Journal of Applied Analysis and Computation Website:http://www.jaac-online.com
Volume 14, Number 3, June 2024, 1374–1394 DOI:10.11948/20230158

THE FIRST THREE ORDER MELNIKOV FUNCTIONS FOR GENERAL PIECEWISE HAMILTONIAN SYSTEMS WITH A NON-REGULAR SEPARATION LINE

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Abstract This paper focuses on the first three order Melnikov functions of general planar piecewise Hamiltonian systems under the piecewise perturbations with a non-regular separation line. By using the first three order Melnikov functions, we obtain the exact upper bounds of the number of limit cycles bifurcated from two different piecewise linear near-Hamiltonian systems.

Keywords Melnikov functions, separation line, limit cycles.

MSC(2010) 34C05, 34C07, 37G15.

1. Introduction

In recent years, the qualitative theory of piecewise smooth systems has received a large amount of attention due to its application in real life. The problem of the number and distribution of limit cycles of piecewise smooth differential systems is one of the important issues, which is closely related to the weak Hilbert's 16th problem. For a piecewise system, the separation line plays a significant role in determining the number of limit cycles. Braga and Mello in [2] proposed that "Given $n \in N$, there is a piecewise linear system with two zones in the plane with exactly n limit cycles."

In the present paper, we would like to focus on the higher order Melnikov theory for planar piecewise differential systems, which is one of the important tools to deal with the weak Hilbert's 16th problem aiming at finding more limit cycles. The algorithm of higher order Melnikov functions for smooth differential systems is proposed by Françoise [7]. In recent decades, researchers extended the Melnikov theory into the non-smooth case. But due to the complex calculations and iterations, there are fewer articles related to higher order Melnikov functions. When the separation line is a straight line, the authors in [15] derived the first order Melnikov function for piecewise Hamiltonian systems, and [14, 18] gave the second order Melnikov function for piecewise Hamiltonian systems in different forms. When the separation line is non-regular, we proposed in [19] the first and second order Melnikov functions for piecewise Hamiltonian systems. Recently, [11] considered the third order Melnikov function of a special piecewise Hamiltonian system and gave an application

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of the first and second order Melnikov functions. The authors in [4] also calculated the higher order Melnikov functions of an elementary center under piecewise linear perturbations. Some other papers also consider Melnikov or averaging functions for different cases of piecewise differential systems, see references [1,3,6,8,12,13,16,17].

Inspired by the above articles, we derive the explicit formula of the first three order Melnikov functions for the general piecewise Hamiltonian systems when the separation line is formed by two semi-straight lines starting from the origin forming an angle $\theta \in (0, \pi]$. And as applications, we consider the linear perturbations of two concrete different piecewise linear systems where $\theta \in (0, \pi)$ and $\theta = \pi$, respectively.

More specificly, we consider the perturbed system as follows,

$$dH + \epsilon\omega_1 + \epsilon^2\omega_2 + \epsilon^3\omega_3 = 0 \tag{1.1}$$

where

$$H(x, y) = \begin{cases} H^+(x, y), \ (x, y) \in \Sigma^+, \\ H^-(x, y), \ (x, y) \in \Sigma^-, \end{cases}$$

and

$$\omega_i = \begin{cases} \omega_i^+ = p_i^+(x, y)dy - q_i^+(x, y)dx, \ (x, y) \in \Sigma^+, \\ \omega_i^- = p_i^-(x, y)dy - q_i^-(x, y)dx, \ (x, y) \in \Sigma^-, \end{cases}$$

with $H^{\pm}(x, y)$, $p_i^{\pm}(x, y)$, $q_i^{\pm}(x, y)$, $i = 1, 2, 3$ being polynomials of degree n in x and y. $\Sigma^{\pm}$ are the sectors separated by the two semi-straight lines, and Σ^+ corresponds to the angle $\theta \in (0, \pi]$. Without loss of generality, the separation lines are denoted by $y = k_0x$ and $y = k_1x$. The unperturbed system (1.1) with $\epsilon = 0$ is a piecewise Hamiltonian system.

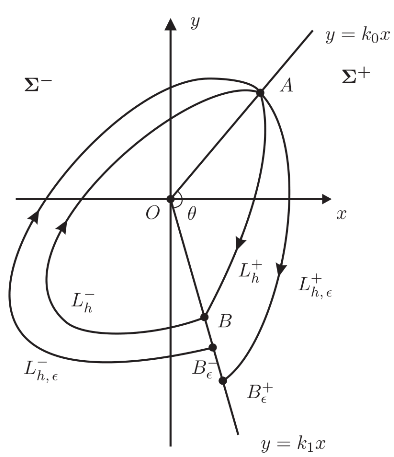

Figure 1. The perturbed system (1.1).

Assume that system $(1.1)|_{\epsilon=0}$ has a family of period orbits around the origin denoted by $L_h = L_h^+ \cup L_h^-$ for $h \in (\alpha, \beta)$, here $L_h^+(resp.\, L_h^-)$ represents the arc orbit beginning from $A(h)(resp.\,B(h))$ and ending at $B(h)(resp.\,(A(h)))$ on $(x,\,y)\in\Sigma^+(resp.(x,\,y)\in\Sigma^-)$. This implies, there exist two points $A(h)=(a(h),\,k_0a(h))$ and $B(h)=(b_0(h),\,k_1b_0(h))$, denoted simply by $\mathbf{A}$ and $\mathbf{B}$, such that

$$H^+(\mathbf{A})=H^+(\mathbf{B})=h,\qquad H^-(\mathbf{A})=H^-(\mathbf{B}). \tag{1.2}$$

Without loss of generality, suppose that L_h has a clockwise orientation, see Figure 1. Let $L^+_{h,\,\epsilon}(resp.\,L^-_{h,\,\epsilon})$ be the solution of system (1.1) defined in $\Sigma^\pm$ which starts from $A(h)(resp.\,B^-_\epsilon(h))$ and ends at $B^+_\epsilon(h)(resp.\,A(h))$, here $B^\pm_\epsilon(h)=(b^\pm_\epsilon(h),\,k_1b^\pm_\epsilon(h))$, denoted simply by $\mathbf{B}^\pm_\epsilon$. Expand $b^\pm_\epsilon(h)$ at $\epsilon=0$ and let $b^\pm_\epsilon(h)=b^\pm_0(h)+\epsilon b^\pm_1(h)+\frac{1}{2}\epsilon^2b^\pm_2(h)+\cdots$. Naturally we can define the displacement function as

$$b^+_\epsilon(h)-b^-_\epsilon(h)=\epsilon M_1(h)+\epsilon^2M_2(h)+\cdots, \tag{1.3}$$

where

$$M_k(h)=\frac{1}{k!}(b^+_k(h)-b^-_k(h)),\quad k\geq 1, \tag{1.4}$$

which is called the $k-th$ order Melnikov function. Such assumptions and definitions see also in [18, 19], etc.

This paper is organized as follows. The main results and related definitions are presented in Section 2. The expressions of the first three order Melnikov functions for the general piecewise Hamiltonian systems are derived in Section 3. The formula is applied to consider piecewise linear perturbations of two different Hamiltonian systems in Section 4. Some comments and open problems are proposed in Section 5.

2. The main results

In this section, we shall give our main results.

Definition 2.1 (Corresponding function)**.** Let $\varphi^\pm_t$ be a flow of the vector field $\chi^\pm_0$ defined by system $(1.1)|_{\epsilon=0}$ starting from $(x^\pm_0,\,y^\pm_0)$, we denote the corresponding multiple-valued function as follows, $\psi^\pm_1=-1$, and

$$\psi^\pm_2(x,\,y)=\int_0^{T^\pm(x,\,y)}div(\chi^\pm_1)\circ(\varphi^\pm_t(x^\pm_0,\,y^\pm_0))dt,$$
$$\psi^\pm_3(x,\,y)=\int_0^{T^\pm(x,\,y)}div(\chi^\pm_2)\circ(\varphi^\pm_t(x^\pm_0,\,y^\pm_0))dt,$$

where the vector fields $\chi^\pm_1=(P^\pm_1,\,Q^\pm_1)$, $\chi^\pm_2=(P^\pm_2,\,Q^\pm_2)$ correspond to the $1-$forms $\Omega^\pm_1=-\psi^\pm_1\omega^\pm_1=P^\pm_1dy-Q^\pm_1dx$ and $\Omega^\pm_2=-\psi^\pm_2\omega^\pm_1-\psi^\pm_1\omega^\pm_2=P^\pm_2dy-Q^\pm_2dx$, here $(x,\,y)=\varphi^\pm(T^\pm(x,\,y),\,x^\pm_0,\,y^\pm_0)$.

According to Definition 2.1, we have the following decomposition:

Lemma 2.1 ($1-$form decomposition)**.** *According to the definitions of* $\Omega^\pm_1$, $\Omega^\pm_2$ *and* $\psi^\pm_2(x,\,y)$, $\psi^\pm_3(x,\,y)$, *there exist multiple-valued analytic functions* $R^\pm_2(x,\,y)$ *and* $R^\pm_3(x,\,y)$ *satisfying*

$$\Omega^\pm_1=\psi^\pm_2(x,\,y)dH^\pm(x,\,y)+dR^\pm_2(x,\,y),\quad \Omega^\pm_2=\psi^\pm_3(x,\,y)dH^\pm(x,\,y)+dR^\pm_3(x,\,y).$$

Finally, we give the explicit expressions of the first three order Melnikov functions:

Theorem 2.1. *For system (1.1) with* $\theta \in (0, \pi)$*, if* $M_1(h) = M_2(h) \equiv 0$*, then the third order Melnikov function* $M_3(h)$ *is given as*

$$\begin{aligned} M_3(h) = &\frac{1}{K_1^+(\mathbf{B})}\Big(\int_{L_h^+}(\psi_3^+\omega_1^+ + \psi_2^+\omega_2^+ + \psi_1^+\omega_3^+) - [J_{31}^+(\mathbf{B})b_1^+ + \frac{1}{2}(J_{21}^+(\mathbf{B})b_2^+ \\ &+ J_{22}^+(\mathbf{B}){b_1^+}^2)] - (\frac{1}{2}K_2^+(\mathbf{B})b_1^+b_2^+ + \frac{1}{3!}K_3^+(\mathbf{B}){b_1^+}^3)\Big) \\ &+ \frac{1}{K_1^-(\mathbf{B})}\Big(\int_{L_h^-}(\psi_3^-\omega_1^- + \psi_2^-\omega_2^- + \psi_1^-\omega_3^-) + [J_{31}^-(\mathbf{B})b_1^- + \frac{1}{2}(J_{21}^-(\mathbf{B})b_2^- \\ &+ J_{22}^-(\mathbf{B}){b_1^-}^2)] + (\frac{1}{2}K_2^-(\mathbf{B})b_1^-b_2^- + \frac{1}{3!}K_3^-(\mathbf{B}){b_1^-}^3)\Big). \end{aligned} \tag{2.1}$$

Here $K_i^{\pm}(x, y) = (\frac{\partial}{\partial x} + k_1\frac{\partial}{\partial y})^i H^{\pm}(x, y)$*, and* $J_{ji}^{\pm}(x, y) = (\frac{\partial}{\partial x} + k_1\frac{\partial}{\partial y})^i R_j^{\pm}(x, y)$ *for* $i = 1, 2, 3$, $j = 2, 3$.

Remark 2.1. It is worth mentioning that the first and second order Melnikov functions for system (1.1) have been given in [19], here their more symmetrical expressions are presented in the following.

(i) The first order Melnikov function $M_1(h)$ is given as

$$M_1(h) = \frac{1}{K_1^+(\mathbf{B})}\int_{L_h^+} -\omega_1^+ + \frac{1}{K_1^-(\mathbf{B})}\int_{L_h^-} -\omega_1^-.$$

(ii) If $M_1(h) \equiv 0$, then the second order Melnikov function $M_2(h)$ is given as

$$\begin{aligned} M_2(h) = &\frac{1}{K_1^+(\mathbf{B})}\int_{L_h^+}(\psi_2^+\omega_1^+ + \psi_1^+\omega_2^+) - \frac{1}{K_1^+(\mathbf{B})}(J_{21}^+(\mathbf{B})b_1^+ + \frac{1}{2}K_2^+(\mathbf{B}){b_1^+}^2) \\ &+ \frac{1}{K_1^-(\mathbf{B})}\int_{L_h^-}(\psi_2^-\omega_1^- + \psi_1^-\omega_2^-) + \frac{1}{K_1^-(\mathbf{B})}(J_{21}^-(\mathbf{B})b_1^- + \frac{1}{2}K_2^-(\mathbf{B}){b_1^-}^2). \end{aligned}$$

Corollary 2.1. *For system (1.1) with* $\theta = \pi$*, the following statements hold.*

(i) The first order Melnikov function $M_1(h)$ *is given as*

$$M_1(h) = \frac{1}{H_y^+(\mathbf{B})}\int_{L_h^+} -\omega_1^+ + \frac{1}{H_y^-(\mathbf{B})}\int_{L_h^-} -\omega_1^-.$$

(ii) If $M_1(h) \equiv 0$*, then the second order Melnikov function* $M_2(h)$ *is given as*

$$\begin{aligned} M_2(h) = &\frac{1}{H_y^+(\mathbf{B})}\int_{L_h^+}(\psi_2^+\omega_1^+ + \psi_1^+\omega_2^+) - \frac{1}{H_y^+(\mathbf{B})}(R_{2y}^+(\mathbf{B})b_1^+ + \frac{1}{2}H_{yy}^+(\mathbf{B}){b_1^+}^2) \\ &+ \frac{1}{H_y^-(\mathbf{B})}\int_{L_h^-}(\psi_2^-\omega_1^- + \psi_1^-\omega_2^-) + \frac{1}{H_y^-(\mathbf{B})}(R_{2y}^-(\mathbf{B})b_1^- + \frac{1}{2}H_{yy}^-(\mathbf{B}){b_1^-}^2). \end{aligned}$$

(iii) If $M_1(h) = M_2(h) \equiv 0$*, then the third order Melnikov function* $M_3(h)$ *is*

given as

$$
\begin{aligned}
&M_3(h)\\
=&\frac{1}{H_y^+(\mathbf{B})}\Big(\int_{L_h^+}(\psi_3^+\omega_1^+ + \psi_2^+\omega_2^+ + \psi_1^+\omega_3^+) - [R_{3y}^+(\mathbf{B})b_1^+ + \frac{1}{2}(R_{2y}^+(\mathbf{B})b_2^+ + R_{2yy}^+(\mathbf{B}){b_1^+}^2)]\\
&- (\frac{1}{2}H_{yy}^+(\mathbf{B})b_1^+b_2^+ + \frac{1}{3!}H_{yyy}^+(\mathbf{B}){b_1^+}^3)\Big) + \frac{1}{H_y^-(\mathbf{B})}\Big(\int_{L_h^-}(\psi_3^-\omega_1^- + \psi_2^-\omega_2^- + \psi_1^-\omega_3^-)\\
&+ [R_{3y}^-(\mathbf{B})b_1^- + \frac{1}{2}(R_{2y}^-(\mathbf{B})b_2^- + R_{2yy}^-(\mathbf{B}){b_1^-}^2)] + \frac{1}{2}H_{yy}^-(\mathbf{B})b_1^-b_2^- + \frac{1}{3!}H_{yyy}^-(\mathbf{B}){b_1^-}^3\Big).
\end{aligned}
$$

3. The Algorithm for higher order Melnikov functions

In this section, we give the proof of main theorems.

Proof of Lemma 2.1. For $k = 1, 2$, noting that $\psi_{k+1}^\pm(x, y)$ are the integrals of the functions $div(\chi_k^\pm)$ along the trajectories of $\chi_0^\pm$. Hence, it is straightforward to obtain $\chi_0^\pm \circ (\psi_{k+1}^\pm) = div(\chi_k^\pm)$ from the derivatives of $\psi_{k+1}^\pm(x, y)$ along the trajectories of the vector field $\chi_0^\pm$, which implies

$$d\psi_{k+1}^\pm \wedge dH^\pm = \chi_0^\pm \circ (\psi_{k+1}^\pm)dx \wedge dy = div(\chi_k^\pm)dx \wedge dy.$$

It is easy to get $div(\chi_k^\pm)dx \wedge dy = d\Omega_k^\pm$. Therefore we have

$$d\psi_{k+1}^\pm \wedge dH^\pm = d(\psi_{k+1}^\pm dH^\pm) = d\Omega_k^\pm,$$

and the 1-forms $\Omega_k^\pm - \psi_{k+1}^\pm dH^\pm$ are closed. The domain on which it is defined is connected, so there exist unique functions $R_{k+1}^\pm(x, y)$ satisfying

$$\Omega_k^\pm = \psi_{k+1}^\pm dH^\pm + dR_{k+1}^\pm.$$

Then we completed the proof. □

Proof of Theorem 2.1. By expanding $H^\pm(\mathbf{B}_\epsilon^\pm)$ in variable ϵ at the point $\mathbf{B}$, we obtain

$$
\begin{aligned}
&H^\pm(\mathbf{B}_\epsilon) - H^\pm(\mathbf{A})\\
&= H^\pm(\mathbf{B}) - H^\pm(\mathbf{A}) + \Big(\Delta b^\pm\frac{\partial}{\partial x} + k_1\Delta b^\pm\frac{\partial}{\partial y}\Big)H^\pm(\mathbf{B}) + \frac{1}{2}\Big(\Delta b^\pm\frac{\partial}{\partial x}\\
&\quad + k_1\Delta b\frac{\partial}{\partial y}\Big)^2 H^\pm(\mathbf{B}) + \frac{1}{3!}\Big(\Delta b^\pm\frac{\partial}{\partial x} + k_1\Delta b^\pm\frac{\partial}{\partial y}\Big)^3 H^\pm(\mathbf{B}) + O((\Delta b^\pm)^4)\\
&= \epsilon l_1^\pm + \epsilon^2 l_2^\pm + \epsilon^3 l_3^\pm + O(\epsilon^4),
\end{aligned}
\tag{3.1}
$$

where $\Delta b^\pm = \epsilon b_1^\pm + \frac{1}{2}\epsilon^2 b_2^\pm + \frac{1}{3!}\epsilon^3 b_3^\pm + O(\epsilon^4)$ and

$$
\begin{aligned}
l_1^\pm &= K_1^\pm(\mathbf{B})b_1^\pm,\\
l_2^\pm &= \frac{1}{2}K_1^\pm(\mathbf{B})b_2^\pm + \frac{1}{2}K_2^\pm(\mathbf{B}){b_1^\pm}^2,\\
l_3^\pm &= \frac{1}{3!}K_1^\pm(\mathbf{B})b_3^\pm + \frac{1}{2}K_2^\pm(\mathbf{B})b_1^\pm b_2^\pm + \frac{1}{3!}K_3^\pm(\mathbf{B}){b_1^\pm}^3.
\end{aligned}
\tag{3.2}
$$

In the following, we will restrict ourselves to $M_3(h)$. We first focus on the following equation

$$(-\psi_1^{\pm} - \epsilon\psi_2^{\pm} - \epsilon^2\psi_3^{\pm})(dH^{\pm} + \epsilon\omega_1^{\pm} + \epsilon^2\omega_2^{\pm} + \epsilon^3\omega_3^{\pm}) = 0.$$

It can be rewritten as

$$-\psi_1^{\pm}dH^{\pm} - \epsilon\psi_2^{\pm}dH^{\pm} - \epsilon^2\psi_3^{\pm}dH^{\pm} + \epsilon\Omega_1^{\pm} + \epsilon^2\Omega_2^{\pm} = \epsilon^3(\psi_3^{\pm}\omega_1^{\pm} + \psi_2^{\pm}\omega_2^{\pm} + \psi_1^{\pm}\omega_3^{\pm}) + O(\epsilon^4).$$

Taking the decomposition in Theorem 2.1 into consideration, we have

$$dH^{\pm} + \epsilon dR_2^{\pm} + \epsilon^2 dR_3^{\pm} = \epsilon^3(\psi_3^{\pm}\omega_1^{\pm} + \psi_2^{\pm}\omega_2^{\pm} + \psi_1^{\pm}\omega_3^{\pm}) + O(\epsilon^4).$$

Next we only consider the right subsystem, and the left subsystem is similar to follow. By integrating the above equations along $L_{h,\epsilon}^{+}$, we have

$$\int_{L_{h,\epsilon}^{+}} dH^{+} + \int_{L_{h,\epsilon}^{+}} \sum_{i=1}^{2} \epsilon^i dR_{i+1}^{+} = \int_{L_{h,\epsilon}^{+}} \epsilon^3(\psi_3^{+}\omega_1^{+} + \psi_2^{+}\omega_2^{+} + \psi_1^{+}\omega_3^{+}) + O(\epsilon^4), \quad (3.3)$$

which implies

$$\begin{aligned} &H^{+}(\mathbf{B}_\epsilon^{+}) - H^{+}(\mathbf{A}) + \sum_{i=1}^{2} \epsilon^i (R_{i+1}^{+}(\mathbf{B}_\epsilon^{+}) - R_{i+1}^{+}(\mathbf{A})) \\ =&\epsilon^3 \int_{L_h^{+}} (\psi_3^{+}\omega_1^{+} + \psi_2^{+}\omega_2^{+} + \psi_1^{+}\omega_3^{+}) + O(\epsilon^4). \end{aligned} \quad (3.4)$$

Expand $R_{i+1}^{+}(\mathbf{B}_\epsilon^{+})$ in variable ϵ at $\mathbf{B}(b_0(h),\, k_1 b_0(h))$ as follows,

$$\begin{aligned} R_{i+1}^{+}(\mathbf{B}_\epsilon^{+}) =& R_{i+1}^{+}(\mathbf{B}) + \Big(\Delta b^{\pm}\frac{\partial}{\partial x} + k_1\Delta b^{\pm}\frac{\partial}{\partial y}\Big)R_{i+1}^{+}(\mathbf{B}) + \frac{1}{2}\Big(\Delta b^{\pm}\frac{\partial}{\partial x} \\ &+ k_1\Delta b^{\pm}\frac{\partial}{\partial y}\Big)^2 R_{i+1}^{+}(\mathbf{B}) + \frac{1}{3!}\Big(\Delta b^{\pm}\frac{\partial}{\partial x} + k_1\Delta b^{\pm}\frac{\partial}{\partial y}\Big)^3 R_{i+1}^{+}(\mathbf{B}) \\ &+ O((\Delta b^{\pm})^4) \\ =& R_{i+1}^{+}(\mathbf{B}) + \epsilon R_{i+1,1}^{+}(\mathbf{B}) + \cdots + \epsilon^n R_{i+1,n}^{+}(\mathbf{B}) + O(\epsilon^{n+1}), \end{aligned} \quad (3.5)$$

where $\Delta b^{\pm} = \epsilon b_1^{\pm} + \frac{1}{2}\epsilon^2 b_2^{\pm} + \frac{1}{3!}\epsilon^3 b_3^{\pm} + O(\epsilon^4)$ and

$$\begin{aligned} R_{i+1,1}^{+}(\mathbf{B}) &= J_{i+1,1}^{+}(\mathbf{B})b_1^{+}, \\ R_{i+1,2}^{+}(\mathbf{B}) &= \frac{1}{2}J_{i+1,1}^{+}(\mathbf{B})b_2^{+} + \frac{1}{2}J_{i+1,2}^{+}(\mathbf{B}){b_1^{+}}^2, \\ R_{i+1,3}^{+}(\mathbf{B}) &= \frac{1}{3!}J_{i+1,1}^{+}(\mathbf{B})b_3^{+} + \frac{1}{2}J_{i+1,2}^{+}(\mathbf{B})b_1^{+}b_2^{+} + \frac{1}{3!}J_{i+1,3}^{+}(\mathbf{B}){b_1^{+}}^3. \end{aligned}$$

Taking derivative three times in succession with respect to ϵ, we can obtain the following equation

$$l_3^{+} + R_{2,2}^{+}(\mathbf{B}) + R_{3,1}^{+}(\mathbf{B}) = \int_{L_h^{+}} (\psi_3^{+}\omega_1^{+} + \psi_2^{+}\omega_2^{+} + \psi_1^{+}\omega_3^{+}),$$

which displays

$$l_3^{+} = \int_{L_h^{+}} (\psi_3^{+}\omega_1^{+} + \psi_2^{+}\omega_2^{+} + \psi_1^{+}\omega_3^{+}) - (R_{2,2}^{+}(\mathbf{B}) + R_{3,1}^{+}(\mathbf{B})). \quad (3.6)$$

According to the formula (3.2), we can derive

$$\begin{aligned} b_3^+ = & \frac{3!}{K_1^+(\mathbf{B})}\Big(\int_{L_h^+}(\psi_3^+\omega_1^+ + \psi_2^+\omega_2^+ + \psi_1^+\omega_3^+) - (R_{2,\,2}^+(\mathbf{B}) + R_{3,\,1}^+(\mathbf{B}))\Big) \\ & - \frac{1}{3!}(\frac{1}{2}K_2^+(\mathbf{B})b_1^+b_2^+ + \frac{1}{3!}K_3^+(\mathbf{B}){b_1^+}^3)). \end{aligned} \tag{3.7}$$

Similarly, when we consider the left subsystem, by integrating the above equations along $-L_{h,\,\epsilon}^-$ from $\mathbf{A}$ to $\mathbf{B}_\epsilon^-$, we have

$$\begin{aligned} b_3^- = & \frac{3!}{K_1^-(\mathbf{B})}\Big(\int_{L_h^-} -(\psi_3^-\omega_1^- + \psi_2^-\omega_2^- + \psi_1^-\omega_3^-) - (R_{2,\,2}^-(\mathbf{B}) + R_{3,\,1}^-(\mathbf{B}))\Big) \\ & - \frac{1}{3!}(\frac{1}{2}K_2^-(\mathbf{B})b_1^-b_2^- + \frac{1}{3!}K_3^-(\mathbf{B}){b_1^-}^3)). \end{aligned} \tag{3.8}$$

Finally, it is easy to obtain that $M_3(h) = \frac{1}{3!}(b_3^+ - b_3^-)$, hence we completed the proof. □

Proof of Corollary 2.1. The proof of Corollary 2.1 is analogous to that of Theorem 2.1. It is worth noticing that $K_i^\pm(x,\,y) = \frac{\partial^i H^\pm(x,\,y)}{\partial y^i}$ and $J_{ji}^\pm(x,\,y) = \frac{\partial^i R_j^\pm(x,\,y)}{\partial y^i}$ for $\theta = \pi$. □

4. Applications

In this section, we shall consider two piecewise Hamiltonian systems with different separation lines, that is, the non-regular separation line with $\theta \in (0,\,\pi)$ and the straight line with $\theta = \pi$.

4.1. An application for $\theta \in (0,\,\pi)$

As an application, we consider a linear perturbation of a piecewise linear Hamiltonian system. One of the subsystems is a linear center and the other is a constant differential system, and the corresponding Hamiltonian functions are $H(x,\,y) = ax^2 + by^2 + cxy + dx + ey$ with $-c^2 + 4ab > 0$ and $H(x,\,y) = px + qy$. This system has been studied in [19] by using the first two order Melnikov functions. Here we shall consider the same system by using the third order Melnikov function and the aim is to find if there are more limit cycles or not by considering linear perturbations up to higher order in ϵ.

More specificly, consider the perturbed system as follows:

$$dH + \epsilon\omega_1 + \epsilon^2\omega_2 + \epsilon^3\omega_3 = 0 \tag{4.1}$$

where $H(x,\,y)$ is defined by:

$$H(x,y) = \begin{cases} H^-(x,\,y) = ax^2 + by^2 + cxy + dx + ey & (x,y) \in \Sigma^-, \\ H^+(x,\,y) = px + qy & (x,y) \in \Sigma^+, \end{cases}$$

and

$$\omega_i = \begin{cases} \omega_i^- = f_i^-(x,\,y)dy - g_i^-(x,\,y)dx & (x,y) \in \Sigma^-, \\ \omega_i^+ = f_i^+(x,\,y)dy - g_i^+(x,\,y)dx & (x,y) \in \Sigma^+, \end{cases}$$

with the condition of $-c^2+4ab>0$ and the definition of $\Sigma^{\pm}$ are the same as system (1.1). Here $i=1,\ 2,\ 3$ and $f_i^{\pm}(x,\ y)$, $g_i^{\pm}(x,\ y)$ are linear polynomials. Assume that the unperturbed system (4.1) with $\epsilon=0$ has a family of period orbits.

Theorem 4.1. *For system (4.1), when* $M_1(h)=M_2(h)\equiv 0$ *and* $M_3(h)\not\equiv 0$, *the upper bound of the number of limit cycles bifurcated from the period orbits is* 5 *(taking into account their multiplicities) by using the third order Melnikov function, and the upper bound can be reached.*

Corollary 4.1. *For system (4.1) with* $\omega_2^{\pm}=\omega_3^{\pm}=0$, *when* $M_1(h)=M_2(h)\equiv 0$ *and* $M_3(h)\not\equiv 0$, *the upper bound of the number of limit cycles bifurcated from the period orbits is 2 (taking into account their multiplicities) by using the third order Melnikov function and the upper bound can be reached.*

Before proving Theorem 4.1, we first give a theorem of the normal form of system (4.1), which is proved in [19].

Theorem 4.2. [19] *The unperturbed system* $(4.1)|_{\epsilon=0}$ *can be written as a normalized canonical form as*

$$\begin{cases} \dot{x}=H_y, \\ \dot{y}=-H_x, \end{cases} \tag{4.2}$$

where

$$H(x,y)=\begin{cases} H^-(x,y)=(x+1)^2+y^2=h^2 \quad (x,y)\in\Sigma^-, \\ H^+(x,y)=x=x_0 \quad (x,y)\in\Sigma^+. \end{cases}$$

Here $\Sigma^{\pm}$ *are the same as system (1.1), but it is worth mentioning that in system (4.2),* $k_1=-k_0$, *and we replace* k_0 *and* k_1 *by* k *and* $-k$, *see Figure 2.*

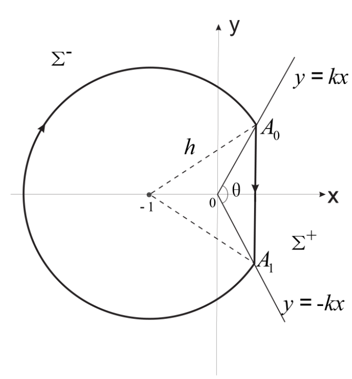

Figure 2. Unperturbed system (4.2).

Naturally, the perturbed system (4.1) can be transformed into the following system:

$$\begin{cases} \dot{x}=H_y+\sum\limits_{i=1}^{3}\varepsilon^i f_i(x,y), \\ \dot{y}=-H_x+\sum\limits_{i=1}^{3}\varepsilon^i g_i(x,y), \end{cases} \tag{4.3}$$

where

$$H(x,y)=\begin{cases}H^-(x,y)=(x+1)^2+y^2=h^2 & (x,y)\in\Sigma^-,\\ H^+(x,y)=x=x_0 & (x,y)\in\Sigma^+,\end{cases}$$

$$f_i(x,y)=\begin{cases}f_i^-(x,y)=a_{i1}^-x+a_{i2}^-y+a_{i0}^- & (x,y)\in\Sigma^-,\\ f_i^+(x,y)=a_{i1}^+x+a_{i2}^+y+a_{i0}^+ & (x,y)\in\Sigma^+,\end{cases}$$

and

$$g_i(x,y)=\begin{cases}g_i^-(x,y)=b_{i1}^-x+b_{i2}^-y+b_{i0}^- & (x,y)\in\Sigma^-,\\ g_i^+(x,y)=b_{i1}^+x+b_{i2}^+y+b_{i0}^+ & (x,y)\in\Sigma^+.\end{cases}$$

The first and second order Melnikov functions have been considered in [19]. In order to successfully introduce the calculation of the third order Melnikov function, we briefly display the associated results of the first two order Melnikov functions.

Noting that the unperturbed system (4.3) with $\epsilon=0$ has a family of period orbits, we have

$$x_0=\frac{\sqrt{(1+k^2)h^2-k^2}-1}{k^2+1}\quad\text{and}\quad\alpha=\arcsin\left(\frac{kx_0}{h}\right).$$

In [19] we have given the first order Melnikov function as follows,

$$M_1(x_0)=\frac{A_0(\pi-\alpha)+A_1x_0+A_2x_0^2+A_3x_0^3}{2(x_0+1)+2k^2x_0} \tag{4.4}$$

with

$$\begin{aligned}A_0&=a_{11}^-+b_{12}^-,\\ A_1&=4ka_{10}^++2\pi(a_{11}^-+b_{12}^-)+k(a_{11}^-+b_{12}^--2a_{10}^-),\\ A_2&=\pi(1+k^2)(a_{11}^-+b_{12}^-)+4k(a_{10}^+k^2+a_{10}^++a_{11}^+)-a_{11}^-k+b_{12}^-k,\\ A_3&=4k(k^2+1)a_{11}^+,\end{aligned}$$

and when $M_1(x_0)\equiv 0$, the second order Melnikov function is given as follows,

$$M_2(x_0)=-\frac{B_1x_0+B_2x_0^2+B_3x_0^3+B_4x_0^4-B_0(k^2x_0^2+x_0^2+2x_0+1)(\pi-\alpha)}{6(k^2x_0+x_0+1)} \tag{4.5}$$

where

$$\begin{aligned}B_0&=3(a_{21}^-+b_{22}^-),\\ B_1&=-3k(4a_{10}^+b_{10}^+-2a_{10}^+b_{10}^-+4a_{20}^+-2a_{20}^-+a_{21}^-+b_{22}^-),\\ B_2&=(12a_{10}^+a_{12}^+-12a_{10}^+b_{10}^+-6a_{10}^+a_{12}^--12a_{20}^+)k^3+(-12a_{10}^+b_{10}^+-12a_{10}^+b_{11}^+\\ &\quad+6a_{10}^+b_{11}^--12a_{20}^+-12a_{21}^++3a_{21}^--3b_{22}^-)k,\\ B_3&=4k(3a_{10}^+a_{12}^+k^4+3a_{10}^+a_{12}^+k^2-3a_{10}^+b_{11}^+k^2-a_{12}^+b_{12}^+k^2-3a_{21}^+k^2-3a_{10}^+b_{11}^+-3a_{21}^+),\\ B_4&=-4a_{12}^+b_{12}^+k^3(k^2+1).\end{aligned}$$

If $M_1(x_0) = M_2(x_0) \equiv 0$, then $A_0 = A_1 = A_2 = A_3 = B_0 = B_1 = B_2 = B_3 = B_4 = 0$. Consequently, we have

Case 21. $\{a_{11}^+ = 0,\ a_{10}^- = 2a_{10}^+,\ a_{11}^- = -b_{12}^- = (2k^2+2)a_{10}^+,\ a_{12}^+ = 0,\ a_{21}^+ = -a_{10}^+ b_{11}^+,\ a_{20}^- = 2a_{10}^+ b_{10}^+ - a_{10}^+ b_{10}^- + 2a_{20}^+,\ a_{21}^- = -b_{22}^- = 2a_{10}^+ b_{10}^+ k^2 + a_{10}^+ a_{12}^- k^2 + 2a_{20}^+ k^2 + 2a_{10}^+ b_{10}^+ - a_{10}^+ b_{11}^- + 2a_{20}^+\}$;

Case 22. $\{a_{11}^+ = 0,\ a_{10}^- = 2a_{10}^+,\ a_{11}^- = -b_{12}^- = (2k^2+2)a_{10}^+,\ a_{21}^+ = a_{10}^+ a_{12}^+ k^2 - a_{10}^+ b_{11}^+,\ b_{12}^+ = 0,\ a_{20}^- = 2a_{10}^+ b_{10}^+ - a_{10}^+ b_{10}^- + 2a_{20}^+,\ a_{21}^- = -b_{22}^- = 2a_{10}^+ b_{10}^+ k^2 + a_{10}^+ a_{12}^- k^2 + 2a_{20}^+ k^2 + 2a_{10}^+ b_{10}^+ - a_{10}^+ b_{11}^- + 2a_{20}^+\}$.

Lemma 4.1. *For system (4.3), when $M_1(x_0) = M_2(x_0) \equiv 0$, and $M_3(x_0) \not\equiv 0$, the third order Melnikov function has at most 5 isolated zeros, multiplicity taken into account, and the upper bound can be reached.*

Proof. Since the first two order Melnikov functions have already been calculated in detail in [19], here we just display the related results which will be used in the proof. The expressions of $b_1^\pm$ and $b_2^\pm$ are

$$
\begin{aligned}
b_1^+ &= 2ka_{11}^+ x_0^2 + 2ka_{10}^+ x_0,\\
b_1^- &= \frac{1}{2(x_0+1)+2k^2x_0}((k(a_{11}^- - b_{12}^-) - (k^2+1)(a_{11}^- + b_{12}^-)(\pi-\alpha))x_0^2\\
&\quad + 2(k(a_{10}^- - a_{11}^- - b_{12}^-) - (a_{11}^- + b_{12}^-)(\pi-\alpha))x_0 - (a_{11}^- + b_{12}^-)(\pi-\alpha)),\\
b_2^+ &= \frac{2}{3}a_{12}^+ b_{12}^+ k^3 x_0^3 - 2k(a_{10}a_{12}k^2 - a_{10}b_{11}^+ - a_{21}^+)x_0^2 + 2k(a_{10}^{+2}k + a_{10}^+ b_{10}^+ + a_{20}^+)x_0,
\end{aligned}
$$

and

$$
\begin{aligned}
b_2^- &= \frac{1}{2+2x_0(k^2+1)}[(k(4a_{10}^{+2}k^3 - 2a_{10}^+ a_{12}^- k^2 + 4a_{10}^{+2}k + 2a_{10}^+ b_{11}^- + a_{21}^- - b_{22}^-)\\
&\quad + (k(4a_{10}^{+2}k + 2a_{10}^+ b_{10}^- + 2a_{20}^- - a_{21}^- b_{22}^-) - 2(a_{21}^- + b_{22}^-)(\pi-\alpha))x_0\\
&\quad - (k^2+1)(a_{21}^- + b_{22}^-)(\pi-\alpha))x_0^2 - (a_{21}^- + b_{22}^-)(\pi-\alpha)].
\end{aligned}
$$

The corresponding functions $\psi_2^\pm$ are also needed and have been presented as follows:

$$\psi_2^+(x,\, y) = b_{12}^+(kx_0 - y), \qquad \psi_2^-(x,\, y) = 0.$$

Then we have

$$R_{2x}^+(\mathbf{B}) = -b_{12}^+ kx_0 - b_{11}^+ x_0 - b_{10}^+, \quad R_{2y}^+(\mathbf{B}) = -a_{12}^+ kx_0 + a_{10}^+,$$

and

$$R_{2x}^-(\mathbf{B}) = b_{12}^- kx_0 - b_{11}^- x_0 - b_{10}^-, \quad R_{2y}^-(\mathbf{B}) = -a_{12}^- kx_0 + a_{11}^- x_0 + a_{10}^-.$$

In the following, we shall show the calculation of the third order Melnikov function. Firstly, by the definition of corresponding function and the conditions of $M_1(x_0) = M_2(x_0) \equiv 0$, we get

$$\psi_3^-(x,\, y) = \int_0^{T^-(x,\, y)} div(\chi_2^-) \circ (\varphi_t^-(x_0,\, y_0))dt = 0, \tag{4.6}$$

and

$$\psi_3^+(x,\,y) = \int_0^{T^+(x,\,y)} div(\chi_2^+)\circ(\varphi_t^+(x_0,\,y_0))dt$$
$$= -\frac{1}{2}b_{12}^+(a_{11}^+ + 2b_{12}^+)y^2 - (b_{11}^+b_{12}^+x - a_{11}^+b_{12}^+kx_0 - b_{12}^{+2}kx_0 + b_{10}^+b_{12}^+ \tag{4.7}$$
$$+ a_{21}^+ + b_{22}^+)y + kx_0(b_{11}^+b_{12}^+x - \frac{1}{2}a_{11}^+b_{12}^+kx_0 + b_{10}^+b_{12}^+ + a_{21}^+ + b_{22}^+).$$

Secondly, from $M_1(x_0) = M_2(x_0) \equiv 0$ and the decompositions of

$$\Omega_1^{\pm} = \psi_2^{\pm}(x,\,y)dH^{\pm}(x,\,y) + dR_2^{\pm}(x,\,y), \quad \Omega_2^{\pm} = \psi_3^{\pm}(x,\,y)dH^{\pm}(x,\,y) + dR_3^{\pm}(x,\,y),$$

we have

$$R_{2xx}^+(\mathbf{B}) = -b_{11}^+, \quad R_{2yy}^+(\mathbf{B}) = a_{12}^+, \quad R_{2xy}^+(\mathbf{B}) = 0,$$
$$R_{2xx}^-(\mathbf{B}) = -b_{11}^-, \quad R_{2yy}^-(\mathbf{B}) = a_{12}^-, \quad R_{2xy}^-(\mathbf{B}) = 0,$$
$$R_{3x}^+(\mathbf{B}) = 2a_{11}^+b_{12}^+k^2x_0^2 + (-2a_{21}^+k - b_{22}^+k - b_{21}^+)x_0 - b_{20}^+, \tag{4.8}$$
$$R_{3y}^+(\mathbf{B}) = 2(a_{12}^+k - a_{11}^+)b_{12}kx_0^2 + (-2a_{10}^+b_{12}^+k - a_{22}^+k + a_{21}^+)x_0 + a_{20}^+,$$
$$R_{3x}^-(\mathbf{B}) = b_{22}^-kx_0 - b_{21}^-x_0 - b_{20}^-, \quad R_{3y}^-(\mathbf{B}) = -a_{22}^-kx_0 + a_{21}^-x_0 + a_{20}^-.$$

Finally, we consider the third order Melnikov function for Case 21 and Case 22, respectively.

For Case 21, by substituting the above equations into the explicit expression of the third order Melnikov function (2.1), we have

$$M_3(x_0) = \frac{\bar{C}_0 + \bar{C}_1x_0 + \bar{C}_2x_0^2 + \bar{C}_3x_0^3 + \bar{C}_4x_0^4 + \bar{C}_5f_5}{k^2x_0 + x_0 + 1} \tag{4.9}$$

where

$$\bar{C}_0 = 12a_{10}^+b_{10}^{+2} - 6a_{10}^+b_{10}^+b_{10}^- + 12a_{10}^+b_{20}^+ - 6a_{10}^+b_{20}^- + 12a_{20}^+b_{10}^+ - 6a_{20}^+b_{10}^- + 12a_{30}^+ - 6a_{30}^- + 3a_{31}^- + 3b_{32}^-,$$

$$\bar{C}_1 = 12a_{10}^{+3}k^4 + (12a_{10}^{+3} - 12a_{10}^{+2}b_{12}^+ + 12a_{10}^+b_{10}^{+2} + 6a_{10}^+b_{10}^+a_{12}^- - 12a_{10}^+a_{22}^+ + 12a_{10}^+b_{20}^+ + 6a_{10}^+a_{22}^- + 12a_{20}^+b_{10}^+ + 6a_{20}^+a_{12}^- + 12a_{30}^+)k^2 + 12a_{10}^+b_{10}^{+2} + 12a_{10}^+b_{10}^+b_{11}^+ - 6a_{10}^+b_{10}^+b_{11}^- + 12a_{10}^+b_{20}^+ + 12a_{10}^+b_{21}^+ - 6a_{10}^+b_{21}^- + 12a_{20}^+b_{10}^+ + 12a_{20}^+b_{11}^+ - 6a_{20}^+b_{11}^- + 12a_{30}^+ + 12a_{31}^+ - 3a_{31}^- + 3b_{32}^- + \bar{C}_0(1+k^2),$$

$$\bar{C}_2 = -12a_{10}^+(a_{10}^+b_{12}^+ + a_{22}^+)k^4 + 4(-3a_{10}^{+2}b_{12}^+ + 3a_{10}^+b_{10}^+b_{11}^+ + a_{10}^+b_{12}^{+2} - 3a_{10}^+a_{22}^+ + 3a_{10}^+b_{21}^+ + 3a_{20}^+b_{11}^+ + a_{22}^+b_{12}^+ + 3a_{31}^+)k^2 + 12(a_{10}^+b_{10}^+b_{11}^+ + a_{10}^+b_{21}^+ + a_{20}^+b_{11}^+ + a_{31}^+) + \bar{C}_1(1+k^2),$$

$$\bar{C}_3 = (1+k^2)(4k^2b_{12}^+(a_{10}^+b_{12}^+ + a_{22}^+) + \bar{C}_2), \qquad \bar{C}_4 = \bar{C}_3(1+k^2),$$

$$\bar{C}_5 = \frac{1}{2}(a_{31}^- + b_{32}^-), \qquad f_5 = (k^2x_0^2 + x_0^2 + 2x_0 + 1)(\pi - \alpha).$$

Similarly, for Case 22, the third order Melnikov function $M_3(x_0)$ is

$$M_3(x_0) = \frac{\tilde{C}_0 + \tilde{C}_1x_0 + \tilde{C}_2x_0^2 + \tilde{C}_3x_0^3 + \tilde{C}_4x_0^4 + \tilde{C}_5f_5}{k^2x_0 + x_0 + 1} \tag{4.10}$$

where

$$\tilde{C}_0 = 12a_{10}^{+}b_{10}^{+2} - 6a_{10}^{+}b_{10}^{+}b_{10}^{-} + 12a_{10}^{+}b_{20}^{+} - 6a_{10}^{+}b_{20}^{-} + 12a_{20}^{+}b_{10}^{+} - 6a_{20}^{+}b_{10}^{-} + 12a_{30}^{+} - 6a_{30}^{-} + 3a_{31}^{-} + 3b_{32}^{-},$$

$$\tilde{C}_1 = 12a_{10}^{+3}k^4 + (12a_{10}^{+3} - 12a_{10}^{+}a_{12}^{+}b_{10}^{+} + 12a_{10}^{+}b_{10}^{+2} + 6a_{10}^{+}b_{10}^{+}a_{12}^{-} - 12a_{10}^{+}a_{22}^{+} + 12a_{10}^{+}b_{20}^{+} + 6a_{10}^{+}a_{22}^{-} - 12a_{12}^{+}a_{20}^{+} + 12a_{20}^{+}b_{10}^{+} + 6a_{20}^{+}a_{12}^{-} + 12a_{30}^{+})k^2 + 12a_{10}^{+}b_{10}^{+2} + 12a_{10}^{+}b_{10}^{+}b_{11}^{+} - 6a_{10}^{+}b_{10}^{+}b_{11}^{-} + 12a_{10}^{+}b_{20}^{+} + 12a_{10}^{+}b_{21}^{+} - 6a_{10}^{+}b_{21}^{-} + 12a_{20}^{+}b_{10}^{+} + 12a_{20}^{+}b_{11}^{+} - 6a_{20}^{+}b_{11}^{-} + 12a_{30}^{+} + 12a_{31}^{+} - 3a_{31}^{-} + 3b_{32}^{-} + \tilde{C}_0(1+k^2),$$

$$\tilde{C}_2 = (4a_{10}^{+}a_{12}^{+2} - 12a_{10}^{+}a_{12}^{+}b_{10}^{+} - 12a_{10}^{+}a_{22}^{+} - 12a_{12}^{+}a_{20}^{+})k^4 + (-12a_{10}^{+}a_{12}^{+}b_{10}^{+} - 4a_{10}^{+}a_{12}^{+}b_{11}^{+} + 12a_{10}^{+}b_{10}^{+}b_{11}^{+} - 12a_{10}^{+}a_{22}^{+} + 12a_{10}^{+}b_{21}^{+} - 12a_{12}^{+}a_{20}^{+} + 4a_{12}^{+}b_{22}^{+} + 12a_{20}^{+}b_{11}^{+} + 12a_{31}^{+})k^2 + 12a_{10}^{+}b_{10}^{+}b_{11}^{+} + 12a_{10}^{+}b_{21}^{+} + 12a_{20}^{+}b_{11}^{+} + 12a_{31}^{+} + \tilde{C}_1(1+k^2),$$

$$\tilde{C}_3 = (1+k^2)(4a_{10}^{+}a_{12}^{+2}k^4 - 4a_{10}^{+}a_{12}^{+}b_{11}^{+}k^2 + 4a_{12}^{+}b_{22}^{+}k^2 + \tilde{C}_2),$$

$$\tilde{C}_4 = \tilde{C}_3(1+k^2), \qquad \tilde{C}_5 = \frac{1}{2}(a_{31}^{-} + b_{32}^{-}), \qquad f_5 = (k^2x_0^2 + x_0^2 + 2x_0 + 1)(\pi - \alpha).$$

It is obvious that $M_3(x_0)$ has the same generators with $M_2(x_0)$, hence according to Theorem 2.4 in [19], the third order Melnikov function has at most 5 isolated zeros, taking into account their multiplicities. By straightly calculation, we have

$$rank\Big(\frac{\partial(\bar{C}_0, \bar{C}_1, \bar{C}_2, \bar{C}_3, \bar{C}_4, \bar{C}_5)}{\partial(a_{20}^{+}, b_{12}^{+}, b_{11}^{+}, a_{22}^{+}, a_{31}^{-}, k)}\Big) = 6, \quad rank\Big(\frac{\partial(\tilde{C}_0, \tilde{C}_1, \tilde{C}_2, \tilde{C}_3, \tilde{C}_4, \tilde{C}_5)}{\partial(a_{20}^{+}, a_{12}^{+}, b_{10}^{+}, b_{22}^{+}, a_{31}^{-}, k)}\Big) = 6,$$

which means that with proper parameters, the upper bound can be reached. Hence the proof has been finished. □

The proof of Theorem 4.1 and Corollary 4.1. The proof of Theorem 4.1 can be obtained according to Lemma 4.1, Theorem 1.1 [9] and Theorem 3.3 [10], namely, the upper bound of the number of limit cycles bifurcated from the period orbits is 5 (taking into account their multiplicities) for system (4.3).

Corollary 4.1 can be proved similarly by substituting $a_{i0}^{\pm} = a_{i1}^{\pm} = a_{i2}^{\pm} = b_{i0}^{\pm} = b_{i1}^{\pm} = b_{i2}^{\pm} = 0$, $i = 2, 3$ into (4.6)-(4.8). When $M_1(x_0) = M_2(x_0) \equiv 0$, we have

Case 21a. $\{a_{10}^{-} = a_{10}^{+} = a_{11}^{+} = a_{11}^{-} = b_{12}^{-} = a_{12}^{+} = 0\}$;

Case 21b. $\{a_{11}^{+} = b_{11}^{+} = a_{12}^{+} = 0,\ a_{10}^{-} = 2a_{10}^{+},\ a_{11}^{-} = (2k^2+2)a_{10}^{+},\ b_{12}^{-} = -(2k^2+2)a_{10}^{+},\ b_{10}^{-} = 2b_{10}^{+},\ b_{11}^{-} = 2b_{10}^{+}k^2 + a_{12}^{-}k^2 + 2b_{10}^{+}\}$;

Case 22a. $\{a_{10}^{-} = a_{10}^{+} = a_{11}^{+} = a_{11}^{-} = b_{12}^{-} = b_{12}^{+} = 0\}$;

Case 22b. $\{a_{11}^{+} = b_{12}^{+} = 0,\ a_{10}^{-} = 2a_{10}^{+},\ a_{11}^{-} = (2k^2+2)a_{10}^{+},\ b_{12}^{-} = -(2k^2+2)a_{10}^{+},\ b_{11}^{+} = a_{12}^{+}k^2,\ b_{10}^{-} = 2b_{10}^{+},\ b_{11}^{-} = 2b_{10}^{+}k^2 + a_{12}^{-}k^2 + 2b_{10}^{+}\}$.

For Case 21a and Case 22a, $M_3(x_0) \equiv 0$. For Case 22b, we have $M_3(x_0) = \frac{2k^3x_0^2a_{10}^{+3}(k^2+1)}{1+(k^2+1)x_0}$. It has no isolated zero for $x_0 > 0$.

For Case 21b, we have

$$M_3(x_0) = \frac{2}{3}\frac{k^3x_0^2a_{10}^{+}(b_{12}^{+}(k^2+1)x_0^2 - b_{12}^{+}(3a_{10}^{+}k^2 + 3a_{10}^{+} - b_{12}^{+})x_0 + 3a_{10}^{+}(a_{10}^{+}k^2 + a_{10}^{+} - b_{12}^{+}))}{1+(k^2+1)x_0}.$$

It is obvious that for system (4.3) with $\omega_2^{\pm} = \omega_3^{\pm} = 0$, $M_3(x_0)$ has at most 2 isolated zeros taking into account their multiplicities, and the upper bound can be reached with proper parameters. Hence Corollary 4.1 has been proved. □

4.2. An application for $\theta = \pi$

In this subsection, we consider another piecewise linear differential system which is a perturbation of a linear center-center type Hamiltonian system, namely,

$$dH + \epsilon\omega_1 + \epsilon^2\omega_2 + \epsilon^3\omega_3 = 0, \tag{4.11}$$

where

$$H(x, y) = \begin{cases} H^+(x, y) = (x-1)^2 + y^2 = h^2, \, x \geq 0, \\ H^-(x, y) = (x+1)^2 + y^2, \, x < 0, \end{cases}$$

and for $i = 1, 2, 3$,

$$\omega_i = \begin{cases} \omega_i^+ = P_i^+(x, y)dy - Q_i^+(x, y)dx, \, x \geq 0, \\ \omega_i^- = P_i^-(x, y)dy - Q_i^-(x, y)dx, \, x < 0, \end{cases}$$

with

$$P_i^{\pm}(x, y) = a_{i0}^{\pm} + a_{i1}^{\pm}x + a_{i2}^{\pm}y,$$
$$Q_i^{\pm}(x, y) = b_{i0}^{\pm} + b_{i1}^{\pm}x + b_{i2}^{\pm}y.$$

This system has been studied in [5] by using the first order Melnikov function. Here we shall consider the same system by using the first three order Melnikov functions and the aim is also to find if there are more limit cycles or not by considering linear perturbations up to higher order in ϵ.

Obviously, system $(4.11)|_{\epsilon=0}$ has a family of period orbits when $h > 1$. In the following, we give the main theorem about the limit cycles for system (4.11). It is worth mentioning that the number of zeros for the $k-$th order Melnikov function is obtained when the first $k-1$ order Melnikov functions vanish and the $k-$th order Melnikov function is not zero identically.

Theorem 4.3. *For system (4.11), the upper bound of the number of limit cycles bifurcated from the period orbits is* 2, 2, 2 *(taking into account their multiplicities) by using the first three order Melnikov functions, respectively, and two limit cycles can appear with proper parameters.*

Lemma 4.2. *For system (4.11), if the first order Melnikov function $M_1(h)$ is not zero identically, then $M_1(h)$ has at most* 2 *isolated zeros, taking into account their multiplicities, and* 2 *can be reached.*

Proof. For system (4.11), we make different coordinate transformations as $x = -h\cos\theta - 1$, $y = h\sin\theta$, and $x = h\cos\theta + 1$, $y = h\sin\theta$, respectively for the left and right subsystems. Noting that the unperturbed system (4.11) with $\epsilon = 0$ has a family of period orbits, we have $\alpha = \arctan(\sqrt{h^2-1})$, see Figure 3. It is easy

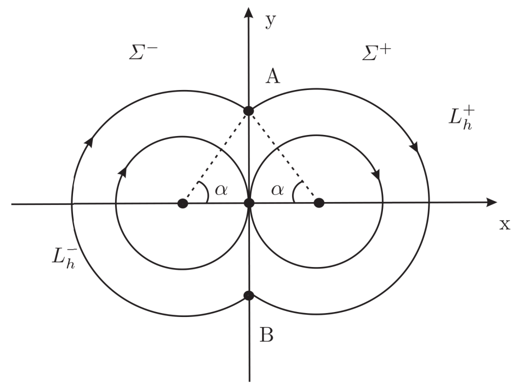

Figure 3. Unperturbed system (4.11)

to calculate the first order Melnikov function. Here we just give the expression of $M_1(h)$ as follows,

$$M_1(h) = b_1^+ - b_1^- = \tilde{A}_0 + \tilde{A}_1 \frac{h^2}{\sqrt{h^2-1}}(\pi - \alpha), \tag{4.12}$$

where

$$\tilde{A}_0 = a_{10}^- - a_{10}^+ - \frac{1}{2}(a_{11}^- + b_{12}^-) - \frac{1}{2}(a_{11}^+ + b_{12}^+),$$
$$\tilde{A}_1 = -\frac{1}{2}(a_{11}^- + b_{12}^-) - \frac{1}{2}(a_{11}^+ + b_{12}^+).$$

Let $t = \sqrt{h^2-1}$, and taking derivative twice in succession leads to

$$M_1'(t) = \tilde{A}_1 \cdot \frac{t^2(\pi - \arctan t) - (\pi - \arctan t) - t}{t^2},$$
$$M_1''(t) = 2\tilde{A}_1 \cdot \frac{t^2(\pi - \arctan t) + (\pi - \arctan t) + t}{t^3(t^2+1)}.$$

Noting that $\pi - \alpha > 0$, it is apparent from $M_1''(t) > 0$ for $t \in (0, +\infty)$ that $M_1(t)$ has at most 2 isolated zeros, taking into account their multiplicities, and it can be reached with proper parameters. □

If $M_1(h) \equiv 0$, then we have $\tilde{A}_0 = \tilde{A}_1 = 0$, namely,

$$\{a_{10}^- = a_{10}^+,\ a_{11}^- + b_{12}^- + a_{11}^+ + b_{12}^+ = 0\}. \tag{4.13}$$

Lemma 4.3. *For system (4.11), when $M_1(h) \equiv 0$ and $M_2(h) \not\equiv 0$, the second order Melnikov function $M_2(h)$ has at most 2 isolated zeros, taking into account their multiplicities, and 2 can be reached.*

Proof. When $M_1(h) \equiv 0$, we have to consider the second order Melnikov function. The first step is calculating the corresponding functions $\psi_2^{\pm}(h, \theta)$ as follows,

$$\psi_2^+(h, \theta) = \int_0^{T^+(h,\theta)} div(\chi_1^+) \circ (\varphi_t^+) dt = -\frac{1}{2}(a_{11}^+ + b_{12}^+)(\theta - \pi + \alpha),$$

where $\theta \in [\alpha - \pi, \pi - \alpha]$, and

$$\psi_2^-(h, \theta) = \int_0^{T^-(h,\theta)} div(\chi_1^-) \circ (\varphi_t^-) dt = \frac{1}{2}(a_{11}^- + b_{12}^-)(\theta - \pi + \alpha),$$

with $\theta \in [\alpha - \pi, \pi - \alpha]$.

The second step focuses on the decomposition aiming at $R_{2y}^{\pm}(\mathbf{B})$, and it gives to

$$\begin{aligned}R_{2y}^+(\mathbf{B}) &= P_1^+(\mathbf{B}) - \psi_2^+ H_y^+(\mathbf{B}) \\ &= P_1^+(\alpha - \pi, h) - \psi_2^+(\alpha - \pi) H_y^+(\mathbf{B}) \\ &= a_{10}^+ - a_{12}^+\sqrt{h^2-1} + 2(a_{11}^+ + b_{12}^+)\sqrt{h^2-1}(\pi - \alpha),\end{aligned}$$

and

$$\begin{aligned}R_{2y}^-(\mathbf{B}) &= P_1^-(\mathbf{B}) - \psi_2^- H_y^-(\mathbf{B}) \\ &= P_1^-(\alpha - \pi, h) - \psi_2^-(\alpha - \pi) H_y^-(\mathbf{B}) \\ &= a_{10}^- - a_{12}^-\sqrt{h^2-1} - 2(a_{11}^- + b_{12}^-)\sqrt{h^2-1}(\pi - \alpha).\end{aligned}$$

Thirdly, according to Corollary 2.1, we have

$$b_2^+ = \frac{2}{H_y^+(\mathbf{B})}\left(\int_{\pi-\alpha}^{\alpha-\pi} \psi_2^+\omega_1^+ - \omega_2^+ - R_{2y}^+(\mathbf{B})b_1^+ - \frac{1}{2}H_{yy}^+(\mathbf{B}){b_1^+}^2\right),$$

and

$$b_2^- = \frac{2}{H_y^-(\mathbf{B})}\left(\int_{\pi-\alpha}^{\alpha-\pi} \psi_2^-\omega_1^- - \omega_2^- - R_{2y}^-(\mathbf{B})b_1^- - \frac{1}{2}H_{yy}^-(\mathbf{B}){b_1^-}^2\right).$$

The first part in above formula can be calculated as follows,

$$\begin{aligned}&\int_{\pi-\alpha}^{\alpha-\pi} \psi_2^+\omega_1^+ - \omega_2^+ \\ &= -\frac{1}{4}(\pi - \alpha)(-a_{11}^+a_{12}^+ - a_{11}^+b_{11}^+ - a_{12}^+b_{12}^+ - b_{11}^+b_{12}^+ - 4a_{21}^+ - 4b_{22}^+)h^2 \\ &\quad - \frac{1}{2}(a_{11}^+ + b_{12}^+)^2(\pi - \alpha)^2h^2 - \frac{1}{2}(\pi - \alpha)(a_{12}^+ - 2b_{10}^+ - b_{11}^+)(a_{11}^+ + b_{12}^+) \\ &\quad + (-\frac{1}{4}a_{11}^+a_{12}^+ + b_{10}^+a_{11}^+ + a_{21}^+ + b_{22}^+ + 2a_{20}^+ - \frac{1}{4}a_{12}^+b_{12}^+ + \frac{3}{4}(b_{11}^+a_{11}^+ + b_{11}^+b_{12}^+) \\ &\quad + b_{10}^+b_{12}^+)\sqrt{h^2-1} - \frac{1}{2}(\pi - \alpha)(a_{11}^+ + b_{12}^+)(2a_{10}^+ + a_{11}^+ + b_{12}^+)\sqrt{h^2-1},\end{aligned}$$

and

$$\begin{aligned}&\int_{\pi-\alpha}^{\alpha-\pi} \psi_2^-\omega_1^- - \omega_2^- \\ &= -\frac{1}{4}(\pi - \alpha)(a_{11}^-a_{12}^- + a_{11}^-b_{11}^- + a_{12}^-b_{12}^- + b_{11}^-b_{12}^- + 4a_{21}^- + 4b_{22}^-)h^2 \\ &\quad - \frac{1}{2}(a_{11}^- + b_{12}^-)^2(\pi - \alpha)^2h^2 + \frac{1}{2}(\pi - \alpha)(a_{12}^- + 2b_{10}^- - b_{11}^-)(a_{11}^- + b_{12}^-) \\ &\quad + (\frac{1}{4}a_{11}^-a_{12}^- + \frac{1}{4}a_{12}^-b_{12}^- + b_{10}^-a_{11}^- - \frac{3}{4}b_{11}^-(a_{11}^- + b_{12}^-) + 2a_{20}^- - a_{21}^- - b_{22}^- \\ &\quad + b_{10}^-b_{12}^-)\sqrt{h^2-1} + \frac{1}{2}(\pi - \alpha)(a_{11}^- + b_{12}^-)(2a_{10}^- - a_{11}^- - b_{12}^-)\sqrt{h^2-1}.\end{aligned}$$

Finally, let $t=\sqrt{h^2-1}$, a routine computation gives rise to

$$M_2(t)=\frac{1}{2}(b_2^+-b_2^-)=-\frac{1}{8t}(B_1f_1+B_2f_2+B_3f_3),$$

where

$$f_1=t,\quad f_2=(\pi-\alpha),\quad f_3=(\pi-\alpha)t^2,$$

and

$$\begin{aligned}B_1=&-4a_{12}^+a_{10}^-+3a_{12}^+a_{11}^-+3a_{12}^+b_{12}^--4b_{10}^+a_{11}^--4b_{10}^+b_{12}^--3b_{11}^+a_{11}^--3b_{11}^+b_{12}^-\\&+4a_{10}^-a_{12}^--3a_{11}^-a_{12}^--4a_{11}^-b_{10}^-+3a_{11}^-b_{11}^--3a_{12}^-b_{12}^--4b_{10}^-b_{12}^-\\&+3b_{11}^-b_{12}^-+8a_{20}^++4a_{21}^++4b_{22}^+-8a_{20}^-+4a_{21}^-+4b_{22}^-,\end{aligned}$$

$$\begin{aligned}B_2=&(a_{11}^-+b_{12}^-)(3a_{12}^+-4b_{10}^+-3b_{11}^+-3a_{12}^--4b_{10}^-+3b_{11}^-)\\&+4a_{21}^++4b_{22}^++4a_{21}^-+4b_{22}^-,\end{aligned}$$

$$B_3=(a_{11}^-+b_{12}^-)(a_{12}^+-b_{11}^+-a_{12}^-+b_{11}^-)+4a_{21}^++4b_{22}^++4a_{21}^-+4b_{22}^-.$$

In the following, rewrite $M_2(t)$ as $M_2(t)=-\frac{B_3}{8}(\frac{B_1}{B_3}+\frac{B_2}{B_3}P(t)+Q(t))$, where $P(t)=\frac{\pi-\alpha}{t}$ and $Q(t)=(\pi-\alpha)t$. The zeros of $M_2(t)$ can be obtained by considering the zeros of $\frac{B_1}{B_3}+\frac{B_2}{B_3}P(t)+Q(t)=0$. Let $\Sigma=\{(P,Q)(t)|t\in(0,+\infty)\}$, and $L=\{(P,Q)|\frac{B_1}{B_3}+\frac{B_2}{B_3}P+Q=0\}$, then the problem is equivalent to finding the number of intersection points of Σ and L in $P-Q$ plane, taking into account their multiplicities. We claim that Σ is convex strictly. The proof of this claim is postponed until Appendix A. Based on the claim, L can have at most two intersection points with Σ, namely, $M_2(t)$ has at most two zeros, taking into account their multiplicities.

On the other hand, it gives by calculating

$$rank\Big(\frac{\partial(B_1,B_2,B_3)}{\partial(a_{10}^-,b_{12}^-,b_{10}^+)}\Big)=3.$$

Thus $M_2(t)$ can have at least 2 simple zeros, taking into account their multiplicities, and the upper bound can be reached. □

If $M_1(h)=M_2(h)\equiv 0$, then we have

Case a. $\{a_{10}^+=a_{10}^-,\ a_{11}^+=-b_{12}^+,\ a_{20}^+=a_{20}^--\frac{1}{2}a_{10}^-a_{12}^-+\frac{1}{2}a_{10}^-a_{12}^+,\ a_{21}^+=-b_{22}^+-a_{21}^--b_{22}^-,\ a_{11}^-=-b_{12}^-\}$;

Case b. $\{a_{10}^+=a_{10}^-,\ a_{11}^+=-b_{12}^+-a_{11}^--b_{12}^-,\ a_{12}^+=2b_{10}^++b_{11}^++a_{12}^-+2b_{10}^--b_{11}^-,\ a_{20}^+=a_{20}^-+a_{10}^-b_{10}^++\frac{1}{2}a_{10}^-b_{11}^++a_{10}^-b_{10}^--\frac{1}{2}a_{10}^-b_{11}^-,\ a_{21}^+=-\frac{1}{2}b_{10}^+a_{11}^--\frac{1}{2}a_{11}^-b_{10}^--\frac{1}{2}b_{10}^+b_{12}^--\frac{1}{2}b_{10}^-b_{12}^-b_{22}^+-a_{21}^--b_{22}^-\}$.

Lemma 4.4. *For system (4.11), when $M_1(h)=M_2(h)\equiv 0$ and $M_3(h)\not\equiv 0$, the third order Melnikov function $M_3(h)$ has at most 2 isolated zeros for Case a and Case b, taking into account their multiplicities, and the upper bound can be reached.*

Proof. Because of tedious expressions, we omit the first and second steps for calculating $\psi_3^{\pm}(h,\theta)$, $R_{2yy}^{\pm}(\mathbf{B})$ and $R_{3y}^{\pm}(\mathbf{B})$, and put them in Appendix B. Hence we obtain $M_3(t)$ straightly for Case a:

$$M_3(t)=-\frac{1}{8t}(C_{1a}f_1+C_{2a}f_2+C_{3a}f_3),\tag{4.14}$$

where

$$\begin{aligned}C_{1a} &= 2a_{10}^-a_{12}^-(a_{12}^+ - a_{12}^-) - 4a_{12}^+a_{20}^- + 3a_{12}^+a_{21}^- + 3a_{12}^+b_{22}^- - 4a_{10}^-a_{22}^+ - 4b_{10}^+a_{21}^- - 4b_{10}^+b_{22}^- \\ &\quad - 3b_{11}^+a_{21}^- - 3b_{11}^+b_{22}^- + 4a_{10}^-a_{22}^- + 4a_{12}^-a_{20}^- - 3a_{12}^-a_{21}^- - 3a_{12}^-b_{22}^- - 4a_{21}^-b_{10}^- + 3a_{21}^-b_{11}^- \\ &\quad - 4b_{10}^-b_{22}^- + 3b_{11}^-b_{22}^- + 8a_{30}^+ + 4a_{31}^+ + 4b_{32}^+ - 8a_{30}^- + 4a_{31}^- + 4b_{32}^-,\end{aligned}$$

$$C_{2a} = (a_{21}^- + b_{22}^-)(3a_{12}^+ - 4b_{10}^+ - 3b_{11}^+ - 3a_{12}^- - 4b_{10}^- + 3b_{11}^-) + 4a_{31}^+ + 4b_{32}^+ + 4a_{31}^- + 4b_{32}^-,$$

$$C_{3a} = (a_{21}^- + b_{22}^-)(a_{12}^+ - b_{11}^- - a_{12}^- + b_{11}^-) + 4a_{31}^+ + 4b_{32}^+ + 4a_{31}^- + 4b_{32}^-.$$

Similarly, the third order Melnikov function for Case b is calculated as follows,

$$M_3(t) = -\frac{1}{16t}(C_{1b}f_1 + C_{2b}f_2 + C_{3b}f_3), \tag{4.15}$$

where

$$\begin{aligned}C_{1b} &= -3b_{10}^{+2}a_{11}^- - 3b_{10}^{+2}b_{12}^- - 6b_{10}^+b_{11}^+a_{11}^- - 6b_{10}^+b_{11}^+b_{12}^- + 8b_{10}^+a_{10}^-a_{12}^- - 6b_{10}^+a_{11}^-a_{12}^- \\ &\quad - 10b_{10}^+a_{11}^-b_{10}^- + 6b_{10}^+a_{11}^-b_{11}^- - 6b_{10}^+a_{12}^-b_{12}^- - 10b_{10}^+b_{10}^-b_{12}^- + 6b_{10}^+b_{11}^-b_{12}^- - 3b_{11}^{+2}a_{11}^- \\ &\quad - 3b_{11}^{+2}b_{12}^- + 4b_{11}^+a_{10}^-a_{12}^- - 3b_{11}^+a_{11}^-a_{12}^- - 6b_{11}^+a_{11}^-b_{10}^- + 3b_{11}^+a_{11}^-b_{11}^- - 3b_{11}^+a_{12}^-b_{12}^- \\ &\quad - 6b_{11}^+b_{10}^-b_{12}^- + 3b_{11}^+b_{11}^-b_{12}^- - 3b_{12}^{+2}a_{11}^- - 3b_{12}^{+2}b_{12}^- + 4b_{12}^+a_{10}^-a_{11}^- + 4b_{12}^+a_{10}^-b_{12}^- \\ &\quad - 3b_{12}^+a_{11}^{-2} - 6b_{12}^+a_{11}^-b_{12}^- - 3b_{12}^+b_{12}^{-2} + 4a_{10}^-a_{11}^-b_{12}^- + 8a_{10}^-a_{12}^-b_{10}^- - 4a_{10}^-a_{12}^-b_{11}^- \\ &\quad + 4a_{10}^-b_{12}^{-2} - 3a_{11}^{-2}b_{12}^- - 6a_{11}^-a_{12}^-b_{10}^- + 3a_{11}^-a_{12}^-b_{11}^- - 7a_{11}^-b_{10}^{-2} + 6a_{11}^-b_{10}^-b_{11}^- - 3a_{11}^-b_{12}^{-2} \\ &\quad - 6a_{12}^-b_{10}^-b_{12}^- + 3a_{12}^-b_{11}^-b_{12}^- - 7b_{10}^{-2}b_{12}^- + 6b_{10}^-b_{11}^-b_{12}^- - 8a_{22}^+a_{10}^- + 6a_{22}^+a_{11}^- + 6a_{22}^+b_{12}^- \\ &\quad - 16b_{10}^+a_{20}^- + 4b_{10}^+a_{21}^- + 4b_{10}^+b_{22}^- - 8b_{11}^+a_{20}^- - 8b_{20}^+a_{11}^- - 8b_{20}^+b_{12}^- - 6b_{21}^+a_{11}^- - 6b_{21}^+b_{12}^- \\ &\quad + 8a_{10}^-a_{22}^- - 6a_{11}^-a_{22}^- - 8a_{11}^-b_{20}^- + 6a_{11}^-b_{21}^- - 16a_{20}^-b_{10}^- + 8a_{20}^-b_{11}^- + 4a_{21}^-b_{10}^- - 6a_{22}^-b_{12}^- \\ &\quad + 4b_{10}^-b_{22}^- - 8b_{12}^-b_{20}^- + 6b_{12}^-b_{21}^- + 16a_{30}^+ + 8a_{31}^+ + 8b_{32}^+ - 16a_{30}^- + 8a_{31}^- + 8b_{32}^-,\end{aligned}$$

$$\begin{aligned}C_{2b} &= -(a_{11}^- + b_{12}^-)(3b_{10}^{+2} + 6b_{10}^+b_{11}^+ + 6b_{10}^+a_{12}^- + 10b_{10}^+b_{10}^- - 6b_{10}^+b_{11}^- + 3b_{11}^{+2} + 3b_{11}^+a_{12}^- \\ &\quad + 6b_{11}^+b_{10}^- - 3b_{11}^+b_{11}^- + 3b_{12}^{+2} - 4b_{12}^+a_{10}^- + 3b_{12}^+a_{11}^- + 3b_{12}^+b_{12}^- - 4a_{10}^-b_{12}^- + 3a_{11}^-b_{12}^- \\ &\quad + 6a_{12}^-b_{10}^- - 3a_{12}^-b_{11}^- + 7b_{10}^{-2} - 6b_{10}^-b_{11}^-) + 6a_{22}^+a_{11}^- + 6a_{22}^+b_{12}^- + 4b_{10}^+a_{21}^- + 4b_{10}^+b_{22}^- \\ &\quad - 8b_{20}^+a_{11}^- - 8b_{20}^+b_{12}^- - 6b_{21}^+a_{11}^- - 6b_{21}^+b_{12}^- - 6a_{11}^-a_{22}^- - 8a_{11}^-b_{20}^- + 6a_{11}^-b_{21}^- + 4a_{21}^-b_{10}^- \\ &\quad - 6a_{22}^-b_{12}^- + 4b_{10}^-b_{22}^- - 8b_{12}^-b_{20}^- + 6b_{12}^-b_{21}^- + 8a_{31}^+ + 8b_{32}^+ + 8a_{31}^- + 8b_{32}^-,\end{aligned}$$

$$\begin{aligned}C_{3b} &= -(a_{11}^- + b_{12}^-)(b_{10}^{+2} + 2b_{10}^+b_{11}^+ + 2b_{10}^+a_{12}^- + 2b_{10}^+b_{10}^- - 2b_{10}^+b_{11}^- + b_{11}^{+2} + b_{11}^+a_{12}^- + 2b_{11}^+b_{10}^- \\ &\quad - b_{11}^+b_{11}^- + b_{12}^{+2} + b_{12}^+a_{11}^- + b_{12}^+b_{12}^- + a_{11}^-b_{12}^- + 2a_{12}^-b_{10}^- - a_{12}^-b_{11}^- + b_{10}^{-2} - 2b_{10}^-b_{11}^-) \\ &\quad + 2a_{22}^+a_{11}^- + 2a_{22}^+b_{12}^- + 4b_{10}^+a_{21}^- + 4b_{10}^+b_{22}^- - 2b_{21}^+a_{11}^- - 2b_{21}^+b_{12}^- - 2a_{11}^-a_{22}^- + 2a_{11}^-b_{21}^- \\ &\quad + 4a_{21}^-b_{10}^- - 2a_{22}^-b_{12}^- + 4b_{10}^-b_{22}^- + 2b_{12}^-b_{21}^- + 8a_{31}^+ + 8b_{32}^+ + 8a_{31}^- + 8b_{32}^-.\end{aligned}$$

Noting that the generators of $M_3(t)$ are the same with $M_2(t)$, it follows that $M_3(t)$ has at most 2 zeros in Case a and Case b .

On the other hand, it gives by calculating

$$rank\Big(\frac{\partial(C_{1a}, C_{2a}, C_{3a})}{\partial(a_{10}^-, b_{21}^-, b_{32}^-)}\Big) = 3, \quad rank\Big(\frac{\partial(C_{1b}, C_{2b}, C_{3b})}{\partial(a_{10}^-, b_{12}^-, b_{32}^-)}\Big) = 3.$$

Thus $M_3(t)$ can have at least 2 isolated zeros, and 2 can be reached. □

The proof of Theorem 4.3. The proof of Theorem 4.3 can be obtained according to Lemma 4.2, Lemma 4.3, Lemma 4.4, Theorem 1.1 [9] and Theorem 3.3 [10]. □

5. Conclusion

Compared with the results in [3] and [4], we can clearly observe the differences between them by Table 1. When the unperturbed system is an elementary center, Buzzi et al. [3] found one limit cycle by the first and second order Melnikov functions. When considering the center-center type system (4.11), we find two limit cycles by $M_1(h)$. On the other hand, Cardin and Torregrosa [4] obtained 4 and 5 limit cycles by $M_5(h)$ and $M_6(h)$, respectively. Compared with the results in [4], we obtain 4, 5, 5 limit cycles by the first three order Melnikov functions, respectively.

Table 1. The comparison of the upper bounds for different systems

Unperturbed system \ $M_k(h)$	$M_1(h)$	$M_2(h)$	$M_3(h)$	$M_4(h)$	$M_5(h)$	$M_6(h)$	$M_7(h)$
Elementary center [3], $\theta = \pi$	1	1	2	3	3	3	3
System $(4.11)\|_{\epsilon=0}$, $\theta = \pi$	2	2	2	–	–	–	–
Elementary center [4], $\theta \in (0, \pi)$	1	2	2	3	4	5	–
System $(4.1)\|_{\epsilon=0}$, $\theta \in (0, \pi)$	4	5	5	–	–	–	–

It is worth mentioning that for system (4.11), we guess the maximum number of zeros for any order Melnikov function $M_k(h)$ is two. This conjecture is based on the following two facts. On the one hand, we try to calculate the fourth order Melnikov function and get the same generators with $M_2(h)$ and $M_3(h)$. Due to the complex calculation for higher order Melnikov functions, we omit the expression of $M_4(h)$. On the other hand, in other work, we also consider the piecewise linear perturbation of linear center-center type system which the centers of two subsystems are not symmetry. We obtain the maximum number of zeros for any order Melnikov function $M_k(h)$ is also two for such system. Hence we propose a conjecture: for the piecewise linear perturbation of a symmetric linear center-center type system (4.11), the maximum number of zeros is 2 by any order Melnikov function $M_k(h)$.

Similarly, for system (4.1) with $\theta \in (0, \pi)$, we also calculate the fourth order Melnikov function and obtain the same generators with $M_2(h)$ and $M_3(h)$. Hence we guess that the maximum number of zeros is 5 by any order Melnikov function $M_k(h)$. These are both open problems which need us to explore in the future.

Appendix A

In the proof of Lemma 4.3, we give a claim that Σ is convex strictly. Here $\Sigma = \{(P, Q)(t) | t \in (0, +\infty)\}$, where $\alpha = \arctan t$, $P(t) = \frac{\pi-\alpha}{t}$ and $Q(t) = (\pi - \alpha)t$. We are now in a position to show that this claim holds.

Claim. Σ *is convex strictly.*

Proof. A routine computation gives rise to $\frac{d^2Q}{dP^2} = \frac{2(t^2+1)t^3}{((\pi-\alpha)t^2+(\pi-\alpha)+t)^3} S(t)$ where

$$S(t) = (\pi - \alpha)^2(t^2 + 1)^2 + (\pi - \alpha)t^3 - (\pi - \alpha)t - 2t^2.$$

In view of the fact that $\frac{\pi}{2} < \pi - \alpha < \pi$ and $t > 0$, it is easy to obtain that $S(t) > s(t)$ where

$$s(t) = \frac{\pi^2}{4}(t^2+1)^2 + \frac{\pi}{2}t^3 - \pi t - 2t^2.$$

The first two derivatives of $s(t)$ are $s'(t) = (t^2+1)\pi^2 t + \frac{3}{2}\pi t^2 - \pi - 4t$ and $s''(t) = 3\pi^2 t^2 + 3\pi t + \pi^2 - 4$. Obviously, $s''(t) > 0$ and $s'(t)$ has one simple zero $t_0 \approx 0.3567304312$, which implies, $s(t)$ has a minimum value at $t = t_0$. It is straightforward to obtain that $S(t) > s(t) \geq s(t_0) \approx 1.828792213$. It is apparent from $\frac{d^2Q}{dP^2} > 0$ that Σ is convex strictly, hence we completed the proof. □

Appendix B

The expressions of $\psi_3^{\pm}(\theta, h)$, $R^{\pm}_{2yy}(\mathbf{B})$ and $R^{\pm}_{3y}(\mathbf{B})$ are given as follows:

$$\begin{aligned}
\psi_3^{+}(\theta, h) = {} & -\frac{1}{8}(\pi-\alpha-\theta)((a_{11}^{+}+b_{12}^{+})^2(\pi-\alpha-\theta) + a_{11}^{+}a_{12}^{+} - a_{11}^{+}b_{11}^{+} + a_{12}^{+}b_{12}^{+} \\
& - b_{11}^{+}b_{12}^{+} - 4a_{21}^{+} - 4b_{22}^{+}) - \frac{1}{8}\cos(\theta)(a_{11}^{+}+b_{12}^{+})(\cos(\theta)a_{11}^{+} + a_{12}^{+}\sin(\theta) \\
& + b_{11}^{+}\sin(\theta) - \cos(\theta)b_{12}^{+}) - \frac{1}{4h}(a_{11}^{+}+b_{12}^{+})(a_{10}^{+}\cos(\theta) + \cos(\theta)a_{11}^{+} \\
& + b_{10}^{+}\sin(\theta) + b_{11}^{+}\sin(\theta)) - \frac{1}{8h^2}(a_{11}^{+}+b_{12}^{+})(\sqrt{h^2-1}(a_{12}^{+} - 2b_{10}^{+} - b_{11}^{+}) \\
& + 2a_{10}^{+} + a_{11}^{+} + b_{12}^{+}), \\
\psi_3^{-}(\theta, h) = {} & -\frac{1}{8}(\pi-\alpha-\theta)((a_{11}^{-}+b_{12}^{-})^2(\pi-\alpha-\theta) - a_{11}^{-}a_{12}^{-} + a_{11}^{-}b_{11}^{-} - a_{12}^{-}b_{12}^{-} \\
& + b_{11}^{-}b_{12}^{-} + 4a_{21}^{-} + 4b_{22}^{-}) + \frac{1}{8}\cos(\theta)(a_{11}^{-}+b_{12}^{-})(\sin(\theta)a_{12}^{-} + b_{11}^{-}\sin(\theta) \\
& - a_{11}^{-}\cos(\theta) + \cos(\theta)b_{12}^{-}) - \frac{1}{4h}(a_{11}^{-}+b_{12}^{-})(-a_{10}^{-}\cos(\theta) + \cos(\theta)a_{11}^{-} \\
& + b_{10}^{-}\sin(\theta) - b_{11}^{-}\sin(\theta)) + \frac{1}{8h^2}(a_{11}^{-}+b_{12}^{-})(\sqrt{h^2-1}(a_{12}^{-} + 2b_{10}^{-} - b_{11}^{-}) \\
& + 2a_{10}^{-} - a_{11}^{-} - b_{12}^{-}), \\
R_{2yy}^{+}(\mathbf{B}) = {} & -2(\pi-\alpha)(a_{11}^{+}+b_{12}^{+}) + a_{12}^{+} + (a_{11}^{+}+b_{12}^{+})\frac{\sqrt{h^2-1}}{h^2}, \\
R_{2yy}^{-}(\mathbf{B}) = {} & 2(\pi-\alpha)(a_{11}^{-}+b_{12}^{-}) + a_{12}^{-} - (a_{11}^{-}+b_{12}^{-})\frac{\sqrt{h^2-1}}{h^2}, \\
R_{3y}^{-}(\mathbf{B}) = {} & -(\pi-\alpha)^2(a_{11}^{-}+b_{12}^{-})^2\sqrt{h^2-1} - \frac{1}{2}(\pi-\alpha)((a_{11}^{-}a_{12}^{-} + a_{11}^{-}b_{11}^{-} + a_{12}^{-}b_{12}^{-} \\
& + b_{11}^{-}b_{12}^{-} + 4a_{21}^{-} + 4b_{22}^{-})\sqrt{h^2-1} - 2a_{10}^{-}a_{11}^{-} - 2a_{10}^{-}b_{12}^{-}) + \frac{1}{2}(a_{12}^{-} + 2b_{10}^{-} - b_{11}^{-}) \\
& (a_{11}^{-}+b_{12}^{-}) + a_{20}^{-} - a_{22}^{-}\sqrt{h^2-1} - \frac{1}{2h^2}(a_{12}^{-} + 2b_{10}^{-} - b_{11}^{-})(a_{11}^{-}+b_{12}^{-}),
\end{aligned}$$

and

$$\begin{aligned}
R_{3y}^{+}(\mathbf{B}) = {} & -(\pi-\alpha)^2(a_{11}^{+}+b_{12}^{+})^2\sqrt{h^2-1} + \frac{1}{2}(\pi-\alpha)((a_{11}^{+}a_{12}^{+} + a_{11}^{+}b_{11}^{+} + a_{12}^{+}b_{12}^{+} + b_{11}^{+}b_{12}^{+} \\
& + 4a_{21}^{+} + 4b_{22}^{+})\sqrt{h^2-1} - 2a_{10}^{+}a_{11}^{+} - 2a_{10}^{+}b_{12}^{+}) - \frac{1}{2}(a_{12}^{+} - 2b_{10}^{+} - b_{11}^{+})(a_{11}^{+}+b_{12}^{+}) \\
& + a_{20}^{+} - a_{22}^{+}\sqrt{h^2-1} + \frac{1}{2h^2}(a_{12}^{+} - 2b_{10}^{+} - b_{11}^{+})(a_{11}^{+}+b_{12}^{+}).
\end{aligned}$$

Acknowledgements

The authors are grateful to the anonymous referees and editors for their useful comments and suggestions.

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