_1)-Tu(t_2)\big|\\
\leq&\frac{t_2^{n-1}-t_1^{n-1}}{|\beta|}\Big[\sum_{j=1}^{n-2}\sum_{i=2}^{n}\frac{|\lambda_i||\eta_j|}{|\lambda_1|}\int_0^{\xi_j}\frac{(\xi_j-qs)^{(\alpha_1-\alpha_i-1)}}{\Gamma_q(\alpha_1-\alpha_i)}\big|u(s)\big|d_qs\\
&+\sum_{j=1}^{n-2}\frac{|\eta_j|}{|\lambda_1|}\int_0^{\xi_j}\frac{(\xi_j-qs)^{(\alpha_1-1)}}{\Gamma_q(\alpha_1)}\big|f(s,u_(s),u_(\tau s))\big|d_qs\\
&+\sum_{i=2}^{n}\frac{|\lambda_i|}{|\lambda_1|}\int_0^{1}\frac{(1-qs)^{(\alpha_1-\alpha_i-1)}}{\Gamma_q(\alpha_1-\alpha_i)}\big|u(s)\big|d_qs\\
&+\frac{1}{|\lambda_1|}\int_0^{1}\frac{(1-qs)^{(\alpha_1-1)}}{\Gamma_q(\alpha_1)}\big|f(s,u(s),u(\tau s))\big|d_qs\Big]\\
&+\sum_{i=2}^{n}\frac{|\lambda_i|}{|\lambda_1|}\int_0^{t_1}\frac{(t_2-qs)^{(\alpha_1-\alpha_i-1)}-(t_1-qs)^{(\alpha_1-\alpha_i-1)}}{\Gamma_q(\alpha_1-\alpha_i)}\big|u(s)\big|d_qs\\
&+\sum_{i=2}^{n}\frac{|\lambda_i|}{|\lambda_1|}\int_{t_1}^{t_2}\frac{(t_2-qs)^{(\alpha_1-\alpha_i-1)}}{\Gamma_q(\alpha_1-\alpha_i)}\big|u(s)\big|d_qs\\
&+\frac{1}{|\lambda_1|}\int_0^{t_1}\frac{(t_2-qs)^{(\alpha_1-1)}-(t_1-qs)^{(\alpha_1-1)}}{\Gamma_q(\alpha_1)}\big|f(s,u(s),u(\tau s))\big|d_qs\\
&+\frac{1}{|\lambda_1|}\int_{t_1}^{t_2}\frac{(t_2-qs)^{(\alpha_1-1)}}{\Gamma_q(\alpha_1)}\big|f(s,u(s),u(\tau s))\big|d_qs\\
\leq&\frac{t_2^{n-1}-t_1^{n-1}}{|\beta|}\Big[\sum_{j=1}^{n-2}\sum_{i=2}^{n}\frac{|\lambda_i||\eta_j|}{|\lambda_1|}\int_0^{\xi_j}\frac{(\xi_j-qs)^{(\alpha_1-\alpha_i-1)}}{\Gamma_q(\alpha_1-\alpha_i)}r d_qs\\
&+\sum_{j=1}^{n-2}\frac{|\eta_j|}{|\lambda_1|}\int_0^{\xi_j}\frac{(\xi_j-qs)^{(\alpha_1-1)}}{\Gamma_q(\alpha_1)}\|\phi_f\|d_qs\\
&+\sum_{i=2}^{n}\frac{|\lambda_i|}{|\lambda_1|}\int_0^{1}\frac{(1-qs)^{(\alpha_1-\alpha_i-1)}}{\Gamma_q(\alpha_1-\alpha_i)}r d_qs+\frac{1}{|\lambda_1|}\int_0^{1}\frac{(1-qs)^{(\alpha_1-1)}}{\Gamma_q(\alpha_1)}\|\phi_f\|d_qs\Big]\\
&+\sum_{i=2}^{n}\frac{|\lambda_i|}{|\lambda_1|}\int_0^{t_1}\frac{(t_2-qs)^{(\alpha_1-\alpha_i-1)}-(t_1-qs)^{(\alpha_1-\alpha_i-1)}}{\Gamma_q(\alpha_1-\alpha_i)}r d_qs\\
&+\sum_{i=2}^{n}\frac{|\lambda_i|}{|\lambda_1|}\int_{t_1}^{t_2}\frac{(t_2-qs)^{(\alpha_1-\alpha_i-1)}}{\Gamma_q(\alpha_1-\alpha_i)}r d_qs\\
&+\frac{1}{|\lambda_1|}\int_0^{t_1}\frac{(t_2-qs)^{(\alpha_1-1)}-(t_1-qs)^{(\alpha_1-1)}}{\Gamma_q(\alpha_1)}\|\phi_f\|d_qs\\
&+\frac{1}{|\lambda_1|}\int_{t_1}^{t_2}\frac{(t_2-qs)^{(\alpha_1-1)}}{\Gamma_q(\alpha_1)}\|\phi_f\|d_qs
\end{aligned}
$$

$$\leq \frac{t_2^{n-1}-t_1^{n-1}}{|\beta|}\Big[\sum_{j=1}^{n-2}\sum_{i=2}^{n}\frac{|\lambda_i||\eta_j|}{|\lambda_1|}\frac{\xi_j^{\alpha_1-\alpha_i}}{\Gamma_q(\alpha_1-\alpha_i+1)}r+\sum_{j=1}^{n-2}\frac{|\eta_j|}{|\lambda_1|}\frac{\xi_j^{\alpha_1}}{\Gamma_q(\alpha_1+1)}\|\phi_f\|$$
$$+\sum_{i=2}^{n}\frac{|\lambda_i|}{|\lambda_1|}\frac{1}{\Gamma_q(\alpha_1-\alpha_i+1)}r+\frac{1}{|\lambda_1|}\frac{1}{\Gamma_q(\alpha_1+1)}\|\phi_f\|\Big]$$
$$+\sum_{i=2}^{n}\frac{|\lambda_i|}{|\lambda_1|}\frac{-(t_2-qt_2)^{(\alpha_1-\alpha_i)}+(t_1-qt_1)^{(\alpha_1-\alpha_i)}+t_2^{\alpha_1-\alpha_i}-t_1^{\alpha_1-\alpha_i}}{\Gamma_q(\alpha_1-\alpha_i+1)}r$$
$$+\frac{1}{|\lambda_1|}\frac{-(t_2-qt_2)^{(\alpha_1)}+(t_1-qt_1)^{(\alpha_1)}+t_2^{\alpha_1}-t_1^{\alpha_1}}{\Gamma_q(\alpha_1+1)}\|\phi_f\|.$$

Then we can see that $\big|Tu(t_1)-Tu(t_2)\big|\to 0$ as $t_1\to t_2$.

(iii) We prove that T is continuous. Assume $\{u_n\}\subset\Omega$ and $\|u_n-u\|\to 0(n\to +\infty)$. Then,

$$\big\|Tu_n-Tu\big\|$$
$$\leq \sup_{t\in[0,1]}\Big\{\frac{t^{n-1}}{|\beta|}\Big[\sum_{j=1}^{n-2}\sum_{i=2}^{n}\frac{|\lambda_i||\eta_j|}{|\lambda_1|}\int_0^{\xi_j}\frac{(\xi_j-qs)^{(\alpha_1-\alpha_i-1)}}{\Gamma_q(\alpha_1-\alpha_i)}\big|u_n(s)-u(s)\big|d_qs$$
$$+\sum_{j=1}^{n-2}\frac{|\eta_j|}{|\lambda_1|}\int_0^{\xi_j}\frac{(\xi_j-qs)^{(\alpha_1-1)}}{\Gamma_q(\alpha_1)}\big|f(s,u_n(s),u_n(\tau s))-f(s,u(s),u(\tau s))\big|d_qs$$
$$+\sum_{i=2}^{n}\frac{|\lambda_i|}{|\lambda_1|}\int_0^{1}\frac{(1-qs)^{(\alpha_1-\alpha_i-1)}}{\Gamma_q(\alpha_1-\alpha_i)}\big|u_n(s)-u(s)\big|d_qs$$
$$+\frac{1}{|\lambda_1|}\int_0^{1}\frac{(1-qs)^{(\alpha_1-1)}}{\Gamma_q(\alpha_1)}\big|f(s,u_n(s),u_n(\tau s))-f(s,u(s),u(\tau s))\big|d_qs\Big]$$
$$+\sum_{i=2}^{n}\frac{|\lambda_i|}{|\lambda_1|}\int_0^{t}\frac{(t-qs)^{(\alpha_1-\alpha_i-1)}}{\Gamma_q(\alpha_1-\alpha_i)}\big|u_n(s)-u(s)\big|d_qs$$
$$+\frac{1}{|\lambda_1|}\int_0^{t}\frac{(t-qs)^{(\alpha_1-1)}}{\Gamma_q(\alpha_1)}\big|f(s,u_n(s),u_n(\tau s))-f(s,u(s),u(\tau s))\big|d_qs\Big\}.$$

From f is continuous on $[0,1]\times\mathbb{R}\times\mathbb{R}$ and $\|u_n-u\|\to 0(n\to+\infty)$, we obtain $\big\|Tu_n-Tu\big\|\to 0$ as $n\to+\infty$.

(iiii) We demonstrate that there exists an open subset $W\subset C([0,1],\mathbb{R})$, for any $u\in\partial W$ and $\lambda\in(0,1)$, such that $u\neq\lambda T(u)$. We suppose that there exist $u\in C([0,1],\mathbb{R})$ and $\lambda\in(0,1)$, such that $u=\lambda T(u)$. For any $t\in[0,1]$, from (A_3), we get

$$\big\|u\big\|=\big\|\lambda Tu\big\|$$
$$\leq \sup_{t\in[0,1]}\Big\{\frac{t^{n-1}}{|\beta|}\Big[\sum_{j=1}^{n-2}\sum_{i=2}^{n}\frac{|\lambda_i||\eta_j|}{|\lambda_1|}\int_0^{\xi_j}\frac{(\xi_j-qs)^{(\alpha_1-\alpha_i-1)}}{\Gamma_q(\alpha_1-\alpha_i)}\big|u(s)\big|d_qs$$
$$+\sum_{j=1}^{n-2}\frac{|\eta_j|}{|\lambda_1|}\int_0^{\xi_j}\frac{(\xi_j-qs)^{(\alpha_1-1)}}{\Gamma_q(\alpha_1)}\big|f(s,u_(s),u_(\tau s))\big|d_qs$$
$$+\sum_{i=2}^{n}\frac{|\lambda_i|}{|\lambda_1|}\int_0^{1}\frac{(1-qs)^{(\alpha_1-\alpha_i-1)}}{\Gamma_q(\alpha_1-\alpha_i)}\big|u(s)\big|d_qs$$

$$
\begin{aligned}
&+\frac{1}{|\lambda_1|}\int_0^1\frac{(1-qs)^{(\alpha_1-1)}}{\Gamma_q(\alpha_1)}\big|f(s,u(s),u(\tau s))\big|d_qs\Big]\\
&+\sum_{i=2}^{n}\frac{|\lambda_i|}{|\lambda_1|}\int_0^t\frac{(t-qs)^{(\alpha_1-\alpha_i-1)}}{\Gamma_q(\alpha_1-\alpha_i)}\big|u(s)\big|d_qs\\
&+\frac{1}{|\lambda_1|}\int_0^t\frac{(t-qs)^{(\alpha_1-1)}}{\Gamma_q(\alpha_1)}\big|f(s,u(s),u(\tau s))\big|d_qs\Big\}\\
\le&\sup_{t\in[0,1]}\Big\{\frac{t^{n-1}}{|\beta|}\Big[\sum_{j=1}^{n-2}\sum_{i=2}^{n}\frac{|\lambda_i||\eta_j|}{|\lambda_1|}\int_0^{\xi_j}\frac{(\xi_j-qs)^{(\alpha_1-\alpha_i-1)}}{\Gamma_q(\alpha_1-\alpha_i)}rd_qs\\
&+\sum_{j=1}^{n-2}\frac{|\eta_j|}{|\lambda_1|}\int_0^{\xi_j}\frac{(\xi_j-qs)^{(\alpha_1-1)}}{\Gamma_q(\alpha_1)}\|\phi_f\|d_qs\\
&+\sum_{i=2}^{n}\frac{|\lambda_i|}{|\lambda_1|}\int_0^1\frac{(1-qs)^{(\alpha_1-\alpha_i-1)}}{\Gamma_q(\alpha_1-\alpha_i)}rd_qs+\frac{1}{|\lambda_1|}\int_0^1\frac{(1-qs)^{(\alpha_1-1)}}{\Gamma_q(\alpha_1)}\|\phi_f\|d_qs\Big]\\
&+\sum_{i=2}^{n}\frac{|\lambda_i|}{|\lambda_1|}\int_0^t\frac{(t-qs)^{(\alpha_1-\alpha_i-1)}}{\Gamma_q(\alpha_1-\alpha_i)}rd_qs+\frac{1}{|\lambda_1|}\int_0^t\frac{(t-qs)^{(\alpha_1-1)}}{\Gamma_q(\alpha_1)}\|\phi_f\|d_qs\Big\}\\
\le&\frac{1}{|\beta|}\Big[\sum_{j=1}^{n-2}\sum_{i=2}^{n}\frac{|\lambda_i||\eta_j|}{|\lambda_1|}\frac{\xi_j^{\alpha_1-\alpha_i}}{\Gamma_q(\alpha_1-\alpha_i+1)}r+\sum_{j=1}^{n-2}\frac{|\eta_j|}{|\lambda_1|}\frac{\xi_j^{\alpha_1}}{\Gamma_q(\alpha_1+1)}\|\phi_f\|\\
&+\sum_{i=2}^{n}\frac{|\lambda_i|}{|\lambda_1|}\frac{1}{\Gamma_q(\alpha_1-\alpha_i+1)}r+\frac{1}{|\lambda_1|}\frac{1}{\Gamma_q(\alpha_1+1)}\|\phi_f\|\Big]\\
&+\sum_{i=2}^{n}\frac{|\lambda_i|}{|\lambda_1|}\frac{1}{\Gamma_q(\alpha_1-\alpha_i+1)}r+\frac{1}{|\lambda_1|}\frac{1}{\Gamma_q(\alpha_1+1)}\|\phi_f\|\\
=&\sum_{j=1}^{n-2}\sum_{i=2}^{n}\frac{|\lambda_i|r}{|\beta||\lambda_1|\Gamma_q(\alpha_1-\alpha_i+1)}\big(|\eta_j|\xi_j^{\alpha_1-\alpha_i}+1+|\beta|\big)\\
&+\sum_{j=1}^{n-2}\frac{\|\phi_f\|}{|\beta||\lambda_1|\Gamma_q(\alpha_1+1)}\big(|\eta_j|\xi_j^{\alpha_1}+1+|\beta|\big)\\
:=&m.
\end{aligned}
$$

Let $W=\{u\in C([0,1],\mathbb{R}):\|u\|\le m+1\}$, there is no $u\in\partial W$, such that $u=\lambda T(u)$. From Theorem 2.2, boundary value problem (1.1) has at least a solution on W. This completes the proof. □

4. The stability of fractional q-difference equation

Lemma 4.1. *If $u\in C([0,1],\mathbb{R})$ is solution of*

$$
\begin{cases}\sum\limits_{i=1}^{n}\lambda_i{}^CD_q^{\alpha_i}u(t)=f(t,u(t),u(\tau t))+\zeta(t), & t\in[0,1],\\ u(0)=0,\frac{d_q^ju(0)}{d_qu^j}=0,u(1)=\sum\limits_{j=1}^{n-2}\eta_ju(\xi_j).\end{cases}\tag{4.1}
$$

Then, u satisfies the following inequality

$$\big|u(t) - Tu(t)\big| \le m_1\varepsilon,$$

where

$$m_1 = \frac{1}{|\beta||\lambda_1|\Gamma_q(\alpha_1+1)}\Big(\sum_{j=1}^{n-2}|\eta_j|\xi_j^{\alpha_1} + 1 + |\beta|\Big).$$

Proof. If $u \in C([0,1],\mathbb{R})$ is the solution of (4.1), then

$$\begin{aligned}u(t) =&Tu(t) + \frac{t^{n-1}}{\beta}\Big[\sum_{j=1}^{n-2}\frac{\eta_j}{\lambda_1}\int_0^{\xi_j}\frac{(\xi_j-qs)^{(\alpha_1-1)}}{\Gamma_q(\alpha_1)}\zeta(s)d_qs \\ &- \frac{1}{\lambda_1}\int_0^1\frac{(1-qs)^{(\alpha_1-1)}}{\Gamma_q(\alpha_1)}\zeta(s)d_qs\Big] + \frac{1}{\lambda_1}\int_0^t\frac{(t-qs)^{(\alpha_1-1)}}{\Gamma_q(\alpha_1)}\zeta(s)d_qs.\end{aligned}$$

In view of Lemma 2.3, we obtain

$$\begin{aligned}\big|u(t) - Tu(t)\big| \le&\frac{t^{n-1}}{|\beta|}\Big[\sum_{j=1}^{n-2}\frac{|\eta_j|}{|\lambda_1|}\int_0^{\xi_j}\frac{(\xi_j-qs)^{(\alpha_1-1)}}{\Gamma_q(\alpha_1)}\zeta(s)d_qs \\ &+ \frac{1}{|\lambda_1|}\int_0^1\frac{(1-qs)^{(\alpha_1-1)}}{\Gamma_q(\alpha_1)}\zeta(s)d_qs\Big] + \frac{1}{|\lambda_1|}\int_0^t\frac{(t-qs)^{(\alpha_1-1)}}{\Gamma_q(\alpha_1)}\zeta(s)d_qs \\ \le&\varepsilon\Big\{\frac{t^{n-1}}{|\beta|}\Big[\sum_{j=1}^{n-2}\frac{|\eta_j|}{|\lambda_1|}\int_0^{\xi_j}\frac{(\xi_j-qs)^{(\alpha_1-1)}}{\Gamma_q(\alpha_1)}d_qs \\ &+ \frac{1}{|\lambda_1|}\int_0^1\frac{(1-qs)^{(\alpha_1-1)}}{\Gamma_q(\alpha_1)}d_qs\Big] + \frac{1}{|\lambda_1|}\int_0^t\frac{(t-qs)^{(\alpha_1-1)}}{\Gamma_q(\alpha_1)}d_qs\Big\} \\ \le&\varepsilon\Big\{\frac{1}{|\beta|}\Big[\sum_{j=1}^{n-2}\frac{|\eta_j|}{|\lambda_1|}\frac{\xi_j^{\alpha_1}}{\Gamma_q(\alpha_1+1)} + \frac{1}{|\lambda_1|}\frac{1}{\Gamma_q(\alpha_1+1)}\Big] + \frac{1}{|\lambda_1|}\frac{1}{\Gamma_q(\alpha_1+1)}\Big\} \\ =&\varepsilon\frac{1}{|\beta||\lambda_1|\Gamma_q(\alpha_1+1)}\Big(\sum_{j=1}^{n-2}|\eta_j|\xi_j^{\alpha_1} + 1 + |\beta|\Big) \\ =&m_1\varepsilon.\end{aligned}$$

This completes the proof. □

Theorem 4.1. *If (A_1) and (A_2) hold and*

$$\begin{aligned}k =&\sum_{j=1}^{n-2}\sum_{i=2}^{n}\frac{|\lambda_i|}{|\beta||\lambda_1|\Gamma_q(\alpha_1-\alpha_i+1)}\big(|\eta_j|\xi_j^{\alpha_1-\alpha_i} + 1 + |\beta|\big) \\ &+ \sum_{j=1}^{n-2}\frac{k_1+k_2}{|\beta||\lambda_1|\Gamma_q(\alpha_1+1)}\big(|\eta_j|\xi_j^{\alpha_1} + 1 + |\beta|\big) \\ <&1.\end{aligned}$$

Then boundary value problem (1.1) is UH stable and GUH stable.

Proof. For each solution $u \in C([0,1],\mathbb{R})$ of (4.1) and the unique solution u^* of (1.1), by Theorem 3.1 and Lemma 4.1, we get

$$\begin{aligned}\|u-u^*\| &= \|u-Tu^*\| = \|u-Tu+Tu-Tu^*\| \\ &\le \|u-Tu\| + \|Tu-Tu^*\| \\ &\le m_1\varepsilon + k\|u-u^*\|.\end{aligned}$$

Therefore,

$$\|u-u^*\| \le \frac{m_1}{1-k}\varepsilon := M_1\varepsilon.$$

Then boundary value problem (1.1) is UH stable. For $\psi(\varepsilon)=\varepsilon$, boundary value problem (1.1) is GUH stable. This completes the proof. □

Lemma 4.2. *If (A_4) holds and $u \in C([0,1],\mathbb{R})$ is solution of*

$$\begin{cases} \sum\limits_{i=1}^{n} \lambda_i {}^C D_q^{\alpha_i} u(t) = f(t,u(t),u(\tau t)) + \zeta(t), \quad t\in[0,1], \\ u(0)=0, \frac{d_q^j u(0)}{d_q u^j} = 0, u(1) = \sum\limits_{j=1}^{n-2} \eta_j u(\xi_j). \end{cases} \tag{4.2}$$

Then, u satisfies the following inequality

$$|u(t)-Tu(t)| \le m_2(\varphi(t)+\rho)\varepsilon,$$

where

$$m_2 = \max\Big\{\frac{1}{|\beta||\lambda_1|\Gamma_q(\alpha_1+1)}\Big(\sum_{j=1}^{n-2}|\eta_j|\xi_j^{\alpha_1}+1\Big), \frac{M}{|\lambda_1|}\Big\}.$$

Proof. If $u \in C([0,1],\mathbb{R})$ is the solution of (4.2), then

$$\begin{aligned} u(t) =& Tu(t) + \frac{t^{n-1}}{\beta}\Big[\sum_{j=1}^{n-2}\frac{\eta_j}{\lambda_1}\int_0^{\xi_j}\frac{(\xi_j-qs)^{(\alpha_1-1)}}{\Gamma_q(\alpha_1)}\zeta(s)d_qs \\ &- \frac{1}{\lambda_1}\int_0^1\frac{(1-qs)^{(\alpha_1-1)}}{\Gamma_q(\alpha_1)}\zeta(s)d_qs\Big] + \frac{1}{\lambda_1}\int_0^t\frac{(t-qs)^{(\alpha_1-1)}}{\Gamma_q(\alpha_1)}\zeta(s)d_qs. \end{aligned}$$

By virtue of Lemma 2.4 and (A_4), we obtain

$$\begin{aligned} |u(t)-Tu(t)| \le& \frac{t^{n-1}}{|\beta|}\Big[\sum_{j=1}^{n-2}\frac{|\eta_j|}{|\lambda_1|}\int_0^{\xi_j}\frac{(\xi_j-qs)^{(\alpha_1-1)}}{\Gamma_q(\alpha_1)}\zeta(s)d_qs \\ &+ \frac{1}{|\lambda_1|}\int_0^1\frac{(1-qs)^{(\alpha_1-1)}}{\Gamma_q(\alpha_1)}\zeta(s)d_qs\Big] + \frac{1}{|\lambda_1|}\int_0^t\frac{(t-qs)^{(\alpha_1-1)}}{\Gamma_q(\alpha_1)}\zeta(s)d_qs \\ \le& \varepsilon\Big\{\frac{t^{n-1}}{|\beta|}\Big[\sum_{j=1}^{n-2}\frac{|\eta_j|}{|\lambda_1|}\int_0^{\xi_j}\frac{(\xi_j-qs)^{(\alpha_1-1)}}{\Gamma_q(\alpha_1)}\varphi(s)d_qs \\ &+ \frac{1}{|\lambda_1|}\int_0^1\frac{(1-qs)^{(\alpha_1-1)}}{\Gamma_q(\alpha_1)}\varphi(s)d_qs\Big] + \frac{1}{|\lambda_1|}\int_0^t\frac{(t-qs)^{(\alpha_1-1)}}{\Gamma_q(\alpha_1)}\varphi(s)d_qs\Big\} \\ \le& \varepsilon\Big\{\frac{\rho}{|\beta|}\Big[\sum_{j=1}^{n-2}\frac{|\eta_j|}{|\lambda_1|}\frac{\xi_j^{\alpha_1}}{\Gamma_q(\alpha_1+1)} + \frac{1}{|\lambda_1|}\frac{1}{\Gamma_q(\alpha_1+1)}\Big] + \frac{M}{|\lambda_1|}\varphi(t)\Big\} \end{aligned}$$

$$=\varepsilon\left[\frac{1}{|\beta||\lambda_1|\Gamma_q(\alpha_1+1)}\Big(\sum_{j=1}^{n-2}|\eta_j|\xi_j^{\alpha_1}+1\Big)\rho+\frac{M}{|\lambda_1|}\varphi(t)\right]$$
$$\leq m_2(\varphi(t)+\rho)\varepsilon,$$

where $m_2=\max\left\{\frac{1}{|\beta||\lambda_1|\Gamma_q(\alpha_1+1)}\Big(\sum_{j=1}^{n-2}|\eta_j|\xi_j^{\alpha_1}+1\Big),\frac{M}{|\lambda_1|}\right\}$. This completes the proof.

□

Theorem 4.2. *If* (A_1), (A_2) *and* (A_4) *hold and*

$$k=\sum_{j=1}^{n-2}\sum_{i=2}^{n}\frac{|\lambda_i|}{|\beta||\lambda_1|\Gamma_q(\alpha_1-\alpha_i+1)}\big(|\eta_j|\xi_j^{\alpha_1-\alpha_i}+1+|\beta|\big)$$
$$+\sum_{j=1}^{n-2}\frac{k_1+k_2}{|\beta||\lambda_1|\Gamma_q(\alpha_1+1)}\big(|\eta_j|\xi_j^{\alpha_1}+1+|\beta|\big)<1.$$

Then boundary value problem (1.1) *is* UHR *stable and* $GUHR$ *stable.*

Proof. For each solution $u\in C([0,1],\mathbb{R})$ of (4.2) and the unique solution u^* of (1.1), by Theorem 3.1 and Lemma 4.2, we get

$$\big\|u-u^*\big\|=\big\|u-Tu^*\big\|=\big\|u-Tu+Tu-Tu^*\big\|$$
$$\leq\big\|u-Tu\big\|+\big\|Tu-Tu^*\big\|$$
$$\leq m_2(\varphi(t)+\rho)\varepsilon+k\big\|u-u^*\big\|.$$

Therefore,

$$\big\|u-u^*\big\|\leq\frac{m_2(\varphi(t)+\rho)}{1-k}\varepsilon:=M_2(\varphi(t)+\rho)\varepsilon.$$

Then boundary value problem (1.1) is UHR stable. For $\varepsilon=1$, boundary value problem (1.1) is $GUHR$ stable. This completes the proof. □

5. Example

Example 5.1. Consider the following boundary value problem:

$$\begin{cases}20^{C}D_{\frac{1}{2}}^{3.8}u(t)+\sum_{i=2}^{4}\frac{10(-1)^{i+1}}{3i-2}{}^{C}D_{\frac{1}{2}}^{\frac{3.8}{i+2}}u(t)=\frac{M}{e^t+t^3}\Big(\frac{4|u(t)|}{2t+|u(t)|}-\frac{3|u(\frac{t}{8})|}{1+e^{t^2}t^4+|u(\frac{t}{8})|}-\sin t\Big),\\ u(0)=0,\frac{d_q^ju(0)}{d_qu^j}=0,u(1)=\sum_{j=1}^{2}3ju(0.1j),\quad t\in[0,1],\end{cases}\tag{5.1}$$

where $\alpha_1=3.8$, $\alpha_i=\frac{3.8}{i+2}$, $i=2,3,4$, $\lambda_1=20$, $\lambda_i=\frac{(-1)^{i+1}10}{3i-2}$, $\tau=\frac{1}{8}$, $\eta_j=3j$, $\xi_j=0.1j$, $j=1,2$. Hence,

$$\big|f(t,u_1(t),u_1(\tau t))-f(t,u_2(t),u_2(\tau t))\big|$$
$$\leq|M|\Big(\frac{2}{3}\big|u_1(t)-u_2(t)\big|+3\big|u_1(\frac{t}{8})-u_2(\frac{t}{8})\big|\Big)$$
$$\leq|M|\Big(\frac{2}{3}\big\|u_1-u_2\big\|+3\big\|u_1-u_2\big\|\Big)$$
$$\leq3|M|\|u_1-u_2\|,$$

where

$$\beta = 1 - \sum_{j=1}^{n-2} \eta_j \xi_j^{n-1} = \sum_{j=1}^{2} 3j(0.1j)^3 = 0.949 \neq 0.$$

Therefore, if $M < 24.6086$, we can get $k < 1$. From Theorem 3.1, boundary value problem (5.1) has a unique solution.

Let $\varphi(t) = t^2 + 1$ and $\rho = 3$, $t \in [0,1]$. We can obtain

$$I_{\frac{1}{2}}^{3.8}\zeta(t) = \frac{2t^{5.8}}{\Gamma_{\frac{1}{2}}(6.8)} + \frac{t^{3.8}}{\Gamma_{\frac{1}{2}}(4.8)} \leq \frac{t^2+1}{\Gamma_{\frac{1}{2}}(4.8)}.$$

Then, in view of Theorem 4.2, boundary value problem (5.1) is UHR stable and $GUHR$ stable. Moreover, by using Theorem 4.1, boundary value problem (5.1) is UH stable and GUH stable, which can be seen in Figure 1.

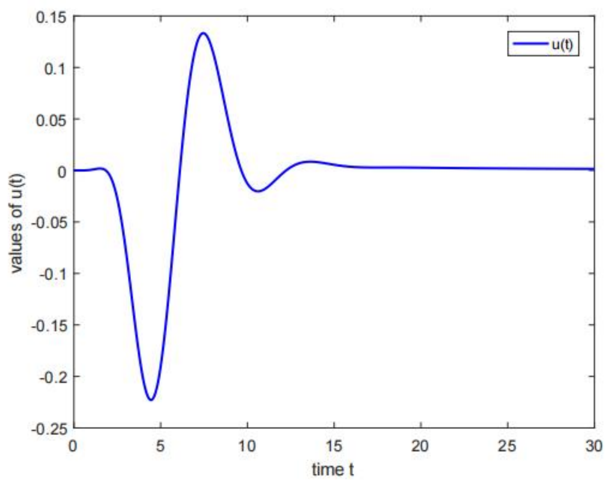

Figure 1. Illustration of proof.

Example 5.2. Consider the following boundary value problem:

$$\begin{cases} {}^{C}D_{\frac{1}{2}}^{8.9}u(t) + \sum\limits_{i=2}^{9} 10^{1-i}\,{}^{C}D_{\frac{1}{2}}^{\frac{1}{i}}u(t) = \frac{3t^5}{e^t}\Big(\frac{t}{e^{-t}t^4+5|u(t)|+\sin t|u(\frac{t}{8})|} \\ \qquad\qquad\qquad\qquad\qquad\qquad - \frac{e^t}{3e^{-2t}t+e^{t^4}t^4|u(t)|+|u(\frac{t}{8})|} + \cos e^{-t}t\Big), \\ u(0) = 0, \frac{d_q^j u(0)}{d_q u^j} = 0, u(1) = \sum\limits_{j=1}^{7} \frac{1}{3^j}u(\frac{1}{2^j}), \quad t \in [0,1], \end{cases} \tag{5.2}$$

where $\alpha_1 = 8.9$, $\alpha_i = \frac{1}{i}$, $i = 2, \cdots, 9$, $\lambda_1 = 1$, $\lambda_i = 10^{1-i}$, $\tau = \frac{1}{8}$, $\eta_j = \frac{1}{3^j}$, $\xi_j = \frac{1}{2^j}$, $j = 1, \cdots, 7$. Then,

$$\begin{aligned} &\big|f(t,u(t),u(\tau t))\big| \\ =&\Big|\frac{3t^5}{e^t}\Big(\frac{t}{e^{-t}t^4+5|u(t)|+\sin t|u(\frac{t}{8})|} - \frac{e^t}{3e^{-2t}t+e^{t^4}t^4|u(t)|+|u(\frac{t}{8})|} + \cos e^{-t}t\Big)\Big| \end{aligned}$$

$$\le \frac{3t^5}{e^t}\Big(\frac{t}{e^{-t}t^4+5|u(t)|+\sin t|u(\frac{t}{8})|}+\frac{e^t}{3e^{-2t}t+e^{t^4}t^4|u(t)|+|u(\frac{t}{8})|}+\cos e^{-t}t\Big)$$
$$\le \frac{3t^5}{e^t}\Big(\frac{t}{e^{-t}t^4}+\frac{e^t}{3e^{-2t}t}+e^{-t}\Big)$$
$$\le 3t^2+t^4e^{2t}+3t^5=\phi_f(t),$$

where

$$\beta = 1-\sum_{j=1}^{n-2}\eta_j\xi_j^{n-1}=\sum_{j=1}^{7}\frac{1}{3^j}(\frac{1}{2^j})^8=0.9987\neq 0 \quad \text{and} \quad r>0.0003.$$

Therefore, by Theorem 3.2, boundary value problem (5.2) has at least a solution.

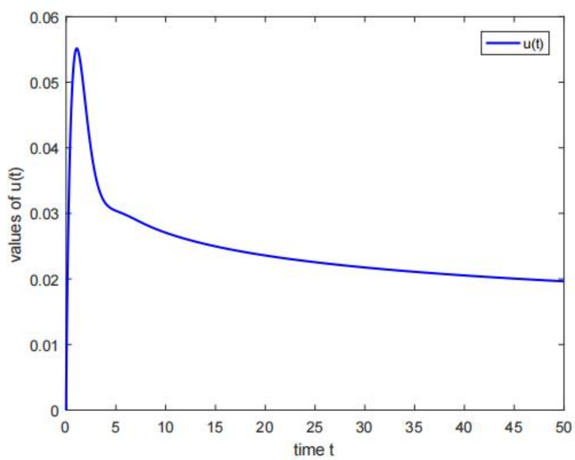

Figure 2. Illustration of proof.

6. Conclusion

In this paper, we study multi-term fractional delay q-difference equation, where the delay is proportional delay. In view of Banach contraction mapping principle and Leray-Schauder nonlinear alternative theorem, we obtain the existence and uniqueness of solutions. In addition, we give four different functional stability results, including UH stability, GUH stability, UHR stability and $GUHR$ stability. The results of existence, uniqueness and stability of solutions obtained in this paper can also be applied to other types of delays, such as continuous delay and discrete delay. Moreover, we can obtain different equations by assigning different values to parameters, for example, when $\lambda_1 = 1$ and $\lambda_i = 0$, $i = 2, \cdots, n$, it is a single operator fractional delay q-difference equation.

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