Journal of Applied Analysis and Computation
Volume 14, Number 3, June 2024, 1222–1268
Website:http://www.jaac-online.com
DOI:10.11948/20230023

HIGHER-ORDER UNIFORM CONVERGENCE AND ORDER REDUCTION ANALYSIS OF A NOVEL FRACTIONAL-STEP FMM FOR SINGULARLY PERTURBED 2D PARABOLIC PDES WITH TIME-DEPENDENT BOUNDARY DATA*

Narendra Singh Yadav[1] and Kaushik Mukherjee[2,†]

Abstract The aim of this paper is to develop and analyze a cost-effective high-order efficient numerical method for a class of two-dimensional singularly perturbed parabolic convection-diffusion problems with non-homogeneous time-dependent boundary data. To achieve the goal, we develop an efficient fractional-step fitted mesh method that combines the fractional implicit-Euler method with an alternative evaluation of the boundary data for discretization in time, and consists of a new hybrid finite difference method for discretization in space. In the case of fully discrete numerical approximation of evolutionary PDEs, in particular, with time-dependent boundary conditions; it is noticed that the classical evaluation of the boundary data usually causes the order reduction in time; and it becomes severe when the fractional-step method is used. In this regard, the novelty of the current algorithm, other than its higher-order accuracy, is that the method can eliminate the order-reduction in time before and after the extrapolation with an appropriate evaluation of the boundary data; and at the same time, it can reduce the computational cost for solving the multi-dimensional problem by using the fractional-step method that converts the multi-dimensional system into two-independent one-dimensional subsystems at each time level. To accomplish this, we discretize the spatial domain using a layer-adapted non-uniform rectangular mesh and the time domain by an equidistant mesh. Stability and ε-uniform convergence result of the fully discrete scheme are established in the supremum-norm. Moreover, the Richardson extrapolation technique is implemented solely in the time direction to enhance the order of convergence in time. Finally, numerical results are presented with the default and alternative choice for the boundary data to validate the theoretical findings.

Keywords Singularly perturbed 2D parabolic convection-diffusion PDEs, non-homogeneous boundary data, fractional-step fitted mesh method, Richardson extrapolation technique, layer-adapted Shishkin mesh, order reduction analysis and higher-order convergence.

MSC(2010) 65M06, 65M12, 65M15.

[†]The corresponding author.
[1]Department of Mathematics, SRM University AP, Andhra Pradesh, India
[2]Department of Mathematics, Indian Institute of Space Science and Technology (IIST), Thiruvananthapuram-695547, Kerala, India
[*]This paper is dedicated to Professor Natesan Srinivasan on the occasion of his 55th birthday.
Email: narendrasingh.y@srmap.edu.in, singh.maths1993@gmail.com(N. S. Yadav), kaushik@iist.ac.in, mathkaushik@gmail.com(K. Mukherjee)

1. Introduction

Computational investigation of multi-dimensional singularly perturbed partial differential equations (SPPDEs) has always been of great interest to engineers and scientists due to wide applications of SPPDEs in real-life. As a prominent example, one can consider time-dependent advection-dispersion PDEs which are used to model several complex phenomena appeared in hydrological and environmental sciences and engineering. In this regard, we refer the article [2] dealing with the solute transport phenomenon associated with groundwater and surface water flow system. Another important example includes the mathematical modeling of anti-cancer drug transportation into tumor cells governed by the time-dependent advection-diffusion PDEs (see [18]) arising in mathematical biology. Computational analysis of this type of mathematical model enables to predict spatio-temporal distributions of drugs within the tumor tissue; and indeed plays a vital role for improving the chemotherapy treatment by identifying the specific cell or compound-related factors that prevent drug penetration through tumors. It is worth mentioning that the above mentioned PDEs turn out to be SPPDEs whenever dispersion or diffusion coefficient is substantially smaller than the advection term; and the corresponding model inherently becomes more complex due to the presence of the boundary layer phenomena. Consequently, analyzing such PDEs becomes computationally challenging task; and thus requires development of robust numerical techniques to solve them efficiently.

1.1. The continuous problem

In this paper, we focus on the following class of two-dimensional singularly perturbed parabolic IBVPs with non-homogeneous time-dependent boundary data posed on the domain $\mathtt{D} = \mathtt{G} \times (0, T] = (0, 1)^2 \times (0, T]$; $\overline{\mathtt{G}} = [0, 1]^2$:

$$\begin{cases} \Big(\dfrac{\partial}{\partial t} + \mathtt{L}_\varepsilon\Big) u(x, y, t) = g(x, y, t), \quad (x, y, t) \in \mathtt{D}, \\[2ex] u(x, y, 0) = \mathtt{q}_0(x, y), \quad (x, y) \in \overline{\mathtt{G}}, \\[2ex] u(x, y, t) = \mathtt{s}(x, y, t), \quad (x, y, t) \in \partial\mathtt{G} \times (0, T], \end{cases} \tag{1.1}$$

where

$$\begin{cases} \mathtt{L}_\varepsilon u = -\varepsilon \Delta u + \vec{v}(x, y, t) \cdot \vec{\nabla} u + b(x, y, t) u, \\[2ex] \vec{v}(x, y, t) = \big(v_1(x, y, t), v_2(x, y, t)\big), \end{cases}$$

and ε is a small parameter such that $\varepsilon \in (0, 1]$. The coefficients $\vec{v}(x, y, t)$, $b(x, y, t)$ and the source term $g(x, y, t)$ are considered to be sufficiently smooth with

$$v_1(x, y, t) \geq \mathtt{m}_1 > 0, \quad v_2(x, y, t) \geq \mathtt{m}_2 > 0, \quad b(x, y, t) \geq \beta \geq 0, \quad \text{on}\, \overline{\mathtt{D}}. \tag{1.2}$$

We set $\mathtt{L}_\varepsilon = \mathtt{L}_{1,\varepsilon} + \mathtt{L}_{2,\varepsilon}$, where the differential operators $\mathtt{L}_{1,\varepsilon}$, $\mathtt{L}_{2,\varepsilon}$ are defined by

$$\begin{cases} \mathtt{L}_{1,\varepsilon} u = -\varepsilon \dfrac{\partial^2 u}{\partial x^2} + v_1(x, y, t) \dfrac{\partial u}{\partial x} + b_1(x, y, t) u, \\[2ex] \mathtt{L}_{2,\varepsilon} u = -\varepsilon \dfrac{\partial^2 u}{\partial y^2} + v_2(x, y, t) \dfrac{\partial u}{\partial y} + b_2(x, y, t) u, \end{cases}$$

with $g = g_1 + g_2,\ b = b_1 + b_2$ with $b_i \geq \beta_i \geq 0$, for $i = 1, 2$. We further assume that the initial and boundary data of the problem are sufficiently smooth functions and also assume that necessary compatibility conditions hold among them in order to $u(x, y, t) \in \mathcal{C}^{4+\gamma}(\overline{\mathsf{D}})$, which has continuous derivatives up to fourth-order in space and second-order in time. The existence of the solution $u(x, y, t)$ of the IBVP (1.1)-(1.2) follows from [Chapter IV, §5] of the book [19] by Ladyzenskaja et al. The necessary compatibility requirements are given in [6]. The solution of the IBVP (1.1)-(1.2) generally possesses exponential layers of width $O(\varepsilon)$ at the outflow boundaries $x = 1$ and $y = 1$ (see [22, 28]).

1.2. Existing works

The fitted mesh methods(FMMs), which utilize a specified difference operator on special layer-resolving meshes, are often considered as a well-known methodology for solving SPPDEs to overcome the limitations of the classical methods which can not capture the solution accurately on the equidistant meshes unless the mesh size becomes very small (*i.e.*, $O(\varepsilon)$). To know about the variety of FMMs, one can see the books [13, 28]. In this regard, couple of articles [4, 14, 15, 24] dealing with efficient FMMs for 1D non-stationary problems can also be referred. Further, one can recall different FMMs analyzed in [21, 30] for 2D stationary problems. For 2D non-stationary problems, one can consider the implicit FMM analyzed in [5] which utilizes the classical finite difference scheme for the spatial derivatives and the backward Euler method for the time derivative. Even though reduction in the order of convergence does not occur; but the specified method leads to a pentadiagonal linear system, at each time step. Consequently, the computational cost of that scheme becomes quite high.

On the contrary, the fractional-step method reduces the computational cost remarkably because only tridiagonal linear systems need to be solved at each time level. We now highlight couple of research outcomes related to the fractional-step fitted mesh methods (FSFMMs) for solving multi-dimensional evolutionary SPPDEs. In connection with the fractional implicit Euler methods, for instance, we refer [20] for singularly perturbed 2D parabolic reaction-diffusion IBVP and [9, 12] for 2D parabolic convection-diffusion IBVP; [23] for 2D parabolic convection-diffusion problem with two parameters and [10] for 2D parabolic problem having degenerated convective components. These methods are uniformly convergent with first-order accurate in time. Further, we refer contributions of Clavero et al. in [3] and Mukherjee and Natesan in [26] to develop higher-order (with respect to both space and time) FSFMMs for singularly perturbed 2D parabolic IBVPs by using the Peaceman-Rachford fractional-step method. However, the above cited articles are devoted to the study of evolutionary SPPDEs with homogeneous boundary conditions and therefore, do not discuss how to overcome the order reduction phenomenon occurred in case of the time-dependent boundary conditions.

In the case of fully discrete numerical approximation of evolutionary PDEs with non-homogeneous boundary conditions (in particular, time-dependent); it is observed that the classical evaluation of the boundary data usually causes the order reduction in time while using one step method for time integration (see, the article [1]). Moreover, the similar evaluation drastically reduces the order of convergence when the fractional step method is used as can be seen in Section 6. In the literature there are relatively few research articles which deal with numerical approximation of singularly perturbed 2D parabolic PDEs with time-dependent boundary conditions. In this regard, we cite recent contributions of Clavero and Jorge in [6, 7] and [8] for implementation of the fractional implicit Euler method in the temporal direction and the classical finite difference schemes in the spatial direction to develop FSFMMs, respectively for singularly perturbed 2D parabolic scalar problems and systems of convection-diffusion type with time-dependent boundary conditions.

1.3. Present work

The goal of this article is to construct and analyze an efficient FSFMM followed by the Richardson extrapolation technique to obtain ε-uniformly convergent higher-order space-time accurate numerical solution for a class of singularly perturbed 2D parabolic IBVPs of the form (1.1)-(1.2) with time-dependent boundary conditions; and also to analyze the order reduction phenomenon before and after the extrapolation. The proposed FSFMM combines the fractional implicit-Euler method with an alternative evaluation of the boundary data for discretization in the temporal direction; and consists of a new hybrid finite difference method (a suitable combination of the modified central difference scheme and the midpoint upwind scheme based on $\varepsilon \leq \|v_l\| N^{-1}$ and $\varepsilon > \|v_l\| N^{-1},\ l = 1, 2$) for discretization in the spatial directions. Note that 1D version of this hybrid method is previously analyzed in [31] for singularly perturbed 1D parabolic IBVPs. For this purpose, we discretize the spatial domain using a non-uniform layer-adapted rectangular mesh (tensor-product of 1D piecewise-uniform Shishkin mesh) and the time domain by an equidistant mesh.

The novelty of the proposed finite difference method is that it can produce higher-order spatial accurate numerical solution across the different regions both in x and y-directions. More precisely, the spatial accuracy is at least two in the outer region and almost two in the layer regions, irrespective of the smaller and larger values of the parameter ε; as can be seen from Tables 9 and 10. It is worth mentioning that the associated convergence analysis also includes both the cases, *i.e.* $\varepsilon \leq \|v_l\| N^{-1}$ but also for $\varepsilon > \|v_l\| N^{-1},\ l = 1, 2$. In addition to this, by implementing the temporal Richardson extrapolation, *i.e.*, the Richardson extrapolation solely for the time variable, we enhance the order of convergence in the temporal direction. As a result, we show that one can achieve globally almost second-order accurate numerical solution (considering both space and time) provided the suitable evaluation of the boundary data is considered. To the best of our knowledge, hardly any research contribution is available in the literature dealing with a higher-order space-time accurate numerical scheme that combines a fractional-step numerical method together with the temporal Richardson extrapolation for solving 2D parabolic convection-diffusion SPPDEs, particularly, with time-dependent boundary conditions.

To achieve the goal, we perform the error analysis for the proposed fractional-step method in two-steps, which discretizes first in time and then in space. For the time semidiscrete case, we consider the suitable choice of the boundary data (mentioned in (3.3)) other than the classical choice to avoid the order reduction in time, as like [6]. Apart from this, we derive asymptotic behavior of the analytical solution of the semidiscrete problem (3.4) by incorporating the non-homogeneous boundary data, which is indeed an essential part of the error estimate for the spatial discretization. However, it is important to note that in [6] the two-step analysis is carried out in the reverse order which firstly converts the IBVP into the semidiscrete

IVP via spatial discretization and later into the fully-discrete problem via temporal discretization. For the spatial discretization of the IBVP, it utilizes the convergence result of the upwind finite difference scheme proved in [5], because the pentadiagonal structure of the upwind scheme can be further decomposed into the tridiagonal structure in the x and y-direction. However, we can not consider such technique to achieve higher-order spatial accuracy using the newly proposed finite difference scheme because of the use of finite difference operators, particularly associated with $\mathtt{L}^{n+1}_{1,N,mup}, \mathtt{L}^{n+1}_{2,N,mup}$. In this regard, the present error analysis plays a quite significant role.

In the literature, one can find couple of articles which utilize the Richardson extrapolation technique to obtain the higher-order accurate numerical solutions for stationary and non-stationary singularly perturbed differential equations. For instance, we refer [11, 25, 29] for parabolic convection-diffusion IBVPs and [27] for stationary convection-diffusion BVP. In those cited articles, the implementation of the extrapolation technique particularly for the parabolic IBVPs requires doubling the mesh-intervals both in the spatial and the temporal directions. It is worth noting that the temporal Richardson extrapolation considered in the article has less computational cost as it requires doubling the mesh-intervals only in the temporal direction. In fact, due to the Richardson extrapolation technique only for the time variable, one does not require to choose sufficiently small Δt for enhancing temporal accuracy of the proposed fractional-step numerical method.

1.4. Outline and notations

The rest of this paper is structured as follows: Section 2 presents a priori bounds for the analytical solution and its derivatives. In Section 3, we introduce the time semidiscrete scheme, that uses the fractional implicit-Euler method together with an appropriate evaluation of the boundary data and prove its uniform convergence. The fully discrete scheme is introduced in Section 4 and the description of an appropriate rectangular mesh is also given. Here, we deduce parameter-uniform convergence result of the fully discrete scheme. In Section 5, we discuss about the temporal Richardson extrapolation. Finally, the numerical results are presented in Section 6 for two test examples. Here, we compare the accuracy of the present method with the implicit-upwind method given in [6]. The conclusion of this paper is provided in Section 7.

In this paper, for each integer $\ell \geq 0$, $\mathcal{C}^{\ell}(\mathtt{D})$ denotes the set of functions which are continuously differentiable up to order ℓ in the domain $\mathtt{D}$; and $\mathcal{C}^{\gamma}(\mathtt{D})$ denotes the set of functions which are Hölder continuous in $\mathtt{D}$ with exponent γ, for $\gamma \in (0,1)$. Then, for each integer $\ell \geq 1$, the parabolic Hölder space $\mathcal{C}^{\ell+\gamma}(\mathtt{D})$ is defined as

$$\mathcal{C}^{\ell+\gamma}(\mathtt{D}) := \left\{ \mathtt{f} : \frac{\partial^{j+k}\mathtt{f}}{\partial x^j \partial t^k} \in \mathcal{C}^{\gamma}(\mathtt{D}), \quad \forall\, j,k \in \mathbb{N} \cup \{0\} \text{ and with } 0 \leq j + 2k \leq \ell \right\}.$$

Further, $\left\| \cdot \right\|_{\mathtt{D}}$ denotes the maximum norm over $\mathtt{D}$; and in case if the domain is obvious, $\mathtt{D}$ is omitted. Note that the above definitions and notations are often used with $\overline{\mathtt{D}}, \partial\mathtt{D}, \overline{\mathtt{D}}^{N,\Delta t}, \partial\mathtt{D}^{N,\Delta t}$. Throughout the paper, C denotes a generic positive constant which is independent of the parameter ε and the discretization parameters N and M.

2. Asymptotic behavior for the analytical solution

We decompose the solution $u(x,y,t)$ such that

$$u(x,y,t) = v(x,y,t) + w(x,y,t), \quad (x,y,t) \in \bar{\mathtt{D}},$$

where v is the regular component and w is the singular component. Again, we consider the decomposition

$$w(x,y,t) = w_1(x,y,t) + w_2(x,y,t) + w_{11}(x,y,t), \quad (x,y,t) \in \bar{\mathtt{D}},$$

where $w_1,\ w_2$ are the exponential layers near the sides $x = 1$ and $y = 1$ of $\bar{\mathtt{G}}$, respectively; and w_{11} is the corner layer near the point (1,1). Following the approach given in [9], one can show that the components of $u(x,y,t)$ satisfy the following bounds:

$$\Big|\frac{\partial^{j+k} v(x,y,t)}{\partial x^{j_1}\partial y^{j_2}\partial t^k}\Big| \le C, \tag{2.1}$$

$$\Big|\frac{\partial^{j+k} w_1(x,y,t)}{\partial x^{j_1}\partial y^{j_2}\partial t^k}\Big| \le C\varepsilon^{-j_1} \exp\Big(-\frac{\mathtt{m}_1(1-x)}{\varepsilon}\Big), \tag{2.2}$$

$$\Big|\frac{\partial^{j+k} w_2(x,y,t)}{\partial x^{j_1}\partial y^{j_2}\partial t^k}\Big| \le C\varepsilon^{-j_2} \exp\Big(-\frac{\mathtt{m}_2(1-y)}{\varepsilon}\Big), \tag{2.3}$$

$$\Big|\frac{\partial^{j+k} w_{11}(x,y,t)}{\partial x^{j_1}\partial y^{j_2}\partial t^k}\Big| \le C\varepsilon^{-j} \min\Big\{\exp\Big(-\frac{\mathtt{m}_1(1-x)}{\varepsilon}\Big), \exp\Big(-\frac{\mathtt{m}_2(1-y)}{\varepsilon}\Big)\Big\}, \tag{2.4}$$

where $\forall\ j_1, j_2, k \in \mathbb{N}\cup\{0\},\ j = j_1 + j_2,\ 0 \le j + 2k \le 4$ and $(x,y,t) \in \bar{\mathtt{D}}$.

Lemma 2.1. *The derivatives of the solution $u(x,y,t)$ of the IBVP (1.1)-(1.2) satisfy the following bounds:*

$$\Big|\frac{\partial^{k} u(x,y,t)}{\partial t^k}\Big| \le C, \tag{2.5}$$

$$\Big|\frac{\partial^{j_1} u(x,y,t)}{\partial x^{j_1}}\Big| \le C\varepsilon^{-j_1} \exp\Big(-\frac{\mathtt{m}_1(1-x)}{\varepsilon}\Big), \tag{2.6}$$

$$\Big|\frac{\partial^{j_2} u(x,y,t)}{\partial y^{j_2}}\Big| \le C\varepsilon^{-j_2} \exp\Big(-\frac{\mathtt{m}_2(1-y)}{\varepsilon}\Big), \tag{2.7}$$

where $j = j_1 + j_2,\ 0 \le j + 2k \le 4$ and $(x,y,t) \in \bar{\mathtt{D}}$.

3. The time semidiscrete problem

In this section, we describe the numerical method to discretize the continuous problem (1.1)-(1.2) in the temporal direction. We discuss the stability, and provide the error analysis by proposing a suitable choice of the time-dependent boundary data instead of evaluating them classically in order to avoid the order reduction phenomena.

We consider an equidistant mesh, denoted by $\Lambda^{\Delta t} := \{t_n\}_{n=0}^{M}$ on the temporal domain $[0,T]$ with M mesh-intervals such that $\Delta t = t_n - t_{n-1} = T/M, \quad n = 1,\dots,M$. Let $u^n(x,y) \approx u(x,y,t_n)$. Then, the semidiscrete problem obtained by utilizing the fractional implicit-Euler method which can be written as two-half scheme is given below:

(*i*) (initial condition)

$u^0(x,y) = \mathtt{q}_0(x,y), \quad (x,y) \in \overline{\mathtt{G}},$

(*ii*) (first half)

$$\begin{cases} (\mathtt{I} + \Delta t \mathtt{L}_{1,\varepsilon}^{n+1}) u^{n+1/2}(x,y) = u^n(x,y) + \Delta t g_1(x,y,t_{n+1}), \quad (x,y) \in \mathtt{G}, \\ u^{n+1/2}(x,y) = \mathtt{s}^{n+1/2}(x,y), \quad (x,y) \in \{0,1\} \times [0,1], \end{cases} \tag{3.1}$$

(*iii*) (second half)

$$\begin{cases} (\mathtt{I} + \Delta t \mathtt{L}_{2,\varepsilon}^{n+1}) u^{n+1}(x,y) = u^{n+1/2}(x,y) + \Delta t g_2(x,y,t_{n+1}), \ (x,y) \in \mathtt{G}, \\ u^{n+1}(x,y) = \mathtt{s}^{n+1}(x,y), \quad (x,y) \in [0,1] \times \{0,1\}, \end{cases}$$

for $n = 0, \ldots, M-1$, where the operators $\mathtt{L}_{1,\varepsilon}^{n+1}$ and $\mathtt{L}_{2,\varepsilon}^{n+1}$ are defined by

$$\begin{cases} \mathtt{L}_{1,\varepsilon}^{n+1} \equiv -\varepsilon \frac{\partial^2}{\partial x^2} + v_1(x,y,t_{n+1}) \frac{\partial}{\partial x} + b_1(x,y,t_{n+1}), \\ \mathtt{L}_{2,\varepsilon}^{n+1} \equiv -\varepsilon \frac{\partial^2}{\partial y^2} + v_2(x,y,t_{n+1}) \frac{\partial}{\partial y} + b_2(x,y,t_{n+1}). \end{cases}$$

The classical choice of the boundary conditions is given by

$$\begin{cases} \mathtt{s}^{n+1/2}(x,y) = \mathtt{s}(x,y,t_{n+1}), \quad (x,y) \in \{0,1\} \times [0,1], \\ \mathtt{s}^{n+1}(x,y) = \mathtt{s}(x,y,t_{n+1}), \quad (x,y) \in [0,1] \times \{0,1\}. \end{cases} \tag{3.2}$$

Here, we propose an alternative choice of the boundary data which is given by

$$\begin{cases} \quad \mathtt{s}^{n+1/2}(x,y) \\ = (\mathtt{I} + \Delta t \mathtt{L}_{2,\varepsilon}^{n+1}) \mathtt{s}(x,y,t_{n+1}) - \Delta t g_2(x,y,t_{n+1}), \ (x,y) \in \{0,1\} \times [0,1], \\ \mathtt{s}^{n+1}(x,y) = \mathtt{s}(x,y,t_{n+1}), \quad (x,y) \in [0,1] \times \{0,1\}. \end{cases} \tag{3.3}$$

Remark 3.1. If the the natural choice of the boundary data is used instead of the alternative choice (3.3), order reduction occurs in the global error. For a detailed discussion of alternative choice of the boundary data , we refer the reader to [6].

It is easy to check that the operators $(\mathtt{I} + \Delta t \mathtt{L}_{1,\varepsilon}^{n+1})$ and $(\mathtt{I} + \Delta t \mathtt{L}_{2,\varepsilon}^{n+1})$ satisfy the following maximum principle.

Lemma 3.1 (Maximum principle)**.** *Let the function* $\psi \in \mathtt{C}^0(\overline{\mathtt{G}}) \cap \mathtt{C}^2(\mathtt{G})$ *such that* $\psi(x,y) \leq 0$ *on* $\partial\mathtt{G}$ *and* $(\mathtt{I} + \Delta t \mathtt{L}_{k,\varepsilon}^{n+1})\psi(x,y) \leq 0, \quad k = 1,2,$ *for all* $(x,y) \in \mathtt{G}$*. Then, it holds* $\psi(x,y) \leq 0$ *for all* $(x,y) \in \overline{\mathtt{G}}$.

As a consequence of Lemma 3.1, we obtain the following result which ensures the stability of the time semidiscrete scheme (3.1).

Lemma 3.2. *Let the function* $\mathcal{Z} \in \mathcal{C}^0(\overline{\mathtt{G}}) \cap \mathcal{C}^2(\mathtt{G})$*. Then, we have*

$$\|\mathcal{Z}\|_{\overline{\mathtt{G}}} \le \|\mathcal{Z}\|_{\partial\mathtt{G}} + \frac{1}{1+\beta_k \Delta t}\|(\mathtt{I} + \Delta t \mathtt{L}_{k,\varepsilon}^{n+1})\mathcal{Z}\|_{\overline{\mathtt{G}}}.$$

3.1. Error analysis

Let us denote $\widetilde{e}^{n+1}$ as the local truncation error of the time semidiscrete scheme (3.1) at the time t_{n+1}, i.e., $\widetilde{e}^{n+1}(x,y) = \widetilde{u}^{n+1}(x,y) - u(x,y,t_{n+1})$, where $\widetilde{u}^{n+1}$ is the solution of the following auxiliary problem

$$
\begin{aligned}
&(i)\ u^0(x,y) = \mathtt{q}_0(x,y), \quad (x,y) \in \overline{\mathtt{G}},\\
&(ii)\ \begin{cases} (\mathtt{I} + \Delta t \mathtt{L}_{1,\varepsilon}^{n+1})\widetilde{u}^{n+1/2}(x,y) = u(x,y,t_n) + \Delta t g_1(x,y,t_{n+1}), & (x,y) \in \mathtt{G},\\ \widetilde{u}^{n+1/2}(x,y) = \mathtt{s}^{n+1/2}(x,y), & (x,y) \in \{0,1\} \times [0,1], \end{cases}\\
&(iii)\ \begin{cases} (\mathtt{I} + \Delta t \mathtt{L}_{2,\varepsilon}^{n+1})\widetilde{u}^{n+1}(x,y) = \widetilde{u}^{n+1/2}(x,y) + \Delta t g_2(x,y,t_{n+1}), & (x,y) \in \mathtt{G},\\ \widetilde{u}^{n+1}(x,y) = \mathtt{s}^{n+1}(x,y), & (x,y) \in [0,1] \times \{0,1\}, \end{cases}
\end{aligned}
\tag{3.4}
$$

for $n = 0, \dots, M-1$.

Lemma 3.3 (Local error)**.** *Under the alternative boundary data of* $\mathtt{s}^{n+1/2}$ *and* $\mathtt{s}^{n+1}$ *given in (3.3), the local error* $\widetilde{e}^{n+1}$ *at the time level* t_{n+1} *satisfies that*

$$\|\widetilde{e}^{n+1}\|_{\overline{\mathtt{G}}} \le C(\Delta t)^2. \tag{3.5}$$

Proof. The proof follows from [[6], Lemma 1]. □

Let us denote $e^{n+1}(x,y)$ as the global error of the time semidiscrete scheme (3.1) at time t_{n+1} as usual *i.e.*, $e^{n+1}(x,y) = u(x,y,t_{n+1}) - u^{n+1}(x,y)$. The following result shows that the fractional implicit-Euler method converges uniformly with first-order accurate in time.

Theorem 3.1 (Global error)**.** *Under the alternative boundary data of* $\mathtt{s}^{n+1/2}$ *and* $\mathtt{s}^{n+1}$ *given in (3.3), the global error* e^{n+1} *satisfies that*

$$\sup_{(n+1)\Delta t \le T} \|e^{n+1}\|_{\overline{\mathtt{G}}} \le C\Delta t.$$

Proof. By making use of the auxiliary problem (3.4), and invoking Lemmas 3.2 and 3.3, we obtain the desired estimate of the global error. □

4. The fully discrete problem

On the spatial domain $\overline{\mathtt{G}} = [0,1]^2$, we construct a rectangular mesh $\overline{\mathtt{G}}^N = \overline{\mathtt{G}}_x^N \times \overline{\mathtt{G}}_y^N \subset \overline{\mathtt{G}}$, having $(N+1)^2$ mesh point as depicted in Fig 1. Here, $N \ge 4$ is an even positive integer; and $\overline{\mathtt{G}}_x^N$ and $\overline{\mathtt{G}}_y^N$ denote the appropriate piecewise-uniform Shishkin meshes, respectively in the x and y directions. The detail construction of $\overline{\mathtt{G}}_x^N$ is given below. We divide the spatial domain $[0,1]$ into two sub-intervals

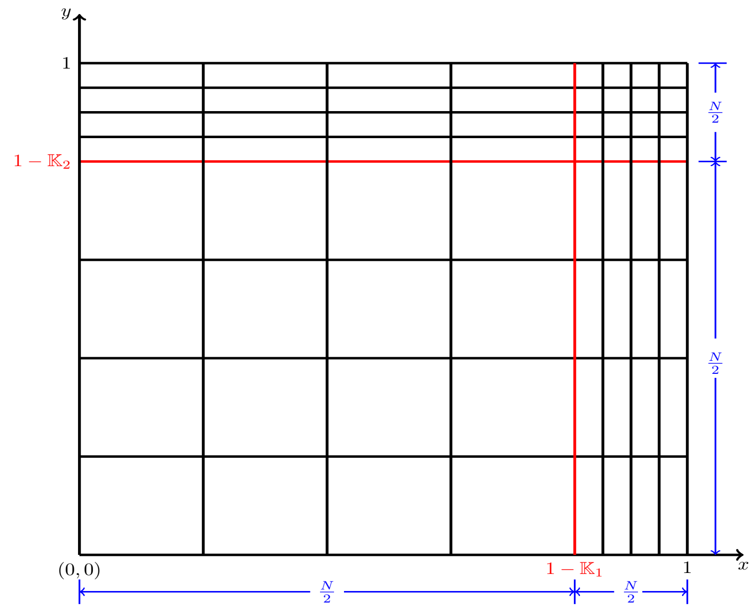


Figure 1. Shishkin mesh in the spatial direction

as $[0, 1-\mathbb{K}_1]$ and $[1-\mathbb{K}_1, 1]$, where the transition parameter $\mathbb{K}_1$ is defined by $\mathbb{K}_1 = \min\left\{\frac{1}{2}, \mathbb{K}_{1,0}\varepsilon \ln N\right\}$, and $\mathbb{K}_{1,0}$ is a positive constant. In the analysis, we only consider non-uniform mesh and for that we assume that $\mathbb{K}_1 = \mathbb{K}_{1,0}\varepsilon \ln N$. Now, on each sub-interval we introduce equidistant mesh with $N/2$ mesh-intervals such that $\overline{\mathtt{G}}_x^N = \{0 = x_0, x_1, \ldots, x_{N/2} = 1-\mathbb{K}_1, \ldots, x_N = 1\}$. Analogously, we define $\overline{\mathtt{G}}_y^N$ such that $\overline{\mathtt{G}}_y^N = \{0 = y_0, y_1, \ldots, y_{N/2} = 1-\mathbb{K}_2, \ldots, y_N = 1\}$, where $\mathbb{K}_2 = \min\left\{\frac{1}{2}, \mathbb{K}_{2,0}\varepsilon \ln N\right\}$.

Further, the mesh widths in the spatial directions are denoted by

$$\begin{cases} h_{x_i} = x_i - x_{i-1}, \quad 1 \le i \le N, \quad h_{y_j} = y_j - y_{j-1}, \quad 1 \le j \le N, \\ \widehat{h}_{x_i} = h_{x_i} + h_{x_{i+1}}, \quad 1 \le i \le N-1, \quad \widehat{h}_{y_j} = h_{y_j} + h_{y_{j+1}}, \quad 1 \le j \le N-1. \end{cases}$$

Let $h_{x_i} = H_1 = \frac{2(1-\mathbb{K}_1)}{N}, \quad 1 \le i \le N/2$ and $h_{y_j} = H_2 = \dfrac{2(1-\mathbb{K}_2)}{N}, \quad 1 \le j \le N/2$. Also let $h_{x_i} = h_1 = \frac{2\mathbb{K}_1}{N}, \quad N/2+1 \le i \le N$ and $h_{y_j} = h_2 = \frac{2\mathbb{K}_2}{N}, \quad N/2+1 \le j \le N$.

4.1. Proposed fully discrete scheme

For a given function $\Psi_{i,j}^n = \Psi(x_i, y_j, t_n)$, defined on the mesh $\overline{\mathtt{D}}^{N,\Delta t} = \overline{\mathtt{G}}^N \times \Lambda^{\Delta t}$, we define $\Psi_{i-\frac{1}{2},j}^n = \dfrac{\Psi_{i,j}^n + \Psi_{i-1,j}^n}{2}$, $\Psi_{i,j-\frac{1}{2}}^n = \dfrac{\Psi_{i,j}^n + \Psi_{i,j-1}^n}{2}$. Also, we define $v_{1,i-1/2,j}^n = \dfrac{v_{1,i,j}^n + v_{1,i-1,j}^n}{2}$, $v_{2,i,j-1/2}^n = \dfrac{v_{2,i,j}^n + v_{2,i,j-1}^n}{2}$; and $b_{1,i-1/2,j}^n$, $b_{2,i,j-1/2}^n$, $g_{1,i-1/2,j}^n$,

$g^n_{2,i,j-1/2}$ are defined similarly. Let us denote $\mathtt{G}^N_x = \overline{\mathtt{G}}^N_x \cap (0,1)$ and $\mathtt{G}^N_y = \overline{\mathtt{G}}^N_y \cap (0,1)$. Further, the difference operators denoted by $D^+_x, D^-_x, D^*_x, \delta^2_x$ in the x-direction, and the backward difference operator denoted by D^-_t in the t-direction, are defined as follows:

$$\begin{cases} D^+_x \psi^n_{i,j} = \dfrac{\psi^n_{i+1,j} - \psi^n_{i,j}}{h_{x_i}}, \quad D^-_x \psi^n_{i,j} = \dfrac{\psi^n_{i,j} - \psi^n_{i-1,j}}{h_{x_i}}, \\ D^*_x \psi^n_{i,j} = \dfrac{h_{x_i}}{\widehat{h}_{x_i}} D^+_x \psi^n_{i,j} + \dfrac{h_{i+1}}{\widehat{h}_{x_i}} D^-_x \psi^n_{i,j}, \\ \delta^2_x \psi^n_{i,j} = \dfrac{2}{\widehat{h}_{x_i}} (D^+_x \psi^n_{i,j} - D^-_x \psi^n_{i,j}), \quad \delta^2_y \psi^n_{i,j} = \dfrac{2}{\widehat{h}_{y_j}} (D^+_y \psi^n_{i,j} - D^-_y \psi^n_{i,j}), \\ \text{and} \quad D^-_t \psi^n_{i,j} = \dfrac{\psi^n_{i,j} - \psi^{n-1}_{i,j}}{\Delta t}. \end{cases}$$

Similarly for the y-direction, we define the difference operators denoted by D^+_y, D^-_y, D^*_y, δ^2_y. In order to construct the fully discrete scheme (4.1) for the IBVP (1.1)-(1.2), we consider spatial discretization of (3.1) in each half by a new hybrid finite difference scheme. The scheme is composed of a modified central difference scheme whenever $\varepsilon > \|v_i\| N^{-1}, i = 1,2$; and a combination of the midpoint upwind scheme in the outer region and the modified central difference scheme in the layer region whenever $\varepsilon \le \|v_i\| N^{-1}, i = 1,2$. In this regard, the associated difference operators $\mathtt{L}^{n+1}_{1,N,mcd}, \mathtt{L}^{n+1}_{1,N,mup}, \mathtt{L}^{n+1}_{2,N,mcd}, \mathtt{L}^{n+1}_{2,N,mup}$ are given by

$$\begin{cases} \quad \mathtt{L}^{n+1}_{1,N,mcd} U^{n+1/2}_{i,j} \\ = -\varepsilon \delta^2_x U^{n+1/2}_{i,j} + v_1(x_i, y_j, t_{n+1}) D^*_x U^{n+1/2}_{i,j} + b_1(x_i, y_j, t_{n+1}) U^{n+1/2}_{i,j}, \\ \mathtt{L}^{n+1}_{1,N,mup} U^{n+1/2}_{i,j} = -\varepsilon \delta^2_x U^{n+1/2}_{i,j} + v^{n+1}_{1,i-\frac{1}{2},j} D^-_x U^{n+1/2}_{i,j} + b^{n+1}_{1,i-\frac{1}{2},j} U^{n+1/2}_{i-\frac{1}{2},j}, \\ \mathtt{L}^{n+1}_{2,N,mcd} U^{n+1}_{i,j} = -\varepsilon \delta^2_y U^{n+1}_{i,j} + v_2(x_i, y_j, t_{n+1}) D^*_y U^{n+1}_{i,j} + b_2(x_i, y_j, t_{n+1}) U^{n+1}_{i,j}, \\ \mathtt{L}^{n+1}_{2,N,mup} U^{n+1}_{i,j} = -\varepsilon \delta^2_y U^{n+1}_{i,j} + v^{n+1}_{2,i,j-\frac{1}{2}} D^-_y U^{n+1}_{i,j} + b^{n+1}_{2,i,j-\frac{1}{2}} U^{n+1}_{i,j-\frac{1}{2}}. \end{cases}$$

On the mesh $\overline{\mathtt{D}}^{N,\Delta t}$, the fully discrete scheme takes the following form:

$$(i) \ U^0_{i,j} = \mathtt{q}_0(x_i, y_j), \quad \text{for } i,j = 0,1,\ldots,N,$$

$$(ii) \begin{cases} U^{n+1/2}_{i,j} + \Delta t \mathtt{L}^{n+1}_{1,N,mcd} U^{n+1/2}_{i,j} = U^n_{i,j} + \Delta t g_1(x_i, y_j, t_{n+1}), \\ \text{for } 1 \le i \le N/2, \ y_j \in \mathtt{G}^N_y \text{ and when } \varepsilon > \|v_1\| N^{-1}, \\ U^{n+1/2}_{i-\frac{1}{2},j} + \Delta t \mathtt{L}^{n+1}_{1,N,mup} U^{n+1/2}_{i,j} = U^n_{i-\frac{1}{2},j} + \Delta t g^{n+1}_{1,i-1/2,j}, \\ \text{for } 1 \le i \le N/2, \ y_j \in \mathtt{G}^N_y \text{ and when } \varepsilon \le \|v_1\| N^{-1}, \\ U^{n+1/2}_{i,j} + \Delta t \mathtt{L}^{n+1}_{1,N,mcd} U^{n+1/2}_{i,j} = U^n_{i,j} + \Delta t g_1(x_i, y_j, t_{n+1}), \\ \text{for } N/2 < i \le N-1, \ y_j \in \mathtt{G}^N_y, \\ U^{n+1/2}_{i,j} = \mathtt{s}^{n+1/2}(x_i, y_j), \quad \text{for } i = 0, N, \ y_j \in \overline{\mathtt{G}}^N_y, \end{cases} \tag{4.1}$$

$$
(iii)\quad \begin{cases}
U_{i,j}^{n+1} + \Delta t \mathtt{L}_{2,N,mcd}^{n+1} U_{i,j}^{n+1} = U_{i,j}^{n+1/2} + \Delta t g_2(x_i, y_j, t_{n+1}),\\
\text{for } 1 \le j \le N/2,\ x_i \in \mathtt{G}_x^N \text{ and when } \varepsilon > \|v_2\| N^{-1},\\
U_{i,j-\frac{1}{2}}^{n+1} + \Delta t \mathtt{L}_{2,N,mup}^{n+1} U_{i,j}^{n+1} = U_{i,j-\frac{1}{2}}^{n+1/2} + \Delta t g_{2,i,j-1/2}^{n+1},\\
\text{for } 1 \le j \le N/2,\ x_i \in \mathtt{G}_x^N \text{ and when } \varepsilon \le \|v_2\| N^{-1},\\
U_{i,j}^{n+1} + \Delta t \mathtt{L}_{2,N,mcd}^{n+1} U_{i,j}^{n+1} = U_{i,j}^{n+1/2} + \Delta t g_2(x_i, y_j, t_{n+1}),\\
\text{for } N/2 < j \le N-1,\ x_i \in \mathtt{G}_x^N,\\
U_{i,j}^{n+1} = \mathtt{s}^{n+1}(x_i, y_j), \quad \text{for } j = 0, N,\ x_i \in \overline{\mathtt{G}}_x^N,
\end{cases}
$$

where $\mathtt{s}^{n+1/2}, \mathtt{s}^{n+1}$ are defined in (3.3). Let $\rho_{x_i} = \left(\varepsilon - \dfrac{v_1(x_i, y_j, t_{n+1}) h_{x_i}}{2}\right)$ and $\rho_{y_j} = \left(\varepsilon - \dfrac{v_2(x_i, y_j, t_{n+1}) h_{y_j}}{2}\right)$. Then, after rearranging the terms in (4.1), we obtain the following form of the difference scheme:

$$
\begin{cases}
U_{i,j}^0 = \mathtt{q}_0(x_i, y_j), \quad \text{for } (x_i, y_j) \in \overline{\mathtt{G}}^N,\\
\begin{cases}
\mathtt{L}_{1,\varepsilon}^{N,\Delta t} U_{i,j}^{n+1/2} \equiv \mu_{x_i}^- U_{i-1,j}^{n+1/2} + \mu_{x_i}^c U_{i,j}^{n+1/2} + \mu_{x_i}^+ U_{i+1,j}^{n+1/2} = \mathtt{F}_1^{\Delta t}(x_i, y_j),\\
\text{for } 1 \le i \le N-1,\ y_j \in \mathtt{G}_y^N,\\
U_{i,j}^{n+1/2} = \mathtt{s}^{n+1/2}(x_i, y_j), \quad \text{for } i = 0, N,\ y_j \in \overline{\mathtt{G}}_y^N,\\
\mathtt{L}_{2,\varepsilon}^{N,\Delta t} U_{i,j}^{n+1} \equiv \mu_{y_j}^- U_{i,j-1}^{n+1} + \mu_{y_j}^c U_{i,j}^{n+1} + \mu_{y_j}^+ U_{i,j+1}^{n+1} = \mathtt{F}_2^{\Delta t}(x_i, y_j),\\
\text{for } 1 \le j \le N-1,\ x_i \in \mathtt{G}_x^N,\\
U_{i,j}^{n+1} = \mathtt{s}^{n+1}(x_i, y_j), \quad \text{for } j = 0, N,\ x_i \in \overline{\mathtt{G}}_x^N,
\end{cases}\\
\text{for } n = 0, \ldots, M-1,
\end{cases}
\tag{4.2}
$$

where the right hand side terms $\mathtt{F}_1^{\Delta t}(x_i, y_j), \mathtt{F}_2^{\Delta t}(x_i, y_j)$ in (4.2) are respectively given by

$$
\mathtt{F}_1^{\Delta t}(x_i, y_j) = \begin{cases}
\frac{1}{2}(U_{i-1,j}^n + \Delta t\, g_{1,i-1,j}^{n+1}) + \frac{1}{2}(U_{i,j}^n + \Delta t\, g_{1,i,j}^{n+1}),\\
\text{for } 1 \le i \le N/2, \text{ and when } \varepsilon \le \|v_1\| N^{-1}, \quad y_j \in \mathtt{G}_y^N,\\
U_{i,j}^n + \Delta t\, g_{1,i,j}^{n+1}, \text{ for } 1 \le i \le N/2, \text{ and when } \varepsilon > \|v_1\| N^{-1},\ y_j \in \mathtt{G}_y^N,\\
U_{i,j}^n + \Delta t\, g_{1,i,j}^{n+1}, \text{ for } N/2 < i \le N-1,\ y_j \in \mathtt{G}_y^N,
\end{cases}
\tag{4.3}
$$

and

$$\mathtt{F}_2^{\Delta t}(x_i,y_j)=\begin{cases}\frac{1}{2}(U_{i,j-1}^{n+1/2}+\Delta t\, g_{2,i,j-1}^{n+1})+\frac{1}{2}(U_{i,j}^{n+1/2}+\Delta t\, g_{2,i,j}^{n+1}),\\ \text{for } 1\le j\le N/2, \text{ and when } \varepsilon\le\|v_2\|N^{-1}, \quad x_i\in\mathtt{G}_x^N,\\ U_{i,j}^{n+1/2}+\Delta t\, g_{2,i,j}^{n+1}, \quad \text{for } 1\le j\le N/2,\\ \text{and when } \varepsilon>\|v_2\|N^{-1},\ x_i\in\mathtt{G}_x^N,\\ U_{i,j}^{n+1/2}+\Delta t\, g_{2,i,j}^{n+1}, \quad \text{for } N/2<j\le N-1,\ x_i\in\mathtt{G}_x^N.\end{cases} \tag{4.4}$$

Here, the coefficients $\mu_{z_k}^-,\mu_{z_k}^c,\mu_{z_k}^+$ for $z_k=x_i,\ k=i$ or $z_k=y_j,\ k=j$ are given by

$$\begin{cases}\mu_{z_k}^-=\Delta t\,\mu_{mcd,z_k}^-,\ \ \mu_{z_k}^c=\Delta t\,\mu_{mcd,z_k}^c+1,\ \ \mu_{z_k}^+=\Delta t\,\mu_{mcd,z_k}^+,\\ \text{for } 1\le k\le N/2, \ \text{ and when } \varepsilon>\|v_l\|N^{-1},\\ \mu_{z_k}^-=\Delta t\,\mu_{mup,z_k}^-+\frac{1}{2},\ \ \mu_{z_k}^c=\Delta t\,\mu_{mup,z_k}^c+\frac{1}{2},\ \ \mu_{z_k}^+=\Delta t\,\mu_{mup,z_k}^+,\\ \text{for } 1\le k\le N/2, \ \text{ and when } \varepsilon\le\|v_l\|N^{-1},\\ \mu_{z_k}^-=\Delta t\,\mu_{mcd,z_k}^-,\ \ \mu_{z_k}^c=\Delta t\,\mu_{mcd,z_k}^c+1,\ \ \mu_{z_k}^+=\Delta t\,\mu_{mcd,z_k}^+,\\ \text{for } N/2<k\le N-1.\end{cases} \tag{4.5}$$

For $k=i$,

$$\begin{cases}\mu_{mup,x_i}^-=-\dfrac{2\varepsilon}{\widetilde{h}_{x_i}h_{x_i}}-\dfrac{v_{1,i-\frac{1}{2},j}^{n+1}}{h_{x_i}}+\dfrac{b_{1,i-\frac{1}{2},j}^{n+1}}{2},\\ \mu_{mup,x_i}^c=\dfrac{2\varepsilon}{h_{x_i}h_{x_{i+1}}}+\dfrac{v_{1,i-\frac{1}{2},j}^{n+1}}{h_{x_i}}+\dfrac{b_{1,i-\frac{1}{2},j}^{n+1}}{2},\\ \mu_{mup,x_i}^+=-\dfrac{2\varepsilon}{\widetilde{h}_{x_i}h_{x_{i+1}}},\end{cases}$$

$$\text{and}\quad\begin{cases}\mu_{mcd,x_i}^-=-\dfrac{2\rho_{x_i}}{\widetilde{h}_{x_i}h_{x_i}}-\dfrac{v_{1,i,j}^{n+1}}{h_{x_i}},\\ \mu_{mcd,x_i}^c=\dfrac{2\rho_{x_i}}{h_{x_i}h_{x_{i+1}}}+\dfrac{v_{1,i,j}^{n+1}}{h_{x_i}}+b_{1,i,j}^{n+1},\\ \mu_{mcd,x_i}^+=-\dfrac{2\rho_{x_i}}{\widetilde{h}_{x_i}h_{x_{i+1}}}.\end{cases}$$

Similarly, for $k=j$, $\mu_{mup,y_j}^-,\mu_{mup,y_j}^c,\mu_{mup,y_j}^+,\mu_{mcd,y_j}^-,\mu_{mcd,y_j}^c,\mu_{mcd,y_j}^+$ are defined. We show that the difference operators $\mathtt{L}_{1,\varepsilon}^{N,\Delta t}$, $\mathtt{L}_{2,\varepsilon}^{N,\Delta t}$ defined in (4.2) satisfy the following discrete maximum principle. Let $\mathtt{G}^N=\overline{\mathtt{G}}^N\cap\mathtt{G}$ and $\partial\mathtt{G}^N=\overline{\mathtt{G}}^N\setminus\mathtt{G}^N$.

Lemma 4.1 (Discrete maximum principle)**.** *Suppose that the following conditions hold for $N \geq N_0$:*

$$\frac{N}{\ln N} > \mathbb{K}_{l,0}\|v_l\| \quad \text{and} \quad \mathtt{m}_l N \geq \Big(\|b_l\| + \frac{1}{\Delta t}\Big), \quad l = 1, 2, \tag{4.6}$$

where N_0 is a positive integer. If any mesh function $Z_{i,j} = Z(x_i, y_j)$ defined on $\overline{\mathtt{G}}^N$ satisfies that $Z_{i,j} \leq 0$ on $\partial\mathtt{G}^N$ and $\mathtt{L}_{l,\varepsilon}^{N,\Delta t} Z_{i,j} \leq 0$, $l = 1, 2$, in $\mathtt{G}^N$, then it implies that $Z_{i,j} \leq 0$ for all i, j.

Proof. To prove this, one requires to adopt the approach given in [31] by considering $\varepsilon > \|v_l\|N^{-1}$ and $\varepsilon \leq \|v_l\|N^{-1}$. □

4.2. Error analysis

In the beginning, we study the asymptotic behavior of the analytical solution of the semidiscrete problem (3.4) and its derivatives. This will be used later to derive the bounds of the truncation errors $\mathcal{T}^{N,\Delta t}_{x_i,\widetilde{u}^{n+1/2}}$ and $\mathcal{T}^{N,\Delta t}_{y_j,\widetilde{u}^{n+1}}$. For this purpose, at first, we deduce a-priori bounds for $\widetilde{u}^{n+1/2}(x, y)$ and its derivatives in the x-direction; and also for $\widetilde{u}^{n+1}(x, y)$ and its derivatives in the y-direction. From Lemma 3.2, it is clear that $\|\widetilde{u}^{n+1/2}\| \leq C$ and $\|\widetilde{u}^{n+1}\| \leq C$, since $u(x, y, t_n), g_1, g_2, \mathtt{s}^{n+1/2}$ and $\mathtt{s}^{n+1}$ are ε-uniformly bounded. For the proof of Lemma 4.2, apart from the requirement of ε-uniform boundedness and smoothness criteria on the given data, we also need certain compatibility conditions at $(0, t_n)$ and $(1, t_n)$ as mentioned in (4.16). Note that, the following result is established by extending the approach of [9], which incorporates the non-homogeneous boundary data $\mathtt{s}^{n+1/2}$, $\mathtt{s}^{n+1}$; and also by applying the argument of Kellogg and Tsan technique [17].

Lemma 4.2. *The solutions $\widetilde{u}^{n+1/2}(x, y)$ and $\widetilde{u}^{n+1}(x, y)$ of the time semidiscrete scheme (3.4) and their derivatives satisfy that*

$$\Big|\frac{\partial^j \widetilde{u}^{n+1/2}(x, y)}{\partial x^j}\Big| \leq C\big(1 + \varepsilon^{-j}\exp(-\mathtt{m}_1(1 - x)/\varepsilon)\big), \quad j = 0, 1, 2, 3, 4, \tag{4.7}$$

and

$$\Big|\frac{\partial^j \widetilde{u}^{n+1}(x, y)}{\partial y^j}\Big| \leq C\big(1 + \varepsilon^{-j}\exp(-\mathtt{m}_2(1 - y)/\varepsilon)\big), \quad j = 0, 1, 2, 3, 4, \tag{4.8}$$

for all $(x, y) \in \overline{\mathtt{G}}$.

Proof. For concise presentation, we provide detailed derivation of the result (4.7) for $\widetilde{u}^{n+1/2}(x, y)$.

Part-I: Here, we prove the result (4.7). Consider the auxiliary BVP

$$(\mathtt{I} + \Delta t\mathtt{L}^{n+1}_{1,\varepsilon})\zeta(x, y) = -\mathtt{L}^{n+1}_{1,\varepsilon}u(x, y, t_n) + g_1(x, y, t_{n+1}) \equiv \mathcal{H}_1(x, y), \tag{4.9}$$

where $\zeta(x, y) = \dfrac{\widetilde{u}^{n+1/2}(x, y) - u(x, y, t_n)}{\Delta t}$, with boundary conditions:

$$\begin{aligned}\zeta(0, y) &= \frac{\widetilde{u}^{n+1/2}(0, y) - u(0, y, t_n)}{\Delta t}, \\ &= \frac{(\mathtt{I} + \Delta t\mathtt{L}^{n+1}_{2,\varepsilon})\mathtt{s}(0, y, t_{n+1}) - \Delta t g_2(0, y, t_{n+1}) - \mathtt{s}(0, y, t_n)}{\Delta t}, \\ &= \mathtt{L}^{n+1}_{2,\varepsilon}\mathtt{s}(0, y, t_{n+1}) - g_2(0, y, t_{n+1}) + \frac{\partial \mathtt{s}(0, y, t_n)}{\partial t} + O(\Delta t),\end{aligned} \tag{4.10}$$

$$\zeta(1,y) = \mathtt{L}_{2,\varepsilon}^{n+1}\mathtt{s}(1,y,t_{n+1}) - g_2(1,y,t_{n+1}) + \frac{\partial \mathtt{s}(1,y,t_n)}{\partial t} + O(\Delta t). \tag{4.11}$$

Therefore, (4.9)-(4.11) reduces to the following form:

$$\begin{cases} (\mathtt{I} + \Delta t \mathtt{L}_{1,\varepsilon}^{n+1})\zeta(x,y) = \mathcal{H}_1(x,y), \\ \zeta(0,y) = \mathtt{L}_{2,\varepsilon}^{n+1}\mathtt{s}(0,y,t_{n+1}) - g_2(0,y,t_{n+1}) + \dfrac{\partial \mathtt{s}(0,y,t_n)}{\partial t} + O(\Delta t), \\ \zeta(1,y) = \mathtt{L}_{2,\varepsilon}^{n+1}\mathtt{s}(1,y,t_{n+1}) - g_2(1,y,t_{n+1}) + \dfrac{\partial \mathtt{s}(1,y,t_n)}{\partial t} + O(\Delta t). \end{cases} \tag{4.12}$$

We see that the boundary conditions of problem (4.12) are $(\varepsilon, \Delta t)$-uniformly bounded. Since $|\mathtt{L}_{1,\varepsilon}^{n+1} u(x,y,t_n)| \le C$, we have $|\mathcal{H}_1(x,y)| \le C$. Hence, applying Lemma 3.2 , we obtain that $|\zeta(x,y)| \le C$. We have

$$\begin{cases} \mathtt{L}_{1,\varepsilon}^{n+1}\widetilde{u}^{n+1/2}(x,y) = -\zeta(x,y) + g_1(x,y,t_{n+1}), \quad (x,y) \in \mathtt{G}, \\ \widetilde{u}^{n+1/2}(0,y) = (\mathtt{I} + \Delta t \mathtt{L}_{2,\varepsilon}^{n+1})\mathtt{s}(0,y,t_{n+1}) - \Delta t g_2(0,y,t_{n+1}), \\ \widetilde{u}^{n+1/2}(1,y) = (\mathtt{I} + \Delta t \mathtt{L}_{2,\varepsilon}^{n+1})\mathtt{s}(1,y,t_{n+1}) - \Delta t g_2(0,y,t_{n+1}). \end{cases} \tag{4.13}$$

Using the argument of Kellogg and Tsan technique [17], one can obtain that

$$\left| \frac{\partial \widetilde{u}^{n+1/2}(x,y)}{\partial x} \right| \le C\Big[1 + \varepsilon^{-1} \exp(-\mathtt{m}_1(1-x)/\varepsilon)\Big], \quad (x,y) \in \overline{\mathtt{G}}. \tag{4.14}$$

Let $\zeta_1(x,y) = \mathtt{L}_{1,\varepsilon}^{n+1}\zeta(x,y)$, which satisfies that

$$\begin{cases} (\mathtt{I} + \Delta t \mathtt{L}_{1,\varepsilon}^{n+1})\zeta_1(x,y) = -(\mathtt{L}_{1,\varepsilon}^{n+1})^2 u(x,y,t_n) + \mathtt{L}_{1,\varepsilon}^{n+1} g_1(x,y,t_{n+1}) \equiv \mathcal{H}_2(x,y), \\ \zeta_1(0,y) = -\dfrac{\zeta(0,y)}{\Delta t} + \dfrac{1}{\Delta t}\Big[g_1(0,y,t_{n+1}) - \mathtt{L}_{1,\varepsilon}^{n+1} u(0,y,t_n)\Big], \\ \zeta_1(1,y) = -\dfrac{\zeta(1,y)}{\Delta t} + \dfrac{1}{\Delta t}\Big[g_1(1,y,t_{n+1}) - \mathtt{L}_{1,\varepsilon}^{n+1} u(1,y,t_n)\Big]. \end{cases} \tag{4.15}$$

Since $|(\mathtt{L}_{1,\varepsilon}^{n+1})^2 u(x,y,t_n)| \le C$, we have $|\mathcal{H}_2(x,y)| \le C$. Now, from the compatibility conditions

$$\frac{\partial \mathtt{s}(x,y,t)}{\partial t} = -\mathtt{L}_\varepsilon \mathtt{s}(x,y,t) + g(x,y,t), \quad (x,y) \in \{0,1\} \times \{0,1\} \times (0,T],$$

one can obtain that

$$\begin{aligned} \frac{\partial \mathtt{s}(0,y,t_n)}{\partial t} &= -\mathtt{L}_\varepsilon^n \mathtt{s}(0,y,t_n) + g(0,y,t_n), \\ \frac{\partial \mathtt{s}(1,y,t_n)}{\partial t} &= -\mathtt{L}_\varepsilon^n \mathtt{s}(1,y,t_n) + g(1,y,t_n). \end{aligned} \tag{4.16}$$

By using the equations (4.15) and (4.16), we get

$$\begin{cases} (\mathtt{I} + \Delta t \mathtt{L}_{1,\varepsilon}^{n+1})\zeta_1(x,y) = \mathcal{H}_2(x,y), \\ \zeta_1(0,y) = \dfrac{\partial g(0,y,t_n)}{\partial t} - \mathtt{L}_{2,\varepsilon}^{n+1} \dfrac{\partial \mathtt{s}(0,y,t_n)}{\partial t} + C_1, \\ \zeta_1(1,y) = \dfrac{\partial g(1,y,t_n)}{\partial t} - \mathtt{L}_{2,\varepsilon}^{n+1} \dfrac{\partial \mathtt{s}(1,y,t_n)}{\partial t} + C_2, \end{cases} \tag{4.17}$$

where C_1 and C_2 are independent of ε and Δt. We see that $\mathcal{H}_2(x,y) = -(\mathtt{L}_{1,\varepsilon}^{n+1})^2 u(x,y,t_n)+\mathtt{L}_{1,\varepsilon}^{n+1}g_1(x,y,t_{n+1})$ is bounded (ε-uniformly) and boundary conditions are $(\varepsilon,\Delta t)$-uniformly bounded. Hence, applying Lemma 3.2 , we obtain that $|\zeta_1(x,y)| \le C$. Afterwards, one can deduce that

$$\Big|\frac{\partial \zeta(x,y)}{\partial x}\Big| \le C\Big[1+\varepsilon^{-1}\exp(-\mathtt{m}_1(1-x)/\varepsilon)\Big], \quad (x,y)\in \overline{\mathtt{G}}, \tag{4.18}$$

by invoking Kellogg and Tsan technique [17] to the following BVP:

$$\begin{cases} \mathtt{L}_{1,\varepsilon}^{n+1}\zeta(x,y) = \zeta_1(x,y), \\ \zeta(0,y) = \mathtt{L}_{2,\varepsilon}^{n+1}\mathtt{s}(0,y,t_{n+1}) - g_2(0,y,t_{n+1}) + \dfrac{\partial \mathtt{s}(0,y,t_n)}{\partial t} + O(\Delta t), \\ \zeta(1,y) = \mathtt{L}_{2,\varepsilon}^{n+1}\mathtt{s}(1,y,t_{n+1}) - g_2(1,y,t_{n+1}) + \dfrac{\partial \mathtt{s}(1,y,t_n)}{\partial t} + O(\Delta t). \end{cases} \tag{4.19}$$

Now, differentiate (4.13) with respect to x, we have that $\overline{\zeta}(x,y) = \dfrac{\partial \widetilde{u}^{n+1/2}}{\partial x}$ satisfies the following problem

$$\begin{cases} \mathtt{L}_{1,\varepsilon}^{n+1}\overline{\zeta}(x,y) = \mathcal{H}_3(x,y), \\ \overline{\zeta}(0,y) = C_1, \quad \overline{\zeta}(1,y) = C_2\varepsilon^{-1}, \end{cases} \tag{4.20}$$

where $\mathcal{H}_3(x,y) = -\dfrac{\partial \zeta(x,y)}{\partial x} + \dfrac{\partial g_1(x,y,t_{n+1})}{\partial x} - \dfrac{\partial v_1(x,y,t_{n+1})}{\partial x}\dfrac{\partial \widetilde{u}^{n+1/2}}{\partial x} - \dfrac{\partial b_1(x,y,t_{n+1})}{\partial x}\widetilde{u}^{n+1/2}(x,y)$ and we obtain that

$$\big|\mathcal{H}_3(x,y)\big| \le C\Big[1+\varepsilon^{-1}\exp(-\mathtt{m}_1(1-x)/\varepsilon)\Big], \quad (x,y)\in\overline{\mathtt{G}}.$$

Again, using the argument of Kellogg and Tsan technique [17] for (4.20), we get

$$\Big|\frac{\partial \overline{\zeta}(x,y)}{\partial x}\Big| = \Big|\frac{\partial^2 \widetilde{u}^{n+1/2}(x,y)}{\partial x^2}\Big| \le C\Big[1+\varepsilon^{-2}\exp(-\mathtt{m}_1(1-x)/\varepsilon)\Big], \quad (x,y)\in\overline{\mathtt{G}}.$$

To establish the result for $j=3$, we follow the similar procedure. Firstly, we consider the function $\zeta_2(x,y) = (\mathtt{L}_{1,\varepsilon}^{n+1})^2\zeta(x,y)$ as the solution of the following BVP:

$$\begin{cases} (\mathtt{I}+\Delta t\mathtt{L}_{1,\varepsilon}^{n+1})\zeta_2(x,y) = -(\mathtt{L}_{1,\varepsilon}^{n+1})^3 u(x,y,t_n) + (\mathtt{L}_{1,\varepsilon}^{n+1})^2 g_1(x,y,t_{n+1}) \equiv \mathcal{H}_4(x,y), \\ \zeta_2(0,y) = \dfrac{1}{\Delta t}\Big[-\mathtt{L}_{1,\varepsilon}^{n+1}\zeta(0,y) + \Big(\mathtt{L}_{1,\varepsilon}^{n+1}g_1(0,y,t_{n+1}) - (\mathtt{L}_{1,\varepsilon}^{n+1})^2 u(0,y,t_n)\Big)\Big], \\ \zeta_2(1,y) = \dfrac{1}{\Delta t}\Big[-\mathtt{L}_{1,\varepsilon}^{n+1}\zeta(1,y) + \Big(\mathtt{L}_{1,\varepsilon}^{n+1}g_1(1,y,t_{n+1}) - (\mathtt{L}_{1,\varepsilon}^{n+1})^2 u(1,y,t_n)\Big)\Big]. \end{cases} \tag{4.21}$$

We simplify the boundary conditions of the problem (4.21) by using the compatibility conditions (4.16), to get

$$\begin{cases} (\mathtt{I}+\Delta t\mathtt{L}_{1,\varepsilon}^{n+1})\zeta_2(x,y) = \mathcal{H}_4(x,y), \\ \zeta_2(0,y) = \Big(\dfrac{1}{2}\mathtt{L}_{1,\varepsilon}^{n+1}\dfrac{\partial^2 \mathtt{s}}{\partial t^2}\Big)(0,y,t_{n+1}) + \Big((\mathtt{L}_{1,\varepsilon}^{n+1})^2\dfrac{\partial \mathtt{s}}{\partial t}\Big)(0,y,t_{n+1}) + O(\Delta t), \\ \zeta_2(1,y) = \Big(\dfrac{1}{2}\mathtt{L}_{1,\varepsilon}^{n+1}\dfrac{\partial^2 \mathtt{s}}{\partial t^2}\Big)(1,y,t_{n+1}) + \Big((\mathtt{L}_{1,\varepsilon}^{n+1})^2\dfrac{\partial \mathtt{s}}{\partial t}\Big)(1,y,t_{n+1}) + O(\Delta t). \end{cases} \tag{4.22}$$

We see that $\mathcal{H}_4(x,y) = -(\mathtt{L}_{1,\varepsilon}^{n+1})^3 u(x,y,t_n) + (\mathtt{L}_{1,\varepsilon}^{n+1})^2 g_1(x,y,t_{n+1})$ is bounded (ε-uniformly) and boundary conditions are $(\varepsilon,\Delta t)$-uniformly bounded. Hence, applying Lemma 3.2 , we obtain that $|\zeta_2(x,y)| \le C$. Now, similar arguments can be applied for the following BVP:

$$\begin{cases} \mathtt{L}_{1,\varepsilon}^{n+1}\zeta_1(x,y) = \zeta_2(x,y), \\ \zeta_1(0,y) = \dfrac{\partial g(0,y,t_n)}{\partial t} - \mathtt{L}_{2,\varepsilon}^{n+1}\dfrac{\partial \mathtt{s}(0,y,t_n)}{\partial t} + C_1, \\ \zeta_1(1,y) = \dfrac{\partial g(1,y,t_n)}{\partial t} - \mathtt{L}_{2,\varepsilon}^{n+1}\dfrac{\partial \mathtt{s}(1,y,t_n)}{\partial t} + C_2, \end{cases} \tag{4.23}$$

to prove that

$$\left|\frac{\partial^2 \zeta(x,y)}{\partial^2 x}\right| \le C\Big[1+\varepsilon^{-2}\exp(-\mathtt{m}_1(1-x)/\varepsilon)\Big], \quad (x,y)\in \overline{\mathtt{G}}. \tag{4.24}$$

Now, we differentiate (4.20) with respect to x, we consider that $\bar{\zeta}_1(x,y) = \dfrac{\partial^2 \widetilde{u}^{n+1/2}(x,y)}{\partial x^2}$ satisfies the following problem:

$$\begin{cases} \mathtt{L}_{1,\varepsilon}^{n+1}\bar{\zeta}_1(x,y) = \mathcal{H}_5(x,y), \\ \bar{\zeta}_1(0,y) = C_1, \quad \bar{\zeta}_1(1,y) = C_2\varepsilon^{-2}, \end{cases} \tag{4.25}$$

where $|\mathcal{H}_5(x,y)| \le C\Big[1+\varepsilon^{-2}\exp(-\mathtt{m}_1(1-x)/\varepsilon)\Big], \quad (x,y)\in\overline{\mathtt{G}}$. From the argument of Kellogg and Tsan technique [17] for (4.25), we get

$$\left|\frac{\partial\bar{\zeta}_1(x,y)}{\partial x}\right| = \left|\frac{\partial^3\widetilde{u}^{n+1/2}(x,y)}{\partial x^3}\right| \le C\Big[1+\varepsilon^{-3}\exp(-\mathtt{m}_1(1-x)/\varepsilon)\Big], \quad (x,y)\in\overline{\mathtt{G}}.$$

Similar way one can obtain the bound for $j=4$.

We now derive the bound of $\widetilde{u}^{n+1/2}(x,y)$ with respect to y by differentiating the auxiliary BVP (3.4) at the first half with respect to y, and we get

$$\begin{cases} (\mathtt{I}+\Delta t\mathtt{L}_{1,\varepsilon}^{n+1})\dfrac{\partial\widetilde{u}^{n+1/2}(x,y)}{\partial y} = \dfrac{\partial u(x,y,t_n)}{\partial y} + \Delta t\dfrac{\partial g_1(x,y,t_{n+1})}{\partial y} \\ \qquad\qquad - \dfrac{\partial v_1(x,y,t_{n+1})}{\partial y}\dfrac{\partial\widetilde{u}^{n+1/2}(x,y)}{\partial x} \\ \qquad\qquad - \dfrac{\partial b_1(x,y,t_{n+1})}{\partial y}\widetilde{u}^{n+1/2}(x,y), \\ \dfrac{\partial\widetilde{u}^{n+1/2}(0,y)}{\partial y} = (\mathtt{I}+\Delta t\mathtt{L}_2^{n+1})\dfrac{\partial\mathtt{s}(0,y,t_{n+1})}{\partial y} + \Delta t\dfrac{\partial v_2(0,y,t_{n+1})}{\partial y}\dfrac{\partial\mathtt{s}(0,y,t_{n+1})}{\partial y} \\ \qquad + \Delta t\frac{\partial b_2(0,y,t_{n+1})}{\partial y}\mathtt{s}(0,y,t_{n+1}) - \Delta t\frac{\partial g_2(0,y,t_{n+1})}{\partial y}, \quad y\in[0,1], \\ \dfrac{\partial\widetilde{u}^{n+1/2}(1,y)}{\partial y} = (\mathtt{I}+\Delta t\mathtt{L}_2^{n+1})\dfrac{\partial\mathtt{s}(1,y,t_{n+1})}{\partial y} + \Delta t\dfrac{\partial v_2(1,y,t_{n+1})}{\partial y}\dfrac{\partial\mathtt{s}(1,y,t_{n+1})}{\partial y} \\ \qquad + \Delta t\dfrac{\partial b_2(1,y,t_{n+1})}{\partial y}\mathtt{s}(1,y,t_{n+1}) - \Delta t\dfrac{\partial g_2(1,y,t_{n+1})}{\partial y}, \quad y\in[0,1]. \end{cases} \tag{4.26}$$

The following bounds are proven by using the bounds of $\dfrac{\partial^j \widetilde{u}^{n+1/2}(x,y)}{\partial x^j}$ for $j = 0,1,2,3,4$,

$$\Big|\frac{\partial^j \widetilde{u}^{n+1/2}(x,y)}{\partial y^j}\Big| \le C\Big[1+\varepsilon^{-j}\exp(-\mathtt{m}_2(1-y)/\varepsilon)\Big], \quad (x,y)\in \overline{\mathtt{G}},\ \text{for } j=0,1,2,3,4. \tag{4.27}$$

Part-II: We suppose that, based on prior technical criterion,

$$\|\mathtt{L}_{2,\varepsilon}^{n+1}\widetilde{u}^{n+1/2}(x,y)\|_{\overline{\mathtt{G}}} \le C, \quad \|(\mathtt{L}_{2,\varepsilon}^{n+1})^2\widetilde{u}^{n+1/2}(x,y)\|_{\overline{\mathtt{G}}} \le C,$$
$$\|(\mathtt{L}_{2,\varepsilon}^{n+1})^3\widetilde{u}^{n+1/2}(x,y)\|_{\overline{\mathtt{G}}} \le C.$$

Then, the proof of the result (4.8) for $\widetilde{u}^{n+1}(x,y)$ follows similarly as like the previous case. □

Lemma 4.3. *The exact solutions $\widetilde{u}^{n+1/2}(x,y)$ and $\widetilde{u}^{n+1}(x,y)$ of the time semidiscrete scheme (3.4) can be decomposed as*

$$\begin{cases} \widetilde{u}^{n+1/2}(x,y) = \widetilde{p}^{n+1/2}(x,y) + \gamma_1\widetilde{q}^{n+1/2}(x,y), \\ \widetilde{u}^{n+1}(x,y) = \widetilde{p}^{n+1}(x,y) + \gamma_2\widetilde{q}^{n+1}(x,y), \end{cases}$$

where $y\in(0,1)$ the components of $\widetilde{u}^{n+1/2}(x,y)$ satisfy

$$\begin{cases} \Big|\dfrac{\partial^j \widetilde{p}^{n+1/2}}{\partial x^j}\Big| \le C\Big(1+\varepsilon^{-j+1}\exp\Big(-\dfrac{\mathtt{m}_1(1-x)}{\varepsilon}\Big)\Big),\ j=0,1,2,3,4, \\ \widetilde{q}^{n+1}(x,y) = \exp(-v_1(1,y,t_{n+1})(1-x)/\varepsilon),\ \gamma_1 = \dfrac{\varepsilon}{v_1(1,y,t_{n+1})}\dfrac{d\widetilde{u}^{n+1}}{dx}(1,y), \end{cases}$$

and for $x\in(0,1)$ the components of $\widetilde{u}^{n+1}(x,y)$ satisfy

$$\begin{cases} \Big|\dfrac{\partial^j \widetilde{p}^{n+1}}{\partial y^j}\Big| \le C\Big(1+\varepsilon^{-j+1}\exp\Big(-\dfrac{\mathtt{m}_2(1-y)}{\varepsilon}\Big)\Big),\ j=0,1,2,3,4, \\ \widetilde{q}^{n+1}(x,y) = \exp(-v_2(x,1,t_{n+1})(1-y)/\varepsilon),\ \gamma_2 = \dfrac{\varepsilon}{v_2(x,1,t_{n+1})}\dfrac{d\widetilde{u}^{n+1}}{dy}(x,1). \end{cases}$$

Proof. The proof can be carried out by using Lemma 4.2. See the approach given in ([9], Appendix A). □

Next, we state several important lemmas which will be used in the next section.

Lemma 4.4. *Consider the following mesh functions $\Theta_{l,k}(\lambda_l)$ with $l=1,2$*

$$\begin{cases} \Theta_{1,k}(\lambda_1) = \displaystyle\prod_{j=k+1}^{N}\Big(1+\frac{\lambda_1 h_{x_k}}{\varepsilon}\Big)^{-1}, \quad \text{for } 0\le k\le N-1, \quad \Theta_{1,N}(\lambda_1)=1, \\ \Theta_{2,k}(\lambda_2) = \displaystyle\prod_{j=k+1}^{N}\Big(1+\frac{\lambda_2 h_{y_k}}{\varepsilon}\Big)^{-1}, \quad \text{for } 0\le k\le N-1, \quad \Theta_{2,N}(\lambda_2)=1, \end{cases}$$

where λ_l is a positive constant. Then, we have the following inequalities:

(i) If $\lambda_l < \mathtt{m}_l/2$, then

$$\begin{cases} \exp\big(-\mathtt{m}_1(1-x_k)/\varepsilon\big) \le \Theta_{1,k}(\lambda_1), & \text{for } 0 \le k \le N-1, \\ \exp\big(-\mathtt{m}_2(1-y_k)/\varepsilon\big) \le \Theta_{2,k}(\lambda_2), & \text{for } 0 \le k \le N-1, \end{cases} \tag{4.28}$$

$$(ii)\ \Theta_{l,N/2}(\lambda_l) \le CN^{-\lambda_l \mathbb{K}_{l,0}}, \tag{4.29}$$

for some constant C.

Proof. Use the arguments given in [26, Lemma 5] for the proof of (i) and (ii). □

Lemma 4.5. *If $\lambda_l < \mathtt{m}_l/2,\ l = 1,2$, then under the hypothesis (4.6) of Lemma 4.1, it follows that*

$$\mathtt{L}_{l,\varepsilon}^{N,\Delta t}\Theta_{l,k}(\lambda_l) \ge \begin{cases} C\Delta t\varepsilon^{-1}\Theta_{2,k}(\lambda_l), & \text{for } 1 \le k \le N/2, \text{ and when } \varepsilon > \|v_l\|N^{-1}, \\ C\Delta t {H_l}^{-1}\Theta_{l,k}(\lambda_l), & \text{for } 1 \le k \le N/2, \text{ and when } \varepsilon \le \|v_l\|N^{-1}, \\ C\Delta t\varepsilon^{-1}\Theta_{l,k}(\lambda_l), & \text{for } N/2 < k \le N-1. \end{cases}$$

Proof. The argument given in [31, Lemma 12] can be used to prove this lemma. □

4.2.1. Error due to spatial discretization

In order to estimate the spatial error related to the fully discrete scheme (4.1), we consider the spatial discretization of the auxiliary problem (3.4) using the new finite difference scheme as described in Section 4.1. Hence, we obtain the following discrete problem:

$$\begin{cases} \widetilde{U}_{i,j}^0 = \mathtt{q}_0(x_i,y_j), \quad 0 \le i,j \le N, \\ \begin{cases} \mathtt{L}_{1,\varepsilon}^{N,\Delta t}\widetilde{U}_{i,j}^{n+1/2} \equiv \mu_{x_i}^-\widetilde{U}_{i-1,j}^{n+1/2} + \mu_{x_i}^c\widetilde{U}_{i,j}^{n+1/2} + \mu_{x_i}^+\widetilde{U}_{i+1,j}^{n+1/2} = \widetilde{\mathtt{F}}_1^{\Delta t}(x_i,y_j), \\ \text{for } 1 \le i,j \le N-1, \\ \widetilde{U}^{n+1/2}(x,y) = \mathtt{s}^{n+1/2}(x,y), \quad (x,y) \in \{0,1\} \times \overline{\mathtt{G}}_y^N, \\ \mathtt{L}_{2,\varepsilon}^{N,\Delta t}\widetilde{U}_{i,j}^{n+1} \equiv \mu_{y_j}^-\widetilde{U}_{i,j-1}^{n+1} + \mu_{y_j}^c\widetilde{U}_{i,j}^{n+1} + \mu_{y_j}^+\widetilde{U}_{i,j+1}^{n+1} = \widetilde{\mathtt{F}}_2^{n+1}(x_i,y_j), \\ \text{for } 1 \le i,j \le N-1, \\ \widetilde{U}^{n+1}(x,y) = \mathtt{s}^{n+1}(x,y), \quad (x,y) \in \overline{\mathtt{G}}_x^N \times \{0,1\}, \end{cases} \\ \text{for } n = 0,\dots,M-1, \end{cases} \tag{4.30}$$

where the coefficients $\mu_{x_i}^-, \mu_{x_i}^c, \mu_{x_i}^+, \mu_{y_j}^-, \mu_{y_j}^c, \mu_{y_j}^+$ are described in (4.5); and the terms $\widetilde{\mathtt{F}}_1^{\Delta t}(x_i,y_j), \widetilde{\mathtt{F}}_2^{\Delta t}(x_i,y_j)$ are respectively given by

$$\widetilde{\mathtt{F}}_1^{\Delta t}(x_i,y_j)=\begin{cases}\frac{1}{2}(u(x_{i-1},y_j,t_n)+\Delta t\, g_{1,i-1,j}^{n+1})+\frac{1}{2}(u(x_i,y_j,t_n)+\Delta t\, g_{1,i,j}^{n+1}),\\ \text{for } 1\le i\le N/2, \text{ and when } \varepsilon\le\|v_1\|N^{-1}, \quad y_j\in\mathtt{G}_y^N,\\ u(x_i,y_j,t_n)+\Delta t\, g_{1,i,j}^{n+1}, \quad \text{for } 1\le i\le N/2, \text{ and when } \varepsilon>\|v_1\|N^{-1},\\ y_j\in\mathtt{G}_y^N,\\ u(x_i,y_j,t_n)+\Delta t\, g_{1,i,j}^{n+1}, \quad \text{for } N/2<i\le N-1, \quad y_j\in\mathtt{G}_y^N,\end{cases} \tag{4.31}$$

and

$$\widetilde{\mathtt{F}}_2^{\Delta t}(x_i,y_j)=\begin{cases}\frac{1}{2}(\widetilde{U}_{i,j-1}^{n+1/2}+\Delta t\, g_{2,i,j-1}^{n+1})+\frac{1}{2}(\widetilde{U}_{i,j}^{n+1/2}+\Delta t\, g_{2,i,j}^{n+1}),\\ \text{for } 1\le j\le N/2, \text{ and when } \varepsilon\le\|v_2\|N^{-1}, \quad x_i\in\mathtt{G}_x^N,\\ \widetilde{U}_{i,j}^{n+1/2}+\Delta t\, g_{2,i,j}^{n+1}, \quad \text{for } 1\le j\le N/2, \text{ and when } \varepsilon>\|v_2\|N^{-1},\\ x_i\in\mathtt{G}_x^N,\\ \widetilde{U}_{i,j}^{n+1/2}+\Delta t\, g_{2,i,j}^{n+1}, \quad \text{for } N/2<j\le N-1, \quad x_i\in\mathtt{G}_x^N.\end{cases} \tag{4.32}$$

At first, we derive the estimate for the error $\left|\widetilde{U}_{i,j}^{n+1/2}-\widetilde{u}^{n+1/2}(x_i,y_j)\right|$. Here, for the discrete problem (4.30), the local truncation error at the first half is defined as

$$\mathcal{T}_{x_i,\widetilde{u}^{n+1/2}}^{N,\Delta t}=\mathtt{L}_{1,\varepsilon}^{N,\Delta t}[\widetilde{U}_{i,j}^{n+1/2}-\widetilde{u}^{n+1/2}(x_i,y_j)],$$

$$=\begin{cases}\mu_{x_i}^-\widetilde{U}_{i-1,j}^{n+1/2}+\mu_{x_i}^c\widetilde{U}_{i,j}^{n+1/2}+\mu_{x_i}^+\widetilde{U}_{i+1,j}^{n+1/2}\\ -\Big(\widetilde{u}^{n+1/2}(x_i,y_j)+\Delta t\mathtt{L}_{1,\varepsilon}^{n+1}\widetilde{u}^{n+1/2}(x_i,y_j)\Big),\\ \text{for } 1\le i\le N/2, \text{ and when } \varepsilon>\|v_1\|N^{-1},\\ \mu_{x_i}^-\widetilde{U}_{i-1,j}^{n+1/2}+\mu_{x_i}^c\widetilde{U}_{i,j}^{n+1/2}+\mu_{x_i}^+\widetilde{U}_{i+1,j}^{n+1/2}\\ -\frac{1}{2}\Big(\widetilde{u}^{n+1/2}(x_i,y_j)+\Delta t\mathtt{L}_{1,\varepsilon}^{n+1}\widetilde{u}^{n+1/2}(x_i,y_j)\Big)\\ -\dfrac{1}{2}\Big(\widetilde{u}^{n+1/2}(x_{i-1},y_j)+\Delta t\mathtt{L}_{1,\varepsilon}^{n+1}\widetilde{u}^{n+1/2}(x_{i-1},y_j)\Big),\\ \text{for } 1\le i\le N/2, \text{ when } \varepsilon\le\|v_1\|N^{-1},\\ \mu_{x_i}^-\widetilde{U}_{i-1,j}^{n+1/2}+\mu_{x_i}^c\widetilde{U}_{i,j}^{n+1/2}+\mu_{x_i}^+\widetilde{U}_{i+1,j}^{n+1/2}\\ -\Big(\widetilde{u}^{n+1/2}(x_i,y_j)+\Delta t\mathtt{L}_{1,\varepsilon}^{n+1}\widetilde{u}^{n+1/2}(x_i,y_j)\Big),\\ \text{for } N/2<i\le N-1,\end{cases}$$

$$= \begin{cases} \Delta t\Big[\mathtt{L}^{n+1}_{1,N,mcd}\widetilde{u}^{n+1/2}(x_i,y_j) - (\mathtt{L}^{n+1}_{1,\varepsilon}\widetilde{u}^{n+1/2})(x_i,y_j)\Big], \\ \text{for } 1 \le i \le N/2, \;\; \text{and when } \varepsilon > \|v_1\| N^{-1}, \\ \\ \Delta t\Big[\mathtt{L}^{n+1}_{1,N,mup}\widetilde{u}^{n+1/2}(x_i,y_j) - (\mathtt{L}_{1,\varepsilon}\widetilde{u}^{n+1/2})_{i-1/2,j}\Big], \\ \text{for } 1 \le i \le N/2, \;\; \text{and when } \varepsilon \le \|v_1\| N^{-1}, \\ \\ \Delta t\Big[\mathtt{L}^{n+1}_{1,N,mcd}\widetilde{u}^{n+1/2}(x_i,y_j) - (\mathtt{L}^{n+1}_{1,\varepsilon}\widetilde{u}^{n+1/2})(x_i,y_j)\Big], \\ \text{for } N/2 < i \le N-1, \end{cases}$$

$$= \Delta t\, \mathcal{T}^N_{x_i,\widetilde{u}^{n+1/2}}. \tag{4.33}$$

Therefore, $\mathcal{T}^N_{x_i,\widetilde{u}}$ denotes the truncation error corresponding to the stationary singularly perturbed problem, and is obtained by approximating the differential operator $\mathtt{L}_{1,\varepsilon}$ with respect to the spatial variable-x using the proposed finite difference scheme. Since $\widetilde{u}^{n+1/2}$ can be decomposed into $\widetilde{p}^{n+1/2}$ and $\widetilde{q}^{n+1/2}$ as mentioned in Lemma 4.3, we rewrite $\mathcal{T}^N_{x_i,\widetilde{u}^{n+1/2}}$, as

$$\mathcal{T}^N_{x_i,\widetilde{u}^{n+1/2}} = \mathcal{T}^N_{x_i,\widetilde{p}^{n+1/2}} + \gamma_1 \mathcal{T}^N_{x_i,\widetilde{q}^{n+1/2}}. \tag{4.34}$$

Now, for fixed $y \in (0,1)$, by making use of the Taylor's theorem on the functions $\widetilde{p}^{n+1/2}(x,y)$, $\widetilde{q}^{n+1/2}(x,y)$; and considering the remainder term in the integral form, we obtain the following result.

Lemma 4.6. (i) *For $1 \le i \le N/2$, $\mathcal{T}^N_{x_i,\widetilde{p}^{n+1/2}}$ satisfies that*

$$\Big|\mathcal{T}^N_{x_i,\widetilde{p}^{n+1/2}}\Big| \le \begin{cases} Ch_{x_i}\Big[\varepsilon \displaystyle\int_{x_{i-1}}^{x_{i+1}} \Big|\frac{\partial^4 \widetilde{p}^{n+1/2}(s,y)}{\partial s^4}\Big| ds + \int_{x_{i-1}}^{x_{i+1}} \Big|\frac{\partial^3 \widetilde{p}^{n+1/2}(s,y)}{\partial s^3}\Big| ds\Big], \\ \text{for } 1 \le i < N/2, \;\text{and when } \varepsilon > \|v_1\| N^{-1}, \\ \\ C\Big[\varepsilon \displaystyle\int_{x_{i-1}}^{x_{i+1}} \Big|\frac{\partial^3 \widetilde{p}^{n+1/2}(s,y)}{\partial s^3}\Big| ds + h_{x_i}\int_{x_{i-1}}^{x_{i+1}} \Big|\frac{\partial^3 \widetilde{p}^{n+1/2}(s,y)}{\partial s^3}\Big| ds\Big], \\ \text{for } i = N/2, \;\text{and when } \varepsilon > \|v_1\| N^{-1}, \end{cases} \tag{4.35}$$

and

$$\begin{aligned} &\Big|\mathcal{T}^N_{x_i,\widetilde{p}^{n+1/2}}\Big| \\ \le& \Big[C\varepsilon \int_{x_{i-1}}^{x_{i+1}} \Big|\frac{\partial^3 \widetilde{p}^{n+1/2}(s,y)}{\partial s^3}\Big| ds \\ &+ Ch_{x_i}\int_{x_{i-1}}^{x_i} \Big(\Big|\frac{\partial^3 \widetilde{p}^{n+1/2}(s,y)}{\partial s^3}\Big| + \Big|\frac{\partial^2 \widetilde{p}^{n+1/2}(s,y)}{\partial s^2}\Big| + \Big|\frac{\partial \widetilde{p}^{n+1/2}(s,y)}{\partial s}\Big|\Big) ds\Big], \\ &\text{for } 1 \le i \le N/2, \;\text{and when } \varepsilon \le \|v_1\| N^{-1}. \end{aligned} \tag{4.36}$$

(ii) *For $N/2 < i \le N-1$, $\mathcal{T}^N_{x_i,\widetilde{p}^{n+1/2}}$ satisfies that*

$$\left|\mathcal{T}^N_{x_i,\widetilde{p}^{n+1/2}}\right| \le C h_{x_i}\left[\varepsilon\int_{x_{i-1}}^{x_{i+1}}\left|\frac{\partial^4\widetilde{p}^{n+1/2}(s,y)}{\partial s^4}\right|ds + \int_{x_{i-1}}^{x_{i+1}}\left|\frac{\partial^3\widetilde{p}^{n+1/2}(s,y)}{\partial s^3}\right|ds\right]. \tag{4.37}$$

Similarly, $\mathcal{T}^N_{x_i,\widetilde{q}^{n+1/2}}$ satisfies the bounds as stated in (4.35)-(4.37).

Next, we furnish a brief outline to derive the required estimate of the truncation error $\mathcal{T}^N_{x_i,\widetilde{u}^{n+1/2}}$. For detailed derivation, we refer to the proof of [[31], Lemma 10]. At first, we obtain the following estimates of $\mathcal{T}^N_{x_i,\widetilde{p}^{n+1/2}}$ and $\mathcal{T}^N_{x_i,\widetilde{q}^{n+1/2}}$ by applying Lemmas 4.3 and 4.6.

(i) Let $1 \le i < N/2$.

(a) When $\varepsilon > \|v_1\|N^{-1}$, we have

$$|\mathcal{T}^N_{x_i,\widetilde{p}^{n+1/2}}| \le C\Big[H_1^2 + H_1^2\varepsilon^{-2}\exp(-\mathtt{m}_1(1-x_i)/\varepsilon)\Big],$$

$$|\mathcal{T}^N_{x_i,\widetilde{q}^{n+1/2}}| \le C\Big[H_1^2\varepsilon^{-3}\exp(-\mathtt{m}_1(1-x_i)/\varepsilon)\Big].$$

(b) When $\varepsilon \le \|v_1\|N^{-1}$, we have

$$|\mathcal{T}^N_{x_i,\widetilde{p}^{n+1/2}}| \le C\Big[\varepsilon H_1 + H_1^2 + \exp(-\mathtt{m}_1(1-x_{i+1})/\varepsilon)\Big],$$

$$|\mathcal{T}^N_{x_i,\widetilde{q}^{n+1/2}}| \le C\Big[H_1^{-1}\exp(-\mathtt{m}_1(1-x_{i+1})/\varepsilon)\Big].$$

(ii) Let $i = N/2$.

(a) When $\varepsilon > \|v_1\|N^{-1}$, we have

$$|\mathcal{T}^N_{x_i,\widetilde{p}^{n+1/2}}| \le C\Big[\varepsilon H_1 + H_1^2 + \exp(-\mathtt{m}_1(1-x_i)/\varepsilon)\Big],$$

$$|\mathcal{T}^N_{x_i,\widetilde{q}^{n+1/2}}| \le C\Big[\varepsilon^{-1}\exp(-\mathtt{m}_1(1-x_i)/\varepsilon)\Big].$$

(b) When $\varepsilon \le \|v_1\|N^{-1}$, we have

$$\begin{aligned}|\mathcal{T}^N_{x_i,\widetilde{p}^{n+1/2}}| \le & C\Big[\varepsilon H_1 + H_1^2 + \exp(-\mathtt{m}_1(1-x_{i+1})/\varepsilon)\\ & + H_1\varepsilon^{-1}\exp(-\mathtt{m}_1(1-x_i)/\varepsilon)\Big],\\ |\mathcal{T}^N_{x_i,\widetilde{q}^{n+1/2}}| \le & C\Big[\varepsilon^{-1}\exp(-\mathtt{m}_1(1-x_i)/\varepsilon)\Big].\end{aligned}$$

(iii) Let $N/2 < i \le N-1$. We have

$$|\mathcal{T}^N_{x_i,\widetilde{p}^{n+1/2}}| \le C\Big[h_1^2 + h_1^2\varepsilon^{-3}\exp(-\mathtt{m}_1(1-x_i)/\varepsilon)\Big],$$

$$|\mathcal{T}^N_{x_i,\widetilde{q}^{n+1/2}}| \le C\Big[h_1^2\varepsilon^{-3}\exp(-\mathtt{m}_1(1-x_i)/\varepsilon)\Big].$$

Thus, invoking the above estimates of $\mathcal{T}^N_{x_i,\widetilde{p}^{n+1/2}}$ and $\mathcal{T}^N_{x_i,\widetilde{q}^{n+1/2}}$ in the decomposition of $\mathcal{T}^N_{x_i,\widetilde{u}^{n+1/2}}$ given in (4.34), one can deduce that

$$\left|\mathcal{T}^N_{x_i,\widetilde{u}^{n+1/2}}\right| \leq \begin{cases} C\Big[H_1^2 + \varepsilon^{-1}\exp(-\mathtt{m}_1(1-x_i)/\varepsilon)\Big], \\ \text{for } 1 \leq i < N/2, \text{ and when } \varepsilon > \|v_1\|N^{-1}, \\ C\Big[\varepsilon H_1 + H_1^2 + \varepsilon^{-1}\exp(-\mathtt{m}_1(1-x_i)/\varepsilon)\Big], \\ \text{for } i = N/2, \text{ and when } \varepsilon > \|v_1\|N^{-1}, \\ C\Big[\varepsilon H_1 + H_1^2 + H_1^{-1}\exp(-\mathtt{m}_1(1-x_{i+1})/\varepsilon)\Big], \\ \text{for } 1 \leq i < N/2, \text{ and when } \varepsilon \leq \|v_1\|N^{-1}, \\ C\Big[\varepsilon H_1 + H_1^2 + \varepsilon^{-1}\exp(-\mathtt{m}_1(1-x_{i+1})/\varepsilon)\Big], \\ \text{for } i = N/2, \text{ and when } \varepsilon \leq \|v_1\|N^{-1}, \\ C\Big[h_1^2 + h_1^2\varepsilon^{-3}\exp(-\mathtt{m}_1(1-x_i)/\varepsilon)\Big], \quad \text{for } N/2 < i \leq N-1, \end{cases}$$

and henceforth, the estimate of $\mathcal{T}^{N,\Delta t}_{x_i,\widetilde{u}^{n+1/2}}$ follows immediately from (4.33).

We now pursue the error analysis at the first half by considering two parts. We assume that $\lambda_1 < \mathtt{m}_1/2$ and let $y_j \in \overline{\mathtt{G}}^N_y$.

Part-I: When $\varepsilon > \|v_1\|N^{-1}$, for sufficiently large C, we consider the following discrete function:

$$\mathcal{A}_{1,i}(\lambda_1) = C\Big[H_1^2\Big(\mathcal{J}_{x_i} + \mathcal{K}_{\mathbb{K}_1,x_i}\Big) + \Theta_{1,i}(\lambda_1)\Big], \quad \text{for } 0 \leq i \leq N.$$

Here,

$$\mathcal{J}_{x_i} = (1+x_i) \quad \text{and} \quad \mathcal{K}_{\mathbb{K}_1,x_i} = \begin{cases} \dfrac{x_i}{1-\mathbb{K}_1}, & \text{for } 0 \leq i \leq N/2, \\ 1, & \text{for } N/2 \leq i \leq N. \end{cases}$$

It follows that

$$\mathtt{L}^{N,\Delta t}_{1,\varepsilon}\mathcal{K}_{\mathbb{K}_1,x_i} \geq \begin{cases} \dfrac{v^{n+1}_{1,i}}{1-\mathbb{K}_1}, & \text{for } 1 \leq i < N/2, \\ \dfrac{2\varepsilon N + v^{n+1}_{1,i}h_1 N}{2(1-\mathbb{K}_1)}, & \text{for } i = N/2, \\ 0, & \text{for } N/2 < i \leq N-1. \end{cases} \tag{4.38}$$

Then, the inequalities (4.28), (4.38) and Lemma 4.5 imply that

$$\mathtt{L}^{N,\Delta t}_{1,\varepsilon}\mathcal{A}_{1,i}(\lambda_1) \geq |\mathcal{T}^{N,\Delta t}_{x_i,\widetilde{u}^{n+1/2}}|, \quad \text{for } 1 \leq i \leq N-1.$$

Thus, invoking Lemma 4.1 for $x_i \in \overline{\mathtt{G}}^N_x$, and the inequality (4.29), we have

$$\left|\widetilde{U}^{n+1/2}_{i,j} - \widetilde{u}^{n+1/2}(x_i,y_j)\right| \leq C\Big[N^{-2} + N^{-\lambda_1,\mathbb{K}_{1,0}}\Big], \quad \text{for } 0 \leq i \leq N/2. \tag{4.39}$$

When $\varepsilon \leq \|v_1\|N^{-1}$, for sufficiently large C, we consider the following discrete function:

$$\mathcal{B}_{1,i}(\lambda_1) = \begin{cases} C\Big[(\varepsilon H_1 + H_1^2)\mathcal{J}_{x_i} + \Theta_{1,i+1}(\lambda_1)\Big], & \text{for } 0 \leq i \leq N/2, \\ C\Big[(\varepsilon H_1 + H_1^2)\mathcal{J}_{x_i} + \Big(1+\lambda_1 h_1 \varepsilon^{-1}\Big)\Theta_{1,i}(\lambda_1)\Big], & \text{for } N/2 < i \leq N. \end{cases}$$

Since the assumption (4.6) yields $\mathtt{m}_1 h_1/\varepsilon < 2$, the inequality (4.28) and Lemma 4.5 imply that

$$\mathtt{L}_{1,\varepsilon}^{N,\Delta t}\mathcal{B}_{1,i}(\lambda_1) \geq |\mathcal{T}^{N,\Delta t}_{x_i,\widetilde{u}^{n+1/2}}|, \quad \text{for } 1 \leq i \leq N-1,$$

and henceforth, by making use of Lemma 4.1 for $x_i \in \overline{\mathtt{G}}_x^N$, and the inequality (4.29), we have

$$\big|\widetilde{U}_{i,j}^{n+1/2} - \widetilde{u}^{n+1/2}(x_i,y_j)\big| \leq C\Big[(\varepsilon + N^{-1})N^{-1} + N^{-\lambda_1,\mathbb{K}_{1,0}}\Big], \quad \text{for } 0 \leq i \leq N/2. \tag{4.40}$$

Part-II: For sufficiently large C, we consider the following discrete function:

$$\mathcal{C}_{1,i}(\lambda_1) = C\Big[(N^{-2} + N^{-\lambda_1\mathbb{K}_{1,0}})\mathcal{J}_{x_i} + h_1^2\varepsilon^{-2}\Theta_{1,i}(\lambda_1)\Big], \quad \text{for } N/2 \leq i \leq N.$$

Then, arguing similarly for $x_i \in [1-\mathbb{K}_1, 1]\bigcap \overline{\mathtt{G}}_x^N$, we get

$$\big|\widetilde{U}_{i,j}^{n+1/2} - \widetilde{u}^{n+1/2}(x_i,y_j)\big| \leq \mathcal{C}_{1,j}(\lambda_1) \leq C\Big(\mathbb{K}_{1,0}^2 N^{-2}\ln^2 N + N^{-\lambda_1\mathbb{K}_{1,0}}\Big),$$
$$\text{for } N/2+1 \leq i \leq N. \tag{4.41}$$

Therefore, by combining (4.39), (4.40) and (4.41), we obtain the following estimate at $(n+\frac{1}{2})^{th}$ time level.

Lemma 4.7. *Let $y_j \in \overline{\mathtt{G}}_y^N$. If $\lambda_1 < \mathtt{m}_1/2$, the error associated with the discrete problem (4.30) at $(n+1/2)^{th}$ time level satisfies the following estimate:*

$$\big|\widetilde{U}_{i,j}^{n+1/2} - \widetilde{u}^{n+1/2}(x_i,y_j)\big| \leq \begin{cases} C\Big((N^{-1}+\chi_{1,\varepsilon})N^{-1} + N^{-\lambda_1\mathbb{K}_{1,0}}\Big), & \text{for } x_i \in [0, 1-\mathbb{K}_1] \cap \overline{\mathtt{G}}_x^N, \\ C\Big(\mathbb{K}_{1,0}^2 N^{-2}\ln^2 N + N^{-\lambda_1\mathbb{K}_{1,0}}\Big), & \text{for } x_i \in (1-\mathbb{K}_1, 1] \cap \overline{\mathtt{G}}_x^N, \end{cases} \tag{4.42}$$

where

$$\chi_{1,\varepsilon} = \begin{cases} \varepsilon, & \text{when } \varepsilon \leq \|v_1\|N^{-1}, \\ 0, & \text{when } \varepsilon > \|v_1\|N^{-1}. \end{cases}$$

Next, we proceeds to estimate the error $\big|\widetilde{U}_{i,j}^{n+1} - \widetilde{u}^{n+1}(x_i,y_j)\big|$. Here, for the discrete problem (4.30), the local truncation error at the second half is defined as

$$\mathcal{T}^{N,\Delta t}_{y_j,\widetilde{u}^{n+1}} = \mathtt{L}_{2,\varepsilon}^{N,\Delta t}[\widetilde{U}_{i,j}^{n+1} - \widetilde{u}^{n+1}(x_i,y_j)]$$

$$
= \begin{cases} \widetilde{U}_{i,j}^{n+1/2} - \widetilde{u}^{n+1/2}(x_i, y_j) + \Delta t\Big[\mathtt{L}_{2,\varepsilon}^{n+1}\widetilde{u}^{n+1}(x_i, y_j) - \mathtt{L}_{2,N,mcd}^{n+1}\widetilde{u}^{n+1}(x_i, y_j)\Big], \\ \text{for } 1 \le j \le N/2, \text{ when } \varepsilon > \|v_2\|N^{-1}, \\ \frac{1}{2}\Big(\widetilde{U}_{i,j}^{n+1/2} - \widetilde{u}^{n+1/2}(x_i, y_j)\Big) + \frac{1}{2}\Big(\widetilde{U}_{i,j-1}^{n+1/2} - \widetilde{u}^{n+1/2}(x_i, y_{j-1})\Big) \\ +\Delta t\Big[(\mathtt{L}_{2,\varepsilon}^{n+1}\widetilde{u}^{n+1})_{i,j-1/2} - \mathtt{L}_{2,N,mup}^{n+1}\widetilde{u}^{n+1}(x_i, y_j)\Big], \\ \text{for } 1 \le j \le N/2, \text{ when } \varepsilon \le \|v_2\|N^{-1}, \\ \widetilde{U}_{i,j}^{n+1/2} - \widetilde{u}^{n+1/2}(x_i, y_j) + \Delta t\Big[\mathtt{L}_{2,\varepsilon}^{n+1}\widetilde{u}^{n+1}(x_i, y_j) - \mathtt{L}_{2,N,mcd}^{n+1}\widetilde{u}^{n+1}(x_i, y_j)\Big], \\ \text{for } N/2 < j \le N-1, \end{cases}
$$

$$
= \Delta t\, \mathcal{T}_{y_j,\widetilde{u}^{n+1}}^{N} + \begin{cases} \widetilde{U}_{i,j}^{n+1/2} - \widetilde{u}^{n+1/2}(x_i, y_j), \\ \text{for } 1 \le j \le N/2, \text{ when } \varepsilon > \|v_2\|N^{-1}, \\ \frac{1}{2}\Big(\widetilde{U}_{i,j}^{n+1/2} - \widetilde{u}^{n+1/2}(x_i, y_j)\Big) \\ +\frac{1}{2}\Big(\widetilde{U}_{i,j-1}^{n+1/2} - \widetilde{u}^{n+1/2}(x_i, y_{j-1})\Big), \\ \text{for } 1 \le j \le N/2, \text{ when } \varepsilon \le \|v_2\|N^{-1}, \\ \widetilde{U}_{i,j}^{n+1/2} - \widetilde{u}^{n+1/2}(x_i, y_j), \quad \text{for } N/2 < j \le N-1. \end{cases} \tag{4.43}
$$

Here,

$$
\mathcal{T}_{y_j,\widetilde{u}^{n+1}}^{N} = \begin{cases} (\mathtt{L}_{2,\varepsilon}^{n+1}\widetilde{u}^{n+1})(x_i, y_j) - \mathtt{L}_{2,N,mcd}^{n+1}\widetilde{u}^{n+1}(x_i, y_j) \\ \text{for } 1 \le j \le N/2, \text{ and when } \varepsilon > \|v_2\|N^{-1}, \\ (\mathtt{L}_{2,\varepsilon}^{n+1}\widetilde{u}^{n+1})_{i,j-1/2} - \mathtt{L}_{2,N,mup}^{n+1}\widetilde{u}^{n+1}(x_i, y_j), \\ \text{for } 1 \le j \le N/2, \text{ and when } \varepsilon \le \|v_2\|N^{-1}, \\ (\mathtt{L}_{2,\varepsilon}^{n+1}\widetilde{u}^{n+1})(x_i, y_j) - \mathtt{L}_{2,N,mcd}^{n+1}\widetilde{u}^{n+1}(x_i, y_j), \quad \text{for } N/2 < j \le N-1. \end{cases}
$$

Therefore, $\mathcal{T}_{y_j,\widetilde{u}}^{N}$ denotes the truncation error corresponding to the stationary singularly perturbed problem, and and is obtained by approximating the differential operator $\mathtt{L}_{2,\varepsilon}$ with respect to the spatial variable-y using the proposed finite difference scheme. Since $\widetilde{u}^{n+1}$ can be decomposed into $\widetilde{p}^{n+1}$ and $\widetilde{q}^{n+1}$ as mentioned in Lemma 4.3, we rewrite $\mathcal{T}_{x_i,\widetilde{u}^{n+1}}^{N}$, as

$$
\mathcal{T}_{y_j,\widetilde{u}^{n+1}}^{N} = \mathcal{T}_{y_j,\widetilde{p}^{n+1}}^{N} + \gamma_2 \mathcal{T}_{y_j,\widetilde{q}^{n+1}}^{N}.
$$

Now, for fixed $x \in (0,1)$, by making use of the Taylor's theorem on the functions $\widetilde{p}^{n+1}(x,y)$, $\widetilde{q}^{n+1}(x,y)$; and considering the remainder term in the integral form, we obtain the following result.

Lemma 4.8. (i) For $1 \le j \le N/2$, $\mathcal{T}^N_{y_j,\widetilde{p}^{n+1}}$ satisfies that

$$\left|\mathcal{T}^N_{y_j,\widetilde{p}^{n+1}}\right| \le \begin{cases} Ch_{y_j}\left[\varepsilon\int_{y_{j-1}}^{y_{j+1}}\left|\dfrac{\partial^4\widetilde{p}^{n+1}(x,\tilde{s})}{\partial\tilde{s}^4}\right|d\tilde{s} + \int_{y_{j-1}}^{y_{j+1}}\left|\dfrac{\partial^3\widetilde{p}^{n+1}(x,\tilde{s})}{\partial\tilde{s}^3}\right|d\tilde{s}\right], \\ \text{for } 1 \le j < N/2, \text{ and when } \varepsilon > \|v_2\|N^{-1}, \\ C\left[\varepsilon\int_{y_{j-1}}^{y_{j+1}}\left|\dfrac{\partial^3\widetilde{p}^{n+1}(x,\tilde{s})}{\partial\tilde{s}^3}\right|d\tilde{s} + h_{y_j}\int_{y_{j-1}}^{y_{j+1}}\left|\dfrac{\partial^3\widetilde{p}^{n+1}(x,\tilde{s})}{\partial\tilde{s}^3}\right|d\tilde{s}\right], \\ \text{for } j = N/2, \text{ and when } \varepsilon > \|v_2\|N^{-1}, \end{cases} \tag{4.44}$$

and

$$\begin{aligned}&\left|\mathcal{T}^N_{y_j,\widetilde{p}^{n+1}}\right| \\ \le&\Big[C\varepsilon\int_{y_{j-1}}^{y_{j+1}}\left|\frac{\partial^3\widetilde{p}^{n+1}(x,\tilde{s})}{\partial\tilde{s}^3}\right|d\tilde{s} \\ &+ Ch_{y_j}\int_{y_{j-1}}^{y_j}\Big(\left|\frac{\partial^3\widetilde{p}^{n+1}(x,\tilde{s})}{\partial\tilde{s}^3}\right| + \left|\frac{\partial^2\widetilde{p}^{n+1}(x,\tilde{s})}{\partial\tilde{s}^2}\right| + \left|\frac{\partial\widetilde{p}^{n+1}(x,\tilde{s})}{\partial\tilde{s}}\right|\Big)d\tilde{s}\Big], \\ &\text{for } 1 \le j \le N/2, \text{ and when } \varepsilon \le \|v_2\|N^{-1}. \end{aligned} \tag{4.45}$$

(ii) For $N/2 < j \le N-1$, $\mathcal{T}^N_{y_j,\widetilde{p}^{n+1}}$ satisfies that

$$\left|\mathcal{T}^N_{y_j,\widetilde{p}^{n+1}}\right| \le Ch_{y_j}\left[\varepsilon\int_{y_{j-1}}^{y_{j+1}}\left|\frac{\partial^4\widetilde{p}^{n+1}(x,\tilde{s})}{\partial\tilde{s}^4}\right|d\tilde{s} + \int_{y_{j-1}}^{y_{j+1}}\left|\frac{\partial^3\widetilde{p}^{n+1}(x,\tilde{s})}{\partial\tilde{s}^3}\right|d\tilde{s}\right]. \tag{4.46}$$

Similarly, $\mathcal{T}^N_{y_j,\widetilde{q}^{n+1}}$ satisfies the bounds as stated in (4.44)-(4.46).

Afterwards, by making use of Lemmas 4.3 and 4.8; and arguing analogously to the $(n+1)^{th}$ time level, we deduce that

$$|\mathcal{T}^N_{y_j,\widetilde{u}^{n+1}}| \le \begin{cases} C\left[H_2^2 + \varepsilon^{-1}\exp(-\mathtt{m}_2(1-y_j)/\varepsilon)\right], \\ \text{for } 1 \le j < N/2, \text{ and when } \varepsilon > \|v_2\|N^{-1}, \\ C\left[\varepsilon H_2 + H_2^2 + \varepsilon^{-1}\exp(-\mathtt{m}_2(1-y_j)/\varepsilon)\right], \\ \text{for } j = N/2, \text{ and when } \varepsilon > \|v_2\|N^{-1}, \\ C\left[\varepsilon H_2 + H_2^2 + H_2^{-1}\exp(-\mathtt{m}_2(1-y_{j+1})/\varepsilon)\right], \\ \text{for } 1 \le j < N/2, \text{ and when } \varepsilon \le \|v_2\|N^{-1}, \\ C\left[\varepsilon H_2 + H_2^2 + \varepsilon^{-1}\exp(-\mathtt{m}_2(1-y_{j+1})/\varepsilon)\right], \\ \text{for } j = N/2, \text{ and when } \varepsilon \le \|v_2\|N^{-1}, \\ C\left[h_2^2 + h_2^2\varepsilon^{-3}\exp(-\mathtt{m}_2(1-y_j)/\varepsilon)\right], \quad \text{for } N/2 < j \le N-1. \end{cases}$$

Now, invoking the above estimate of $\mathcal{T}^{N}_{y_j,\widetilde{u}^{n+1}}$ in (4.43), we get the following bounds of the local truncation error for the $(n+1)^{th}$ time level. For $1 \le j < N/2$ and when $\varepsilon > \|v_2\|N^{-1}$, we have

$$\left|\mathcal{T}^{N,\Delta t}_{y_j,\widetilde{u}^{n+1}}\right| \le \Delta t \left|\mathcal{T}^{N}_{y_j,\widetilde{u}^{n+1}}\right| + \begin{cases} C\Big((N^{-1}+\chi_{1,\varepsilon})N^{-1} + N^{-\lambda_1 \mathbb{K}_{1,0}}\Big), & \text{for } 1 \le i \le N/2, \\ C\Big(\mathbb{K}_{1,0}^2 N^{-2}\ln^2 N + N^{-\lambda_1 \mathbb{K}_{1,0}}\Big), & \text{for } N/2 < i < N, \end{cases}$$

$$\le \begin{cases} C\Big((N^{-1}+\chi_{1,\varepsilon})N^{-1} + N^{-\lambda_1 \mathbb{K}_{1,0}}\Big) \\ +C\Delta t\Big[H_2^2 + \varepsilon^{-1}\exp(-\mathtt{m}_2(1-y_j)/\varepsilon)\Big], & \text{for } 1 \le i \le N/2, \\ C\Big(\mathbb{K}_{1,0}^2 N^{-2}\ln^2 N + N^{-\lambda_1 \mathbb{K}_{1,0}}\Big) \\ +C\Delta t\Big[H_2^2 + \varepsilon^{-1}\exp(-\mathtt{m}_2(1-y_j)/\varepsilon)\Big], & \text{for } N/2 < i < N. \end{cases}$$

Similarly, for $j = N/2$ and when $\varepsilon > \|v_2\|N^{-1}$,

$$\left|\mathcal{T}^{N,\Delta t}_{y_j,\widetilde{u}^{n+1}}\right| \le \begin{cases} C\Big((N^{-1}+\chi_{1,\varepsilon})N^{-1} + N^{-\lambda_1 \mathbb{K}_{1,0}}\Big) \\ +C\Delta t\Big[\varepsilon H_2 + H_2^2 + \varepsilon^{-1}\exp(-\mathtt{m}_2(1-y_{j+1})/\varepsilon)\Big], & \text{for } 1 \le i \le N/2, \\ C\Big(\mathbb{K}_{1,0}^2 N^{-2}\ln^2 N + N^{-\lambda_1 \mathbb{K}_{1,0}}\Big) \\ +C\Delta t\Big[\varepsilon H_2 + H_2^2 + \varepsilon^{-1}\exp(-\mathtt{m}_2(1-y_{j+1})/\varepsilon)\Big], & \text{for } N/2 < i < N. \end{cases}$$

Next, for $1 \le j < N/2$ and when $\varepsilon \le \|v_2\|N^{-1}$, we have

$$|\mathcal{T}^{N,\Delta t}_{y_j,\widetilde{u}^{n+1}}| \le \Delta t \left|\mathcal{T}^{N}_{y_j,\widetilde{u}^{n+1}}\right| + \begin{cases} C\Big((N^{-1}+\chi_{1,\varepsilon})N^{-1} + N^{-\lambda_1 \mathbb{K}_{1,0}}\Big), & \text{for } 1 \le i \le N/2, \\ C\Big(\mathbb{K}_{1,0}^2 N^{-2}\ln^2 N + N^{-\lambda_1 \mathbb{K}_{1,0}}\Big), & \text{for } N/2 < i < N, \end{cases}$$

$$\le \begin{cases} C\Big((N^{-1}+\chi_{1,\varepsilon})N^{-1} + N^{-\lambda_1 \mathbb{K}_{1,0}}\Big) \\ +C\Delta t\Big[\varepsilon H_2 + H_2^2 + H_2^{-1}\exp(-\mathtt{m}_2(1-y_{j+1})/\varepsilon)\Big], & \text{for } 1 \le i \le N/2, \\ C\Big(\mathbb{K}_{1,0}^2 N^{-2}\ln^2 N + N^{-\lambda_1 \mathbb{K}_{1,0}}\Big) \\ +C\Delta t\Big[\varepsilon H_2 + H_2^2 + H_2^{-1}\exp(-\mathtt{m}_2(1-y_{j+1})/\varepsilon)\Big], & \text{for } N/2 < i < N. \end{cases}$$

Similarly, for $j = N/2$ and when $\varepsilon \le \|v_2\|N^{-1}$,

$$|\mathcal{T}^{N,\Delta t}_{y_j,\widetilde{u}^{n+1}}| \le \begin{cases} C\Big((N^{-1}+\chi_{1,\varepsilon})N^{-1} + N^{-\lambda_1\mathbb{K}_{1,0}}\Big) \\ +C\Delta t\Big[\varepsilon H_2 + H_2^2 + \varepsilon^{-1}\exp(-\mathtt{m}_2(1-y_{j+1})/\varepsilon)\Big], \ \text{for } 1 \le i \le N/2, \\ C\Big(\mathbb{K}_{1,0}^2 N^{-2}\ln^2 N + N^{-\lambda_1\mathbb{K}_{1,0}}\Big) \\ +C\Delta t\Big[\varepsilon H_2 + H_2^2 + \varepsilon^{-1}\exp(-\mathtt{m}_2(1-y_{j+1})/\varepsilon)\Big], \ \text{for } N/2 < i < N. \end{cases}$$

Finally, for $N/2 < j < N$, we have

$$|\mathcal{T}^{N,\Delta t}_{y_j,\widetilde{u}^{n+1}}| \le \Delta t\Big|\mathcal{T}^{N}_{y_j,\widetilde{u}^{n+1}}\Big| + \begin{cases} C\Big((N^{-1}+\chi_{1,\varepsilon})N^{-1} + N^{-\lambda_1\mathbb{K}_{1,0}}\Big), \text{for } 1 \le i \le N/2, \\ C\Big(\mathbb{K}_{1,0}^2 N^{-2}\ln^2 N + N^{-\lambda_1\mathbb{K}_{1,0}}\Big), \quad \text{for } N/2 < i < N, \end{cases}$$

$$\le \begin{cases} C\Big((N^{-1}+\chi_{1,\varepsilon})N^{-1} + N^{-\lambda_1\mathbb{K}_{1,0}}\Big) \\ +C\Delta t\Big[h_2^2 + h_2^2\varepsilon^{-3}\exp(-\mathtt{m}_2(1-y_j)/\varepsilon)\Big], \ \text{for } 1 \le i \le N/2, \\ C\Big(\mathbb{K}_{1,0}^2 N^{-2}\ln^2 N + N^{-\lambda_1\mathbb{K}_{1,0}}\Big) \\ +C\Delta t\Big[h_2^2 + h_2^2\varepsilon^{-3}\exp(-\mathtt{m}_2(1-y_j)/\varepsilon)\Big], \ \text{for } N/2 < i < N. \end{cases}$$

In the following, we consider two parts: $x_i \in [0, 1-\mathbb{K}_1] \cap \overline{\mathtt{G}}_x^N$ and $x_i \in (1-\mathbb{K}_1, 1] \cap \overline{\mathtt{G}}_x^N$.

Part-I: Here, we consider two subparts.

(a) When $\varepsilon > \|v_2\|N^{-1}$, for sufficiently large C, we consider the following discrete function

$$\begin{aligned} &\mathcal{A}_{2,j}(\lambda_2) \\ =&C\Big[((N^{-1}+\chi_{1,\varepsilon})N^{-1} + N^{-\lambda_1\mathbb{K}_{1,0}})\mathcal{J}_{y_j} + H_2^2\mathcal{K}_{\mathbb{K}_2,y_j} + \Theta_{2,j}(\lambda_2)\Big], \quad \text{for } 0 \le j \le N. \end{aligned}$$

Then, Lemma 4.5, and the inequality (4.28) yield that

$$\mathtt{L}^{N,\Delta t}_{2,\varepsilon}\mathcal{A}_{2,j}(\lambda_2) \ge |\mathcal{T}^{N,\Delta t}_{y_j,\widetilde{u}^{n+1}}|, \quad \text{for } 1 \le j \le N-1.$$

Thus, invoking Lemma 4.1 for $y_j \in \overline{\mathtt{G}}_y^N$, and the inequality (4.29), we have

$$\begin{aligned} &\big|\widetilde{U}^{n+1}_{i,j} - \widetilde{u}^{n+1}(x_i,y_j)\big| \\ \le &C\Big[(N^{-1}+\chi_{1,\varepsilon})N^{-1} + N^{-2} + N^{-\lambda_1\mathbb{K}_{1,0}} + N^{-\lambda_2\mathbb{K}_{2,0}}\Big], \quad \text{for } 0 \le j \le N/2. \end{aligned}$$

When $\varepsilon \le \|v_2\|N^{-1}$, for sufficiently large C, we consider the following discrete function

$$\mathcal{B}_{2,j}(\lambda_2) = \begin{cases} C\Big[((N^{-1}+\chi_{1,\varepsilon})N^{-1}+N^{-\lambda_1\mathbb{K}_{1,0}})\mathcal{J}_{y_j} \\ +(\varepsilon H_2+H_2^2)\mathcal{J}_{y_j}+\Theta_{2,j+1}(\lambda_2)\Big], \quad \text{for } 0\le j\le N/2, \\ C\Big[((N^{-1}+\chi_{1,\varepsilon})N^{-1}+N^{-\lambda_1\mathbb{K}_{1,0}})\mathcal{J}_{y_j} \\ +(\varepsilon H_2+H_2^2)\mathcal{J}_{y_j}+\Big(1+\lambda_2 h_2\varepsilon^{-1}\Big)\Theta_{2,j}(\lambda_2)\Big], \quad \text{for } N/2\le j\le N. \end{cases}$$

Again, by making use of Lemma 4.1 for $y_j \in \overline{\mathtt{G}}_y^N$, the inequalities (4.29) and $\mathtt{m}_2 h_2/\varepsilon < 2$, we deduce for $0 \le j \le N/2$ that

$$\begin{aligned} &\big|\widetilde{U}_{i,j}^{n+1}-\widetilde{u}^{n+1}(x_i,y_j)\big| \\ \le & C\Big[(N^{-1}+\chi_{1,\varepsilon})N^{-1}+\varepsilon H_2+H_2^2+N^{-\lambda_1\mathbb{K}_{1,0}}+\Theta_{2,N/2}(\lambda_2)\Big] \\ \le & C\Big((N^{-1}+\chi_{1,\varepsilon})N^{-1}+(\varepsilon+N^{-1})N^{-1}+N^{-\lambda_1\mathbb{K}_{1,0}}+N^{-\lambda_2\mathbb{K}_{2,0}}\Big), \end{aligned}$$

(b) For sufficiently large C, we consider the following discrete function

$$\begin{aligned} \mathcal{C}_{2,j}(\lambda_2) =& C\Big[((N^{-1}+\chi_{1,\varepsilon})N^{-1}+N^{-\lambda_1\mathbb{K}_{1,0}}+N^{-\lambda_1\mathbb{K}_{2,0}})\mathcal{J}_{y_j}+h_2^2\varepsilon^{-2}\Theta_{2,j}(\lambda_2)\Big], \\ & N/2\le j\le N, \end{aligned}$$

and arguing similarly for $y_j \in [1-\mathbb{K}_2, 1]\bigcap \overline{\mathtt{G}}_y^N$, we have

$$\begin{aligned} &\big|\widetilde{U}_{i,j}^{n+1}-\widetilde{u}^{n+1}(x_i,y_j)\big| \\ \le & C\Big[((N^{-1}+\chi_{1,\varepsilon})N^{-1}+\mathbb{K}_{2,0}^2N^{-2}\ln^2 N+N^{-\lambda_1\mathbb{K}_{1,0}}+N^{-\lambda_2\mathbb{K}_{2,0}})\Big], \\ & N/2+1\le j\le N. \end{aligned}$$

Part-II: As like the previous part, one can suitably choose $\mathcal{A}_{2,j}(\lambda_2)$, for $0 \le j \le N$, and when $\varepsilon > \|v_2\|N^{-1}$, $\mathcal{B}_{2,j}(\lambda_2)$, for $0 \le j \le N$, and when $\varepsilon \le \|v_2\|N^{-1}$, $\mathcal{C}_{2,j}(\lambda_2)$, for $N/2 \le j \le N$, and arguing similar way as in the previous case, one can obtain the desired result. Therefore, from the above derivations, we deduce the following estimate at $(n+1)^{th}$ time level.

Lemma 4.9. *If $\lambda_l < \mathtt{m}_l/2,\ l = 1, 2$, the error associated with the discrete problem (4.30) at $(n+1)^{th}$ time level satisfies the following estimate:*

$$\big|\widetilde{U}_{i,j}^{n+1}-\widetilde{u}^{n+1}(x_i,y_j)\big|$$

$$
\leq \begin{cases} C\Big((N^{-1}+\chi_{1,\varepsilon})N^{-1}+(N^{-1}+\chi_{2,\varepsilon})N^{-1}+N^{-\lambda_1\mathbb{K}_{1,0}}+N^{-\lambda_2\mathbb{K}_{2,0}}\Big), \\ \text{for } (x_i,y_j)\in\Big([0,1-\mathbb{K}_1]\cap\overline{\mathtt{G}}_x^N\Big)\times\Big([0,1-\mathbb{K}_2]\cap\overline{\mathtt{G}}_y^N\Big), \\ C\Big((N^{-1}+\chi_{1,\varepsilon})N^{-1}+\mathbb{K}_{2,0}^2N^{-2}\ln^2 N+N^{-\lambda_1\mathbb{K}_{1,0}}+N^{-\lambda_2\mathbb{K}_{2,0}}\Big), \\ \text{for } (x_i,y_j)\in\Big([0,1-\mathbb{K}_1]\cap\overline{\mathtt{G}}_x^N\Big)\times\Big((1-\mathbb{K}_2,1]\cap\overline{\mathtt{G}}_y^N\Big), \\ C\Big((N^{-1}+\chi_{2,\varepsilon})N^{-1}+\mathbb{K}_{1,0}^2N^{-2}\ln^2 N+N^{-\lambda_1\mathbb{K}_{1,0}}+N^{-\lambda_2\mathbb{K}_{2,0}}\Big), \\ \text{for } (x_i,y_j)\in\Big((1-\mathbb{K}_1,1]\cap\overline{\mathtt{G}}_x^N\Big)\times\Big([0,1-\mathbb{K}_2]\cap\overline{\mathtt{G}}_y^N\Big), \\ C\Big((\mathbb{K}_{1,0}^2+\mathbb{K}_{2,0}^2)N^{-2}\ln^2 N+N^{-\lambda_1\mathbb{K}_{1,0}}+N^{-\lambda_2\mathbb{K}_{2,0}}\Big), \\ \text{for } (x_i,y_j)\in\Big((1-\mathbb{K}_1,1]\cap\overline{\mathtt{G}}_x^N\Big)\times\Big((1-\mathbb{K}_2,1]\cap\overline{\mathtt{G}}_y^N\Big), \end{cases}
$$

where

$$
\chi_{1,\varepsilon}=\begin{cases}\varepsilon, & \text{when } \varepsilon\leq\|v_1\|N^{-1}\\ 0, & \text{when } \varepsilon>\|v_1\|N^{-1},\end{cases} \quad\text{and}\quad \chi_{2,\varepsilon}=\begin{cases}\varepsilon, & \text{when } \varepsilon\leq\|v_2\|N^{-1}\\ 0, & \text{when } \varepsilon>\|v_2\|N^{-1}.\end{cases}
$$

Corollary 4.1. *Lemma 4.9 implies that for fixed* $\mathbb{K}_{l,0}\geq 2/\lambda_l,\ l=1,2,$ *and for some constant* C

$$
\big|\widetilde{U}_{i,j}^{n+1}-\widetilde{u}^{n+1}(x_i,y_j)\big|\leq\begin{cases} C(N^{-1}+\chi_{1,\varepsilon})N^{-1}+C(N^{-1}+\chi_{2,\varepsilon})N^{-1}, \\ \text{for } (x_i,y_j)\in\Big([0,1-\mathbb{K}_1]\times[0,1-\mathbb{K}_2]\Big)\cap\overline{\mathtt{G}}^N, \\ CN^{-2}\ln^2 N, \quad \text{otherwise}.\end{cases} \tag{4.47}
$$

4.2.2. Uniform convergence of the fully discrete scheme

We define $E^{n+1}(x_i,y_j)=[U_{i,j}^{n+1}-u(x_i,y_j,t_{n+1})],\quad \text{for } (x_i,y_j)\in\overline{\mathtt{G}}^N$, as the global error related to the fully discrete scheme (4.1) at the time level t_{n+1}. Now, to show the ε-uniform convergence of the fully discrete scheme (4.1), we rewrite the global error in the following form:

$$
E^{n+1}(x_i,y_j)=\widetilde{e}^{n+1}(x_i,y_j)+\widetilde{E}^{n+1}(x_i,y_j)+[U_{i,j}^{n+1}-\widetilde{U}_{i,j}^{n+1}]. \tag{4.48}
$$

Here, $\widetilde{e}^{n+1}(x_i,y_j)=[\widetilde{u}^{n+1}(x_i,y_j)-u(x_i,y_j,t_{n+1})]$ and $\widetilde{E}^{n+1}(x_i,y_j)=[\widetilde{U}_{i,j}^{n+1}-\widetilde{u}^{n+1}(x_i,y_j)]$, respectively, denote the local error related to the time semidiscrete scheme and the spatial discretization of the auxiliary problem (3.4) at the time level t_{n+1}. The term $[U_{i,j}^{n+1}-\widetilde{U}_{i,j}^{n+1}]$ can be written as the solution of the following systems:

$$
\begin{cases}\mathtt{L}_{1,\varepsilon}^{N,\Delta t}\mathtt{L}_{2,\varepsilon}^{N,\Delta t}R^{n+1}(x_i,y_j)=U_{i,j}^n-u(x_i,y_j,t_n)+O(\Delta t)^2, \quad (x_i,y_j)\in\mathtt{G}^N,\\ R^{n+1}(x_i,y_j)=0, \quad (x_i,y_j)\in\partial\mathtt{G}^N,\end{cases}
$$

where $R^{n+1}(x_i,y_j) = [U_{i,j}^{n+1} - \widetilde{U}_{i,j}^{n+1}]$, and by employing the discrete maximum principle in Lemma 4.1, we obtain that

$$\left\|\left\{R^{n+1}(x_i,y_j)\right\}_{i,j}\right\| \le \left\|\left\{E^{n}(x_i,y_j)\right\}_{i,j}\right\| + C(\Delta t)^2. \tag{4.49}$$

Afterwards, from (4.48) and (4.49), we obtain that

$$\begin{aligned}\left\|\left\{E^{n+1}(x_i,y_j)\right\}_{i,j}\right\| \le& \left\|\left\{\widetilde{e}^{n+1}(x_i,y_j)\right\}_{i,j}\right\| + \left\|\left\{\widetilde{E}^{n+1}(x_i,y_j)\right\}_{i,j}\right\| \\ &+ \left\|\left\{E^{n}(x_i,y_j)\right\}_{i,j}\right\| + C(\Delta t)^2, \text{for } (x_i,y_j)\in\overline{\mathtt{G}}^N.\end{aligned} \tag{4.50}$$

Now, by invoking the estimate obtained in Lemma 3.3 and the estimate (4.47) in (4.50), with the assumption that $N^{-\theta} \le C\Delta t$, $0<\theta<1$, we obtain the following estimate of the global error.

Theorem 4.1 (Global error)**.** *Assume that the conditions given in (4.6) hold for $N \ge N_0$. Then, if $\lambda_l < \mathtt{m}_l/2$, $\mathbb{K}_{l,0} \ge 2/\lambda_l$, $l=1,2$, the global error associated with the fully discrete scheme (4.1) at time level t_{n+1}, satisfies the following estimate:*

$$\begin{aligned}&\left\|\left\{U_{i,j}^{n+1}\right\}_{i,j} - \left\{u(x_i,y_j,t_{n+1})\right\}_{i,j}\right\| \\ &\le \begin{cases} C\Big(N^{-2+\theta} + \chi_{1,\varepsilon}N^{-1+\theta} + \chi_{2,\varepsilon}N^{-1+\theta} + \Delta t\Big), \\ \text{for } (x_i,y_j)\in\Big([0,1-\mathbb{K}_1]\times[0,1-\mathbb{K}_2]\Big)\cap\overline{\mathtt{G}}^N, \\ C\Big(N^{-2+\theta}\ln^2 N + \Delta t\Big), \quad \text{for otherwise,}\end{cases}\end{aligned} \tag{4.51}$$

where N and Δt are such that $N^{-\theta} \le C\Delta t$ with $0<\theta<1$.

Remark 4.1. The error estimate (4.51) implies that the global error takes the form

$$U_{i,j}^{n+1} - u(x_i,y_j,t_{n+1}) = \begin{cases} O(N^{-2+\theta}) + O(\Delta t), \\ \text{for } (x_i,y_j)\in\Big([0,1-\mathbb{K}_1]\times[0,1-\mathbb{K}_2]\Big)\cap\overline{\mathtt{G}}^N, \\ O(N^{-2+\theta}\ln^2 N) + O(\Delta t), \quad \text{for otherwise,}\end{cases} \tag{4.52}$$

not only for $\varepsilon \le \|v_l\|N^{-1}$ but also for $\varepsilon > \|v_l\|N^{-1}$, $l=1,2$. Note that the temporal accuracy in (4.52) holds under the alternative boundary data given in (3.3).

Remark 4.2. It is important to note that two-step discretization analysis followed in [6] can not be extended to achieve higher-order spatial accuracy using the proposed finite difference scheme, and consequently, the theoretical restriction $N^{-\theta} \le C\Delta t$, $0<\theta<1$ persists in Theorem 4.1 according to the present error analysis. However, it is worth mentioning that the same theoretical restriction does not appear in the numerical results of the proposed numerical method.

5. Error analysis for temporal Richardson extrapolation

In this section, we analyze the Richardson extrapolation in the time variable in order to improve the order of uniform convergence in the temporal direction established in Theorem 3.1 so that the proposed algorithm can produce higher-order accurate numerical solution at low computational cost. On the domain $[0, T]$, we construct a fine mesh, denoted by $\Lambda^{\Delta t/2} = \{\widetilde{t}_n\}_{n=0}^{2M}$, by bisecting each mesh interval of $\Lambda^{\Delta t}$. So, $\widetilde{t}_{n+1} - \widetilde{t}_n = T/2M = \Delta t/2$ is the step-size. Let $U^{N,\Delta t}(x_i, y_j, t_{n+1})$ and $U^{N,\Delta t/2}(x_i, y_j, \widetilde{t}_{n+1})$ be the respective solutions of the fully discrete problem (4.1) on the mesh $\overline{\mathtt{G}}^N \times \Lambda^{\Delta t}$ and $\overline{\mathtt{G}}^N \times \Lambda^{\Delta t/2}$. Then, from (4.52), on $\overline{\mathtt{G}}^N \times \Lambda^{\Delta t}$ we have

$$\begin{cases} \quad \Big(2U^{N,\Delta t/2}(x_i, y_j, t_{n+1}) - U^{N,\Delta t}(x_i, y_j, t_{n+1})\Big) - u(x_i, y_j, t_{n+1}) \\ = o(\Delta t) + O(N^{-2+\theta}) \\ = O(\Delta t)^k + O(N^{-2+\theta}), \quad (x_i, y_j) \in \Big([0, 1 - \mathbb{K}_1] \times [0, 1 - \mathbb{K}_2]\Big) \cap \overline{\mathtt{G}}^N, \\ \quad \Big(2U^{N,\Delta t/2}(x_i, y_j, t_{n+1}) - U^{N,\Delta t}(x_i, y_j, t_{n+1})\Big) - u(x_i, y_j, t_{n+1}) \\ = o(\Delta t) + O(N^{-2+\theta} \ln^2 N) \\ = O(\Delta t)^k + O(N^{-2+\theta} \ln^2 N), \quad \text{otherwise}, \quad \text{for some } k > 1. \end{cases} \tag{5.1}$$

Remark 5.1. We set $\Big(2U^{N,\Delta t/2}(x_i, y_j, t_{n+1}) - U^{N,\Delta t}(x_i, y_j, t_{n+1})\Big)$ as the extrapolation formula so that the time accuracy can be improved from $O(\Delta t)$ to $O(\Delta t)^2$, which requires a proof to establish. However, it is clear from (5.1) that the spatial accuracy remains unchanged due to the Richardson extrapolation only in time variable.

Now, let $u^{\Delta t}(x, y, t_{n+1})$ and $u^{\Delta t/2}(x, y, \widetilde{t}_{n+1})$ be the respective solutions of the time-semidiscrete problem (3.1) on the mesh $\overline{\mathtt{G}} \times \wedge^{\Delta t}$ and $\overline{\mathtt{G}} \times \wedge^{\Delta t/2}$, such that $u^{\Delta t}(x_i, y_j, t_{n+1}) \approx U^{N,\Delta t}(x_i, y_j, t_{n+1})$ and $u^{\Delta t/2}(x_i, y_j, \widetilde{t}_{n+1}) \approx U^{N,\Delta t/2}(x_i, y_j, \widetilde{t}_{n+1})$, $\quad (x_i, y_j) \in \overline{\mathtt{G}}^N$. From Theorem 3.1, we have

$$u(x_i, y_j, t_{n+1}) - \Big(2u^{\Delta t}(x_i, y_j, t_{n+1}) - u^{\Delta t}(x_i, y_j, t_{n+1})\Big)$$

$$= O(\Delta t)^k,\ k > 1,\ (x_i, y_j, t_{n+1}) \in \overline{\mathtt{G}}^N \times \Lambda^{\Delta t}, \tag{5.2}$$

which shows that the term $O(\Delta t)^k$ appeared in (5.1) is due to the time-semidiscrete approximation. Hence, we define the temporal extrapolation formula associated with the time-semidiscrete problem 3.1 by

$$u^{\Delta t}_{extp}(x, y, t_{n+1}) = \Big(2u^{\Delta t/2}(x, y, t_{n+1}) - u^{\Delta t}(x, y, t_{n+1})\Big), \quad (x, y, t_{n+1}) \in \overline{\mathtt{G}} \times \wedge^{\Delta t}, \tag{5.3}$$

Thus, to determine the exact value of k for $O(\Delta t)^k$ in (5.2), we focus on analyzing the error $\big(u - u^{\Delta t}_{extp}\big)$ corresponding to the semidiscrete problem 3.1. Utilizing the global error in Theorem 3.1, one can show that when $\Delta t \to 0$, the following relation holds for the global error of the time semidiscrete scheme (3.1):

$$u^{\Delta t}(x,y,t_{n+1}) = u(x,y,t_{n+1}) + \Delta t \mathcal{P}(x,y,t_{n+1}) + \mathcal{R}(x,y,t_{n+1}), \tag{5.4}$$

where $\mathcal{P}$ is a certain smooth function defined on $\overline{\mathtt{G}} \times \Lambda^{\Delta t}$ and independent of Δt ; $\mathcal{R}$ is the remainder term defined on $\overline{\mathtt{G}} \times \wedge^{\Delta t}$. We begin by assuming that the expansion (5.4) is valid. We substitute $u^{\Delta t}(x,y,t_{n+1})$ in (3.1) and obtain that

$$\begin{cases} u(x,y,0) + \Delta t \mathcal{P}(x,y,0) + \mathcal{R}(x,y,0) = \mathtt{q}_0(x,y), \quad (x,y) \in \overline{\mathtt{G}}, \\ \quad \left(\mathtt{I} + \Delta t \mathtt{L}_{1,\varepsilon}^{n+1}\right)\Big[\left(\mathtt{I} + \Delta t \mathtt{L}_{2,\varepsilon}^{n+1}\right)\left(u(x,y,t_{n+1}) + \Delta t \mathcal{P}(x,y,t_{n+1}) + \mathcal{R}(x,y,t_{n+1})\right) \\ \quad -\Delta t g_2(x,y,t_{n+1})\Big] \\ = u(x,y,t_n) + \Delta t \mathcal{P}(x,y,t_n) + \mathcal{R}(x,y,t_n) + \Delta t g_1(x,y,t_{n+1}), \quad \text{in } \mathtt{G}, \\ u(x,y,t_{n+1}) + \Delta t \mathcal{P}(x,y,t_{n+1}) + \mathcal{R}(x,y,t_{n+1}) = \mathtt{s}(x,y,t_{n+1}), \\ (x,y) \in \partial \mathtt{G} \times \Lambda^{\Delta t}, \ n = 0,1,\ldots,M-1. \end{cases} \tag{5.5}$$

By following the approach in [16] to the problem (5.5), we get the function $\mathcal{P}(x,y,t)$ is the solution of the following IBVP:

$$\begin{cases} \left(\dfrac{\partial}{\partial t} + \mathtt{L}_\varepsilon\right)\mathcal{P}(x,y,t) = \dfrac{1}{2}\dfrac{\partial^2 u(x,y,t)}{\partial t^2} + \mathtt{L}_{1,\varepsilon} g_2(x,y,t) - \mathtt{L}_{1,\varepsilon}\mathtt{L}_{2,\varepsilon} u(x,y,t), \\ (x,y,t) \in \mathtt{D}, \\ \mathcal{P}(x,y,0) = 0, \quad (x,y) \in \overline{\mathtt{G}}, \\ \mathcal{P}(x,y,t) = 0, \quad \text{in } \partial\mathtt{G} \times (0,T]. \end{cases} \tag{5.6}$$

Since $\dfrac{1}{2}\dfrac{\partial^2 u(x,y,t)}{\partial t^2} + \mathtt{L}_{1,\varepsilon} g_2(x,y,t) - \mathtt{L}_{1,\varepsilon}\mathtt{L}_{2,\varepsilon} u(x,y,t)$ is ε-uniformly bounded, under the sufficient continuity and compatibility conditions, one can derive that $\|\mathcal{P}(x,y,t)\|_{\overline{\mathtt{D}}} \le C$. Due to the extrapolation in the time direction, we only need to know the derivative bound of the solution $\mathcal{P}$ with respect to time. To establish the bounds of the derivatives up to second order in time in Lemma 5.1, we require $\mathcal{P}(x,y,t) \in \mathcal{C}^{4+\gamma}(\overline{\mathtt{D}})$.

Lemma 5.1. *The function $\mathcal{P}(x,y,t)$ solution of (5.6) satisfies the bounds*

$$\left|\frac{\partial^k \mathcal{P}(x,y,t)}{\partial t^k}\right| \le C, \quad k = 0,1,2.$$

Proof. We decompose the solution $\mathcal{P}(x,y,t)$ such that

$$\mathcal{P}(x,y,t) = \Phi(x,y,t) + \Xi(x,y,t), \quad (x,y,t) \in \overline{\mathtt{D}},$$

where Φ is the regular component and Ξ is the singular component. Again, we consider the decomposition

$$\Xi(x,y,t) = \Xi_1(x,y,t) + \Xi_2(x,y,t) + \Xi_{11}(x,y,t), \quad (x,y,t) \in \overline{\mathtt{D}},$$

where Ξ_1, Ξ_2 are the exponential layers near the sides $x = 1$ and $y = 1$ of $\overline{\mathtt{G}}$, respectively; and Ξ_{11} is the corner layer near the point $(1,1)$. Following the approach given in [9], one can obtain desired derivative bound of $\mathcal{P}$ with respect to time. □

Lemma 5.2. *The remainder term $\mathcal{R}(x,y,t)$ given in (5.4), satisfies the bound*

$$\left|\mathcal{R}(x,y,t_n)\right| \le C(\Delta t)^2,\ 0 \le n \le M. \tag{5.7}$$

Proof. From the equation (5.5), we get

$$\begin{cases} \mathcal{R}(x,y,0) = 0, \quad (x,y) \in \overline{\mathtt{G}}, \\ \quad u(x,y,t_{n+1}) + \Delta t\mathcal{P}(x,y,t_{n+1}) + \Delta t\mathtt{L}_\varepsilon^{n+1} u(x,y,t_{n+1}) + (\Delta t)^2\mathtt{L}_\varepsilon^{n+1}\mathcal{P}(x,y,t_{n+1}) \\ \quad +\big(\mathtt{I} + \Delta t\mathtt{L}_{1,\varepsilon}^{n+1}\big)\big(\mathtt{I} + \Delta t\mathtt{L}_{2,\varepsilon}^{n+1}\big)\mathcal{R}(x,y,t_{n+1}) \\ = u(x,y,t_n) + \Delta t\mathcal{P}(x,y,t_n) + \mathcal{R}(x,y,t_n) + \Delta t g(x,y,t_{n+1}) \\ \quad +(\Delta t)^2\mathtt{L}_{1,\varepsilon}^{n+1} g_2(x,y,t_{n+1}) - (\Delta t)^2\mathtt{L}_{1,\varepsilon}^{n+1}\mathtt{L}_{2,\varepsilon}^{n+1} u(x,y,t_{n+1}) \\ \quad -(\Delta t)^3\mathtt{L}_{1,\varepsilon}^{n+1}\mathtt{L}_{2,\varepsilon}^{n+1}\mathcal{P}(x,y,t_{n+1}), \\ \mathcal{R}(x,y,t_{n+1}) = 0, \quad (x,y,t_{n+1}) \in \partial\mathtt{G} \times \Lambda^{\Delta t}. \end{cases} \tag{5.8}$$

Using (1.1) and the Taylor-series expansion of the function u with respect to time variable t, we have

$$\begin{aligned} &u(x,y,t_n) \\ =&u(x,y,t_{n+1}) - \Delta t\frac{\partial u(x,y,t_{n+1})}{\partial t} + \frac{\Delta t^2}{2}\frac{\partial^2 u(x,y,t_{n+1})}{\partial t^2} + O(\Delta t^3), \\ &(\mathtt{I} + \Delta t\mathtt{L}_\varepsilon^{n+1})u(x,y,t_{n+1}) \\ =&u(x,y,t_n) + \Delta t g(x,y,t_{n+1}) - \frac{\Delta t^2}{2}\frac{\partial^2 u(x,y,t_{n+1})}{\partial t^2} + O(\Delta t^3), \end{aligned}$$

which implies that by employing the time-semidiscrete scheme (3.1)

$$\begin{aligned} &\big(\mathtt{I} + \Delta t\mathtt{L}_\varepsilon^{n+1}\big)\big(u^{n+1}(x,y) - u(x,y,t_{n+1})\big) \\ =&u^n(x,y) - u(x,y,t_n) + \Delta t^2\mathtt{L}_{1,\varepsilon}^{n+1} g_2(x,y,t_{n+1}) \\ &- \Delta t^2\mathtt{L}_{1,\varepsilon}^{n+1}\mathtt{L}_{2,\varepsilon}^{n+1} u^{n+1}(x,y) + \frac{\Delta t^2}{2}\frac{\partial^2 u(x,y,t_{n+1})}{\partial t^2} + O(\Delta t^3). \end{aligned} \tag{5.9}$$

Again, using (5.6) and the Taylor-series expansion of the function $\mathcal{P}$ with respect to time variable t, we have

$$
\begin{aligned}
&\mathcal{P}(x,y,t_n)\\
=&\mathcal{P}(x,y,t_{n+1})-\Delta t\frac{\partial\mathcal{P}(x,y,t_{n+1})}{\partial t}+\frac{\Delta t^2}{2}\frac{\partial^2\mathcal{P}(x,y,t_{n+1})}{\partial t^2}+O(\Delta t^3),\\
&(\mathtt{I}+\Delta t\mathtt{L}_{\varepsilon}^{n+1})\mathcal{P}(x,y,t_{n+1})\\
=&\mathcal{P}(x,y,t_n)-\frac{\Delta t^2}{2}\frac{\partial^2\mathcal{P}(x,y,t_{n+1})}{\partial t^2}+O(\Delta t^3)\\
&+\Delta t\left[\frac{1}{2}\frac{\partial^2 u(x,y,t_{n+1})}{\partial t^2}+\mathtt{L}_{1,\varepsilon}^{n+1}g_2(x,y,t_{n+1})-\mathtt{L}_{1,\varepsilon}^{n+1}\mathtt{L}_{2,\varepsilon}^{n+1}u(x,y,t_{n+1})\right],
\end{aligned}
$$

which implies that

$$
\begin{aligned}
&(\mathtt{I}+\Delta t\mathtt{L}_{1,\varepsilon}^{n+1})(\mathtt{I}+\Delta t\mathtt{L}_{2,\varepsilon}^{n+1})\mathcal{P}(x,y,t_{n+1})\\
=&\mathcal{P}(x,y,t_n)-\frac{\Delta t^2}{2}\frac{\partial^2\mathcal{P}(x,y,t_{n+1})}{\partial t^2}+O(\Delta t^3)\\
&+\Delta t\left[\frac{1}{2}\frac{\partial^2 u(x,y,t_{n+1})}{\partial t^2}+\mathtt{L}_{1,\varepsilon}^{n+1}g_2(x,y,t_{n+1})-\mathtt{L}_{1,\varepsilon}^{n+1}\mathtt{L}_{2,\varepsilon}^{n+1}u(x,y,t_{n+1})\right]\\
&+\Delta t^2\mathtt{L}_{1,\varepsilon}^{n+1}\mathtt{L}_{2,\varepsilon}^{n+1}\mathcal{P}(x,y,t_{n+1}).
\end{aligned}
\tag{5.10}
$$

Thus, utilizing (5.8), (5.9), and (5.10), the remainder term in (5.4) satisfies the following IBVP:

$$
\begin{cases}
\mathcal{R}(x,y,0)=0, \quad (x,y)\in\overline{\mathtt{G}},\\
\big(\mathtt{I}+\Delta t\mathtt{L}_{2,\varepsilon}^{n+1}\big)\big(\mathtt{I}+\Delta t\mathtt{L}_{1,\varepsilon}^{n+1}\big)\mathcal{R}(x,y,t_{n+1})=\mathcal{R}(x,y,t_n)+O(\Delta t)^3, \quad (x,y)\in\mathtt{G},\\
\mathcal{R}(x,y,t_{n+1})=0, \quad (x,y,t_{n+1})\in\partial\mathtt{G}\times\Lambda^{\Delta t}.
\end{cases}
\tag{5.11}
$$

Finally, using the above relation recursively and by invoking the stability in Lemma 3.2, we obtain the desired bound of the remainder term. □

Theorem 5.1. *Let $u^{\Delta t}(x,y,t_{n+1})$ and $u^{\Delta t/2}(x,y,\widetilde{t}_{n+1})$ be the respective solutions of the time-semidiscrete problem (3.1) on the mesh $\overline{\mathtt{G}}\times\wedge^{\Delta t}$ and $\overline{\mathtt{G}}\times\wedge^{\Delta t/2}$; and let $u(x,y,t_{n+1})$ be the exact solution of the IBVP (1.1)-(1.2) on the mesh $\overline{\mathtt{G}}\times\wedge^{\Delta t}$. Then the error due to the temporal extrapolation $u_{extp}^{\Delta t}(x,y,t_{n+1})$ defined by (5.3) satisfies that $\left|u_{extp}^{\Delta t}(x,y,t_{n+1})-u(x,y,t_{n+1})\right|\le C(\Delta t)^2,\ (x,y,t_{n+1})\in\overline{\mathtt{G}}\times\wedge^{\Delta t}$.*

Proof. From (5.4) and (5.7), we have

$$
u(x,y,t_{n+1})=u^{\Delta t}(x,y,t_{n+1})-\Delta t\mathcal{P}(x,y,t_{n+1})+O(\Delta t)^2,\ (x,y,t_{n+1})\in\overline{\mathtt{G}}^N\times\Lambda^{\Delta t}.
$$

Similarly, we have

$$
u(x,y,\widetilde{t}_{n+1})
$$

$$=u^{\Delta t/2}(x,y,\widetilde{t}_{n+1})-(\Delta t/2)\mathcal{P}(x,y,\widetilde{t}_{n+1})+O(\Delta t)^2,\ (x,y,\widetilde{t}_{n+1})\in\overline{\mathtt{G}}^N\times\Lambda^{\Delta t/2}.$$

Now, using the above two expressions, we obtain the desired result. □

6. Numerical experiments

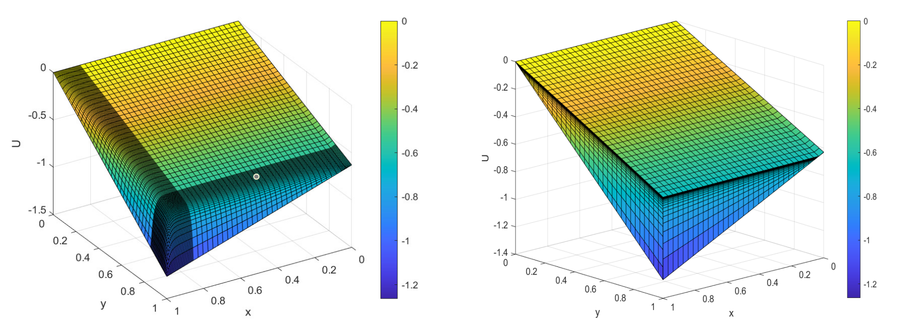
In this section, we present the numerical results obtained using the proposed algorithm for two test examples of the form (1.1)-(1.2). In the numerical experiments, the reaction term is decomposed in the form $b_1(x,y,t)=b_2(x,y,t)=b(x,y,t)/2$; and the right-hand side is decomposed in the form $g(x,y,t)=g_1(x,y,t)+g_2(x,y,t)$, where $g_2(x,y,t)=g(x,0,t)+y(g(x,1,t)-g(x,0,t))$, $g_1(x,y,t)=g(x,y,t)-g_2(x,y,t)$. We refer [6] for a detailed discussion aobut this decomposition and other types of decomposition. To show that the theoretical findings match very well with the computational results, we choose $\mathbb{K}_{1,0}=\mathbb{K}_{2,0}=4.2$, and implement the Thomas algorithm to solve the tridiagonal linear systems resulting from our methods. The numerical results are also compared with the implicit upwind method [6].

6.1. Test examples

Example 6.1. Consider the following parabolic IBVP:

$$\begin{cases}\dfrac{\partial u}{\partial t}-\varepsilon\Delta u+\dfrac{\partial u}{\partial x}+\dfrac{\partial u}{\partial y}+(1+t^2xy)u=g(x,y,t), & \text{in } \mathtt{G}\times(0,1],\\ u(x,y,0)=0, \quad \text{in } \overline{\mathtt{G}}, & \\ u(x,y,t)=(e^{-t}-1)(1+x)y, & (x,y,t)\in\partial\mathtt{G}\times(0,1],\end{cases}\tag{6.1}$$

where $g(x,y,t)=[1+rt^2xy][\Phi(x)\Phi(y)-(1+x)y]+rm_2[\Phi(x)+\Phi(y)]-r(1+x+y)$ and $\Phi(z)=m_1+m_2z+\exp(-(1-z)/\varepsilon)$, $m_1=-\exp(-1/\varepsilon)$, $m_2=-1-m_1$ and $r=1-e^{-t}$.



(a) $\varepsilon=2^{-6}$ (b) $\varepsilon=2^{-14}$

Figure 2. Graphs of numerical solution for Example 6.1 $N=64,\ M=32$ at $t=1$.

In Fig 2, we draw surface plot of numerical solution for Example 6.1 and it shows that the solution generates boundary layers closer to the outflow boundaries

$x = 1$ and $y = 1$. As we are not acquainted with the exact solution of Example 6.1, we calculate the maximum point-wise errors $\widehat{e}_\varepsilon^{N,\Delta t}$ corresponding to the proposed numerical method before and after extrapolation, respectively by

$$\max_{0\le i\le N}\max_{0\le j\le N}\max_{0\le n\le M}\left|U^{N,\Delta t}(x_i,y_j,t_n)-\widehat{U}^{2N,\Delta t/2}(x_i,y_j,t_n)\right|,$$

and

$$\max_{0\le i\le N}\max_{0\le j\le N}\max_{0\le n\le M}\left|U_{extp}^{N,\Delta t}(x_i,y_j,t_n)-\widehat{U}_{extp}^{2N,\Delta t/2}(x_i,y_j,t_n)\right|,$$

and the corresponding orders of convergence are calculated by $\widehat{r}_\varepsilon^{N,\Delta t} = \log_2\left(\dfrac{\widehat{e}_\varepsilon^{N,\Delta t}}{\widehat{e}_\varepsilon^{2N,\Delta t/2}}\right)$.

Here, $\widehat{U}^{2N,\Delta t/2}(x_i,y_j,t_n)$ and $\widehat{U}_{extp}^{2N,\Delta t/2}(x_i,y_j,t_n)$, respectively denote the numerical solution and the extrapolated numerical solution obtained at $(x_i,y_j,t_n)\in \widehat{\mathtt{D}}^{2N,\Delta t/2} = \widehat{\mathtt{G}}^{2N}\times \Lambda^{\Delta t/2}$, where $\Delta t/2 = T/2M$, and $\widehat{\mathtt{G}}^{2N}$ is a piecewise-uniform Shishkin mesh with $2N$ mesh-intervals in both the x- and y-directions and having the same transition parameter η_l, $l = 1,2$, as that of $\overline{\mathtt{G}}^N$ such that the $(i^{\text{th}}, j^{\text{th}})$ point of $\overline{\mathtt{G}}^N$ becomes $(2i^{\text{th}}, 2j^{\text{th}})$ point of $\widehat{\mathtt{G}}^{2N}$, for $i,j = 0,1,\dots N$. Finally, for each N and Δt, we compute the quantities $\widehat{e}^{N,\Delta t}$ and $\widehat{r}^{N,\Delta t}$ analogously to $e^{N,\Delta t}$ and $r^{N,\Delta t}$.

Furthermore, we compute the local errors for the first example. Because, we do not know the exact solution, to approximate $\widetilde{U}^{N,\Delta t}(x_i,y_j,t_n)$ we use one step of the fully discrete scheme given in (4.1) and we replace the numerical solution $U^{N,\Delta t}(x_i,y_j,t_{n-1})$ by the numerical solution obtained on the finest mesh, which is a sufficiently good approximation to $u(x_i,y_j,t_{n-1})$. Finally, the local errors are computed by

$$\widehat{e}_{loc}^{N,\Delta t} = \max_{0\le i\le M}\max_{0\le j\le M}\max_{0\le n\le M}|\widetilde{U}^{N,\Delta t}(x_i,y_j,t_n) - u_{2048,1024}(x_i,y_j,t_n)|,$$

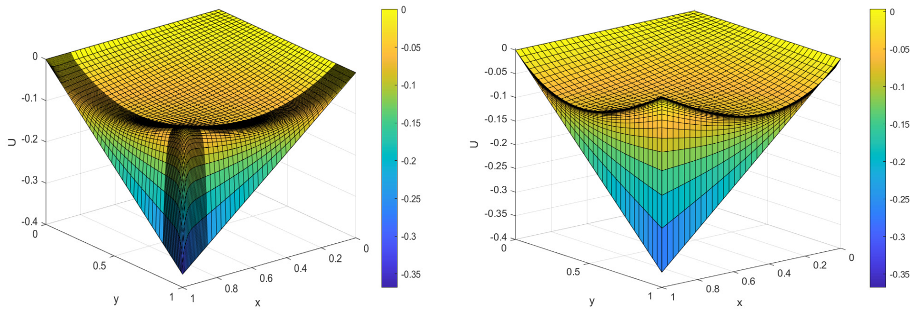
where $\widetilde{U}^{N,\Delta t}(x_i,y_j,t_n)$ is solution of the discrete problem (4.30)-(4.32) and the corresponding local order of convergence computed by $\widehat{r}_{loc}^{N,\Delta t} = \dfrac{\log(\widehat{e}_{loc}^{N,\Delta t}/\widehat{e}_{loc}^{N,\Delta t/2})}{\log 2}$. Note that the corresponding orders of consistency are given by $\widehat{r}_{loc}^{N,\Delta t} - 1$.

Example 6.2. Consider the following parabolic IBVP:

$$\begin{cases} \dfrac{\partial u}{\partial t} - \varepsilon\Delta u + (1 + xt(1-xt))\dfrac{\partial u}{\partial x} + (1+y(1-y))\dfrac{\partial u}{\partial y} = g(x,y,t), \text{ in } \mathtt{G}\times(0,1], \\ u(x,y,0) = \mathtt{q}_0(x,y), \quad \text{in } \overline{\mathtt{G}}, \\ u(x,y,t) = \mathtt{s}(x,y,t), \quad \text{in } \partial\mathtt{G}\times(0,T], \end{cases} \tag{6.2}$$

where $g, \mathtt{q}_0, \mathtt{s}$ are such that the exact solution is

$$\begin{aligned} u(x,y,t) = &\exp(-t)\Big[\Big(\frac{1-\exp(-(1-x)/\varepsilon)}{1-\exp(-1/\varepsilon)} - \cos(\frac{\pi x}{2})\Big) \\ &\times \Big(\frac{1-\exp(-(1-y)/\varepsilon)}{1-\exp(-1/\varepsilon)} - \cos(\frac{\pi y}{2})\Big) - xy\Big]. \end{aligned}$$



(a) $\varepsilon = 2^{-6}$ (b) $\varepsilon = 2^{-14}$

Figure 3. Graphs of numerical solution for Example 6.2 $N = 64,\ M = 32$ at $t = 1$.

In Fig 3, we draw surface plot of numerical solution for Example 6.2 and it shows that the solution generates boundary layers closer to the outflow boundaries $x = 1$ and $y = 1$. Global errors are displayed to demonstrate the uniform convergence of the method. For each ε, we calculate the maximum point-wise errors $e_\varepsilon^{N,\Delta t}$ corresponding to the proposed numerical method before and after extrapolation, respectively by

$$\max_{0\le i\le N}\max_{0\le j\le N}\max_{0\le n\le M}\left|U^{N,\Delta t}(x_i,y_j,t_n)-u(x_i,y_j,t_n)\right|,$$

and

$$\max_{0\le i\le N}\max_{0\le j\le N}\max_{0\le n\le M}\left|U_{extp}^{N,\Delta t}(x_j,y_j,t_n)-u(x_i,y_j,t_n)\right|,$$

and the corresponding orders of convergence are calculated by $r_\varepsilon^{N,\Delta t} = \log_2\left(\dfrac{e_\varepsilon^{N,\Delta t}}{e_\varepsilon^{2N,\Delta t/2}}\right)$. Here, $U^{N,\Delta t}(x_i,y_j,t_n)$ and $U_{extp}^{N,\Delta t}(x_i,y_j,t_n)$, respectively denote the numerical solutions of the proposed method (4.1) obtained at $(x_i,y_j,t_n) \in \overline{\mathtt{D}}^{N,\Delta t}$. Further, for each N and Δt, we calculate the ε-uniform maximum point-wise error and the corresponding ε-uniform order of convergence, respectively by

$$e^{N,\Delta t} = \max_{\varepsilon} e_\varepsilon^{N,\Delta t} \quad \text{and} \quad r^{N,\Delta t} = \log_2\left(\frac{e^{N,\Delta t}}{e^{2N,\Delta t/2}}\right).$$

We also compute the local errors in time to illustrate the numerical behavior of the method. The local errors at the mesh points are computed by

$$e_{loc}^{N,\Delta t} = \max_{0\le i\le M}\max_{0\le j\le M}\max_{0\le n\le M}|\widetilde{U}^{N,\Delta t}(x_i,y_j,t_n)-u(x_i,y_j,t_n)|,$$

where $\widetilde{U}^{N,\Delta t}(x_i,y_j,t_n)$ is solution of the discrete problem (4.30)-(4.32) and the corresponding local order of convergence computed by $r_{loc}^{N,\Delta t} = \dfrac{\log(e_{loc}^{N,\Delta t}/e_{loc}^{N,\Delta t/2})}{\ln 2}$. Note that the corresponding orders of consistency are given by $r_{loc}^{N,\Delta t} - 1$.

6.2. Numerical results and observations

We choose all the values of ε from $\mathbb{S}_\varepsilon = \{2^0, 2^{-2}, \ldots, 2^{-20}\}$, for computation of ε-uniform errors. For different values of ε, N and Δt, we compare the ε-uniform errors computed using the proposed method (4.1) (together with alternative and natural boundary data), and the classical upwind scheme [6] (along with alternative boundary data) in Tables 1 and 2, without the temporal extrapolation, respectively for Examples 6.1 and 6.2. For the sake of clarity, the computed ε-uniform errors in Tables 1 and 2 are depicted in Figs 6 and 7, respectively for Examples 6.1 and 6.2. This represents the ε-uniform convergence of the proposed method (4.1) for both the choice of boundary conditions. Moreover, it shows that proposed method (4.1) furnishes better ε-uniform numerical result in comparison with classical upwind scheme [6]; reflecting the robustness of the proposed method (4.1).

Furthermore, Tables 1 and 2 reflect that the ε-uniform errors of the proposed method (4.1) with alternative boundary conditions (3.3) are smaller than the ε-uniform errors with natural boundary conditions (3.2). To complement this observation, surface plots of the error corresponding to the Examples 6.1 and 6.2 computed at $t = 1$ are also depicted in Figs 4 and 5 for $N = 256$ and $\Delta t = \frac{1}{160}$ with natural and alternative boundary conditions, respectively.

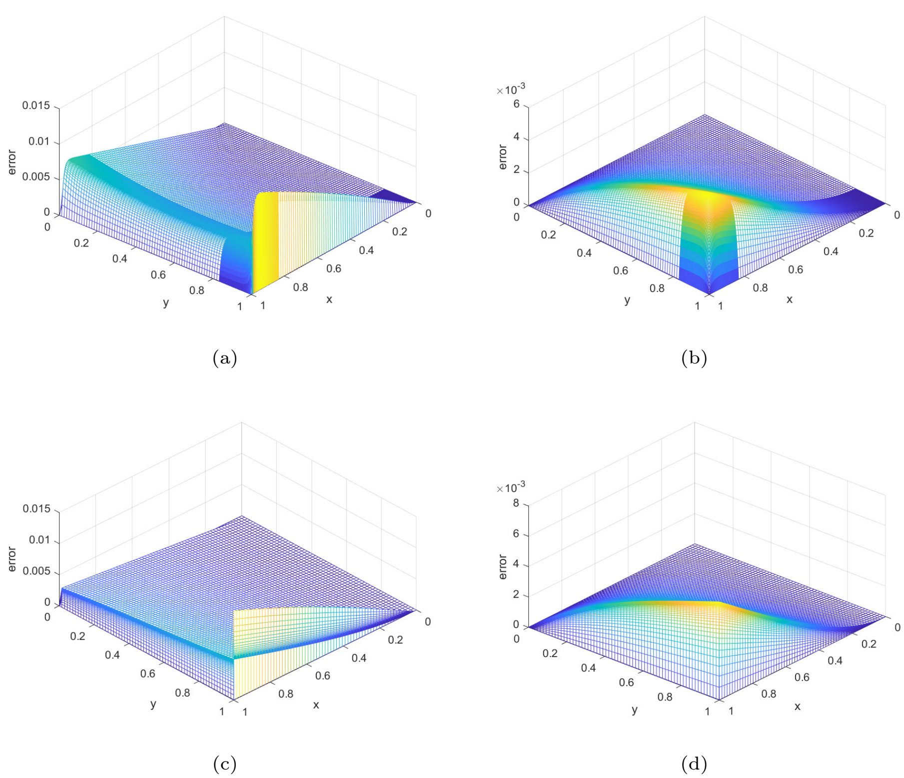
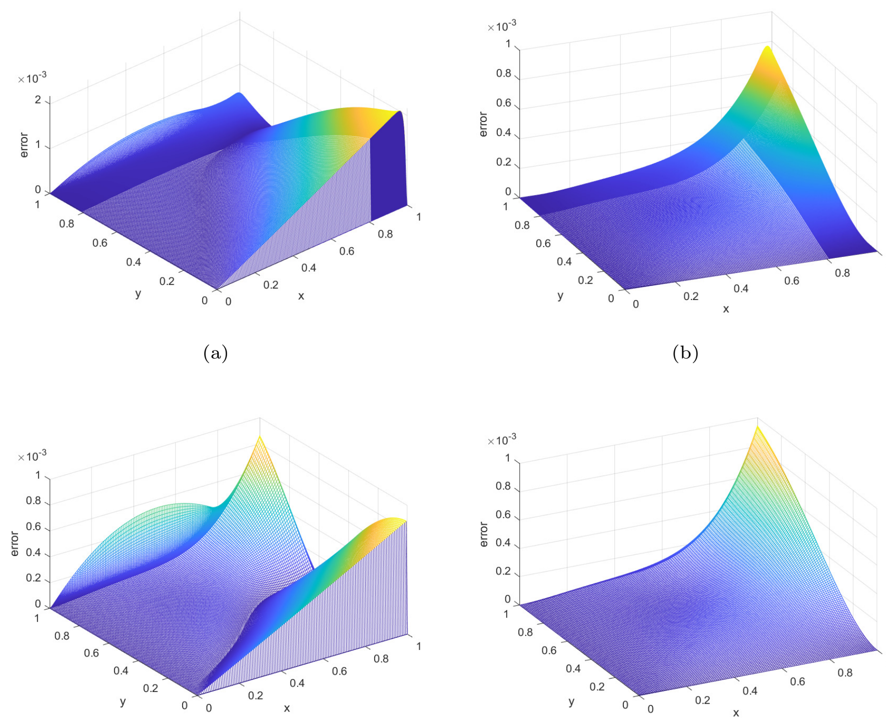


Figure 4. Graphs of error $|U^{N,\Delta t} - U^{2N,\Delta t/2}|$ corresponding to Example 6.1 for $\varepsilon = 2^{-6}$ using (a) natural b.c (b) alternative b.c and for $\varepsilon = 2^{-14}$ using (c) natural b.c (d) alternative b.c at $t = 1$



(a) (b) (c) (d)

Figure 5. Graphs of error $|U^{N,\Delta t} - u|$ corresponding to Example 6.2 for $\varepsilon = 2^{-6}$ using (a) natural b.c (b) alternative b.c and for $\varepsilon = 2^{-14}$ using (c) natural b.c (d) alternative b.c at $t = 1$

Next, we compute the numerical local errors $\widehat{e}_{loc}^{N,\Delta t}, e_{loc}^{N,\Delta t}$ corresponding to the two choices of the boundary data. For this purpose, we fix $N = 512$ for Example 6.1; and to reduce the influence of the spatial error, we increase the discretization parameter from $N = 512$ to $N = 2048$ for Example 6.2. To keep the paper from becoming too long, we display the numerical results only for $\varepsilon = 2^{-3}, 2^{-6}$ from which the differences between the two evaluations of the boundary conditions can be easily noticed.

From Tables 3 and 4, we observe that when the alternative boundary data is chosen, the local errors $\widehat{e}_{loc}^{N,\Delta t}$ are significantly reduced; and the numerical order of consistency ($i.e.$,$(\widehat{r}_{loc}^{N,\Delta t} - 1)$) for Example 6.1 is one, whereas for the classical evaluation the numerical order of consistency is near to zero. Similar observations with respect to the local errors $e_{loc}^{N,\Delta t}$ and the numerical order of consistency ($i.e.$,$(r_{loc}^{N,\Delta t} - 1)$) can be made from Tables 6 and 5 for Example 6.2, in particular when $N = 2048$ is chosen. This indicates that if we increase N further, local order of convergence $r_{loc}^{N,\Delta t}$ will improve gradually as the theory predicts. Theses observations reveal that the option (3.3) for evaluation of the boundary data is evidently better than the conventional one as claimed in Remark 3.1.

Moreover, in order to see the effect of the temporal Richardson extrapolation, we choose a suitably large N to reduce the influence of the spatial error. To make concise presentation, we display numerical results only for $\varepsilon = 2^{-6}, 2^{-20}$ after the temporal extrapolation of the proposed method (4.1) for Examples 6.1 and 6.2, respectively in Tables 7 and 8. This shows the improvement in the temporal order of convergence after employing the Richardson extrapolation in the time variable, as claimed in Theorem 5.1. Simultaneously, Tables 7 and 8 also reflect that the effect of alternative boundary conditions (3.3) after the temporal extrapolation overshadows the effect of natural boundary conditions (3.2), by producing the temporal errors substantially smaller in case of alternative boundary conditions.

The above numerical experiment indicates that by using the temporal Richardson extrapolation, one can check the spatial accuracy by choosing $\Delta t = 1/N$. Following this, in Tables 9 and 10, we compare the region-wise spatial accuracy of the proposed method given in (4.1), for Examples 6.1 and 6.2, respectively . These computational results match very well with the spatial error as stated in Remark 4.1, which shows that the spatial accuracy is at least two in the outer region and almost two in the layer regions, irrespective of the smaller and larger values of the parameter ε.

Table 1. Comparison of ε-uniform errors and order of convergence for Example 6.1 computed using $\Delta t = 1.6/N$ without temporal extrapolation

$\varepsilon \in \mathbb{S}_\varepsilon$	Number of mesh intervals N/ time step size Δt				
	$64\ /\frac{1}{40}$	$128/\frac{1}{80}$	$256/\frac{1}{160}$	$512/\frac{1}{320}$	$1024/\frac{1}{640}$
	upwind scheme with alternative boundary conditions [6]				
$\widehat{e}^{N,\Delta t}$	5.8496e-02	3.9579e-02	2.5215e-02	1.5255e-02	8.8471e-03
$\widehat{r}^{N,\Delta t}$	0.56363	0.65042	0.72501	0.78601	
	proposed method (4.1) with natural boundary conditions (3.2)				
$\widehat{e}^{N,\Delta t}$	3.3637e-02	2.2012e-02	1.2872e-02	7.0903e-03	3.8038e-03
$\widehat{r}^{N,\Delta t}$	0.61179	0.77404	0.86031	0.89842	
	proposed method (4.1) with alternative boundary conditions (3.3)				
$\widehat{e}^{N,\Delta t}$	2.4898e-02	1.2804e-02	6.4916e-03	3.2684e-03	1.6399e-03
$\widehat{r}^{N,\Delta t}$	0.95946	0.97990	0.98999	0.99500	

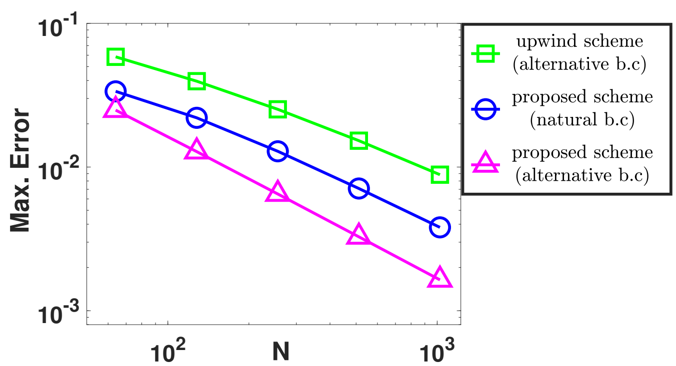


Figure 6. Loglog plot for comparison of the ε-uniform errors $\widehat{e}^{N,\Delta t}$ for Example 6.1.

Table 2. Comparison of ε-uniform errors and order of convergence for Example 6.2 computed using $\Delta t = 1.6/N$ without temporal extrapolation

$\varepsilon \in \mathbb{S}_\varepsilon$	Number of mesh intervals $N/$ time step size Δt				
	$64\ /\frac{1}{40}$	$128/\frac{1}{80}$	$256/\frac{1}{160}$	$512/\frac{1}{320}$	$1024/\frac{1}{640}$
	upwind scheme with alternative boundary conditions [6]				
$e^{N,\Delta t}$	1.1192e-01	7.1853e-02	4.3571e-02	2.5404e-02	1.4408e-02
$r^{N,\Delta t}$	0.63937	0.72168	0.77831	0.81817	
	proposed method (4.1) with natural boundary conditions (3.2)				
$e^{N,\Delta t}$	1.7857e-02	1.0390e-02	5.4965e-03	2.9241e-03	1.4870e-03
$r^{N,\Delta t}$	0.78137	0.91855	0.91050	0.97563	
	proposed method (4.1) with alternative boundary conditions (3.3)				
$e^{N,\Delta t}$	1.3278e-02	4.7100e-03	1.6507e-03	5.8804e-04	2.9925e-04
$r^{N,\Delta t}$	1.4953	1.5126	1.4891	0.97455	

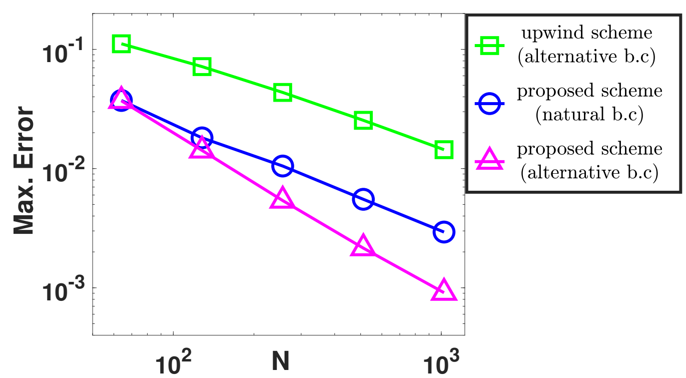


Figure 7. Loglog plot for comparison of the ε-uniform errors $e^{N,\Delta t}$ for Example 6.2

Table 3. Maximum point-wise local errors $\widehat{e}_{loc}^{N,\Delta t}$ and order of convergence $\widehat{r}_{loc}^{N,\Delta t}$ for Example 6.1 with natural boundary conditions(3.2) and without temporal extrapolation

ε	Number of mesh intervals $N = 512$			
	M=16	M=32	M=64	M=128
2^{-3}	4.1048e-02	2.3271e-02	1.2582e-02	6.6195e-03
	0.89873	0.95555	0.98838	
2^{-6}	5.0262e-02	2.6959e-02	1.3901e-02	7.0068e-03
	0.92587	0.99806	1.0663	

7. Conclusion

In this paper, we develop and analyze an efficient FSFMM following the temporal Richardson extrapolation for cost-effective higher-order space-time accurate numerical approximation of a class of two-dimensional singularly perturbed parabolic

Table 4. Maximum point-wise local errors $\widehat{e}_{loc}^{N,\Delta t}$ and order of convergence $\widehat{r}_{loc}^{N,\Delta t}$ for Example 6.1 with alternative boundary conditions (3.3) and without temporal extrapolation

ε	Number of mesh intervals $N = 512$			
	M=16	M=32	M=64	M=128
2^{-3}	2.7796e-03	9.3539e-04	2.8154e-04	7.8802e-05
	1.5713	1.7322	1.8370	
2^{-6}	4.6890e-03	1.4747e-03	4.2654e-04	1.1848e-04
	1.6689	1.7897	1.8481	

Table 5. Maximum point-wise local errors $e_{loc}^{N,\Delta t}$ and order of convergence $r_{loc}^{N,\Delta t}$ for Example 6.2 with natural boundary conditions (3.2) and without temporal extrapolation

	M=8	M=16	M=32	M=64
ε	Number of mesh intervals $N = 512$			
2^{-3}	6.5690e-02	4.0523e-02	2.2828e-02	1.2229e-02
	0.69695	0.82795	0.90053	
2^{-6}	8.5378e-02	4.9119e-02	2.5928e-02	1.3035e-02
	0.79758	0.92175	0.99209	
	Number of mesh intervals $N = 2048$			
2^{-3}	6.6237e-02	4.1036e-02	2.3265e-02	1.2580e-02
	0.69076	0.81872	0.88707	
2^{-6}	8.6643e-02	5.0396e-02	2.7094e-02	1.4024e-02
	0.78179	0.89531	0.95009	

Table 6. Maximum point-wise local errors $e_{loc}^{N,\Delta t}$ and order of convergence $r_{loc}^{N,\Delta t}$ for Example 6.2 with alternative boundary conditions (3.3) and without temporal extrapolation

	M=8	M=16	M=32	M=64
ε	Number of mesh intervals $N = 512$			
2^{-3}	6.0237e-03	2.2568e-03	7.0946e-04	2.0175e-04
	1.4164	1.6695	1.8141	
2^{-6}	7.4980e-03	2.6648e-03	9.1066e-04	3.7428e-04
	1.4925	1.5490	1.2828	
	Number of mesh intervals $N = 2048$			
2^{-3}	6.0232e-03	2.2565e-03	7.0936e-04	2.0171e-04
	1.4164	1.6695	1.8142	
2^{-6}	7.4901e-03	2.6607e-03	7.9614e-04	2.3756e-04
	1.4932	1.7407	1.7447	

Table 7. Effect of temporal extrapolation on global temporal accuracy with natural and alternative boundary conditions for Example 6.1

ε	Number of mesh intervals $N = 2048$			
	M=8	M=16	M=32	M=64
	with natural boundary conditions (3.2)			
2^{-6}	1.9871e-02	6.7257e-03	2.9966e-03	1.6246e-03
	1.5629	1.1664	0.88323	
2^{-20}	1.9935e-02	6.5946e-03	3.1439e-03	1.5293e-03
	1.5960	1.0688	1.0397	
	with alternative boundary conditions (3.3)			
2^{-6}	3.8994e-03	1.0196e-03	2.6615e-04	6.8500e-05
	1.9352	1.9378	1.9581	
2^{-20}	4.4127e-03	1.1497e-03	3.0420e-04	7.9030e-05
	1.9403	1.9182	1.9445	

Table 8. Effect of temporal extrapolation on global temporal accuracy with natural and alternative boundary conditions for Example 6.2

ε	Number of mesh intervals $N = 4096$			
	M=8	M=16	M=32	M=64
	with natural boundary conditions (3.2)			
2^{-6}	3.7679e-03	1.4993e-03	8.0354e-04	4.6111e-04
	1.3295	0.89988	0.80125	
2^{-20}	4.0595e-03	1.3386e-03	5.6366e-04	2.2708e-04
	1.6006	1.2478	1.3116	
	with alternative boundary conditions (3.3)			
2^{-6}	1.4013e-03	4.1762e-04	1.1922e-04	3.2995e-05
	1.7465	1.8086	1.8533	
2^{-20}	1.8114e-03	5.6383e-04	1.6833e-04	4.7957e-05
	1.6838	1.7440	1.8115	

convection-diffusion problems of the form (1.1)-(1.2) with non-homogeneous time-dependent boundary data.

We perform the error analysis for the time semidiscrete case with a suitable choice of the boundary data other than the classical choice to avoid the order reduction in time. We prove that the corresponding fully discrete scheme is ε-uniformly convergent in the discrete supremum norm; and show that the spatial accuracy is at least two in the outer region and almost two in the boundary layer region, regardless of the larger and smaller values of ε. In addition to this, we derive the error estimate associated with temporal Richardson extrapolation and show that it improves the temporal order of convergence from first-order to second-order. As a result, the resulting numerical solution is proved to be globally almost second-order uniformly convergent (considering both space and time) provided the suitable eval-

Table 9. Comparison (region wise) of maximum point-wise errors and order of convergence for Example 6.1, with alternative boundary data and using temporal extrapolation with $\Delta t = \frac{1}{N}$.

	Outer region $[0, 1-\mathbb{K}_1]\times$ $[0, 1-\mathbb{K}_2]$	Right boundary layer region $(1-\mathbb{K}_1, 1]\times$ $[0, 1-\mathbb{K}_2]$	Top boundary layer region $[0, 1-\mathbb{K}_1]\times$ $(1-\mathbb{K}_2, 1]$	Corner layer region $(1-\mathbb{K}_1, 1]\times$ $(1-\mathbb{K}_2, 1]$
	$\varepsilon = 2^{-4} \approx 10^{-1}$			
128	3.0977e-05	5.9927e-05	4.7228e-05	1.5801e-04
	1.9702	2.0044	2.0112	2.0138
256	7.9059e-06	1.4936e-05	1.1716e-05	3.9127e-05
	1.9846	2.0021	2.0068	2.0066
512	1.9977e-06	3.7287e-06	2.9152e-06	9.7372e-06
	1.9922	2.0009	2.0031	2.0036
	$\varepsilon = 2^{-6} \approx 10^{-2}$			
128	5.1342e-05	9.1037e-04	8.9185e-04	1.8120e-03
	2.0527	1.7202	1.7184	1.6129
256	1.2375e-05	2.7631e-04	2.7102e-04	5.9245e-04
	2.0761	1.7659	1.7644	1.6572
512	2.9347e-06	8.1248e-05	7.9776e-05	1.8784e-04
	2.0949	1.8097	1.8085	1.6926
	$\varepsilon = 2^{-14} \approx 10^{-4}$			
128	8.0654e-05	1.4441e-03	1.4442e-03	2.0562e-03
	1.9652	1.6217	1.6219	1.6121
256	2.0656e-05	4.6925e-04	4.6922e-04	6.7262e-04
	1.9776	1.6539	1.6540	1.6594
512	5.2448e-06	1.4912e-04	1.4910e-04	2.1294e-04
	1.9852	1.6918	1.6918	1.6941

uation of the boundary data is considered and it is confirmed by extensive numerical experiments.

Acknowledgements

The authors express their sincere thanks to the anonymous reviewers for their valuable comments and suggestions to improve the quality of the manuscript.

Declarations

The authors declare that they have no conflict of interest.

Table 10. Comparison (region wise) of maximum point-wise errors and order of convergence for Example 6.2, with alternative boundary data and using temporal extrapolation with $\Delta t = \frac{1}{N}$.

	Outer region $[0, 1-\mathbb{K}_1]\times[0, 1-\mathbb{K}_2]$	Right boundary layer region $(1-\mathbb{K}_1, 1]\times[0, 1-\mathbb{K}_2]$	Top boundary layer region $[0, 1-\mathbb{K}_1]\times(1-\mathbb{K}_2, 1]$	Corner layer region $(1-\mathbb{K}_1, 1]\times(1-\mathbb{K}_2, 1]$
	$\varepsilon = 2^{-4} \approx 10^{-1}$			
128	2.4945e-06	2.5924e-06	1.0747e-05	1.6614e-05
	1.9100	1.7768	1.8219	1.8406
256	6.6377e-07	7.5658e-07	3.0398e-06	4.6387e-06
	1.9376	1.9110	1.9095	1.9253
512	1.7327e-07	2.0119e-07	8.0916e-07	1.2213e-06
	1.9561	1.9634	1.9574	1.9667
	$\varepsilon = 2^{-6} \approx 10^{-2}$			
128	2.7049e-06	6.7684e-05	6.1853e-05	1.1266e-04
	2.0548	2.6141	2.7494	2.4630
256	6.5102e-07	1.1055e-05	9.1980e-06	2.0434e-05
	1.9558	2.7964	2.8841	2.614
512	1.6782e-07	1.5912e-06	1.2459e-06	3.3374e-06
	1.9700	2.8600	2.2363	2.7054
	$\varepsilon = 2^{-14} \approx 10^{-4}$			
128	2.1486e-05	1.1012e-03	1.1011e-03	1.5501e-03
	1.9717	1.6725	1.6725	1.6697
256	5.4781e-06	3.4546e-04	3.4540e-04	4.8722e-04
	1.9866	1.7490	1.7490	1.7654
512	1.3823e-06	1.0278e-04	1.0276e-04	1.4332e-04
	1.9938	1.8557	1.8557	1.8813

References

[1] I. Alonso-Mallo, B. Cano and J. C. Jorge, *Spectral-fractional step Runge-Kutta discretizations for initial boundary value problems with time dependent boundary conditions*, Math. Comp., 2004, 73(248), 1801–1825.

[2] V. Batu, *A generalized two-dimensional analytical solution for hydrodynamic dispersion in bounded media with the first-type boundary condition at the source*, Water Resources Research, 1989, 25, 1125–1132.

[3] C. Clavero, J. L. Gracia and J. C. Jorge, *A uniformly convergent alternating direction HODIE finite difference scheme for 2D time-dependent convection-diffusion problems*, IMA J. Numer. Anal., 2006, 26, 155–172.

[4] C. Clavero, J. L. Gracia and F. Lisbona, *Higher-order numerical methods for one-dimensional parabolic singularly perturbed problems with regular layers*, Numer. Methods Partial Differential Equations, 2005, 21, 149–169.

[5] C. Clavero and J. C. Jorge, *Another uniform convergence analysis technique of some numerical methods for parabolic singularly perturbed problems*, Comput. Math. Appl., 2015, 70, 222–235.

[6] C. Clavero and J. C. Jorge, *A fractional step method for 2D parabolic convection-diffusion singularly perturbed problems: Uniform convergence and order reduction*, Numer. Algorithms, 2017, 75, 809–826.

[7] C. Clavero and J. C. Jorge, *Order reduction and uniform convergence of an alternating direction method for solving 2D time dependent convection-diffusion problems*, in Boundary and Interior Layers, Computational and Asymptotic Methods—BAIL 2016, Lect. Notes Comput. Sci. Eng., 2017, 120, 49–61.

[8] C. Clavero and J. C. Jorge, *An efficient numerical method for singularly perturbed time dependent parabolic 2D convection-diffusion systems*, J. Comput. Appl. Math., 2019, 354, 431–444.

[9] C. Clavero, J. C. Jorge, F. Lisbona and G. I. Shishkin, *A fractional step method on a special mesh for the resolution of multidimensional evolutionary convection-diffusion problems*, Appl. Numer. Math., 1998, 27, 211–231.

[10] C. Clavero and J. Vigo-Aguiar, *Numerical approximation of 2D time dependent singularly perturbed convection–diffusion problems with attractive or repulsive turning points*, Appl. Math. Comput., 2018, 317, 223–233.

[11] A. Das and S. Natesan, *Higher-order convergence with fractional-step method for singularly perturbed 2D parabolic convection-diffusion problems on Shishkin mesh*, Comput. Math. Appl., 2018, 75(7), 2387–2403.

[12] A. Das and S. Natesan, *Stability and error analysis of a fully-discrete numerical method for system of 2D singularly perturbed parabolic PDEs*, Comput. Math. Appl., 2022, 110, 135–145.

[13] P. A. Farrell, A. F. Hegarty, J. J. H. Miller, E. O'Riordan and G. I. Shishkin, *Robust Computational Techniques for Boundary Layers*, Chapman & Hall/CRC Press, 2000.

[14] S. Gowrisankar and S. Natesan, *Robust numerical scheme for singularly perturbed convection-diffusion parabolic initial-boundary-value problems on equidistributed grids*, Comput. Phys. Commun., 2014, 185, 2008–2019.

[15] P. W. Hemker, G. I. Shishkin and L. P. Shishkina, *ϵ-uniform schemes with high-order time-accuracy for parabolic singular perturbation problems*, IMA J. Numer. Anal., 2000, 20(1), 99–121.

[16] H. B. Keller, *Numerical Methods for Two-Point Boundary Value Problems*, Dover, New York, 1992.

[17] R. B. Kellogg and A. Tsan, *Analysis of some differences approximations for a singular perturbation problem without turning point*, Math. Comp., 1978, 32(144), 1025–1039.

[18] M. Kim, R. J. Gillies and K. A. Rejniak, *Current advances in mathematical modeling of anti-cancer drug penetration into tumor tissues*, Frontiers in Oncology, 2013, 3, 278.

[19] O. A. Ladyzenskaja, V. A. Solonnikov and N. N. Ural'ceva, *Linear and Quasi-Linear Equations of Parabolic Type*, volume 23 of *Translations of Mathematical Monographs*, American Mathematical Society, 1968.

[20] T. Linss and N. Madden, *Analysis of an alternating direction method applied to singularly perturbed reaction-diffusion problems*, Int. J. Numer. Anal. Model., 2010, 7(3), 507–519.

[21] N. Madden and S. Russell, *A multiscale sparse grid finite element method for a two-dimensional singularly perturbed reaction-diffusion problem*, Adv. Comput. Math., 2015, 41, 987–1014.

[22] J. J. H. Miller, E. O'Riordan and G. I. Shishkin, *Fitted Numerical Methods for Singular Perturbation Problems*, World Scientific, Singapore, 1996.

[23] B. Mrityunjoy, S. Natesan and A. Sendur, *Alternating direction implicit method for singularly perturbed 2D parabolic convection–diffusion–reaction problem with two small parameters*, Int. J. Comput. Math., 2023, 100(2), 253–282.

[24] K. Mukherjee and S. Natesan, *Parameter-uniform hybrid numerical scheme for time-dependent convection-dominated initial-boundary-value problems*, Computing, 2009, 84(3–4), 209–230.

[25] K. Mukherjee and S. Natesan, *Richardson extrapolation technique for singularly perturbed parabolic convection-diffusion problems*, Computing, 2011, 92(1), 1–32.

[26] K. Mukherjee and S. Natesan, *Parameter-uniform fractional step hybrid numerical scheme for 2D singularly perturbed parabolic convection-diffusion problems*, J. Appl. Math. Comput., 2019, 60(1–2), 51–86.

[27] M. C. Natividad and M. Stynes, *Richardson extrapolation for a convection-diffusion problem using a Shishkin mesh*, Appl. Numer. Math., 2023, 45, 315–329.

[28] H. G. Roos, M. Stynes and L. Tobiska, *Robust Numerical Methods for Singularly Perturbed Differential Equations*, Springer-Verlag, Berlin, 2nd edition, 2008.

[29] G. I. Shishkin and L. P. Shishkina, *The Richardson extrapolation technique for quasilinear parabolic singularly perturbed convection-diffusion equations*, in Journal of Physics: Conference Series, 2006, 5, 19. IOP Publishing.

[30] M. Stynes and T. Linss, *A hybrid difference scheme on a Shishkin mesh for linear convection-diffusion problems*, Appl. Numer. Math., 1999, 31, 255–270.

[31] N. S. Yadav and K. Mukherjee, *On ε-uniform higher order accuracy of new efficient numerical method and its extrapolation for singularly perturbed parabolic problems with boundary layer*, Int. J. Appl. Comput. Math., 2021, 7(3). DOI: 10.1007/s40819-021-00979-7.