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ALLEN-CAHN EQUATION BASED ON AN UNCONSTRAINED ORDER PARAMETER WITH SOURCE TERM AND ITS CAHN-HILLIARD LIMIT

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Abstract Our aim in this paper is to study an Allen-Cahn equation based on a microforce balance and unconstrained order parameter, with the introduction of a source term. We first consider the source term, $g(s) = \beta s$, and obtain the existence, uniqueness and regularity of solutions. We prove that, on finite time intervals, the solutions converge to those of the Cahn-Hilliard-Oono equation as a small parameter goes to zero and then to those of the original Cahn-Hilliard equation as $\beta \to 0^+$. Then, we consider another source term and obtain similar results. In this case, we prove that the solutions converge to those of a Cahn-Hilliard equation on finite time intervals as a small parameter goes to zero. We finally give some numerical simulations which confirm the theoretical results.

Keywords Allen-Cahn system, Cahn-Hilliard system, unconstrained order parameter, source term, well-posedness, passage to the limit.

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1. Introduction

The original Cahn-Hilliard equation

$$\frac{\partial \varphi}{\partial t} + \Delta^2 \varphi - \Delta f(\varphi) = 0,$$

was initially proposed in 1958 by John W. Cahn and John E. Hilliard. It is a time evolution equation for the concentration of a material. It was proposed in order to describe phase separation processes (also known as spinodal decomposition) in binary alloy (see [5,6]). Spinodal decomposition is a process by which a mixture of two materials can separate into distinct regions with different material concentration [5].

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This equation has appeared in many other contexts such as: population dynamics (see [7]), thin films (see [23, 25]), topology optimization (see [14, 27]), image processing (see [3, 10, 22]) and tumor growth (see [8, 15, 17]).

Here f is the derivative of a double-well potential F. A thermodynamically relevant potential F is the following logarithmic function which follows from mean-field model:

$$F(s) = \frac{\theta_1}{2}(1 - s^2) + \frac{\theta_2}{2}\Big[(1 - s)\ln\Big(\frac{1 - s}{2}\Big) + (1 + s)\ln\Big(\frac{1 + s}{2}\Big)\Big],$$

$$s \in (-1, 1),\ 0 < \theta_2 < \theta_1,$$

i.e.,

$$f(s) = -\theta_1 s + \frac{\theta_2}{2}\ln\frac{1 + s}{1 - s}, \quad s \in (-1, 1).$$

Although, as this will be the case here, such a function is very often approximated by regular ones, typically,

$$F(s) = \frac{1}{4}(s^2 - 1),$$

leading to the following cubic nonlinear term

$$f(s) = s^3 - s. \tag{1.1}$$

In [20], the author studied the following model:

$$\frac{\partial \varphi}{\partial t} + \Delta^2\varphi - \Delta f(\varphi) + g(\varphi) = 0, \tag{1.2}$$

where

$$g(s) = \beta s,\ \ \beta > 0,$$

and

$$f(s) = s^3 - s.$$

Equation (1.2) is known as the Cahn–Hilliard–Oono equation and was introduced to model long-ranged interactions in the phase separation process (see [20, 26]). In fact, it was also introduced to simplify numerical simulations (see [24]).

Furthermore, the authors in [18] studied the following variant of the Cahn-Hilliard equation:

$$\frac{\partial \varphi}{\partial t} + \Delta^2\varphi - \Delta f(\varphi) + \frac{k\varphi}{k' + |\varphi|} = 0,\ \ k,\ \ k' > 0, \tag{1.3}$$

that models some energy mechanisms (e.g., lactate) in glial cells. Here, a logarithmic nonlinear term f is considered.

Another phase separation model is the Allen-Cahn equation which was initially suggested in 1979 in [1] by Allen and Cahn to describe the ordering of atoms during the process. This equation has been extensively used to study various physical problems, such as crystal growth (see [9]), image segmentation (see [2]) and the motion by mean curvature flows (see [4, 12]).

The original derivations of the Cahn-Hilliard and Allen-Cahn equations were purely phenomenological; however, this purely phenomenological approach is somehow unsatisfactory from a physical point of view. This led Gurtin, Duda, Sarmiento and Fried to propose an approach based on microforce balance (see [11, 16]). Both derivations, in [11, 16], are based on the assumptions that the phase field coincides with the constituent concentration. The essential difference lied in the treatment of this coincidence, Gurtin [16] identifies the concentration with the phase field from the beginning, while the authors in [11] propose a method to introduce internal reactions to maintain the equality between concentration and phase field as an internal constraint.

In [11], the authors formulated two continuum theories, one constrained and the other unconstrained, for constituent migration in bodies with microstructure described by a scalar phase field. The theories are based on the constituent content balance, the microforce balance, and the free-energy imbalance. The Cahn-Hilliard approach arises from the constrained theory, i.e., the concentration is constrained to be equal to the order parameter, while the Allen-Cahn one is based on an unconstrained theory.

More precisely, the authors in [11] derived the following generalized Allen-Cahn system:

$$\begin{aligned} &\frac{\partial \varphi}{\partial t} = \kappa \Delta \mu - \varepsilon \frac{\partial \mu}{\partial t}, \quad \kappa > 0, \\ &\mu = -\alpha \Delta \varphi + f(\varphi), \quad \alpha > 0, \end{aligned} \tag{1.4}$$

where φ characterizes the ordering of atoms, $\chi = \frac{1}{\varepsilon}$ is a constant coupling energy modulus (ε is a small positive parameter), f is the derivative of a double-well potential F and μ is the chemical potential.

The author in [19], prove that if we let ε go to 0^+ (i.e. χ go to $+\infty$), then we recover the original Cahn-Hilliard system,

$$\begin{aligned} &\frac{\partial \varphi}{\partial t} = \kappa \Delta \mu, \\ &\mu = -\alpha \Delta \varphi + f(\varphi). \end{aligned} \tag{1.5}$$

Here, φ and μ satisfy the homogeneous Dirichlet boundary conditions.

Our aim in this paper is to prove the above convergence of Allen-Cahn equation with a source term.

This paper is organized as follows. In Section 2, we state our assumptions on the mathematical problem and give some useful notation. Then, in Section 3, we consider the source term $g(s) = \beta s$. We first prove the existence, uniqueness and regularity of solutions to the Allen-Cahn system. We next prove that, on finite time intervals, these solutions converge, as $\varepsilon \to 0^+$, to those of the Cahn-Hilliard-Oono system and then, as $\beta \to 0^+$, to those of the original Cahn-Hilliard system. Furthermore, we derive error estimates. In Section 4, we consider another source term and obtain similar results. In this case, we prove that the solutions converge, as $\varepsilon \to 0^+$, to those of a Cahn-Hilliard system on finite time intervals and derive error estimates. Finally, in Section 5, we give some numerical simulations which confirm these results.

2. Setting of the problem and notation

We consider the following initial and boundary value problem in a bounded and regular domain $\Omega \subset \mathbb{R}^n$, $n = 1$, 2 or 3, with boundary Γ:

$$\begin{aligned}
&\frac{\partial \varphi}{\partial t} = \Delta\mu - \varepsilon\frac{\partial \mu}{\partial t} - g(\varphi), \ \ \varepsilon > 0, \\
&\mu = -\Delta\varphi + f(\varphi), \\
&\varphi = \mu = 0 \ \text{ on } \Gamma, \\
&\varphi|_{t=0} = \varphi_0, \ \ \mu|_{t=0} = \mu_0.
\end{aligned} \tag{2.1}$$

We take f as the usual cubic nonlinear term,

$$f(s) = s^3 - s, \quad s \in \mathbb{R},$$

and note that

$$f' \geq -1.$$

We also set

$$F(s) = \int_0^s f(\xi)d\xi,$$

and note that

$$F(s) = \frac{1}{4}s^4 - \frac{1}{2}s^2, \ \ s \in \mathbb{R}.$$

In Section 3, we take

$$g(s) = \beta s, \ \ s \in \mathbb{R},$$

where β is a strictly positive constant. We assume that $\beta \in (0, \beta_0)$, where $\beta_0 > 0$.

In Section 4,

$$g(s) = \frac{ks}{k' + |s|}, \ \ k, \ \ k' > 0, \ \ s \in \mathbb{R}. \tag{2.2}$$

Note that

$$-k \leq g(s) \leq k, \ \ s \in \mathbb{R}, \tag{2.3}$$

and

$$0 \leq g'(s) = \frac{kk'}{(k' + |s|)^2} \leq \frac{k}{k'}, \ \ s \in \mathbb{R}. \tag{2.4}$$

As far as the parameter ε is concerned, we assume that $\varepsilon \in (0, \varepsilon_0]$, where $\varepsilon_0 > 0$ satisfies

$$\varepsilon_0 < 1.$$

Setting

$$w = \varphi + \varepsilon\mu,$$

we can rewrite (2.1) as

$$\begin{aligned}
&\frac{\partial w}{\partial t} = \Delta\frac{w - \varphi}{\varepsilon} - g(\varphi), \\
&\frac{w - \varphi}{\varepsilon} = -\Delta\varphi + f(\varphi),
\end{aligned}$$

i.e.,

$$\begin{aligned}
&\varepsilon\frac{\partial w}{\partial t} - \Delta w = -\Delta\varphi - \varepsilon g(\varphi), \\
&w = -\varepsilon\Delta\varphi + \varepsilon f(\varphi) + \varphi.
\end{aligned}$$

We thus consider the following initial and boundary value problem:

$$\varepsilon\frac{\partial w}{\partial t} - \Delta w = -\Delta\varphi - \varepsilon g(\varphi), \tag{2.5a}$$

$$w = -\varepsilon\Delta\varphi + \varepsilon f(\varphi) + \varphi, \tag{2.5b}$$

$$w = \varphi = 0 \text{ on } \Gamma, \tag{2.5c}$$

$$w|_{t=0} = w_0. \tag{2.5d}$$

We have the following

Theorem 2.1. *Let $h \in L^2(\Omega)$. Then, the elliptic problem*

$$\begin{aligned} &-\varepsilon\Delta\varphi + \varepsilon f(\varphi) + \varphi = h, \\ &\varphi = 0 \text{ on } \Gamma, \end{aligned} \tag{2.6}$$

possesses a unique weak solution $\varphi \in H_0^1(\Omega)$. Furthermore, the mapping $\mathcal{A} : L^2(\Omega) \to H_0^1(\Omega)$, $h \mapsto \varphi$, is globally Lipschitz continuous.

Proof. See [19]. □

It follows that we can rewrite (2.5) in the following equivalent form:

$$\begin{aligned} &\varepsilon\frac{\partial w}{\partial t} - \Delta w + \Delta\mathcal{A}(w) + \varepsilon g(\mathcal{A}(w)) = 0, \\ &w = 0 \text{ on } \Gamma, \\ &w|_{t=0} = w_0. \end{aligned} \tag{2.7}$$

Furthermore, once w is known from (2.7), we obtain φ by setting $\varphi = \mathcal{A}(w)$ and we will take $\varphi_0 = \mathcal{A}(w_0)$. We then recover μ by setting $\mu = \frac{1}{\varepsilon}(w - \varphi)$, with $\mu_0 = \frac{1}{\varepsilon}(w_0 - \varphi_0)$. We thus also (formally) recover (2.1).

Notation. We denote by $(\cdot,\cdot)$ the usual L^2-scalar product, with associated norm $\|\cdot\|$. We also set $\|\cdot\|_{-1} = \|(-\Delta)^{-\frac{1}{2}}\cdot\|$, where $(-\Delta)^{-1}$ denotes the inverse of the minus Laplace operator associated with Dirichlet boundary conditions. More generally, we denote by $\|\cdot\|_X$ the norm on the Banach space X.

We note that

$$v \mapsto \|\nabla v\| = \|(-\Delta)^{\frac{1}{2}} v\|$$

and

$$v \mapsto \|v\|_{-1}$$

are norms on $H_0^1(\Omega)$ and $H^{-1}(\Omega) = H_0^1(\Omega)'$, respectively, which are equivalent to the usual norms on these spaces.

Throughout this paper, the same letters c and c' denote (nonnegative or positive) constants which may vary from line to line, or even in a same line.

3. The Case $g(s) = \beta s$

3.1. Well-posedness and regularity results

We first derive a priori estimates for (2.5). These estimates are formal, but they can be justified within a Galerkin scheme.

We multiply (2.5a) by w, integrate over Ω and by parts and have

$$\frac{\varepsilon}{2}\frac{d}{dt}\|w\|^2 + \|\nabla w\|^2 = (\nabla w, \nabla\varphi) - \varepsilon(g(\varphi), w).$$

Note that,

$$|\varepsilon(g(\varphi), w)| \leq \frac{1}{\varepsilon}\|w\|^2 + c\|\nabla\varphi\|^2,$$

we obtain, employing the Cauchy-Schwarz inequality,

$$\varepsilon\frac{d}{dt}\|w\|^2 + \|\nabla w\|^2 \leq \frac{1}{\varepsilon}\|w\|^2 + c\|\nabla\varphi\|^2. \tag{3.1}$$

Multiplying then (2.5b) by $-\Delta\varphi$, we obtain

$$\varepsilon\|\Delta\varphi\|^2 + \varepsilon(f'(\varphi)\nabla\varphi, \nabla\varphi) + \|\nabla\varphi\|^2 = -(w, \Delta\varphi).$$

Noting that

$$\varepsilon(f'(\varphi)\nabla\varphi, \nabla\varphi) + \|\nabla\varphi\|^2 \geq (1-\varepsilon_0)\|\nabla\varphi\|^2,$$

and employing the Cauchy-Schwarz inequality, we find

$$\varepsilon\|\Delta\varphi\|^2 + 2k_1\|\nabla\varphi\|^2 \leq \frac{1}{\varepsilon}\|w\|^2,$$

where $k_1 = 1-\varepsilon_0$, so that

$$\|\Delta\varphi\|^2 \leq \frac{1}{\varepsilon^2}\|w\|^2 \quad \text{and} \quad \|\nabla\varphi\|^2 \leq \frac{1}{2k_1\varepsilon}\|w\|^2. \tag{3.2}$$

In particular, it follows from (3.1) and (3.2) that

$$\varepsilon\frac{d}{dt}\|w\|^2 + \|\nabla w\|^2 \leq \frac{c}{\varepsilon}\|w\|^2.$$

Next, we multiply (2.5a) by $(-\Delta)^{-1}\frac{\partial w}{\partial t}$ and have

$$\frac{1}{2}\frac{d}{dt}\|w\|^2 + \varepsilon\Big\|\frac{\partial w}{\partial t}\Big\|_{-1}^2 = (\varphi, \frac{\partial w}{\partial t}) - \varepsilon(g(\varphi), (-\Delta)^{-1}\frac{\partial w}{\partial t}).$$

Noting that

$$\begin{aligned}(\varphi, \frac{\partial w}{\partial t}) &= ((-\Delta)^{\frac{1}{2}}\varphi, (-\Delta)^{-\frac{1}{2}}\frac{\partial w}{\partial t}) \\ &\leq \frac{\varepsilon}{4}\Big\|\frac{\partial w}{\partial t}\Big\|_{-1}^2 + \frac{1}{\varepsilon}\|\nabla\varphi\|^2,\end{aligned}$$

and

$$\begin{aligned}\varepsilon\beta(\varphi, (-\Delta)^{-1}\frac{\partial w}{\partial t}) &\leq \frac{\varepsilon}{4}\Big\|\frac{\partial w}{\partial t}\Big\|_{-1}^2 + \frac{c}{\varepsilon}\|\varphi\|^2 \\ &\leq \frac{\varepsilon}{4}\Big\|\frac{\partial w}{\partial t}\Big\|_{-1}^2 + \frac{c}{\varepsilon}\|\nabla\varphi\|^2,\end{aligned}$$

we obtain

$$\frac{d}{dt}\|w\|^2 + \varepsilon\Big\|\frac{\partial w}{\partial t}\Big\|_{-1}^2 \leq \frac{c}{\varepsilon}\|\nabla\varphi\|^2,$$

so that, owing to (3.2),

$$\frac{d}{dt}\|w\|^2+\varepsilon\|\frac{\partial w}{\partial t}\|_{-1}^2 \le \frac{c}{\varepsilon^2}\|w\|^2. \tag{3.3}$$

Differentiating now (2.5b) with respect to time, we obtain

$$\frac{\partial w}{\partial t}=-\varepsilon\Delta\frac{\partial \varphi}{\partial t}+\varepsilon f'(\varphi)\frac{\partial \varphi}{\partial t}+\frac{\partial \varphi}{\partial t}, \tag{3.4}$$

$$\frac{\partial \varphi}{\partial t}=0 \text{ on } \Gamma. \tag{3.5}$$

Multiplying (3.4) by $\frac{\partial \varphi}{\partial t}$, we find

$$\varepsilon\|\nabla\frac{\partial \varphi}{\partial t}\|^2+\varepsilon(f'(\varphi)\frac{\partial \varphi}{\partial t},\frac{\partial \varphi}{\partial t})+\|\frac{\partial \varphi}{\partial t}\|^2=(\frac{\partial w}{\partial t},\frac{\partial \varphi}{\partial t}),$$

which yields, proceeding as above,

$$\varepsilon\|\nabla\frac{\partial \varphi}{\partial t}\|^2+2k_1\|\frac{\partial \varphi}{\partial t}\|^2\le\frac{1}{\varepsilon}\|\frac{\partial w}{\partial t}\|_{-1}^2.$$

We multiply (2.5a) by $-\Delta w$ and have

$$\frac{\varepsilon}{2}\frac{d}{dt}\|\nabla w\|^2+\|\Delta w\|^2=(\Delta w,\Delta\varphi)+\varepsilon\beta(\varphi,\Delta w),$$

which yields,

$$\varepsilon\frac{d}{dt}\|\nabla w\|^2+\|\Delta w\|^2\le\frac{c}{\varepsilon^2}\|w\|^2.$$

Multiplying (2.5b) by $\Delta^2\varphi$, we obtain

$$\varepsilon\|\nabla\Delta\varphi\|^2+\varepsilon(\Delta f(\varphi),\Delta\varphi)+\|\Delta\varphi\|^2=-(\nabla w,\nabla\Delta\varphi). \tag{3.6}$$

Note that

$$\Delta f(\varphi)=(3\varphi^2-1)\Delta\varphi+6\varphi\nabla\varphi\cdot\varphi,$$

so that

$$(\Delta f(\varphi),\Delta\varphi)\ge-\|\Delta\varphi\|^2+6\int_\Omega\varphi\nabla\varphi\cdot\nabla\varphi dx. \tag{3.7}$$

Furthermore, we can see that, employing Hölder's inequality, proper Sobolev embeddings and standard elliptic regularity results,

$$\begin{aligned}|\int_\Omega\varphi\nabla\varphi\cdot\nabla\varphi dx| &\le \|\varphi\|_{L^6(\Omega)}\|\nabla\varphi\|\|\nabla\varphi\|_{L^6(\Omega)^n}\|\Delta\varphi\|_{L^6(\Omega)}\\ &\le c\|\Delta\varphi\|^3\|\nabla\Delta\varphi\|.\end{aligned} \tag{3.8}$$

It thus follows from (3.6)-(3.8) that

$$\varepsilon\|\nabla\Delta\varphi\|^2+2k_1\|\Delta\varphi\|^2\le\frac{2}{\varepsilon}\|\nabla w\|^2+c\varepsilon\|\Delta\varphi\|^6$$

and (3.2) yields

$$\varepsilon\|\nabla\Delta\varphi\|^2+2k_1\|\Delta\varphi\|^2\le\frac{2}{\varepsilon}\|\nabla w\|^2+\frac{c}{\varepsilon^5}\|w\|^6. \tag{3.9}$$

Theorem 3.1. *Let $T > 0$ be given.*

1. *We assume that $w_0 \in L^2(\Omega)$. Then, (2.5) possesses a unique weak solution (w, φ) such that*

$$w \in C([0,T]; L^2(\Omega)) \cap L^2(0,T; H_0^1(\Omega)),$$
$$\frac{\partial w}{\partial t} \in L^2(0,T; H^{-1}(\Omega)),$$
$$\varphi \in C([0,T]; H^2(\Omega)_w) \cap L^\infty(0,T; H^2(\Omega) \cap H_0^1(\Omega)),$$

and

$$\frac{\partial \varphi}{\partial t} \in L^2(0,T; H_0^1(\Omega)),$$

where the index w denotes the weak topology.

2. *If we further assume that $w_0 \in H_0^1(\Omega)$, then,*

$$w \in C([0,T]; H_0^1(\Omega)) \cap L^2(0,T; H^2(\Omega) \cap H_0^1(\Omega)),$$
$$\frac{\partial w}{\partial t} \in L^2(0,T; L^2(\Omega)),$$

and

$$\varphi \in C([0,T]; H^3(\Omega)_w) \cap L^\infty(0,T; H^3(\Omega) \cap H_0^1(\Omega)).$$

Furthermore, the solution is strong.

Proof. *Existence:*

We can note that (2.1) is associated with the following weak formulation:

Find $(w, \varphi) : [0,T] \to H_0^1(\Omega)^2$ such that

$$\begin{aligned} &\varepsilon \frac{d}{dt}(w, v) + (\nabla w, \nabla v) = (\nabla \varphi, \nabla v) - \varepsilon (g(\varphi), v), \ \ \forall v \in H_0^1(\Omega), \\ &(w, v) = \varepsilon (\nabla \varphi, \nabla v) + \varepsilon (f(\varphi), v) + (\varphi, v), \ \ \forall v \in H_0^1(\Omega), \\ &w|_{t=0} = w_0 \in L^2(\Omega). \end{aligned} \tag{3.10}$$

Let $0 < \lambda_1 \leq \lambda_2 \leq \cdots$ be the eigenvectors of the minus Laplace operator associated with Dirichlet boundary conditions and $v_1, v_2, \cdots$ be associated eigenvectors such that the v_j's form an orthonormal in $L^2(\Omega)$ and orthogonal in $H_0^1(\Omega)$ basis. We set

$$V_m = Span(v_1, \cdots, v_m), \ \ m \in \mathbb{N},$$

and introduce the following approximated problem, for $m \in \mathbb{N}$ given:

Find $(w_m, \varphi_m) : [0,T] \to V_m^2$ such that

$$\begin{aligned} &\varepsilon \frac{d}{dt}(w_m, v) + (\nabla w_m, \nabla v) = (\nabla \varphi_m, \nabla v) - \varepsilon (g(\varphi_m), v), \ \ \forall v \in V_m, \\ &(w_m, v) = \varepsilon (\nabla \varphi_m, \nabla v) + \varepsilon (f(\varphi_m), v) + (\varphi_m, v), \ \ \forall v \in V_m, \\ &w_m|_{t=0} = w_{0,m}, \end{aligned} \tag{3.11}$$

where $w_{0,m} = P_m w_0$, P_m being the orthogonal projection from $L^2(\Omega)$ into V_m.

As in Theorem (2.1), we can define the mapping $\mathcal{A}_m : V_m \to V_m$ as follows $\varphi_m = \mathcal{A}_m(h)$, $h \in V_m$, where

$$\varepsilon (\nabla \varphi_m, \nabla v) + \varepsilon (f(\varphi_m), v) + (\varphi_m, v) = (h, v), \ \ \forall v \in V_m.$$

Furthermore, we can see that this mapping is globally Lipschitz continuous, in the sense that

$$\|\nabla(\mathcal{A}_m(h_1)-\mathcal{A}_m(h_2))\| \le \frac{c}{\varepsilon^{\frac{1}{2}}}\|h_1-h_2\|,\ \ h_1,h_2\in V_m,$$

and

$$\|\mathcal{A}_m(h_1)-\mathcal{A}_m(h_2)\| \le \frac{c}{k_2^{\frac{1}{2}}}\|h_1-h_2\|,\ \ h_1,h_2\in V_m,$$

where $k_2=\dfrac{1-\varepsilon_0}{2}$.

We can rewrite (3.11) in the equivalent form

$$\begin{aligned}&\varepsilon\frac{d}{dt}(w_m,v)+(\nabla w_m,\nabla v)=(\nabla\mathcal{A}_m(w_m),\nabla v)-\varepsilon(g(\mathcal{A}_m(w_m)),v),\ \ \forall v\in V_m,\\&\varphi_m=\mathcal{A}_m(w_m),\\&w_m|_{t=0}=w_{0,m}.\end{aligned}\tag{3.12}$$

The existence of a local in time solution to (3.12) then follows from the Cauchy-Lipschitz theorem. It follows, noting that the above a priori estimates hold at the approximated level, that w_m is bounded in $L^\infty(0,T;L^2(\Omega))\cap L^2(0,T;H_0^1(\Omega))$ and $\dfrac{\partial w_m}{\partial t}$ is bounded in $L^2(0,T;H^{-1}(\Omega))$, independently of m. Having this, we can pass to the limit in a standard way, owing to Aubin-Lions campactness results, and deduce the existence of solution.

Uniqueness:

Let (w_1,φ_1) and (w_2,φ_2) be two solution corresponding to the initial data $w_{1,0}$ and $w_{2,0}$ respectively, and set $(w,\varphi)=(w_1,\varphi_1)-(w_2,\varphi_2)$ and $w_0=w_{1,0}-w_{2,0}$. Then,

$$\varepsilon\frac{\partial w}{\partial t}-\Delta w=-\Delta(\mathcal{A}(w_1)-\mathcal{A}(w_2))-\varepsilon(g(\mathcal{A}(w_1))-g(\mathcal{A}(w_2))),\tag{3.13a}$$

$$w=0\ \text{ on }\Gamma,\tag{3.13b}$$

$$w|_{t=0}=w_0.\tag{3.13c}$$

Multiplying (3.13a) by w, we have

$$\begin{aligned}&\frac{\varepsilon}{2}\frac{d}{dt}\|w\|^2+\|\nabla w\|^2\\\le&\frac{1}{2}\|\nabla(\mathcal{A}(w_1)-\mathcal{A}(w_2))\|^2+\frac{1}{2}\|\nabla w\|^2+\varepsilon\|g(\mathcal{A}(w_1))-(\mathcal{A}(w_2))\|\|w\|,\end{aligned}$$

which yields

$$\varepsilon\frac{d}{dt}\|w\|^2+\|\nabla w\|^2\le c\|\nabla(\mathcal{A}(w_1)-\mathcal{A}(w_2))\|^2+\frac{c}{\varepsilon}\|w\|^2$$

and, owing to $\|\nabla(\mathcal{A}(w_1)-\mathcal{A}(w_2))\|^2\le\dfrac{c}{\varepsilon}\|w_1-w_2\|^2$,

$$\frac{d}{dt}\|w\|^2\le\frac{c}{\varepsilon^2}\|w\|^2.\tag{3.14}$$

The uniqueness, as well as the continuous dependence with respect to the initial data in the $L^2(\Omega)$-norm, for w follows from (3.14) and Gronwall's lemma. We then immediately deduce similar results for φ, recalling that $\mathcal{A}$ is globally Lipschitz continuous. □

Theorem 3.2. *We assume that* $(\varphi_0, \mu_0) \in (H^3(\Omega) \cap H_0^1(\Omega)) \times H_0^1(\Omega)$, *with* $\mu_0 = -\Delta\varphi_0 + f(\varphi_0)$. *Furthermore, let* $T > 0$ *be given. Then,* (2.1) *possesses a unique strong solution* (φ, μ) *such that*

$$\begin{aligned}
&\varphi \in C([0,T]; H^3(\Omega)_w) \cap L^\infty(0,T; H^3(\Omega) \cap H_0^1(\Omega)),\\
&\frac{\partial \varphi}{\partial t} \in L^2(0,T; H_0^1(\Omega)),\\
&\mu \in C([0,T]; H_0^1(\Omega)) \cap L^2(0,T; H^2(\Omega) \cap H_0^1(\Omega)),
\end{aligned}$$

and

$$\frac{\partial \mu}{\partial t} \in L^2(0,T; L^2(\Omega)).$$

3.2. Convergence to the Cahn-Hilliard system

Our aim in this section is to pass to the limit in (2.1) as ε goes to 0^+ and β goes to 0^+. Note that the limit problem for $\varepsilon = 0$ corresponds to the Cahn–Hilliard-Oono system,

$$\begin{aligned}
&\frac{\partial \varphi^{0,\beta}}{\partial t} = \Delta\mu^{0,\beta} - \beta\varphi^{0,\beta},\\
&\mu^{0,\beta} = -\Delta\varphi^{0,\beta} + f(\varphi^{0,\beta}),\\
&\varphi^{0,\beta} = \mu^{0,\beta} = 0 \ \text{ on } \Gamma,\\
&\varphi^{0,\beta}|_{t=0} = \varphi_0,\ \ \mu^{0,\beta}|_{t=0} = \mu_0 = -\Delta\varphi_0 + f(\varphi_0),
\end{aligned} \tag{3.15}$$

then for $\beta = 0$ corresponds to the Cahn–Hilliard system,

$$\begin{aligned}
&\frac{\partial \varphi^{0,0}}{\partial t} = \Delta\mu^{0,0},\\
&\mu^{0,0} = -\Delta\varphi^{0,0} + f(\varphi^{0,0}),\\
&\varphi^{0,0} = \mu^{0,0} = 0 \ \text{ on } \Gamma,\\
&\varphi^{0,0}|_{t=0} = \varphi_0,\ \ \mu^{0,0}|_{t=0} = \mu_0 = -\Delta\varphi_0 + f(\varphi_0).
\end{aligned} \tag{3.16}$$

In order to accomplish our purpose, we first need to derive estimates on the solutions to (2.1) which are independent of ε and β (we consider here strong solutions as given in Theorem 3.2).

We thus consider the initial and boundary value problem

$$\frac{\partial \varphi^{\varepsilon,\beta}}{\partial t} = \Delta\mu^{\varepsilon,\beta} - \varepsilon\frac{\partial \mu^{\varepsilon,\beta}}{\partial t} - g(\varphi^{\varepsilon,\beta}), \tag{3.17a}$$

$$\mu^{\varepsilon,\beta} = -\Delta\varphi^{\varepsilon,\beta} + f(\varphi^{\varepsilon,\beta}), \tag{3.17b}$$

$$\varphi^{\varepsilon,\beta} = \mu^{\varepsilon,\beta} = 0 \ \text{on } \Gamma, \tag{3.17c}$$

$$\varphi^{\varepsilon,\beta}|_{t=0} = \varphi_0,\ \ \mu^{\varepsilon,\beta}|_{t=0} = \mu_0 = -\Delta\varphi_0 + f(\varphi_0). \tag{3.17d}$$

Note that the constants below may depend on ε_0 and β_0, but they are independent of ε and β.

We multiply (3.17a) by $\mu^{\varepsilon,\beta}$ and have

$$\frac{\varepsilon}{2}\frac{d}{dt}\|\mu^{\varepsilon,\beta}\|^2 + \|\nabla\mu^{\varepsilon,\beta}\|^2 = -(\frac{\partial\varphi^{\varepsilon,\beta}}{\partial t}, \mu^{\varepsilon,\beta}) - \beta(\varphi^{\varepsilon,\beta}, \mu^{\varepsilon,\beta}). \tag{3.18}$$

Note that multiplying (3.17b) by $\frac{\partial\varphi^{\varepsilon,\beta}}{\partial t}$ we obtain

$$(\frac{\partial\varphi^{\varepsilon,\beta}}{\partial t}, \mu^{\varepsilon,\beta}) = \frac{1}{2}\frac{d}{dt}\|\nabla\varphi^{\varepsilon,\beta}\|^2 + \frac{d}{dt}\int_\Omega F(\varphi^{\varepsilon,\beta})dx. \tag{3.19}$$

Combining (3.18) and (3.19), we find

$$\begin{aligned}&\frac{d}{dt}\big(\varepsilon\|\mu^{\varepsilon,\beta}\|^2 + \|\nabla\varphi^{\varepsilon,\beta}\|^2 + 2\int_\Omega F(\varphi^{\varepsilon,\beta})dx\big) + 2\|\nabla\mu^{\varepsilon,\beta}\|^2\\ =& - 2\beta(\varphi^{\varepsilon,\beta}, -\Delta\varphi^{\varepsilon,\beta} + f(\varphi^{\varepsilon,\beta})).\end{aligned}$$

Noting that

$$(f(s), s) \geq c\|s\|^4_{L^4(\Omega)} - c,$$

we find

$$\begin{aligned}&\frac{d}{dt}\big(\varepsilon\|\mu^{\varepsilon,\beta}\|^2 + \|\nabla\varphi^{\varepsilon,\beta}\|^2 + 2\int_\Omega F(\varphi^{\varepsilon,\beta})dx\big) + 2\|\nabla\mu^{\varepsilon,\beta}\|^2 + 2\beta\|\nabla\varphi^{\varepsilon,\beta}\|^2\\ &+ c\beta\|\varphi^{\varepsilon,\beta}\|^4_{L^4(\Omega)} \leq c.\end{aligned} \tag{3.20}$$

Next, we multiply (3.17a) by $\frac{\partial\mu^{\varepsilon,\beta}}{\partial t}$ and have

$$\frac{1}{2}\frac{d}{dt}\|\nabla\mu^{\varepsilon,\beta}\|^2 + \varepsilon\|\frac{\partial\mu^{\varepsilon,\beta}}{\partial t}\|^2 = -(\frac{\partial\varphi^{\varepsilon,\beta}}{\partial t}, \frac{\partial\mu^{\varepsilon,\beta}}{\partial t}) - \beta(\varphi^{\varepsilon,\beta}, \frac{\partial\mu^{\varepsilon,\beta}}{\partial t}). \tag{3.21}$$

Differentiating then (3.17b) with respect to time, we obtain

$$\frac{\partial\mu^{\varepsilon,\beta}}{\partial t} = -\Delta\frac{\partial\varphi^{\varepsilon,\beta}}{\partial t} + f'(\varphi^{\varepsilon,\beta})\frac{\partial\varphi^{\varepsilon,\beta}}{\partial t}. \tag{3.22}$$

Multiplying (3.22) by $\frac{\partial\varphi^{\varepsilon,\beta}}{\partial t}$, we find

$$(\frac{\partial\mu^{\varepsilon,\beta}}{\partial t}, \frac{\partial\varphi^{\varepsilon,\beta}}{\partial t}) \geq \|\nabla\frac{\partial\varphi^{\varepsilon,\beta}}{\partial t}\|^2 - \|\frac{\partial\varphi^{\varepsilon,\beta}}{\partial t}\|^2,$$

which yields, in view of (3.21),

$$\frac{1}{2}\frac{d}{dt}\|\nabla\mu^{\varepsilon,\beta}\|^2 + \varepsilon\|\frac{\partial\mu^{\varepsilon,\beta}}{\partial t}\|^2 + \|\nabla\frac{\partial\varphi^{\varepsilon,\beta}}{\partial t}\|^2 \leq \|\frac{\partial\varphi^{\varepsilon,\beta}}{\partial t}\|^2 - \beta(\varphi^{\varepsilon,\beta}, \frac{\partial\mu^{\varepsilon,\beta}}{\partial t}). \tag{3.23}$$

Using the fact that

$$(\varphi^{\varepsilon,\beta}, \frac{\partial\mu^{\varepsilon,\beta}}{\partial t}) = \frac{\partial}{\partial t}(\varphi^{\varepsilon,\beta}, \mu^{\varepsilon,\beta}) - (\frac{\partial\varphi^{\varepsilon,\beta}}{\partial t}, \mu^{\varepsilon,\beta}),$$

we find

$$\frac{d}{dt}(\|\nabla\mu^{\varepsilon,\beta}\|^2 + 2\beta(\varphi^{\varepsilon,\beta}, \mu^{\varepsilon,\beta})) + 2\varepsilon\|\frac{\partial\mu^{\varepsilon,\beta}}{\partial t}\|^2 + 2\|\nabla\frac{\partial\varphi^{\varepsilon,\beta}}{\partial t}\|^2 \leq c\|\frac{\partial\varphi^{\varepsilon,\beta}}{\partial t}\|^2 + 2\|\mu^{\varepsilon,\beta}\|^2. \tag{3.24}$$

Employing the interpolation inequality

$$\|\frac{\partial\varphi^{\varepsilon,\beta}}{\partial t}\|^2 \le \|\frac{\partial\varphi^{\varepsilon,\beta}}{\partial t}\|_{-1}\|\nabla\frac{\partial\varphi^{\varepsilon,\beta}}{\partial t}\|,$$

we thus have the following differential inequality

$$\frac{d}{dt}(\|\nabla\mu^{\varepsilon,\beta}\|^2+2\beta(\varphi^{\varepsilon,\beta},\mu^{\varepsilon,\beta}))+2\varepsilon\|\frac{\partial\mu^{\varepsilon,\beta}}{\partial t}\|^2+\|\nabla\frac{\partial\varphi^{\varepsilon,\beta}}{\partial t}\|^2 \le c\|\frac{\partial\varphi^{\varepsilon,\beta}}{\partial t}\|_{-1}^2+2\|\mu^{\varepsilon,\beta}\|^2. \tag{3.25}$$

Note that it follows from (3.17a) that

$$\|\frac{\partial\varphi^{\varepsilon,\beta}}{\partial t}\|_{-1} \le \|\nabla\mu^{\varepsilon,\beta}\| + c\varepsilon\|\frac{\partial\mu^{\varepsilon,\beta}}{\partial t}\| + \beta\|\varphi^{\varepsilon,\beta}\|_{-1}.$$

It thus follows that

$$\begin{aligned}&\frac{d}{dt}(\|\nabla\mu^{\varepsilon,\beta}\|^2+2\beta(\varphi^{\varepsilon,\beta},\mu^{\varepsilon,\beta}))+2\varepsilon\|\frac{\partial\mu^{\varepsilon,\beta}}{\partial t}\|^2+\|\nabla\frac{\partial\varphi^{\varepsilon,\beta}}{\partial t}\|^2\\ \le& c\beta^2\|\varphi^{\varepsilon,\beta}\|_{-1}^2+c\|\nabla\mu^{\varepsilon,\beta}\|^2+c\varepsilon^2\|\frac{\partial\mu^{\varepsilon,\beta}}{\partial t}\|^2,\end{aligned} \tag{3.26}$$

which yields, employing the Poincare inequality

$$\|(-\Delta)^{-1}\varphi^{\varepsilon,\beta}\| \le c\|\nabla\varphi^{\varepsilon,\beta}\|,$$

the differential inequality

$$\begin{aligned}&\frac{d}{dt}(\|\nabla\mu^{\varepsilon,\beta}\|^2+2\beta(\varphi^{\varepsilon,\beta},\mu^{\varepsilon,\beta}))+\|\nabla\frac{\partial\varphi^{\varepsilon,\beta}}{\partial t}\|^2+\varepsilon(2-c\varepsilon_0)\|\frac{\partial\mu^{\varepsilon,\beta}}{\partial t}\|^2\\ \le& c\beta^2\|\nabla\varphi^{\varepsilon,\beta}\|^2+c\|\nabla\mu^{\varepsilon,\beta}\|^2.\end{aligned} \tag{3.27}$$

Summing (3.20) and δ_1 times (3.27), where $\delta_1>0$ is small enough, we obtain

$$\begin{aligned}&\frac{d}{dt}\big(\varepsilon\|\mu^{\varepsilon,\beta}\|^2+\|\nabla\varphi^{\varepsilon,\beta}\|^2+\delta_1\|\nabla\mu^{\varepsilon,\beta}\|^2+2\int_\Omega F(\varphi^{\varepsilon,\beta})dx+2\beta\delta_1(\varphi^{\varepsilon,\beta},\mu^{\varepsilon,\beta})\big)\\ &+c\beta\|\varphi^{\varepsilon,\beta}\|_{L^4(\Omega)}^4+\beta(2-c\beta_0\delta_1)\|\nabla\varphi^{\varepsilon,\beta}\|^2+(2-c\delta_1)\|\nabla\mu^{\varepsilon,\beta}\|^2+\|\nabla\frac{\partial\varphi^{\varepsilon,\beta}}{\partial t}\|^2\\ &+\varepsilon\delta_1(2-c\varepsilon_0)\|\frac{\partial\mu^{\varepsilon,\beta}}{\partial t}\|^2 \le c.\end{aligned} \tag{3.28}$$

We can assume, without loss of generality, that

$$2-c\delta_1>0,\ \ 2-c\beta_0\delta_1>0 \text{ and } 2-c\varepsilon_0>0. \tag{3.29}$$

We can now prove the following

Theorem 3.3. *We assume that the assumptions of Theorem 3.2 and (3.29) hold. Then,*

1. *the sequence of solutions $(\varphi^{\varepsilon,\beta},\mu^{\varepsilon,\beta})$ to (3.17) converges to a weak solution, $(\varphi^{0,\beta},\mu^{0,\beta})$, to (3.15) on finite time intervals $[0,T]$, $T>0$, as $\varepsilon\to 0^+$,*
2. *the sequence of solutions $(\varphi^{0,\beta},\mu^{0,\beta})$ to (3.15) converges to a weak solution, $(\varphi^{0,0},\mu^{0,0})$, to (3.16) on finite time intervals $[0,T]$, $T>0$, as $\beta\to 0^+$.*

Furthermore, $(\varphi^{0,0}, \mu^{0,0})$ *is a strong solution to* (3.16).

Proof.

1. We can note that it follows from the above a priori estimates that $\varphi^{\varepsilon,\beta}$ is bounded in $L^\infty(0,T;H_0^1(\Omega))$, $\dfrac{\partial\varphi^{\varepsilon,\beta}}{\partial t}$ is bounded in $L^2(0,T;H_0^1(\Omega))$ and $\mu^{\varepsilon,\beta}$ is bounded in $L^\infty(0,T;H_0^1(\Omega))$, independently of ε, $T>0$ given. Furthermore, $\varepsilon^{\frac{1}{2}}\dfrac{\partial\mu^{\varepsilon,\beta}}{\partial t}$ is bounded in $L^2(0,T;L^2(\Omega))$ and $\beta^{\frac{1}{2}}\varphi^{\varepsilon,\beta}$ is bounded in $L^2(0,T;L^2(\Omega))$, independently of ε.
Then, the solution converges, up to a subsequence which we do not relabel, to a limit function $(\varphi^{0,\beta}, \mu^{0,\beta})$ in the following sense, as $\varepsilon \to 0^+$,
$$\begin{aligned}
&\varphi^{\varepsilon,\beta} \to \varphi^{0,\beta} \text{ in } L^\infty(0,T;H_0^1(\Omega)) \text{ weak - star},\\
&\frac{\partial\varphi^{\varepsilon,\beta}}{\partial t} \to \frac{\partial\varphi^{0,\beta}}{\partial t} \text{ in } L^2(0,T;H_0^1(\Omega)) \text{ weakly},\\
&\mu^{\varepsilon,\beta} \to \mu^{0,\beta} \text{ in } L^2(0,T;H_0^1(\Omega)) \text{ weakly},\\
&\sqrt{\beta}\varphi^{\varepsilon,\beta} \to \sqrt{\beta}\varphi^{0,\beta} \text{ in } L^2(0,T;L^2(\Omega)) \text{ weakly}
\end{aligned}$$
and
$$\varepsilon\frac{\partial\mu^{\varepsilon,\beta}}{\partial t} \to 0 \text{ in } L^2(0,T;L^2(\Omega)) \text{ weakly}.$$
Then we use a classical Aubin-Lions compactness results and have
$$\varphi^{\varepsilon,\beta} \to \varphi^{0,\beta} \text{ a.e. and in } C([0,T];H^{1-\delta}(\Omega)),\ \ \delta>0.$$
We can pass to the limit in a variational formulation associated with (3.17) and prove that $(\varphi^{0,\beta}, \mu^{0,\beta})$ is a weak solution to (3.15). Indeed, passing to the limit in the linear terms is straightforward. As far as the nonlinear term is concerned, we note that
$$\begin{aligned}
\|\varphi^{\varepsilon,\beta}\|_{L^4(\Omega)}^4 &\le 2(f(\varphi^{\varepsilon,\beta}), \varphi^{\varepsilon,\beta}) + 1\\
&\le 2|(\mu^{\varepsilon,\beta}, \varphi^{\varepsilon,\beta})| + 2\|\nabla\varphi^{\varepsilon,\beta}\|^2 + 1\\
&\le \text{ (using the above estimations)}\\
&\le c,
\end{aligned}$$
then
$$\varphi^{\varepsilon,\beta} \to \varphi^{0,\beta} \text{ in } L^4(0,T;L^4(\Omega)),$$
therefore, up to a subsequence which we again do not relabel,
$$(\varphi^{\varepsilon,\beta})^3 \to (\varphi^{0,\beta})^3 \text{ a.e. and in } L^{\frac{4}{3}}(0,T;L^{\frac{4}{3}}(\Omega)).$$
2. Now, taking $\varepsilon \to 0^+$, we can pass to the limit in the above estimations then we note that $\varphi^{0,\beta}$ is bounded in $L^\infty(0,T;H_0^1(\Omega))$, $\dfrac{\partial\varphi^{0,\beta}}{\partial t}$ is bounded in $L^2(0,T;H_0^1(\Omega))$, $\mu^{0,\beta}$ is bounded in $L^2(0,T;H_0^1(\Omega))$ and $\beta^{\frac{1}{2}}\varphi^{\varepsilon,\beta}$ is bounded in $L^2(0,T;L^2(\Omega))$, independently of β. It thus follows from standard Aubin-Lions compactness results that, at least for a subsequence which we do not relabel, there exists $(\varphi^{0,0}, \mu^{0,0})$ such that, as $\beta \to 0^+$,
$$\varphi^{0,\beta} \to \varphi^{0,0} \text{ in } L^\infty(0,T;H_0^1(\Omega)) \text{ weak - star},$$

$$\varphi^{0,\beta} \to \varphi^{0,0} \text{ a.e. and in } C([0,T]; H^{1-\delta}(\Omega)),\ \ \delta > 0,$$
$$\frac{\partial \varphi^{0,\beta}}{\partial t} \to \frac{\partial \varphi^{0,0}}{\partial t} \text{ in } L^2(0,T; H_0^1(\Omega)) \text{ weakly},$$
$$\mu^{0,\beta} \to \mu^{0,0} \text{ in } L^2(0,T; H_0^1(\Omega)) \text{ weakly}$$

and

$$\sqrt{\beta}\varphi^{0,\beta} \to 0 \text{ in } L^2(0,T; L^2(\Omega)) \text{ weakly}.$$

This is sufficient to pass to the limit in a variational formulation associated with (3.17) and prove that $(\varphi^{0,0}, \mu^{0,0})$ is a weak solution to (3.16).

Finally, since $\varphi_0 \in H^3(\Omega) \cap H_0^1(\Omega)$, it is standard to prove that $(\varphi^{0,0}, \mu^{0,0})$ is a strong solution to (3.16) (see [21]). □

Theorem 3.4. *We assume that the assumptions of Theorem 3.2 and* (3.29) *hold. Then, the sequence of strong solutions* $(\varphi^{\varepsilon,\beta}, \mu^{\varepsilon,\beta})$ *to* (3.17) *converges to a strong solution to* (3.16) *on finite time intervals* $[0,T]$*,* $T > 0$*, as* $\varepsilon \to 0^+$ *and* $\beta \to 0^+$*.*

Proof. We can note that it follows from the above a priori estimates that $\varphi^{\varepsilon,\beta}$ is bounded in $L^\infty(0,T; H_0^1(\Omega))$, $\dfrac{\partial \varphi^{\varepsilon,\beta}}{\partial t}$ is bounded in $L^2(0,T; H_0^1(\Omega))$ and $\mu^{\varepsilon,\beta}$ is bounded in $L^2(0,T; H_0^1(\Omega))$, independently of ε and β, $T > 0$ given. Furthermore, $\varepsilon^{\frac{1}{2}} \dfrac{\partial \mu^{\varepsilon,\beta}}{\partial t}$ is bounded in $L^2(0,T; L^2(\Omega))$ and $\beta^{\frac{1}{2}} \varphi^{\varepsilon,\beta}$ is bounded in $L^2(0,T; L^2(\Omega))$, independently of ε and β. It thus follows from standard Aubin-Lions compactness results that, at least for a subsequence which we do not relabel, there exists $(\varphi^{0,0}, \mu^{0,0})$ such that, as $\varepsilon \to 0^+$ and $\beta \to 0^+$,

$$\varphi^{\varepsilon,\beta} \to \varphi^{0,0} \text{ in } L^\infty(0,T; H_0^1(\Omega)) \text{ weak - star},$$
$$\varphi^{\varepsilon,\beta} \to \varphi^{0,0} \text{ a.e. and in } C([0,T]; H^{1-\delta}(\Omega)),\ \ \delta > 0,$$
$$\frac{\partial \varphi^{\varepsilon,\beta}}{\partial t} \to \frac{\partial \varphi^{0,0}}{\partial t} \text{ in } L^2(0,T; H_0^1(\Omega)) \text{ weakly},$$
$$\mu^{\varepsilon,\beta} \to \mu^{0,0} \text{ in } L^2(0,T; H_0^1(\Omega)) \text{ weakly},$$
$$\beta\varphi^{\varepsilon,\beta} \to 0 \text{ in } L^2(0,T; L^2(\Omega)) \text{ weakly}$$

and

$$\varepsilon \frac{\partial \mu^{\varepsilon,\beta}}{\partial t} \to 0 \text{ in } L^2(0,T; L^2(\Omega)) \text{ weakly}.$$

This is sufficient to pass to the limit in a variational formulation associated with (3.17) and prove that $(\varphi^{0,0}, \mu^{0,0})$ is a weak solution to (3.16). Then, since $\varphi_0 \in H^3(\Omega) \cap H_0^1(\Omega)$, it is standard to prove that $(\varphi^{0,0}, \mu^{0,0})$ is a strong solution to (3.16). □

We can derive error estimates and prove the following

Theorem 3.5. *Under the assumptions of Theorem 3.4, there hold, for* $t \in [0,T]$*,* $T > 0$ *given,*

1.

$$\begin{aligned} &\|(\varphi^{\varepsilon,\beta} - \varphi^{0,\beta})(t)\|_{-1}^2 + c\int_0^T \|\nabla(\varphi^{\varepsilon,\beta} - \varphi^{0,\beta})(s)\|^2 ds \\ \le & c\varepsilon(1+T)e^{c'T}(\|\varphi_0\|_{H^1(\Omega)}^2 + \|\varphi_0\|_{L^4(\Omega)}^4 + \|\mu_0\|_{H^1(\Omega)}^2 + 1), \end{aligned} \tag{3.30}$$

where the constants c *and* c' *are independent of* ε*.*

2.

$$\|(\varphi^{0,\beta}-\varphi^{0,0})(t)\|_{-1}^2+c\int_0^T\|\nabla(\varphi^{0,\beta}-\varphi^{0,0})(s)\|^2ds$$
$$\leq c\beta(1+T)e^{c'T}(\|\varphi_0\|_{H^1(\Omega)}^2+\|\varphi_0\|_{L^4(\Omega)}^4+\|\mu_0\|_{H^1(\Omega)}^2+1), \tag{3.31}$$

where the constants c and c' are independent of β.

3.

$$\|(\varphi^{\varepsilon,\beta}-\varphi^{0,0})(t)\|_{-1}^2+c\int_0^T\|\nabla(\varphi^{\varepsilon,\beta}-\varphi^{0,0})(s)\|^2ds$$
$$\leq c\max(\varepsilon,\beta)(1+T)e^{c'T}(\|\varphi_0\|_{H^1(\Omega)}^2+\|\varphi_0\|_{L^4(\Omega)}^4+\|\mu_0\|_{H^1(\Omega)}^2+1), \tag{3.32}$$

where the constants c and c' are independent of ε and β.

Proof.

1. We set $(\varphi_1,\mu_1)=(\varphi^{\varepsilon,\beta},\mu^{\varepsilon,\beta})-(\varphi^{0,\beta},\mu^{0,\beta})$, $0<\varepsilon<\varepsilon_0$ and $0<\beta<\beta_0$, and have

$$(-\Delta)^{-1}\frac{\partial\varphi_1}{\partial t}=-\mu_1-\varepsilon(-\Delta)^{-1}\frac{\partial\mu^{\varepsilon,\beta}}{\partial t}-\beta(-\Delta)^{-1}\varphi_1, \tag{3.33a}$$
$$\mu_1=-\Delta\varphi_1+f(\varphi^{\varepsilon,\beta})-f(\varphi^{0,\beta}), \tag{3.33b}$$
$$\varphi_1=\mu_1=0 \text{ on } \Gamma, \tag{3.33c}$$
$$\varphi_1|_{t=0}=0. \tag{3.33d}$$

Multiplying (3.33a) by φ_1, we obtain

$$\frac{1}{2}\frac{d}{dt}\|\varphi_1\|_{-1}^2+(\mu_1,\varphi_1)=-\varepsilon(\frac{\partial\mu^{\varepsilon,\beta}}{\partial t},(-\Delta)^{-1}\varphi_1)-\beta(\varphi_1,(-\Delta)^{-1}\varphi_1). \tag{3.34}$$

We then multiply (3.33b) by φ_1 to find,

$$(\mu_1,\varphi_1)\geq\|\nabla\varphi_1\|^2-\|\varphi_1\|^2. \tag{3.35}$$

Using (3.35), the Cauchy-Schwarz and Young's inequalities and the interpolation inequality

$$\|\varphi_1\|^2\leq\|\varphi_1\|_{-1}\|\nabla\varphi_1\|,$$

we obtain

$$\frac{d}{dt}\|\varphi_1\|_{-1}^2+c\|\nabla\varphi_1\|^2\leq c'\|\varphi_1\|_{-1}^2+c\varepsilon^2\|\frac{\partial\mu^{\varepsilon,\beta}}{\partial t}\|^2. \tag{3.36}$$

It follows from (3.36) and Gronwall's lemma that

$$\|\varphi_1(t)\|_{-1}^2+c\int_0^T\|\nabla\varphi_1(s)\|^2ds\leq c(1+T)e^{c'T}\int_0^T\varepsilon^2\|\frac{\partial\mu^{\varepsilon,\beta}}{\partial t}(s)\|^2ds. \tag{3.37}$$

We note that it follows from (3.28) and

$$\beta(\varphi^{\varepsilon,\beta},\mu^{\varepsilon,\beta})\geq c(\|\nabla\varphi^{\varepsilon,\beta}\|^2+\|\varphi^{\varepsilon,\beta}\|_{L^4(\Omega)}^4)-c,$$

that

$$\int_0^T \varepsilon \|\frac{\partial \mu^{\varepsilon,\beta}}{\partial t}(s)\|^2 ds \leq c(\|\varphi_0\|^2_{H^1(\Omega)} + \|\varphi_0\|^4_{L^4(\Omega)} + \|\mu_0\|^2_{H^1(\Omega)} + 1),$$

which finally yields

$$\begin{aligned}&\|\varphi_1(t)\|^2_{-1} + c\int_0^T \|\nabla\varphi_1(s)\|^2 ds\\ \leq & c\varepsilon(1+T)e^{c'T}(\|\varphi_0\|^2_{H^1(\Omega)} + \|\varphi_0\|^4_{L^4(\Omega)} + \|\mu_0\|^2_{H^1(\Omega)} + 1).\end{aligned}$$

2. We set $(\varphi_2, \mu_2) = (\varphi^{0,\beta}, \mu^{0,\beta}) - (\varphi^{0,0}, \mu^{0,0})$, $0 < \beta < \beta_0$, and have

$$(-\Delta)^{-1}\frac{\partial \varphi_2}{\partial t} = -\mu_2 - \beta(-\Delta)^{-1}\varphi^{0,\beta}, \tag{3.38a}$$

$$\mu_2 = -\Delta\varphi_2 + f(\varphi^{0,\beta}) - f(\varphi^{0,0}), \tag{3.38b}$$

$$\varphi_2 = \mu_2 = 0 \text{ on } \Gamma, \tag{3.38c}$$

$$\varphi_2|_{t=0} = 0. \tag{3.38d}$$

Multiplying (3.38a) by φ_2, we obtain

$$\frac{1}{2}\frac{d}{dt}\|\varphi_2\|^2_{-1} + (\mu_2, \varphi_2) = -\beta(\varphi^{0,\beta}, (-\Delta)^{-1}\varphi_2). \tag{3.39}$$

We then multiply (3.38b) by φ_2 to find,

$$(\mu_2, \varphi_2) \geq \|\nabla\varphi_2\|^2 - \|\varphi_2\|^2. \tag{3.40}$$

Using (3.40), the Cauchy-Schwarz and Young's inequalities and the interpolation inequality

$$\|\varphi_2\|^2 \leq \|\varphi_2\|_{-1}\|\nabla\varphi_2\|,$$

we obtain

$$\frac{d}{dt}\|\varphi_2\|^2_{-1} + c\|\nabla\varphi_2\|^2 \leq c'\|\varphi_2\|^2_{-1} + c\beta^2\|\nabla\varphi^{0,\beta}\|^2. \tag{3.41}$$

It follows from (3.41) and Gronwall's lemma that

$$\|\varphi_2(t)\|^2_{-1} + c\int_0^T \|\nabla\varphi_2(s)\|^2 ds \leq c(1+T)e^{c'T}\int_0^T \beta^2\|\nabla\varphi^{0,\beta}(s)\|^2 ds. \tag{3.42}$$

We note that it follows from (3.28), as $\varepsilon \to 0^+$,

$$\begin{aligned}&\frac{d}{dt}\big(\|\nabla\varphi^{0,\beta}\|^2 + \|\nabla\mu^{0,\beta}\|^2 + 2\int_\Omega F(\varphi^{0,\beta})dx + 2\beta(\varphi^{0,\beta}, \mu^{0,\beta})\big)\\ &+(2-c)\|\nabla\mu^{0,\beta}\|^2 + 2\beta\|\nabla\varphi^{0,\beta}\|^2 + c\beta\|\varphi^{0,\beta}\|^4_{L^4(\Omega)} + \|\nabla\frac{\partial\varphi^{0,\beta}}{\partial t}\|^2 \leq c.\end{aligned} \tag{3.43}$$

It follows from (3.43) and

$$\beta(\varphi^{0,\beta}, \mu^{0,\beta}) \geq c(\|\nabla\varphi^{0,\beta}\|^2 + \|\varphi^{0,\beta}\|^4_{L^4(\Omega)}) - c,$$

that

$$\int_0^T \beta\|\nabla\varphi^{0,\beta}(s)\|^2 ds \le c(\|\varphi_0\|^2_{H^1(\Omega)} + \|\varphi_0\|^4_{L^4(\Omega)} + \|\mu_0\|^2_{H^1(\Omega)} + 1),$$

which finally yields

$$\begin{aligned}&\|\varphi_2(t)\|^2_{-1} + c\int_0^T \|\nabla\varphi_2(s)\|^2 ds\\ \le& c\beta(1+T)e^{c'T}(\|\varphi_0\|^2_{H^1(\Omega)} + \|\varphi_0\|^4_{L^4(\Omega)} + \|\mu_0\|^2_{H^1(\Omega)} + 1).\end{aligned}$$

3. We set $(\varphi,\mu) = (\varphi^{\varepsilon,\beta},\mu^{\varepsilon,\beta}) - (\varphi^{0,0},\mu^{0,0})$, $0<\varepsilon<\varepsilon_0$ and $0<\beta<\beta_0$, and have

$$(-\Delta)^{-1}\frac{\partial\varphi}{\partial t} = -\mu - \varepsilon(-\Delta)^{-1}\frac{\partial\mu^{\varepsilon,\beta}}{\partial t} - \beta(-\Delta)^{-1}\varphi^{\varepsilon,\beta}, \tag{3.44a}$$

$$\mu = -\Delta\varphi + f(\varphi^{\varepsilon,\beta}) - f(\varphi^{0,0}), \tag{3.44b}$$

$$\varphi = \mu = 0 \text{ on } \Gamma, \tag{3.44c}$$

$$\varphi|_{t=0} = 0. \tag{3.44d}$$

Multiplying (3.44a) by φ, we obtain

$$\frac{1}{2}\frac{d}{dt}\|\varphi\|^2_{-1} + (\mu,\varphi) = -\varepsilon(\frac{\partial\mu^{\varepsilon,\beta}}{\partial t},(-\Delta)^{-1}\varphi) - \beta(\varphi^{\varepsilon,\beta},(-\Delta)^{-1}\varphi). \tag{3.45}$$

We then multiply (3.44b) by φ to find,

$$(\mu,\varphi) \ge \|\nabla\varphi\|^2 - \|\varphi\|^2. \tag{3.46}$$

Combining (3.45) and (3.46), we have

$$\frac{1}{2}\frac{d}{dt}\|\varphi\|^2_{-1} + \|\nabla\varphi\|^2 \le \|\varphi\|^2 - \varepsilon(\frac{\partial\mu^{\varepsilon,\beta}}{\partial t},(-\Delta)^{-1}\varphi) - \beta(\varphi^{\varepsilon,\beta},(-\Delta)^{-1}\varphi),$$

which yields, employing the Cauchy-Schwarz and Young's inequalities and the interpolation inequality

$$\|\varphi\|^2 \le \|\varphi\|_{-1}\|\nabla\varphi\|,$$

the differential inequality

$$\frac{d}{dt}\|\varphi\|^2_{-1} + c\|\nabla\varphi\|^2 \le c'\|\varphi\|^2_{-1} + c\varepsilon^2\|\frac{\partial\mu^{\varepsilon,\beta}}{\partial t}\|^2 + c\beta^2\|\nabla\varphi^{\varepsilon,\beta}\|^2. \tag{3.47}$$

It follows from (3.47) and Gronwall's lemma that

$$\begin{aligned}&\|\varphi(t)\|^2_{-1} + c\int_0^T \|\nabla\varphi(s)\|^2 ds\\ \le& ce^{c'T}(1+T)\int_0^T \big(\varepsilon^2\|\frac{\partial\mu^{\varepsilon,\beta}}{\partial t}(s)\|^2 + \beta^2\|\nabla\varphi^{\varepsilon,\beta}(s)\|^2\big)ds.\end{aligned} \tag{3.48}$$

We note that it follows from (3.28) and

$$\beta(\varphi^{\varepsilon,\beta},\mu^{\varepsilon,\beta}) \ge c(\|\nabla\varphi^{\varepsilon,\beta}\|^2 + \|\varphi^{\varepsilon,\beta}\|^4_{L^4(\Omega)}) - c,$$

that

$$\int_0^T \big(\varepsilon\|\frac{\partial\mu^{\varepsilon,\beta}}{\partial t}(s)\|^2+2\beta\|\nabla\varphi^{\varepsilon,\beta}(s)\|^2\big)ds$$
$$\le c(\|\varphi_0\|^2_{H^1(\Omega)}+\|\varphi_0\|^4_{L^4(\Omega)}+\|\mu_0\|^2_{H^1(\Omega)}+1),$$

which finally yields

$$\|\varphi(t)\|^2_{-1}+c\int_0^T\|\nabla\varphi(s)\|^2ds$$
$$\le c\max(\varepsilon,\beta)(1+T)e^{c'T}(\|\varphi_0\|^2_{H^1(\Omega)}+\|\varphi_0\|^4_{L^4(\Omega)}+\|\mu_0\|^2_{H^1(\Omega)}+1).$$

□

4. The Case $g(s)=\dfrac{ks}{k'+|s|}$

4.1. Well-posedness and regularity results

The estimates below are formal, but they can be justified within a Galerkin scheme.

We multiply (2.5a) by w, integrate over Ω and by parts and have

$$\frac{\varepsilon}{2}\frac{d}{dt}\|w\|^2+\|\nabla w\|^2=(\nabla w,\nabla\varphi)-\varepsilon(g(\varphi),w).$$

Note that,

$$|\varepsilon(g(\varphi),w)|\le\frac{1}{\varepsilon}\|w\|^2+c',$$

we obtain, employing the Cauchy-Schwarz inequality and (3.2),

$$\varepsilon\frac{d}{dt}\|w\|^2+\|\nabla w\|^2\le\frac{c}{\varepsilon}\|w\|^2+c'.$$

Next, we multiply (2.5a) by $(-\Delta)^{-1}\dfrac{\partial w}{\partial t}$ and have

$$\frac{1}{2}\frac{d}{dt}\|w\|^2+\varepsilon\Big\|\frac{\partial w}{\partial t}\Big\|^2_{-1}=(\varphi,\frac{\partial w}{\partial t})-\varepsilon(g(\varphi),(-\Delta)^{-1}\frac{\partial w}{\partial t}),$$

which yields,

$$\frac{d}{dt}\|w\|^2+\varepsilon\Big\|\frac{\partial w}{\partial t}\Big\|^2_{-1}\le\frac{c}{\varepsilon}\|\nabla\varphi\|^2+c',$$

so that, owing to (3.2),

$$\frac{d}{dt}\|w\|^2+\varepsilon\Big\|\frac{\partial w}{\partial t}\Big\|^2_{-1}\le\frac{c}{\varepsilon^2}\|w\|^2+c'. \tag{4.1}$$

Similarly to the previous section, it follows that

Theorem 4.1. *Let $T>0$ be given.*

1. *We assume that $w_0\in L^2(\Omega)$. Then, (2.5) possesses a unique weak solution (w,φ) such that*

$$w\in C([0,T];L^2(\Omega))\cap L^2(0,T;H^1_0(\Omega)),$$

$$\frac{\partial w}{\partial t} \in L^2(0,T;H^{-1}(\Omega)),$$
$$\varphi \in C([0,T];H^2(\Omega)_w) \cap L^\infty(0,T;H^2(\Omega)\cap H_0^1(\Omega)),$$

and
$$\frac{\partial \varphi}{\partial t} \in L^2(0,T;H_0^1(\Omega)),$$

where the index w denotes the weak topology.

2. *If we further assume that $w_0 \in H_0^1(\Omega)$, then,*

$$w \in C([0,T];H_0^1(\Omega)) \cap L^2(0,T;H^2(\Omega)\cap H_0^1(\Omega)),$$
$$\frac{\partial w}{\partial t} \in L^2(0,T;L^2(\Omega)),$$

and
$$\varphi \in C([0,T];H^3(\Omega)_w) \cap L^\infty(0,T;H^3(\Omega)\cap H_0^1(\Omega)).$$

Furthermore, the solution is strong.

Proof. The proof is similar to the one of Theorem 3.1. □

Theorem 4.2. *We assume that $(\varphi_0,\mu_0) \in (H^3(\Omega)\cap H_0^1(\Omega)) \times H_0^1(\Omega)$, with $\mu_0 = -\Delta\varphi_0 + f(\varphi_0)$. Furthermore, let $T > 0$ be given. Then,* (2.1) *possesses a unique strong solution (φ,μ) such that*

$$\varphi \in C([0,T];H^3(\Omega)_w) \cap L^\infty(0,T;H^3(\Omega)\cap H_0^1(\Omega)),$$
$$\frac{\partial \varphi}{\partial t} \in L^2(0,T;H_0^1(\Omega)),$$
$$\mu \in C([0,T];H_0^1(\Omega)) \cap L^2(0,T;H^2(\Omega)\cap H_0^1(\Omega)),$$

and
$$\frac{\partial \mu}{\partial t} \in L^2(0,T;L^2(\Omega)).$$

4.2. Convergence to the Cahn-Hilliard system

We now wish to pass to the limit in (2.1) as ε goes to 0^+. Note that the limit problem for $\varepsilon = 0$ corresponds to the Cahn–Hilliard system,

$$\begin{aligned}
&\frac{\partial \varphi^0}{\partial t} = \Delta\mu^0 - g(\varphi^0),\\
&\mu^0 = -\Delta\varphi^0 + f(\varphi^0),\\
&\varphi^0 = \mu^0 = 0 \text{ on } \Gamma,\\
&\varphi^0|_{t=0} = \varphi_0,\ \mu^0|_{t=0} = \mu_0 = -\Delta\varphi_0 + f(\varphi_0).
\end{aligned} \tag{4.2}$$

In order to accomplish our purpose, we first need to derive estimates on the solutions to (2.1) which are independent of ε (we consider here strong solutions as given in Theorem 4.2).

We thus consider the initial and boundary value problem

$$\frac{\partial \varphi^\varepsilon}{\partial t} = \Delta\mu^\varepsilon - \varepsilon\frac{\partial \mu^\varepsilon}{\partial t} - g(\varphi^\varepsilon), \tag{4.3a}$$

$$\mu^\varepsilon = -\Delta\varphi^\varepsilon + f(\varphi^\varepsilon), \tag{4.3b}$$

$$\varphi^\varepsilon = \mu^\varepsilon = 0 \text{ on } \Gamma, \tag{4.3c}$$

$$\varphi^\varepsilon|_{t=0} = \varphi_0,\ \mu^\varepsilon|_{t=0} = \mu_0 = -\Delta\varphi_0 + f(\varphi_0). \tag{4.3d}$$

Note that the constants below may depend on ε_0, but they are independent of ε.

We multiply (4.3a) by μ^ε and have

$$\frac{\varepsilon}{2}\frac{d}{dt}\|\mu^\varepsilon\|^2 + \|\nabla\mu^\varepsilon\|^2 = -(\frac{\partial\varphi^\varepsilon}{\partial t}, \mu^\varepsilon) - (g(\varphi^\varepsilon), \mu^\varepsilon). \tag{4.4}$$

Note that multiplying (4.3b) by $\frac{\partial\varphi^\varepsilon}{\partial t}$ we obtain

$$(\frac{\partial\varphi^\varepsilon}{\partial t}, \mu^\varepsilon) = \frac{1}{2}\frac{d}{dt}\|\nabla\varphi^\varepsilon\|^2 + \frac{d}{dt}\int_\Omega F(\varphi^\varepsilon)dx. \tag{4.5}$$

Combining (4.4) and (4.5), we find

$$\frac{d}{dt}\big(\varepsilon\|\mu^\varepsilon\|^2 + \|\nabla\varphi^\varepsilon\|^2 + 2\int_\Omega F(\varphi^\varepsilon)dx\big) + 2\|\nabla\mu^\varepsilon\|^2 = (g(\varphi^\varepsilon), \mu^\varepsilon).$$

Using (2.3), we find

$$\frac{d}{dt}\big(\varepsilon\|\mu^\varepsilon\|^2 + \|\nabla\varphi^\varepsilon\|^2 + 2\int_\Omega F(\varphi^\varepsilon)dx\big) + \|\nabla\mu^\varepsilon\|^2 \le c. \tag{4.6}$$

Next, we multiply (4.3a) by $\frac{\partial\mu^\varepsilon}{\partial t}$ and have

$$\frac{1}{2}\frac{d}{dt}\|\nabla\mu^\varepsilon\|^2 + \varepsilon\|\frac{\partial\mu^\varepsilon}{\partial t}\|^2 = -(\frac{\partial\varphi^\varepsilon}{\partial t}, \frac{\partial\mu^\varepsilon}{\partial t}) - (g(\varphi^\varepsilon), \frac{\partial\mu^\varepsilon}{\partial t}). \tag{4.7}$$

Differentiating then (4.3b) with respect to time, we obtain

$$\frac{\partial\mu^\varepsilon}{\partial t} = -\Delta\frac{\partial\varphi^\varepsilon}{\partial t} + f'(\varphi^\varepsilon)\frac{\partial\varphi^\varepsilon}{\partial t}. \tag{4.8}$$

Multiplying (4.8) by $\frac{\partial\varphi^\varepsilon}{\partial t}$, we find

$$(\frac{\partial\mu^\varepsilon}{\partial t}, \frac{\partial\varphi^\varepsilon}{\partial t}) \ge \|\nabla\frac{\partial\varphi^\varepsilon}{\partial t}\|^2 - \|\frac{\partial\varphi^\varepsilon}{\partial t}\|^2,$$

which yields, in view of (4.7),

$$\frac{1}{2}\frac{d}{dt}\|\nabla\mu^\varepsilon\|^2 + \varepsilon\|\frac{\partial\mu^\varepsilon}{\partial t}\|^2 + \|\nabla\frac{\partial\varphi^\varepsilon}{\partial t}\|^2 \le \|\frac{\partial\varphi^\varepsilon}{\partial t}\|^2 - (g(\varphi^\varepsilon), \frac{\partial\mu^\varepsilon}{\partial t}). \tag{4.9}$$

Using the fact that

$$(g(\varphi^\varepsilon), \frac{\partial\mu^\varepsilon}{\partial t}) = \frac{\partial}{\partial t}(g(\varphi^\varepsilon), \mu^\varepsilon) - (\frac{\partial g(\varphi^\varepsilon)}{\partial t}, \mu^\varepsilon),$$

we find

$$\frac{d}{dt}(\|\nabla\mu^\varepsilon\|^2 + 2(g(\varphi^\varepsilon), \mu^\varepsilon)) + 2\varepsilon\|\frac{\partial\mu^\varepsilon}{\partial t}\|^2 + 2\|\nabla\frac{\partial\varphi^\varepsilon}{\partial t}\|^2 \le c\|\frac{\partial\varphi^\varepsilon}{\partial t}\|^2 + |(g'(\varphi^\varepsilon)\frac{\partial\varphi^\varepsilon}{\partial t}, \mu^\varepsilon)|. \tag{4.10}$$

Using

$$|(g'(\varphi^\varepsilon)\frac{\partial\varphi^\varepsilon}{\partial t},\mu^\varepsilon)| \leq c\|\frac{\partial\varphi^\varepsilon}{\partial t}\|^2 + c\|\mu^\varepsilon\|^2,$$

and the interpolation inequality

$$\|\frac{\partial\varphi^\varepsilon}{\partial t}\|^2 \leq \|\frac{\partial\varphi^\varepsilon}{\partial t}\|_{-1}\|\nabla\frac{\partial\varphi^\varepsilon}{\partial t}\|,$$

we thus have the following differential inequality

$$\frac{d}{dt}(\|\nabla\mu^\varepsilon\|^2 + 2(g(\varphi^\varepsilon),\mu^\varepsilon)) + 2\varepsilon\|\frac{\partial\mu^\varepsilon}{\partial t}\|^2 + \|\nabla\frac{\partial\varphi^\varepsilon}{\partial t}\|^2 \leq c\|\frac{\partial\varphi^\varepsilon}{\partial t}\|_{-1}^2 + c\|\mu^\varepsilon\|^2. \quad (4.11)$$

Multiplying (4.3a) by $(-\Delta)^{-1}\frac{\partial\varphi^\varepsilon}{\partial t}$ and using (2.3), we obtain

$$\|\frac{\partial\varphi^\varepsilon}{\partial t}\|_{-1}^2 \leq c_1\|\nabla\mu^\varepsilon\|^2 + c_2\varepsilon^2\|\frac{\partial\mu^\varepsilon}{\partial t}\|^2 + c_3\|\frac{\partial\varphi^\varepsilon}{\partial t}\|_{-1}^2 + c_4,$$

where $1 - c_3 > 0$.

It thus follows that

$$\frac{d}{dt}(\|\nabla\mu^\varepsilon\|^2 + 2(g(\varphi^\varepsilon),\mu^\varepsilon)) + 2\varepsilon\|\frac{\partial\mu^\varepsilon}{\partial t}\|^2 + \|\nabla\frac{\partial\varphi^\varepsilon}{\partial t}\|^2 \leq c + c_1\|\nabla\mu^\varepsilon\|^2 + c_2\varepsilon^2\|\frac{\partial\mu^\varepsilon}{\partial t}\|^2. \quad (4.12)$$

Summing (4.6) and δ_2 times (4.12), where $\delta_2 > 0$ is small enough, we obtain

$$\begin{aligned}&\frac{d}{dt}\big(\varepsilon\|\mu^\varepsilon\|^2 + \|\nabla\varphi^\varepsilon\|^2 + \delta_2\|\nabla\mu^\varepsilon\|^2 + 2\int_\Omega F(\varphi^\varepsilon)dx + 2\delta_2(g(\varphi^\varepsilon),\mu^\varepsilon)\big)\\ &+(1-c_1\delta_2)\|\nabla\mu^\varepsilon\|^2 + \delta_2\|\nabla\frac{\partial\varphi^\varepsilon}{\partial t}\|^2 + \varepsilon\delta_2(2-c_2\varepsilon_0)\|\frac{\partial\mu^\varepsilon}{\partial t}\|^2 \leq c.\end{aligned} \quad (4.13)$$

We can assume, without loss of generality, that

$$1 - c_1\delta_2 > 0 \text{ and } 2 - c_2\varepsilon_0 > 0. \quad (4.14)$$

Theorem 4.3. *We assume that the assumptions of Theorem 4.2 and* (4.14) *hold. Then, the sequence of strong solutions* $(\varphi^\varepsilon,\mu^\varepsilon)$ *to* (4.3) *converges to a strong solution to* (4.2) *on finite time intervals* $[0,T]$, $T>0$, *as* $\varepsilon \to 0^+$.

Proof. We can note that it follows from the above a priori estimates that φ^ε is bounded in $L^\infty(0,T;H_0^1(\Omega))$, $\frac{\partial\varphi^\varepsilon}{\partial t}$ is bounded in $L^2(0,T;H_0^1(\Omega))$ and μ^ε is bounded in $L^2(0,T;H_0^1(\Omega))$, independently of ε, $T>0$ given. Furthermore, $\varepsilon^{\frac{1}{2}}\frac{\partial\mu^\varepsilon}{\partial t}$ is bounded in $L^2(0,T;L^2(\Omega))$, independently of ε. It thus follows from standard Aubin-Lions compactness results that, at least for a subsequence which we do not relabel, there exists (φ^0,μ^0) such that, as $\varepsilon\to 0^+$,

$$\begin{aligned}&\varphi^\varepsilon \to \varphi^0 \text{ in } L^\infty(0,T;H_0^1(\Omega)) \text{ weak - star},\\ &\varphi^\varepsilon \to \varphi^0 \text{ a.e. and in } C([0,T];H^{1-\delta}(\Omega)),\ \delta>0,\\ &\frac{\partial\varphi^\varepsilon}{\partial t} \to \frac{\partial\varphi^0}{\partial t} \text{ in } L^2(0,T;H_0^1(\Omega)) \text{ weakly},\end{aligned}$$

$$\mu^\varepsilon \to \mu^0 \text{ in } L^2(0,T;H_0^1(\Omega)) \text{ weakly}$$

and

$$\varepsilon\frac{\partial\mu^\varepsilon}{\partial t} \to 0 \text{ in } L^2(0,T;L^2(\Omega)) \text{ weakly.}$$

This is sufficient to pass to the limit in a variational formulation associated with (4.3) and prove that (φ^0,μ^0) is a weak solution to (4.2). Then, since $\varphi_0 \in H^3(\Omega)\cap H_0^1(\Omega)$, it is standard to prove that (φ^0,μ^0) is a strong solution to (4.2). □

Theorem 4.4. *Under the assumptions of Theorem 4.3, there holds, for* $t\in[0,T]$, $T>0$ *given,*

$$\begin{aligned}&\|(\varphi^\varepsilon-\varphi^0)(t)\|_{-1}^2 + c\int_0^T \|\nabla(\varphi^\varepsilon-\varphi^0)(s)\|^2 ds\\ \le & c\varepsilon(1+T)e^{c'T}(\|\varphi_0\|_{H^1(\Omega)}^2+\|\varphi_0\|_{L^4(\Omega)}^4+\|\mu_0\|_{H^1(\Omega)}^2+1),\end{aligned} \tag{4.15}$$

where the constants c *and* c' *are independent of* ε *and* β.

Proof. We set $(\varphi,\mu)=(\varphi^\varepsilon,\mu^\varepsilon)-(\varphi^0,\mu^0)$, $0<\varepsilon<\varepsilon_0$, and have

$$(-\Delta)^{-1}\frac{\partial\varphi}{\partial t} = -\mu-\varepsilon(-\Delta)^{-1}\frac{\partial\mu^\varepsilon}{\partial t} - (-\Delta)^{-1}\big(g(\varphi^\varepsilon)-g(\varphi^0)\big), \tag{4.16a}$$

$$\mu = -\Delta\varphi + f(\varphi^\varepsilon)-f(\varphi^0), \tag{4.16b}$$

$$\varphi=\mu=0 \text{ on } \Gamma, \tag{4.16c}$$

$$\varphi|_{t=0}=0. \tag{4.16d}$$

Multiplying (4.16a) by φ, we obtain

$$\frac{1}{2}\frac{d}{dt}\|\varphi\|_{-1}^2 + (\mu,\varphi) = -\varepsilon(\frac{\partial\mu^\varepsilon}{\partial t},(-\Delta)^{-1}\varphi) - (g(\varphi^\varepsilon)-g(\varphi^0),(-\Delta)^{-1}\varphi). \tag{4.17}$$

We then multiply (4.16b) by φ to find,

$$(\mu,\varphi) \ge \|\nabla\varphi\|^2 - \|\varphi\|^2. \tag{4.18}$$

Combining (4.17) and (4.18), we have

$$\frac{1}{2}\frac{d}{dt}\|\varphi\|_{-1}^2 + \|\nabla\varphi\|^2 \le \|\varphi\|^2 - \varepsilon(\frac{\partial\mu^\varepsilon}{\partial t},(-\Delta)^{-1}\varphi) - (g(\varphi^\varepsilon)-g(\varphi^0),(-\Delta)^{-1}\varphi),$$

which yields, employing the Cauchy-Schwarz and Young's inequalities and the interpolation inequality

$$\|\varphi\|^2 \le \|\varphi\|_{-1}\|\nabla\varphi\|,$$

the differential inequality

$$\frac{d}{dt}\|\varphi\|_{-1}^2 + c\|\nabla\varphi\|^2 \le c'\|\varphi\|_{-1}^2 + c\varepsilon^2\|\frac{\partial\mu^\varepsilon}{\partial t}\|^2 + |(g(\varphi^\varepsilon)-g(\varphi^0),(-\Delta)^{-1}\varphi)|. \tag{4.19}$$

We note that

$$\begin{aligned}|(g(\varphi^\varepsilon)-g(\varphi^0),(-\Delta)^{-1}\varphi)| &\le \int_\Omega |(-\Delta)^{-1}\varphi||\varphi| \int_0^1 |g'(s\varphi^\varepsilon+(1-s)\varphi^0)|dsdx\\ &\le c\int_\Omega |(-\Delta)^{-1}\varphi||\varphi|dx\\ &\le c\|\varphi\|_{-1}\|\varphi\|\\ &\le c\|\varphi\|_{-1}^2+c\|\nabla\varphi\|^2,\end{aligned}$$

which yields

$$\frac{d}{dt}\|\varphi\|_{-1}^2+c\|\nabla\varphi\|^2 \le c'\|\varphi\|_{-1}^2+c\varepsilon^2\|\frac{\partial\mu^\varepsilon}{\partial t}\|^2. \tag{4.20}$$

It follows from (4.20) and Gronwall's lemma that

$$\|\varphi(t)\|_{-1}^2+c\int_0^T \|\nabla\varphi(s)\|^2ds \le c(1+T)e^{c'T}\int_0^T \varepsilon^2\|\frac{\partial\mu^\varepsilon}{\partial t}(s)\|^2ds. \tag{4.21}$$

We note that it follows from (4.13) that

$$\int_0^T \varepsilon\|\frac{\partial\mu^\varepsilon}{\partial t}(s)\|^2ds \le c(\|\varphi_0\|_{H^1(\Omega)}^2+\|\varphi_0\|_{L^4(\Omega)}^4+\|\mu_0\|_{H^1(\Omega)}^2+1),$$

which finally yields

$$\|\varphi(t)\|_{-1}^2+c\int_0^T \|\nabla\varphi(s)\|^2ds \le c\varepsilon(1+T)e^{c'T}(\|\varphi_0\|_{H^1(\Omega)}^2+\|\varphi_0\|_{L^4(\Omega)}^4+\|\mu_0\|_{H^1(\Omega)}^2+1).$$

□

Remark 4.1. The Allen-Cahn system endowed with Neumann boundary conditions,

$$\frac{\partial\varphi}{\partial t}=\Delta\mu-\varepsilon\frac{\partial\mu}{\partial t}-g(\varphi),\ \ \varepsilon>0, \tag{4.22a}$$

$$\mu=-\Delta\varphi+f(\varphi), \tag{4.22b}$$

$$\frac{\partial\varphi}{\partial\nu}=\frac{\partial\mu}{\partial\nu}=0 \text{ on } \Gamma, \tag{4.22c}$$

$$\varphi|_{t=0}=\varphi_0,\ \ \mu|_{t=0}=\mu_0, \tag{4.22d}$$

where ν is the unit outer normal, can also be considered. In this case, integrating (4.22a) and owing to the boundary conditions, we obtain

$$\frac{d}{dt}\Big(\langle\varphi\rangle+\varepsilon\langle\mu\rangle\Big)+\langle g(\varphi)\rangle=0, \tag{4.23}$$

where $\langle\cdot\rangle=\dfrac{1}{\mathrm{Vol}(\Omega)}\int_\Omega\cdot\ \mathrm{dx}$. We can then adapt the above proofs to these boundary conditions using (4.23), definition of source term, $g(s)$, and the fact that

$$v\mapsto(\|\bar{v}\|_{-1}^2+\langle v\rangle^2)^{\frac{1}{2}},$$

$$v \mapsto (\|\bar{v}\|^2 + \langle v\rangle^2)^{\frac{1}{2}},$$

$$v \mapsto (\|\nabla v\|^2 + \langle v\rangle^2)^{\frac{1}{2}},$$

and

$$v \mapsto (\|\Delta v\|^2 + \langle v\rangle^2)^{\frac{1}{2}},$$

are norms in $H^{-1}(\Omega)$, $L^2(\Omega)$, $H^1(\Omega)$, and $H^2(\Omega)$, respectively, which are equivalent to the usual ones.

5. Numerical validation

Consider the initial and boundary value problem

$$\frac{\partial\varphi}{\partial t} = \Delta\mu - \varepsilon\frac{\partial\mu}{\partial t} - g(\varphi), \tag{5.1a}$$

$$\mu = -\Delta\varphi + f(\varphi), \tag{5.1b}$$

$$\varphi = \mu = 0 \text{ on } \Gamma, \tag{5.1c}$$

$$\varphi|_{t=0} = \varphi_0,\ \ \mu|_{t=0} = \mu_0 = -\Delta\varphi_0 + f(\varphi_0). \tag{5.1d}$$

For the numerical simulations, we used Newton's algorithm to approach the solution of the nonlinear system. Then, for the obtained linearized problem, we use a P1-finite element for the space discretization. The numerical simulations are performed with the software Freefem++ [13].

In the numerical results presented below, Ω is a $(0,1)\times(0,1)$-rectangle. The triangulation is obtained by dividing Ω into 100×100 rectangles and by dividing each rectangle along the same diagonal. The time step is taken as $\Delta t = 0.00001$. We furthermore take $f(s) = s^3 - s$.

5.1. The case when $g(s) = \beta s$

Figures 4 and 5 show the evolution of $\langle\varphi\rangle = \frac{1}{\mathrm{Vol}(\Omega)}\int_\Omega \varphi dx$, with respect to time, φ being the numerical solution to (5.1).

In Fig. 4, we take $\varphi_0(x,y) = 1 - \sin^2(x+y)$. Fig. 5 corresponds to the initial datum $\varphi_0(x,y) = \frac{\cos(x-y)}{x^6+1}$.

Figures 1 and 2 correspond to the cases $\beta = 0.05$ and $\beta = 0.5$, respectively. The solutions tend to the solution of (5.1) for $\varepsilon = 0$, as ε decreases.

In Figures 3 and 5, we take different values for ε and β. The solutions tend to the solution of (5.1) for $\varepsilon = 0$ and $\beta = 0$, as ε and β decrease.

In Fig. 10, we take $\varphi_0(x,y) = \frac{1}{5(x^6+1)(y^5+0.5)}$ and $\beta = 0.01$. Figures 6, 7, 8 and 9 correspond to the cases $\varepsilon = 0$, 0.00001, 0.0001 and 0.001, respectively. The figures below show the variation of φ, φ being the numerical solution to (5.1), after 100 iterations ($t = 10^{-4}$).

In Fig. 15, φ_0 takes random values between 0 and 0.5. Figures 11, 12, 13 and 14 correspond to the cases $(\varepsilon,\beta) = (0,0)$, $(0.00001, 0.001)$, $(0.0001, 0.01)$ and $(0.001, 0.1)$, respectively. The figures below show the variation of φ, φ being the numerical solution to (5.1), after 15 iterations ($t = 15\times 10^{-5}$).

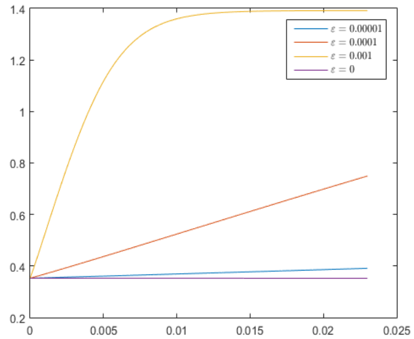

Figure 1. $\beta = 0.05$.

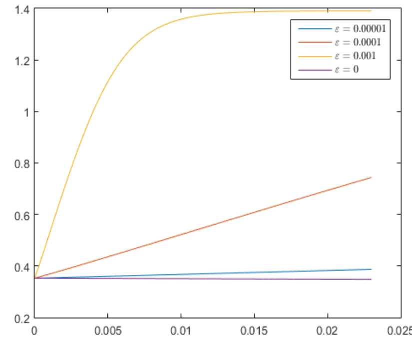

Figure 2. $\beta = 0.5$.

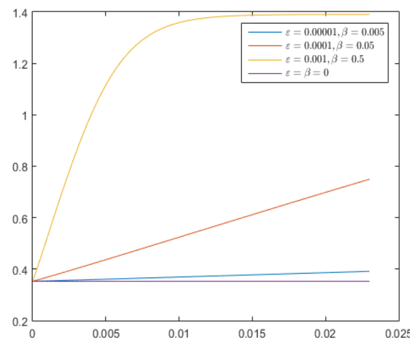

Figure 3. $g(s) = \beta s$.

Figure 4. $\varphi_0(x, y) = 1 - \sin^2(x + y)$.

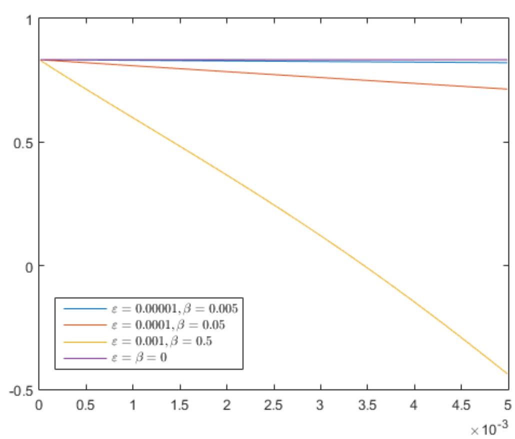

Figure 5. $\varphi_0(x, y) = \dfrac{\cos(x - y)}{x^6 + 1}$.

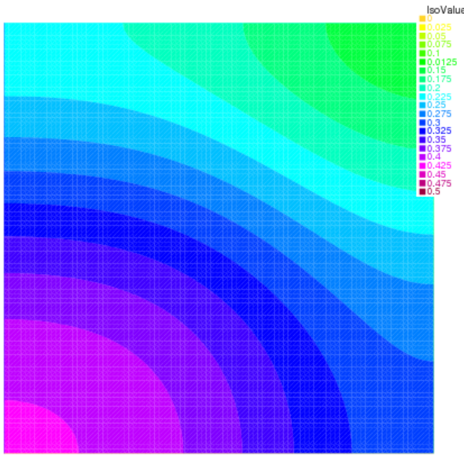

Figure 6. $\varepsilon = 0$.

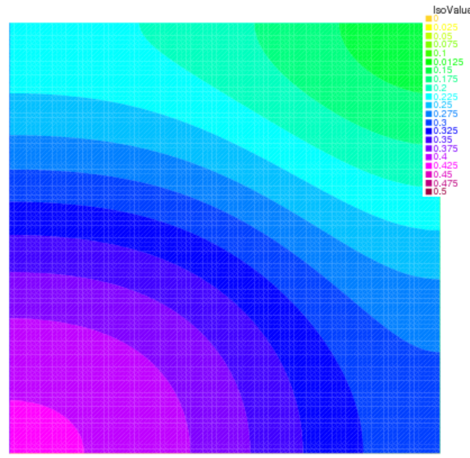

Figure 7. $\varepsilon = 0.00001$.

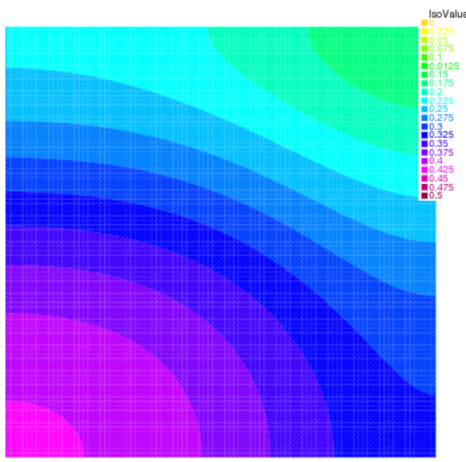

Figure 8. $\varepsilon = 0.0001$.

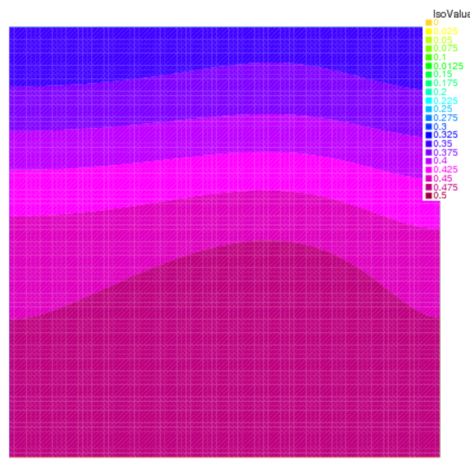

Figure 9. $\varepsilon = 0.001$.

Figure 10. $\varphi_0(x, y) = \dfrac{1}{5(x^6 + 1)(y^5 + 0.5)}$.

5.2. The case when $g(s) = \dfrac{s}{2 + |s|}$

In Fig. 20, we take $\varphi_0(x, y) = 1 - \sin^2(x + y)$. Figures 16, 17, 18 and 19 correspond to the cases $\varepsilon = 0$, 0.00001, 0.0001 and 0.001, respectively. The figures below show the variation of φ, φ being the numerical solution to (5.1), after 300 iterations ($t = 3 \times 10^{-4}$). In Fig. 27, φ_0 takes random values between 0 and 0.5. Figures 21, 22, 23, 24, 25 and 26 correspond to the cases $\varepsilon = 0$, 0.0000001, 0.000001, 0.00001, 0.0001 and 0.001, respectively. The figures below show the variation of φ, φ being the numerical solution to (5.1), after 15 iterations ($t = 15 \times 10^{-5}$).

Table 1 provides the numerical results obtained for different initial datum with different values for ε, where φ^ε is the numerical solution to (5.1). The results support the theoretical results obtained above.

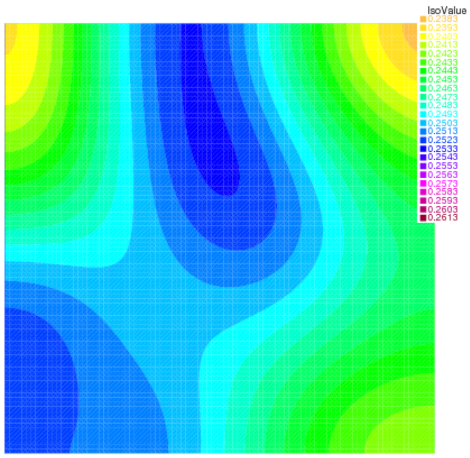

Figure 11. $(\varepsilon, \beta) = (0, 0)$.

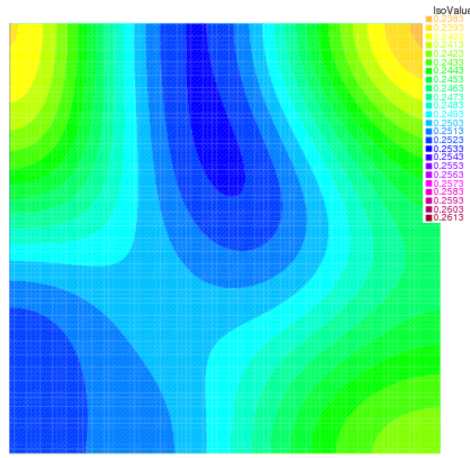

Figure 12. $(\varepsilon, \beta) = (0.00001, 0.001)$.

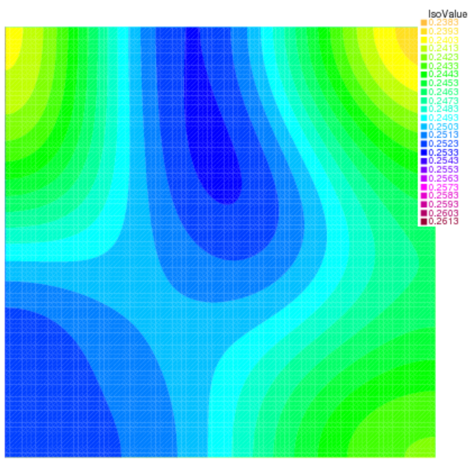

Figure 13. $(\varepsilon, \beta) = (0.0001, 0.01)$.

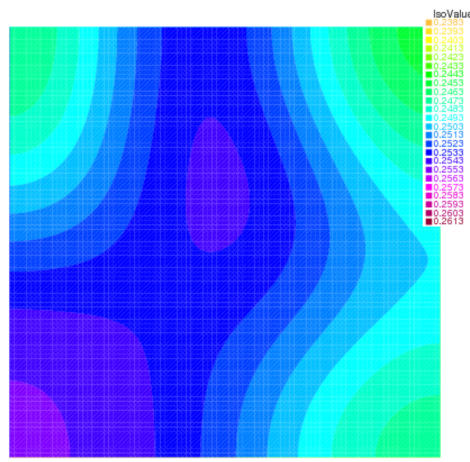

Figure 14. $(\varepsilon, \beta) = (0.001, 0.1)$.

Figure 15. Random initial datum between 0 and 0.5.

Table 1. Numerical results after 200 iterations.

ε \ $\varphi_0(x,y)$	$1 - \sin^2(x+y)$	$\dfrac{\cos(x-y)}{x^6+1}$	$\dfrac{1}{5(x^6+1)(y^5+0.5)}$	Random values between 0 and 0.5
	$\Vert\varphi^\varepsilon - \varphi^0\Vert^2$	$\Vert\varphi^\varepsilon - \varphi^0\Vert^2$	$\Vert\varphi^\varepsilon - \varphi^0\Vert^2$	$\Vert\varphi^\varepsilon - \varphi^0\Vert^2$
0.1	1.07465	6.03614	1.1294	0.564182
0.01	1.07443	6.03514	1.12807	0.449319
0.001	0.119854	0.219703	0.0915962	0.00333638
0.0001	0.0010845	0.00236198	0.000836541	7.98553×10^{-6}
10^{-5}	1.04779×10^{-5}	2.41389×10^{-5}	8.25374×10^{-6}	7.55053×10^{-6}
10^{-6}	1.04367×10^{-7}	2.41971×10^{-7}	8.24235×10^{-8}	9.29424×10^{-7}

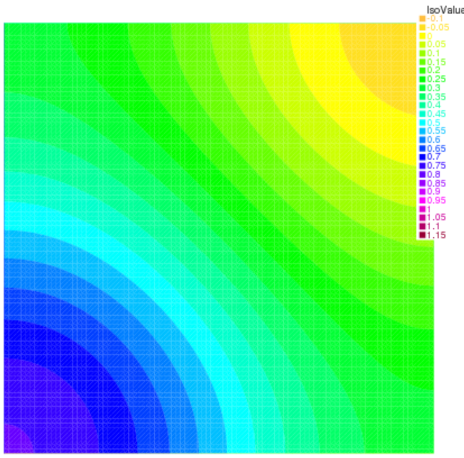

Figure 16. $\varepsilon = 0$.

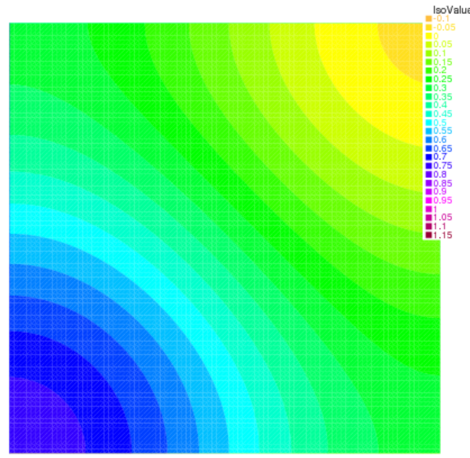

Figure 17. $\varepsilon = 0.00001$.

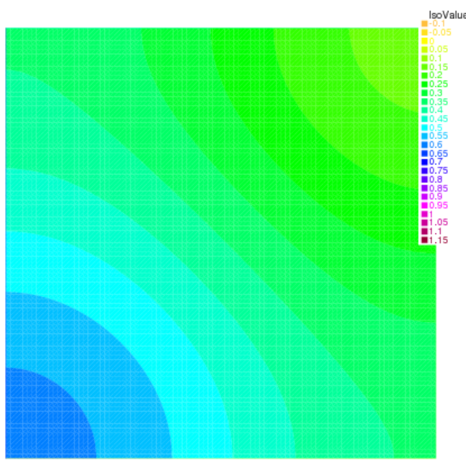

Figure 18. $\varepsilon = 0.0001$.

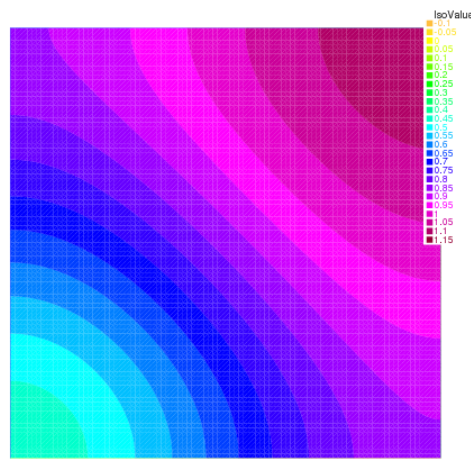

Figure 19. $\varepsilon = 0.001$.

Figure 20. $\varphi_0(x, y) = 1 - \sin^2(x + y)$.

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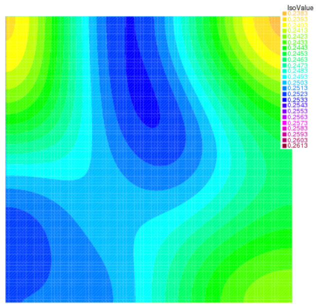

Figure 21. $\varepsilon = 0$.

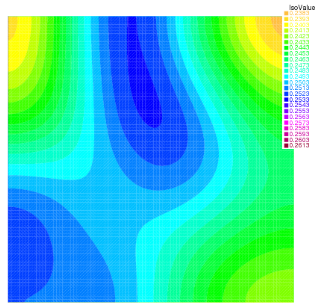

Figure 22. $\varepsilon = 0.0000001$.

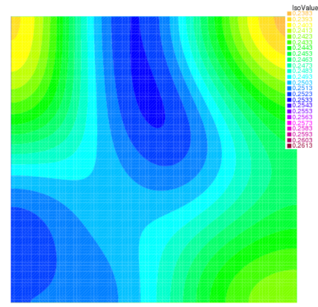

Figure 23. $\varepsilon = 0.000001$.

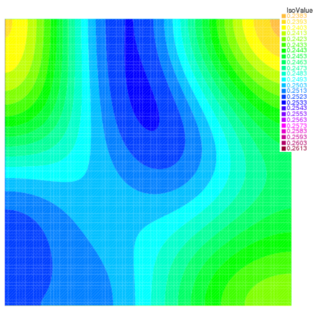

Figure 24. $\varepsilon = 0.00001$.

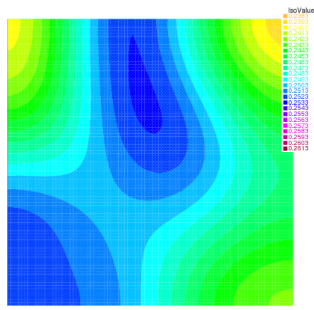

Figure 25. $\varepsilon = 0.0001$.

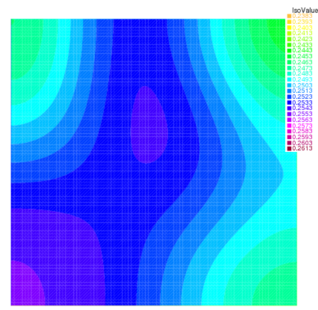

Figure 26. $\varepsilon = 0.001$.

Figure 27. Random initial datum between 0 and 0.5.

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