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PROPAGATING TERRACE IN A PERIODIC REACTION-DIFFUSION EQUATION WITH CONVECTION

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Abstract In this paper, we study the asymptotic properties of the solution for a space periodic reaction-diffusion equation with convection term. Firstly, we introduce the zero-number parameter method to compare the steepness of different solutions, so as to obtain the convergence of solution and the existence of the minimal propagating terrace. To be exact, the minimal propagating terrace is composed of individual pulsating traveling waves. By constructing the ω-limit set, we prove that the existence of pulsating traveling waves. Secondly, the stability theory is a necessary condition for the existence of the propagating terrace. Contrary to conventional conclusions, there we first consider extend the stability theory of the classical reaction-diffusion equation to the reaction-diffusion equation with convection term, and through constructing the Cauchy problem of the initial boundary value to solve the stability problem of the equation solution. Besides, we are especially concerned with the minimal propagating terrace existence, uniqueness, and their spatial structure.

Keywords Spatial period, convective term, pulsating wave, minimal propagating terrace, limit set.

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1. Introduction

In this paper, we consider the following reaction-diffusion equation

$$\begin{cases} \partial_t u(t,x) = \partial_{xx} u(t,x) + b(x)\partial_x u(t,x) + f(x,u(t,x)), \quad x \in \mathbb{R}, \ \ t > 0, \\ u(0,x) = u_0(x), \quad x \in \mathbb{R}, \end{cases} \tag{1.1}$$

where $b(x)$ is the convective term coefficient and satisfies $\underset{x\in\mathbb{R}}{inf}\, b(x) > 0$. Equation (1.1) arise especially in the fields of river dynamics, natural air flow ecology, physics and also medicine and biology (see [3, 5, 12, 30–32, 35, 36]). The function $u(t,x)$ denotes the temperature or air velocity and reaction term $f \in C(\mathbb{R}^2, \mathbb{R})$ satisfies

$$f(x+L,u) \equiv f(x,u) \quad \text{and} \quad f(x,0) \equiv 0,$$

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where $L > 0$ is a constant. Assuming that there exists a positive and L-periodic stationary solution $p(x)$ of (1.1), that is to say $p(x)$ satisfies

$$\begin{cases} p''(x) + b(x)p'(x) + f(x, p(x)) = 0, \ \ x \in \mathbb{R}, \\ p(x) > 0, \ \ p(x + L) \equiv p(x). \end{cases} \tag{1.2}$$

In the past, there are many researches about periodic reaction-diffusion equations. By using comparison principle and establishing the decay rate of pulsating fronts near unstable equilibrium, the uniqueness of monostable pulsating wave front for time periodic reaction-diffusion equations have been studied (see [1,6,13,19–22,38]). The admissible speeds in the space periodic bistable reaction-diffusion equation is considered through upper and lower solution method (see [4, 7, 16–18, 24–28, 33]). In [23], a pulsating front was constructed in special case when the coefficients are close to constants. Yet dealing with more general heterogeneities turned out to be much more difficult, and only recently a pulsating front was constructed in [9] for the one-dimensional case. Besides, when multiple steady states and finite fronts are involved, how do we think about the propagating properties? Clearly, the notion of a single front is not sufficient to understand the dynamics behavior of solutions, thus we instead observe the appearance of a so-called propagating terrace.

The concept of propagating terrace was first put forward by Ducrot et al. in [11]. They considered a spatial periodic reaction-diffusion equation, by using the upper and lower solution method and stability theory proved that the solution converges to a minimal propagating terrace. Giletti and Matano [14] further studied properties of propagating terrace and extended the convergence result to the zero speeds. Then Matano & Du [10] and Giletti & Rossi [15] considered the existence of propagating terrace in high-dimensional case by constructing eadial terrace solutions and using zero-number argument theory and upper and lower solutions method proved that existence of minimal propagating terrace.

To sum up, the propagation properties of parabolic equations and the formation of propagating terrace in space are considered. But for the propagation of the equation solution in time, is there a similar conclusion? Ding and Matano [8] first considered the long time behavior of the equation solutions for time period and gave initial data with compactly support. On this basis, Ding and Matano [9] weakened the condition of initial data and considered the front-like initial data, through stability theory and zero-number argument theory proving the existence of the minimal propagating terrace. In addition, they obtain that the convergence of the time period equation solution can be divided into local convergence or exponential convergence to the minimal propagating terrace in the case of multistable states. For the initial data with compactly support, Wang & Wang [37] considered the initial data with the Heavside type and proved that the existence of minimal propagating terrace.

The above researches were studied by scholars in one-dimensional reaction-diffusion equation for time or space. Different from the above studies, Du and Matano [10] considered the existence of propagating terrace in higher dimensional reaction-diffusion equations. They adopted lengthy argument to construct radial terrace solution $u(r, t)$, by using upper and lower solutions method introduce shifting function $\eta_k(t)$ to prove the existence of propagating terrace. Additionily, Giletti and Rossi [15] studied special case, that is the pulsating solutions for multidimensional bistable or multistable equations. By defining speed function and capturing the iteration at the suitable moment and position, they constructed a discrete pulsating travelling front and completed the proof of the propagating terrace existence.

However, the above studies ignore the influence of convection term (such as water flow, air flow) on propagation dynamics. Inspired by [2, 5, 8, 29, 34, 37], we consider the model (1.1), the presence of convection term increases the complexity of the equation. For the classical parabolic equations, the existing stability theory can be applied, but after adding the convection term to the reaction-diffusion equations, this set of theory will not be appliable. We will construct a Cauchy boundary value problem, by using the strong maximum principle and upper and lower solution method to solve the stability problem and obtain relevant conclusions about propagating terrace.

In order to show the attractiveness of $p(x)$ with respect to at least one compactly supported initial data and propagating terrace consists of not only a single (pulsating) traveling wave. There are two assumptions.

Assumption 1.1. *There exists a solution $u(x,t)$ of (1.1) with compactly supported initial data $0 \leq u_0(x) \leq p(x)$ that converges locally uniformly to $p(x)$ as $t \to \infty$.*

Assumption 1.2. *There exists no L-periodic stationary solution $q(x)$ with $0 < q(x) < p(x)$ that is both isolated from below and stable from below with respect to* (1.1).

To facilitate the definition of the propagating terrace, the following concepts are given.

Definition 1.3. Let $v_1(t,x)$ and $v_2(t,x)$ be two entire solutions of (1.1). We say $v_1(t,x)$ is steeper than $v_2(t,x)$ if and only if for any t_1, t_2 and $x' \in \mathbb{R}$ such that $v_1(t_1,x') = v_2(t_2,x')$, one has

$$v_1(\cdot + t_1, \cdot) \equiv v_2(\cdot + t_2, \cdot) \quad \text{or} \quad \partial_x v_1(t_1, x') < \partial_x v_2(t_2, x').$$

Definition 1.4. There exists two periodic stationary states $p_1(x)$ and $p_2(x)$, by a pulsating traveling front of (1.1) connecting $p_1(x)$ to $p_2(x)$, which mean any entire solution $U(t,x)$ of (1.1) satisfying, for $T > 0$,

$$U(t, x - L) = U(t + T, x)$$

along with the asymptotics

$$\lim_{t\to-\infty} U(t,x) = p_1(\cdot) \text{ and } \lim_{t\to\infty} U(t,x) = p_2(\cdot).$$

There $c := \frac{L}{T} > 0$ is called the average speed of this pulsating traveling wave. Clearly, the form $U(t,x)$ can be written $\hat{U}(x - ct, x)$, where $\hat{U}$ is L-periodic in its second variable and satisfies $U(+\infty, \cdot) = p_1(\cdot)$ and $U(-\infty, \cdot) = p_2(\cdot)$.

Next, we introduce the notion of propagating terrace and some properties of the propagating terrace.

Definition 1.5. A propagating terrace connecting 0 to $p(x)$ is a pair of finite sequences $(p_k)_{0\leq k\leq N}$ and $(U_k)_{1\leq k\leq N}$ satisfies

- Each p_k is a L-periodic stationary solution of (1.1) satisfying

$$p(x) = p_0(x) > p_1(x) > ... > p_N(x) = 0.$$

• For each 1≤ k ≤ N, $U_k(t,x)$ is a pulsating traveling solution of (1.1) connecting $p_k(x)$ to $p_{k-1}(x)$.

• The speed $(c_k)_{1\le k\le N}$ of each $U_k(t,x)$ satisfies $c_1 \le c_2 \le ... \le c_N$.

Moreover, a propagating terrace $\mathcal{F} = ((p_k)_{0\le k\le N}, (U_k)_{1\le k\le N})$ connecting 0 to $p(x)$ is the minimal if it satisfies the following

• For any propagating terrace $\mathcal{F}' = ((q_k)_{0\le k\le N'}, (V_k)_{1\le k\le N'})$, one has

$$\{p_k(x)|0 \le k \le N\} \subset \{q_k(x)|0 \le k \le N'\}.$$

• For each $1 \le k \le N$, $U_k(t,x)$ is steeper than any other traveling wave connecting $p_k(x)$ to $p_{k-1}(x)$.

Proposition 1.6. *A propagation terrace $\mathcal{F} = ((p_k)_{0\le k\le N}, (U_k)_{1\le k\le N})$ is called the minimal propagating terrace in the sense of Definition 1.5, then it is unique up to time shifts.*

Assumption 1.7. *There exists a decomposition between 0 and $p(x)$, that is to say, there exists a finite sequence of solutions $(p_k)_{0\le k\le N}$ of (1.2) such that $p(x) = p_0(x) > p_1(x) > ... > p_N(x) = 0$, there exists a pulsating traveling wave $U_k(t,x)$ connecting $p_k(x)$ to $p_{k-1}(x)$.*

Before giving the final conclusion, we consider (1.1) whose the initial data is Heavside type, that is $u_0(x) = p(x)H(a-x)$, where $a \in \mathbb{R}$ is any constant. Where H is defined by

$$H(x) := \begin{cases} 0 & \text{if } x < 0, \\ 1 & \text{if } x \ge 0. \end{cases}$$

Thus, we have the following results.

Theorem 1.8. *Let Assumption 1.1 and Assumption 1.7 hold. There exists a minimal propagating terrace $((p_k)_{0\le k\le N}, (U_k)_{1\le k\le N})$ in the sense of Definition 1.5. Moreover, it satisfies*

(i) *For any $0 \le k \le N$, the stationary solution $p_k(x)$ is isolated and stable from below with respect to* (1.1).

(ii) *All the $p_k(x)$ and $U_k(t,x)$ are steeper than any other entire solution of (1.1).*

And about the convergence of the solution, we have the following results.

Theorem 1.9. *Let Assumption 1.1 hold and $u(t,x)$ is the solution of (1.1) with $u_0(x)$ satisfying Heavside type. The solution $u(t,x)$ converges as $t \to +\infty$ to the minimal propagating terrace $((p_k)_{0\le k\le N}, (U_k)_{1\le k\le N})$. Moreover, there exist functions $(\gamma_i(t))_{1\le i\le N}$ such that the following statements hold*

(i) $\gamma_i(t) = o(t)$ *as $t \to +\infty$ for $i = 1, ..., N$,*

(ii) *The convergence holds*

$$\lim_{t\to\infty} \sup_{x\in\mathbb{R}} \Big| u(t,x) - \Big(\sum_{i=1}^{N} U_i(t, x-\gamma_i(t)) - \sum_{i=1}^{N} p_i(x) \Big) \Big| = 0.$$

Outline of the paper. In Section 2, we recall the properties of ω-limit set and the zero-number argument theory, and introduce the stability of solutions of (1.2). The asymptotic behavior of solutions of (1.1) with Heaviside type initial data and the proof of Theorem 1.8 and Theorem 1.9 are shown in Section 3.

2. Preliminary

In the section, we will do preliminaries to prove the existence of the propagating terrace. In subsection 2.1, the zero-number parameter theory is introduced to compare steepness between different solutions. In subsection 2.2, the concept and basic properties of ω-limit set are given. The stability theory of stationary solutions of (1.2) is shown in subsection 2.3.

2.1. Zero-number argument properties

In this subsection, we mainly rely on the properties of zero-number argument theory. This zero-number argument method can be used to prove the convergence results of solutions in semilinear parabolic equations, but in this paper we use this zero-number argument method not only to prove the convergence of solutions, but also to prove the existence of the pulsating traveling waves and propagating terrace. First of all, from standard zero-number argument symbol $Z[\cdot]$, we introduce a related concept $SGN[\cdot]$, which will be very useful later in our lemmas.

Definition 2.1. For a real-valued function $w(x)$ defined on $\mathbb{R}$, we define two properties. **(i)** $Z[w(\cdot)]$ is defined as the number of changes in sign of the real valued function w, namely the supremum over all $k \in \mathbb{N}$ such that there exist real numbers $x_1 < x_2 < ... < x_{k+1}$ with

$$w(x_i).w(x_{i+1}) < 0 \text{ for all } i = 1, 2, ..., k.$$

(ii) $SGN[w(\cdot)]$ is the word consisting of $+$ and $-$ and it describes the signs of

$$w(x_1), ..., w(x_{k+1}),$$

where $x_1 < ... < x_{k+1}$ is a sequence defined in $Z[w]$ with maximal k.

In the case of (i), We set $Z[w] = 0$ if $w \not\equiv 0$, and $Z[w] = -1$ if $w \equiv 0$. In the case of (ii), when k does not exist, we set $SGN[w]{=}sgn(w(x))$ if $w \not\equiv 0$ and $w(x) \neq 0$. In this case, sgn is defined as a sign function, and $SGN[0] = [\ ]$, the empty word.

When w only have simple zeros on $\mathbb{R}$, then $Z[w]$ is made up of a number of zeros. We give an example,

$$Z[x^2 - 5] = 2, \ SGN[x^2 - 5] = [+ - +].$$

From the previous definition, we know that the length of $SGN[w]$ is equal to $Z[w] + 1$.

If W and V are made up of $+$ and $-$, we stipulate that $W \rhd V$ (or $W \lhd V$) if V is a subword of W. We give an example,

$$[+-] \rhd V \text{ for } V = [+-], [+], [-], [], \quad \text{but not } [+-] \rhd [-+].$$

Lemma 2.2. *Let $w(t, x) \not\equiv 0$ be a solution to the periodic equation of the following form*

$$\partial_t w = \partial_{xx} w + b(x)\partial_x w + c(t, x)w \quad \text{on a domain} \quad (t_1, t_2) \times \mathbb{R}, \tag{2.1}$$

where $c(t, x)$ is a bounded function. Furthermore,
(i) *$Z[w(t, \cdot)]$ and $SGN[w(t, \cdot)]$ are nonincreasing with respect to t, for any $t' > t$, we have*

$$Z[w(t,\cdot)] \geq Z[w(t',\cdot)], \;\; SGN[w(t,\cdot)] \triangleright SGN[w(t',\cdot)].$$

When $w(t,x)$ can be extended as a continuous function on $[t_1,t_2)\times\mathbb{R}$, we assert that this is also true for $t=t_1$.

(ii) *For some $t' \in (t_1,t_2)$ and $x' \in \mathbb{R}$, if $w(t',x') = \partial_x w(t',x') = 0$, then*

$$Z[w(t,\cdot)] - 2 \geq Z[w(s,\cdot)] \geq 0, \quad \text{for } t \in (t_1,t') \text{ and } s \in (t',t_2),$$

where $Z[w(t,\cdot)] < \infty$.

We know from the statement (ii) that the second inequality is not true when $w \equiv 0$ on $(t_1,t_2)\times\mathbb{R}$. Otherwise, for $t \in (t_1,t_2)$, the function $w(t,x)$ does not vanish on $\mathbb{R}$. The second inequality is due to [2]. However, [2] only solves the equation on the bounded intervals. While in [8], the author uses the maximum principle to generalize it to the infinite interval.

The statement (i) is derived from (ii), when the boundary is a bounded interval. We can prove it by the maximum principle and a topological argument, and by this proof we can prove $SGN[w]$, so that the proof for $Z[w]$ can be derived automatically.

Lemma 2.3. *Let $(w_n)_{n\in\mathbb{N}}$ be a sequence of functions that converge to w pointwise on $\mathbb{R}$, we have*

$$Z[w] \leq \lim_{n\to+\infty} \inf Z[w_n],$$

$$SGN[w] \triangleleft \lim_{n\to+\infty} \inf SGN[w_n].$$

Combining Lemma 2.2 and Lemma 2.3, we can derive the following lemma.

Lemma 2.4. *Let $v_1(t,x)$ and $v_2(t,x)$ be two solutions of (1.1) such that the initial value $v_1(0,x)$ is a piecewise continuous bounded function on $\mathbb{R}$, and the initial value of $v_2(0,x)$ be continuous bounded function on $\mathbb{R}$. We assume that $v_1(0,x)-v_2(0,x)$ changes sign at most a finite number of times on $\mathbb{R}$. Then*

(i) *for $0 \leq t < t' < \infty$,*

$$Z[v_1(t,\cdot)-v_2(t,\cdot)] \geq Z[v_1(t',\cdot)-v_2(t',\cdot)],$$

$$SGN[v_1(t,\cdot)-v_2(t,\cdot)] \triangleright SGN[v_1(t',\cdot)-v_2(t',\cdot)].$$

(ii) *for some $t_1>0$, the graphs of $v_1(t_1,x)$ and $v_2(t_1,x)$ intersect at some point in $\mathbb{R}$, if $v_1(t,x) \not\equiv v_2(t,x)$, then for any t,s with $0 \leq t < t_1 < s$,*

$$Z[v_1(t,\cdot)-v_2(t,\cdot)] - 2 \geq Z[v_1(s,\cdot)-v_2(s,\cdot)] \geq 0.$$

Proof. Let the function $w(t,x) := v_1(t,x) - v_2(t,x)$ be a solution to the equation of the form (2.1), where $c(x,t) := \frac{f(x,v_1)-f(x,v_2)}{v_1-v_2}$ is bounded, so this lemma can be concluded from Lemma (2.2). In addition, when $t=0$, if both $v_1(0,x)$ and $v_2(0,x)$ are continuous functions, then this conclusion is also true. In the general case, when $v_1(0,x)$ is a piecewise continuous function, we can approximate v_1 with a sequence of the solution, say $v_{1,n}$, whose $v_{1,n}(0,x)$ are continuous and satisfy

(i) $sup\|v_{1,n}(0,\cdot)\|_{L^\infty(\mathbb{R})} < \infty$ and $v_{1,n}(0,x) \to v_1(0,x)$,

(ii) $SGN[v_{1,n}(0,\cdot)-v_2(0,\cdot)] = SGN[v_1(0,\cdot)-v_2(0,\cdot)]$ for $n \in \mathbb{N}$.

Then, we conclude that for $t'>0$,

$$SGN[v_1(0,\cdot)-v_2(0,\cdot)] \triangleright SGN[v_{1,n}(t',\cdot)-v_2(t',\cdot)].$$

Finally let $n \to \infty$ and combine Lemma 2.3, we can reach the above conclusion. $\square$

Next we introduce a corollary, which we will use later.

Corollary 2.5. *Suppose u_1 and u_2 are two entire solutions of (1.1), and satisfy*

$$SGN[u_1(t', \cdot) - u_2(t'', \cdot)] \triangleleft [+-] \quad \text{for} \quad t', t'' \in \mathbb{R}.$$

Then in the sense of Definition 1.3, u_1 is steeper than u_2.

Proof. Take any $t', t'' \in \mathbb{R}$. From the above assumption we have

$$Z[u_1(t + t', \cdot) - u_2(t + t'', \cdot)] \leq 1 \text{ for } t \in \mathbb{R}.$$

If $u_1(\cdot + t', \cdot) \not\equiv u_2(\cdot + t'', \cdot)$, it can be known from (ii) of Lemma (2.4) that the graph of $u_1(t, x)$ and $u_2(t, x)$ have at most one intersection point, that is to say, $u_1(t', x) - u_2(t'', x)$ has at most one zero point. We assume the zero is x_1. Then we can know from this zero point and $SGN[u_1 - u_2] \triangleleft [+-]$, which display $\partial_x u_1(t', x_1) < \partial_x u_2(t'', x_1)$. This indicate that $u_1(t, x)$ is steeper than $u_2(t, x)$. $\square$

2.2. Basic properties on the ω-limit set of $\hat{u}(\cdot, \cdot)$.

In this subsection, we summarize some basic properties of ω-limit set, the definition of ω-limit set is different from the standard notion of $\omega(u)$.

Definition 2.6. Let $u(t, x)$ be a bounded solution of the (1.1). We say $w(t, x)$ is a member of the ω-limit set of $\hat{u}$ if it exists a sequences $t_j \to \infty$ and $x_j = k_j L$ for some integers k_j such that

$$u(t + t_j, x + x_j) \to w(t, x), \quad \text{as} \quad j \to \infty. \tag{2.2}$$

Proposition 2.7. *By the standard parabolic estimates, we know that the convergence conclusion above occurs in $x \in C^2$ and $t \in C^1$, so we can easily get that the element $w(t, x)$ of the ω-limit set is an entire solution of (1.1). In addition, We have assume $w(t, x)$ is an element of the ω-limit set, then we have*

$$u(t + t_j, x) \to w(t, x).$$

Take $t' = t + t_j$ and we get

$$u(t' + \tau_j, x) \to w(t + \tau, x),$$

which implies $w(t + \tau, x)$ is also an entire solution for $\tau \in \mathbb{R}$ and $x \in \mathbb{R}$.

Next we will introduce a lemma that will be used repeatedly in our article.

Lemma 2.8. *Let $w_1(t, x)$ is an element of the ω-limit set of $\hat{u}(\cdot, \cdot)$. In the sense of Definition 1.3, $w_1(t, x)$ is steeper than any entire solution of (1.1), which the entire solution is between* 0 *and $p(x)$.*

Proof. Take the sequences $t_j \to \infty$ and $x_j \in \mathbb{R}$. By (2.2), we can get $\hat{u}(t + t_j, x + x_j) \to w_1(t, x)$ *as* $j \to +\infty$. By standard parabolic estimate, this convergence holds in $C_{loc}(\mathbb{R}^2)$. Take $w(t, x)$ as an entire solution between 0 and $p(x)$. We fix $x_1 \in \mathbb{R}$, then $\hat{u}(0, x) \geq w(t, x)$ for $x < x_1$ and $\hat{u}(0, x) \leq w(t, x)$ for $x > x_1$. For $j \in \mathbb{N}$ and $\tau \in \mathbb{R}$, we have

$$Z[\hat{u}(0, \cdot) - w(\tau - t_j, \cdot)] = 1 \text{ and } SGN[\hat{u}(0, \cdot) - w(\tau - t_j, \cdot)] = [+-].$$

From Lemma 2.4, we know for $j \in \mathbb{N}$ and $t \geq -t_j$,

$$Z[\hat{u}(t+t_j,\cdot) - w(\tau+t,\cdot)] \leq 1,$$
$$SGN[\hat{u}(t+t_j,\cdot) - w(\tau+t,\cdot)] \triangleleft [+-].$$

Let $j \to +\infty$, we get

$$SGN[w_1(t,\cdot) - w(\tau+t,\cdot)] \triangleleft [+-].$$

Finally, by Corollary 2.5, we proved that $w_1(t,x)$ is steeper than $w(t,x)$. □

2.3. Stability of solutions of (1.2) connected by stationary solution.

In this subsection, we research the stability of $p_+(x)$ and $p_-(x)$ by giving a pulsating traveling wave $U(t,x)$ to connect them. We will introduce a lemma, and this lemma is going to be important in the rest of the proof.

Lemma 2.9. *Let $p_-(x) < p_+(x)$ be two solutions of (1.2). We assume there is a periodic pulsating traveling wave $U(t,x)$ of (1.1) connecting $p_-(x)$ to $p_+(x)$ with speed $c \in \mathbb{R}$. Then the following statements hold.*
(a) *If $c > 0$, then $p_+(x)$ is stable from below and isolated from below;*

(b) *If $c < 0$, then $p_-(x)$ is stable from above and isolated from above;*

(c) *If $c = 0$, then $p_+(x)$ is stable from below and $p_-(x)$ is stable from above.*

Proof. Let us consider situation (a). Use proof by contradiction, we assume that $p_+(x)$ is unstable from below. Then here exists $R > 0$ sufficiently large such that the following problem

$$\begin{cases} \varphi_t = \varphi_{xx} + b(x)\varphi_x + f(x,\varphi), \quad \text{for} \quad x \in \mathbb{R}, -R < t < R, \\ \varphi(t,x+L) = \varphi(t,x), \quad \text{for} \quad x \in \mathbb{R}, -R < t < R, \\ p_-(x) < \varphi(t,x) < p_+(x), \quad \text{for} \quad x \in \mathbb{R}, -R < t < R, \\ \varphi(\pm R, x) = p_+(x), \quad \text{for} \quad x \in \mathbb{R}, \end{cases}$$

has an entire soiution $\varphi(t,x)$ satisfying

$$\partial_t \varphi(t,x) \geq 0 \quad \text{for} \quad t \in [-R,R], \ \ x \in \mathbb{R}.$$

Clearly, we can find an entire solution $\varphi(t,x)$ satisfies above problem, we only just show that $\varphi(t,x)$ is nondecreasing with respect to time. We will more precisely prove that for $t \in [-R,R]$, $\partial_t\varphi(t,x)$ does not change sign. To prove this statement, we will argue by contradiction by assuming that for some given $t_0 \in [-R,R]$ there exist $x_0 \in \mathbb{R}$ and $x_1 \in \mathbb{R}$ such that

$$\partial_t \varphi(t_0,x_0) > 0 \ \text{ and } \ \partial_t \varphi(t_0,x_1) < 0. \tag{2.3}$$

Let $x' \in \mathbb{R}$ be given. For any $0 < \alpha < p(x')$, let us define $\tau(x^{'},\alpha) := \min\{t > 0 | \hat{u}(t,x^{'}) = \alpha)\}$. It is then clear that for any τ small enough, one has

$$Z[\varphi(t_0+\tau,\cdot) - \varphi(t_0,\cdot)] \geq 1.$$

Besides, recall that $\varphi(t,x)$ is an ω-limit set of $\hat{u}$, and so is $\varphi(\cdot+\tau,\cdot)$ for any τ *in* $\mathbb{R}$. Therefore, they are steeper than each other, and it immediately follows that $\varphi(t_0+\tau,\cdot)\equiv\varphi(t_0,\cdot)$ for each τ small enough. Hence,

$$\partial_t\varphi(t_0,\cdot)\equiv 0,$$

a contradiction together with (2.3). This implies that for any $t\in[-R,R]$, one has

$$SGN[\partial_t\varphi(t,\cdot)] = [\ ] \quad \text{or} \quad [+] \quad \text{or} \quad [-]. \tag{2.4}$$

Let us denote $\Phi := \partial_t\varphi$. It is an entire solution of the linear parabolic equation

$$\partial_t\Phi = \partial_{xx}\Phi + b(x)\partial_x\Phi + \partial_u f(x,\varphi)\Phi.$$

We infer from (2.4) and the strong maximum principle that either $\partial_t\varphi<0$, $\partial_t\varphi>0$ or $\partial_t\varphi\equiv 0$. Due to the definition of $\tau(x^{'},\alpha)$, one has $\partial_t\varphi(0,x^{'})\geq 0$. This completes the proof of the monotonicity in time of $\varphi(t,x)$.

In addition, the pulsating traveling wave $U(t,x)$ satisfies

$$\lim_{t\to-\infty}U(t,x)=p_-(x) \quad \text{and} \quad \lim_{t\to+\infty}U(t,x)=p_+(x) \quad \text{locally uniformly in} \quad x\in\mathbb{R}. \tag{2.5}$$

We know from the formula above that, at $x=0$, there exists $t_0\in\mathbb{R}$ such that $U(t+t_0,0)$ and $\varphi(t,0)$ intersects and we have

$$U(t+t_0,0)\leq\varphi(t,0) \text{ for } t\in[-R,R].$$

Besides, $U(t+t_0,x)<\varphi(t,x)$ for $x\geq 0,\ \ t=\pm R$. By the strong maximum principle, we get

$$U(t+t_0,x)<\varphi(t,x) \quad \text{for} \quad x>0, \ \ -R\leq t\leq R. \tag{2.6}$$

If $c=0$, due to $U(t,x)$ is L-periodic in x and $\varphi(t,x)$ is also L-periodic. There esists β_0 such that $U(t_0+\beta_0,L)=\varphi(\beta_0,L)$, which is a contradiction with (2.6). In $c>0$, since

$$U(t+kT,x)=U(t,x-ckT) \quad \text{for} \quad k\in\mathbb{Z}, \tag{2.7}$$

it shows from (2.5) that $U(t+kT,x)$ converges to $p_+(x)$ locally uniformly in $t\in\mathbb{R}$ *and* $x\in\mathbb{R}$ which contradicts (2.6). Since $\varphi(0,ckT)=\varphi(0,0)<p_+(0)$ *for* $k\in\mathbb{Z}$. Thus $p_+(x)$ is stable from below if $c\geq 0$. In $c\leq 0$, we can prove the similar conclusion that $p_-(x)$ is stable from above. Here we omit the details.

Next we show that $p_+(x)$ is isolated from below in $c>0$. We use proof by contradiction and assume $p_+(x)$ is accumulated from below. Then there exists the sequence $p_j(x)$ of solutions of (1.2) such that $p_j(x)\to p_+(x)$ as $j\to\infty$ and $p_j(x)<p_+(x)$ for $j\in\mathbb{N}$. We can see that

$$\int_0^L \partial_u f(x,p_+(x))dx=0$$

and the following problem

$$\begin{cases} -\partial_{xx}\phi_\lambda+(2\lambda-b(x))\partial_x\phi_\lambda-f_u(x,p_+(x))=\mu(\lambda)\phi_\lambda, x\in\mathbb{R}, \\ \phi_\lambda>0 \quad \text{and} \quad L-periodic. \end{cases} \tag{2.8}$$

has a unique positive solution $\phi\in C^1(\mathbb{R})$.

In order to find a contradiction, we will construct a super-solution of (1.1). Consider the function $v(t,x)$ defined by

$$v(t,x) := \min\left\{p_+(x), e^{-\lambda(x-ct)}\phi_\lambda + p_j\right\} \quad \text{for } t \in [-R,R], \ \ x \in \mathbb{R},$$

the value of c here is between zero and minimum speed and $\lambda > 0$ while ϕ_λ is a solution of (2.8).

Clearly, there exists increasing map function $t \mapsto x(t)$ satisfies

$$\begin{aligned} &v(t,x(t)) = p_+(x(t)), \ \ \forall t \in [-R,R], \\ &v(t,x) < p_+(x), \ \ \forall t \in [-R,R] \quad \text{and} \quad \forall x > x(t). \end{aligned}$$

Next we define

$$D := \{(t,x) | x \geq x(t)\},$$

and straightforwardly compute the following formula:

$$\begin{aligned} &\partial_t v - b(x)\partial_x v - \partial_{xx} v - f(x,v) \\ =&e^{-\lambda(x-ct)}[(b(x)\lambda + c\lambda - \lambda^2)\phi_\lambda + (2\lambda - b(x))\partial_x\phi_\lambda - \partial_{xx}\phi_\lambda] \\ &- \partial_{xx}p_j - f(x, p_j + e^{-\lambda(x-ct)}\phi_\lambda) \\ =&e^{-\lambda(x-ct)}[(b(x)\lambda + c\lambda - \lambda^2)\phi_\lambda + (2\lambda - b(x))\partial_x\phi_\lambda - \partial_{xx}\phi_\lambda] \\ &- \frac{\partial_f}{\partial_u}(x,p_j)e^{-\lambda(x-ct)}\phi_\lambda + o\Big(\min\left\{p_+ - p_j, e^{-\lambda(x-ct)}\phi_\lambda\right\}\Big) \\ =&e^{-\lambda(x-ct)}[(b(x)\lambda + c\lambda - \lambda^2)\phi_\lambda + (2\lambda - b(x))\partial_x\phi_\lambda - \partial_{xx}\phi_\lambda] \\ &- \frac{\partial_f}{\partial_u}(x,p_*)e^{-\lambda(x-ct)}\phi_\lambda + o\Big(\min\left\{p_+ - p_j, e^{-\lambda(x-ct)}\phi_\lambda\right\}\Big) \\ =&e^{-\lambda(x-ct)}(c\lambda + b(x)\lambda - \lambda^2 + \mu(\lambda))\phi_\lambda + o\Big(\min\left\{p_+ - p_j, e^{-\lambda(x-ct)}\phi_\lambda\right\}\Big). \end{aligned}$$

Due to $\mu(\lambda)\phi_\lambda \geq 0$, we have

$$\begin{aligned} &e^{-\lambda(x-ct)}(c\lambda + b(x)\lambda - \lambda^2 + \mu(\lambda))\phi_\lambda + o\Big(\min\left\{p_+ - p_j, e^{-\lambda(x-ct)}\phi_\lambda\right\}\Big) \\ \geq&e^{-\lambda(x-ct)}(c\lambda + b(x)\lambda - \lambda^2)\phi_\lambda + o\Big(\min\left\{p_+ - p_j, e^{-\lambda(x-ct)}\phi_\lambda\right\}\Big). \end{aligned}$$

In addition, since $b(x) \geq 0$ and $\lambda \geq 0$, we can fix $b(x)$ is enough large such that $c + b(x) - \lambda \geq 0$, which implies

$$e^{-\lambda(x-ct)}(c\lambda + b(x)\lambda - \lambda^2)\phi_\lambda + o\Big(\min\left\{p_+ - p_j, e^{-\lambda(x-ct)}\phi_\lambda\right\}\Big) \geq 0,$$

thus, $v(t,x)$ is a super-solution of (1.1) over D.

Let $x^{'} > 0$ be a enough large real number satisfies

$$U(0, x + x^{'}) \leq v(0, x), \text{ for } x \geq x(0).$$

Then by comparison principle, we have

$$U(t, x + x^{'}) \leq v(t, x), \text{ for } t \geq 0, \ x \geq x(t).$$

It shows that there exists some $\epsilon > 0$ such that

$$U(t, x(t) + 2 + x^{'}) \leq v(t, x(t) + 2) \leq p_+(x) - \epsilon \text{ for } t \geq 0. \tag{2.9}$$

Combine (2.5) and (2.7), we obtain

$$U(kT, x(kT) + 2 + x^{'}) = U(0, x(kT) - ckT + 2 + x^{'}) \to p_+(0) \quad \text{as } k \to \infty.$$

This contradicts the (2.9). Thus, $p_+(x)$ is isolated from below if $c > 0$. In $c < 0$, we can conclude that $p_-(x)$ is isolated from above. The proof of Lemma 2.9 is completed. □

3. Existence of minimal terrace and convergence with Heaviside type initial data

In this section, we will prove the existence of the minimal propagating terrace and the convergence under a given initial data of Heaviside type. The Heaviside type function here is a function $u_0(x) = p(0)H(a - x)$ for $a \in \mathbb{R}$.

In subsection 3.1, we show that $\hat{u}(t, x)$ converges to a limit function around a given level set. In addition, we show that this limit function is either a solution of (1.1) or a periodic traveling wave solution of (1.1) connecting two steady-states. Once we have the property about convergence, we can construct the minimal propagating terrace and complete the proof of Theorem 1.8. We assume that there is a decomposition between 0 and $p(x)$ in order to ensure that the propagation of the solution can be completed in finite steps (see subsection 3.2). In subsection 3.3, we will prove the Theorem 1.9.

3.1. Convergence around a given level set.

Let $\hat{u}(t, x)$ be an entire solution of (1.1) with the initial date of the Heavside type. Obviously, $0 \leq \hat{u}(t, x) \leq p(x)$ for $t \in \mathbb{R}, \ x \in \mathbb{R}$ and $\hat{u}(t, x)$ is nondecreasing in $t \in \mathbb{R}$, satisfies

$$\lim_{t \to -\infty} \hat{u}(t, x) = 0 \quad \text{and} \quad \lim_{t \to +\infty} \hat{u}(t, x) = p(x).$$

It is clear that for $k \in \mathbb{N}$, there exists a unique $m_k \in \mathbb{R}$ such that

$$\hat{u}(kT, m_k) = \alpha, \tag{3.1}$$

where $\alpha \in (0, p(0))$ is a constant.

Lemma 3.1. *For $\alpha \in (0, p(0))$, take sequence $(m_k)_{k \in \mathbb{N}}$ by (3.1). We have*

$$\lim_{k \to \infty} \hat{u}(t + kT, x + m_k) := w_\infty(t, x; \alpha). \tag{3.2}$$

The function $w_\infty(t,x;\alpha)$ is a positive entire solution of (1.1) and it is steeper than any entire solution between 0 and $p(x)$. In addition, it is either time heterogeneity or time nondecreasing.

Proof. By standard parabolic estimates, the sequence $\hat{u}(t+kT, x+m_k)_{k\in\mathbb{N}}$ is uniformly bounded. Then it is compactly supported on $C(\mathbb{R}^2)$. There exists a sequence of integer $(k_j)_{j\in\mathbb{N}}$ that makes $k_j \to \infty$ as $j \to \infty$ and

$$\hat{u}(t+kT_j, x+m_{k_j}) \to w_\infty(t,x;\alpha) \quad \text{as} \quad j\to\infty \ \text{ in } \ C(\mathbb{R}^2).$$

Where $w_\infty(t,x;\alpha)$ is not only an entire solution of (1.1) but also a member of the ω-limit set of $\hat{u}(t,x)$ and $w_\infty(0,0;\alpha)=\alpha$. It is obvious that from the maximum principle that $0 < w_\infty(t,x;\alpha) < p(x)$ for $t\in\mathbb{R}$. By Lemma 2.8 we know $w_\infty(t,x;\alpha)$ is steeper than any other entire solutions of (1.1) which between 0 and $p(x)$. This show that $w_\infty(t,x;\alpha)$ does not depend on the choice of (k_j), so the sequence $\hat{u}(t+kT, x+m_k)$ converges to $w_\infty(t,x;\alpha)$ as $k\to\infty$ *in* $C(\mathbb{R}^2)$. Next we will show $w_\infty(t,x;\alpha)$ is either time heterogeneity or time nondecreasing in $t\in\mathbb{R}$. For each $k\in\mathbb{N}$, the function x $\mapsto \hat{u}(t+kT, x+m_k)$ is nondecreasing. Let $k\to\infty$, we have

$$\partial_x w_\infty(t,x;\alpha) \geq 0 \quad \text{for} \ \ t\in\mathbb{R}, \ \ x\in\mathbb{R}.$$

Finally combine strong maximum principle to the equation satisfied by $\partial_x w_\infty(t,x;\alpha)$, we obtain either $\partial_x w_\infty(t,x;\alpha) \not\equiv 0$ or $\partial_x w_\infty(t,x;\alpha) > 0$ for $t\in\mathbb{R}, \ x\in\mathbb{R}$. Thus we complete the Lemma 3.1 proof. □

Lemma 3.2. *There exists constants $0 < c_* < c^* < \infty$ that do not depend on $a\in\mathbb{R}$, such that*

(i) *for each $c > c^*$, $\lim\limits_{t\to\infty} \sup_{x\geq ct} \hat{u}(t,x) = 0$,*

(ii) *for each $0 < c < c_*$, $\lim\limits_{t\to\infty} \sup_{x\leq ct} |\hat{u}(t,x) - p(x)| = 0$.*

Proof. The proof of this Lemma 3.2 we refer to reference [9, Lemma 2.9], here we omit the details.

Next we will define a sequence of $(l_k)_{k\in\mathbb{N}}$:

$$l_k := \begin{cases} m_k - m_{k-1}, & \text{if } k > 1, \\ m_0, & \text{if } k = 0, \end{cases}$$

where $(m_k)_{k\in\mathbb{N}}$ is given by (3.1) and $m_k = \sum_{j=0}^k l_j$. Due to $\hat{u}(kT, m_k) = \alpha \in (0, p(0))$ for $k\in\mathbb{N}$, then from Lemma 3.2, we have

$$c_* kT \leq m_k \leq c^* kT \quad \text{for} \quad k\in\mathbb{N}.$$

Finally,

$$c_* \leq \lim_{k\to\infty}\inf \frac{\sum_{j=0}^k l_j}{kT} \leq \lim_{k\to\infty}\sup \frac{\sum_{j=0}^k l_j}{kT} \leq c^*. \tag{3.3}$$

□

Lemma 3.3. *For given $\alpha\in(0,p(0))$, let $w_\infty(t,x;\alpha)$ be the entire solution which is defined by Lemma 3.1. Then either of the following is true*

(i) *$w_\infty(t,x;\alpha)$ is time heterogeneity, and it is a positive periodic stationary solution,*

(ii) $w_\infty(t,x;\alpha)$ *is nondecreasing in* $t\in\mathbb{R}$*, and it is a pulsating traveling wave.*

Proof. The proof will depend on the sequence $(l_k)_{k\in\mathbb{N}}$. We divide this proof into two parts.

Case (1). There exists a sequence $(k_j)_{j\in\mathbb{N}}$ such that l_{k_j} converging to l_∞ as $j\to\infty$ for $l_\infty\in\mathbb{R}$. In case (1), combine (3.2), we have

$$\begin{aligned} w_\infty(t,x-l_\infty;\alpha) &= \lim_{j\to\infty}\hat{u}(t+k_jT,x+m_{k_j}-l_{k_j}) \\ &= \lim_{j\to\infty}\hat{u}(t+k_jT,x+m_{k_j-1}) \\ &= w_\infty(t+T,x;\alpha). \end{aligned}$$

From Lemma 2.9 we know that w_∞ is time heterogeneity or time nondecreasing. It shows w_∞ is a positive stationary solution when $\partial_x w_\infty\equiv 0$ and a pulsating traveling wave if $\partial_x w_\infty>0$. When $\partial_x w_\infty>0$, c=$\frac{l_\infty}{T}$ is the speed of w_∞. Therefore, when case (1) happens, (i) and (ii) both are holds.

Case (2). there is no subsequence of $(l_k)_{k\in\mathbb{N}}$ converging to a positive constant.

In the second case, we will show that only (i) happens. Since no sequence of $(l_k)_{k\in\mathbb{N}}$ converges to a positive constant. By (3.3), we can find two subsequences converge to 0 and ∞. First we consider the subsequence convergens to 0, this can lead us to

$$w_\infty(t,x;\alpha)=w_\infty(t,x-l_\infty;\alpha), \tag{3.4}$$

then $w_\infty(t,x;\alpha)$ is periodic in space variable for all time.

We know from (i) that we want to show that $w_\infty(t,x;\alpha)$ is stationary, let us use proof by contradiction and assume w_∞ is not stationary. By Lemma 3.1, one has that

$$\partial_t w_\infty(0,x+l_\infty;\alpha)>0.$$

Combine the convergence of $\hat{u}(t+m_kT,x;\alpha)$ to w_∞ as $k\to\infty$, there exists $\delta>0$ for any $0\le t\le\delta$ and $k\in\mathbb{R}$ such that

$$\partial_t\hat{u}(t+m_kT,x+l_\infty;\alpha)\ge\frac{\partial_t w_\infty(0,x+l_\infty;\alpha)}{4}>0.$$

Next, using (3.4), for any $\epsilon>0$, the following conclusion is holds for k large enough

$$\hat{u}(m_kT,x+l_\infty;\alpha)>\alpha-\epsilon.$$

Then we get

$$\hat{u}(\delta+m_kT,x+l_\infty;\alpha)>\alpha-\epsilon+\frac{\partial_t w_\infty(0,x+l_\infty;\alpha)}{4}\delta.$$

Since ϵ is small enough, we get that $\hat{u}(\delta+kT,x+l_\infty;\alpha)>\alpha$, then

$$\delta>m_{k+1}-m_k.$$

In particular, we can obtain that $(l_k)_{k\in\mathbb{N}}$ is bounded, this contradicts our second case which the existence of a sequence going to ∞. The proof of the Lemma 3.3 is completed. $\square$

3.2. Existence of a minimal terrace.

From the preparatory work we have done above, we are now ready to construct a minimal propagating terrace. Before we construct the minimal propagating terrace, here we give two claims.

Claim 3.4. *Let $U_1(t,x) := w_\infty(t,x;\alpha_1)$ be a periodic pulsating traveling wave of (1.1) which connects $p_{k_1}(x)$ to $p(x)$ for $k_1 \in \{1,...,N\}$.*

Proof of Claim 3.4. Firstly, we prove that $U_1(t,x)$ is a periodic pulsating traveling wave. Here we use proof by contradiction and assume it is not true. From the previous Lemma 3.3 we know that $w_\infty(t,x;\alpha_1)$ is time-heterogeneous and it is a solution of (1.2) with $w_\infty(0,0;\alpha_1) = \alpha_1$. Since $p_1(x) < w_\infty(t,x;\alpha) < p(x)$ and let $V_1(t,x)$ is a periodic pulsating traveling wave connecting $p_1(x)$ to $p(x)$. It is clear that $V_1(t,x)$ is steeper than $w_\infty(t,x;\alpha)$. This contradicts the statement of Lemma 2.8. Therefore, $U_1(t,x)$ is a periodic pulsating traveling wave.

Next we will prove that

$$\lim_{t\to+\infty} U_1(t,x) = p_{k_1}(x) \quad \text{locally} \quad \text{uniformly} \quad \text{in} \quad t \in \mathbb{R}.$$

Due to $w_\infty(t,x;\alpha_1)$ is a periodic pulsating traveling wave, we define $\lim\limits_{t\to+\infty} w_\infty(t,x;\alpha_1) := w_\infty(\infty,x)$ and $w_\infty(\infty,x)$ is a solution of (1.2). We have

$$0 \le w_\infty(\infty,x) \le p(x) \ \text{ for } \ t \in \mathbb{R}. \tag{3.5}$$

In view of Definition 1.3 and Lemma 2.8, we know $w_\infty(t,x;\alpha_1)$ is a steepest solution of (1.1) which between 0 and $p(x)$. We also get $w_\infty(\infty,x)$ is steeper than any other entire solution between 0 and $p(x)$. This shows that for any $x \in \mathbb{R}$, $w_\infty(\infty,x)$ and $V_1(t,x)$ can not intersect. Since $w_\infty(\infty,0) > V_1(t_0,0) = \alpha_1 \ for \ t_0 \in \mathbb{R}$, then it follows $w_\infty(\infty,x) > V_1(t,0)$ for $t \in \mathbb{R}$. By comparison principle, we get

$$w_\infty(\infty,x) \ge V_1(t,x) \quad \text{for } \ t \in \mathbb{R}, \ \ x \in \mathbb{R}.$$

Since $w_\infty(\infty,x)$ is L-periodic with respect to the x variable and $V_1(t,x)$ converges to $p(x)$ as $t \to \infty$, we have $w_\infty(\infty,x) \ge p(x)$ for $x \in \mathbb{R}$. Apply the inequality of (3.5), we immediately get $w_\infty(\infty,x) \equiv p(x)$. This implies that

$$\lim_{t\to+\infty} U_1(t,x) = p_{k_1}(x) \ \ \text{locally uniformly in} \ \ t \in \mathbb{R}.$$

In fact, we know from the above analysis that $w_\infty(\infty,x)$ is steeper than any other entire solution of (1.1) between 0 and $p(x)$. Otherwise, there exists $\widetilde{k} \in \{2,...,N\}$ such that $V_{\widetilde{k}}(t,x)$ cross through $w_\infty(\infty,x)$, which is impossible. And that completes the proof of Claim 3.4. □

Claim 3.5. *Assume that for $k_i \in \{1,...,N-1\}$ and $\alpha_i \in (p_{k_i}(0),p(0))$, the function*

$$U_i(t,x) :\equiv w_\infty(t,x;\alpha_i)$$

is a periodic pulsating traveling wave connecting $p_{k_i}(x)$ to $p_{k_{i-1}(x)} > p_{k_i}(x)$. There exists $\alpha_{i+1} \in (0,p_{k_i}(0))$ such that

$$U_{i+1}(t,x) :\equiv w_\infty(t,x;\alpha_{i+1})$$

is a periodic pulsating traveling wave connecting $p_{k_{i+1}(x)} < p_{k_i}(x)$ *to* $p_{k_i}(x)$.

Proof of Claim 3.5. This part of the proof is given in [9] in detail, therefore we omit the details.

The above process ends at $p_{k_i}(x) \equiv 0$. So this is actually a finite steps, which means it is happening in a finite number of steps $i = M$ for $1 \le M \le N$. Then, we can get a decreasing sequence of level sets $(\alpha_i)_{1\le i\le M}$, a decreasing sequence of solutions $(p_{k_i})_{0\le i\le M}$ and a sequence of $(U_i)_{1\le i\le M}$. For each $1 \le i \le M$, $U_i(t, x)$ is a periodic pulsating traveling wave connecting $p_{k_i}(x)$ to $p_{k_{i-1}}(x)$ and satisfy $U_i(0, 0) = \alpha_i$.

For $1 \le i \le M$, we let c_i be the speed of $U_i(t, x)$. We will prove that

$$\mathcal{F} := ((p_{k_i})_{0\le i\le M}, (U_i, c_i)_{1\le i\le M})$$

is a propagating terrace. We use the sequence $(m_{i,k})_{k\in\mathbb{N}}$ to define (3.1) with α replaced by α_i. The (3.3) clearly implies

$$c_i = \lim_{k\to\infty} \frac{m_{i,k}}{kT}. \tag{3.6}$$

Since $\alpha_{i+1} < \alpha_i$ for each $1 \le i \le M - 1$ and $\hat{u}(t, x)$ is increasing in t. We have $\alpha_{i+1,k} < \alpha_{i,k}$ for $k \in \mathbb{N}$. From (3.7), it implies that $c_{i+1} < c_i$. Then the existence of propagating terrace is completed. According to the above analysis, we see that $p_{k_i}(x)$ and $U_i(t, x)$ are steeper than any other entire solution between 0 and $p(x)$, which shows statement (ii) of Theorem 1.8 and the minimality of the propagating terrace $\mathcal{F}$. Lastly, the statement (i) of Theorem 1.8 from Lemma 2.9, we complete the proof. □

3.3. Convergence with Heaviside type initial data

In this section we are going to prove Theorem 1.9. It have proved the existence of minimal propagating terrace in the previous section, we will show that at certain initial data (Heaviside type) and it will attract all solutions of (1.1).

Proof of Theorem 1.9. Let $\mathcal{F} := ((p_i)_{0\le i\le M}, (U_i, c_i)_{1\le i\le M})$ be the minimal propagating terrace connecting 0 and $p(x)$. We know from the proof in the previous part up to some time shift, it is the unique minimal terrace. Moreover, Theorem 1.8 shows that each $p_i(x)$ and $U_i(t, x)$ is steeper than any other entire solution of (1.1) between 0 and $p(x)$. We take $\hat{u}(t, x)$ be a solution of the equation (1.1) with an initial data of Heaviside type initial function. For each $i \in \{1, ..., M\}$, take the sequence $(m_{i,k})_{k\in\mathbb{N}}$ satisfy

$$\hat{u}(kT, m_{i,k}) = U_i(0, 0) \quad \text{for} \quad k \in \mathbb{N},$$

where $(m_{i,k})_{k\in\mathbb{N}}$ converges to $(m_k)_{k\in\mathbb{N}}$ as $i \to \infty$.

We know that $U_i(t, x)$ is the steepest entire solution between 0 and $p(x)$, combine Lemma 3.1 and Lemma 3.3 for $t \ge 0$,

$$\hat{u}(t + kT, x + m_{i,k}) \to U_i(t, x) \quad \text{as} \quad k \to \infty. \tag{3.7}$$

Since

$$U_i(\cdot, \cdot) = U_i(\cdot + mT, \cdot + c_i mT) \quad \text{for} \quad m \in \mathbb{Z}. \tag{3.8}$$

(3.7) shows that

$$\hat{u}(t, x+m_{i,\lfloor t\backslash T\rfloor}) - U_i(t, x+c_i\lfloor t\backslash T\rfloor T) \to 0 \quad \text{as} \quad t\to\infty, \tag{3.9}$$

where $\lfloor t/T\rfloor$ is the step function of t/T, the maximum integer is smaller than t/T.

For each $i\in\{1,...,M\}$, let $\gamma_i:[0,\infty)\to\mathbb{R},\ \ t\mapsto\gamma_i(t)$ be a function satisfying

$$\gamma_i(t) + c_i\lfloor t/T\rfloor T - m_{i,\lfloor t/T\rfloor} \to 0 \quad \text{as} \quad t\to\infty. \tag{3.10}$$

Next we go through each of the statements (i)-(ii) of Theorem 1.9.

From (3.7) and (3.8), we notice $\lim\limits_{k\to\infty}(m_{i,k+1}-m_{i,k}) = c_iT$. This shows that

$$\frac{c_i\lfloor t/T\rfloor T - m_{i,\lfloor t/T\rfloor}}{t} = \frac{\lfloor t/T\rfloor T\Big(c_i - \frac{m_{i,\lfloor t/T\rfloor}}{\lfloor t/T\rfloor T}\Big)}{t} \to 0 \quad \text{as} \quad t\to\infty.$$

Then from (3.10), we get that $\gamma_i(t)/t\to 0$ as $t\to\infty$. Statement (i) is proved.

Finally, let us prove the convergence property of (ii). Since $U_i(t,x)$ satisfies

$$\lim_{t\to-\infty} U_i(t, x+c_it) = p_{i-1}(x) \ \text{ and } \ \lim_{t\to\infty} U_i(t, x+c_it) = p_i(x) \text{ in } t\in\mathbb{R}.$$

For any sufficiently small $\epsilon>0$, there exists $N>0$ such that

$$U_i(t, c_it+N) \le p_i(x) + \frac{\epsilon}{2} \quad \text{and} \quad U_i(t, c_it-N) \ge p_{i-1}(x) - \frac{\epsilon}{2} \quad \text{for} \quad t\in\mathbb{R}. \tag{3.11}$$

By (3.9) and (3.10), we can find a $T^{'}$ large enough that for any $t\ge T^{'}$,

$$|\hat{u} - U_i(t, x-\gamma_i(t))| \le \frac{\epsilon}{2} \ \text{ for } \ c_it+\gamma_i(t)-N \le x \le c_it+\gamma_i(t)+N. \tag{3.12}$$

Together with (3.11) shows that

$$\hat{u}(t, c_it+\gamma_i(t)+N) \le p_i(x)+\epsilon \ \text{ and } \ \hat{u}(t, c_it+\gamma_i(t)-N) \ge p_{i-1}(x)-\epsilon \ \text{ for } \ t\ge T^{'}.$$

Due to $\hat{u}(t,x)$ is increasing in $t\in\mathbb{R}$, we can get for each $i\in\{2,...,M\}$,

$$-\epsilon \le \hat{u}(t,x) - p_{i-1}(x) \le \epsilon \quad \text{for} \quad c_{i-1}t+\gamma_{i-1}(t)+N \le x \le c_it+\gamma_i(t)-N, \ \ t\ge T^{'}, \tag{3.13}$$

and

$$0 < \hat{u}(t,x) \le \epsilon \quad \text{for} \quad x\ge c_Mt+\gamma_N(t)+N, \ \ t\ge T^{'}, \tag{3.14}$$

and that

$$p(x)-\epsilon \le \hat{u}(t,x) \le p(x) \quad \text{for} \quad x\le c_1t+\gamma_1(t)-N, \ \ t\ge T^{'}. \tag{3.15}$$

Applying inequalities (3.12)-(3.15), we get that for $t\ge T^{'}$ and $x\in\mathbb{R}$,

$$\Big|\hat{u}(t,x) - \Big(\sum_{i=1}^{N} U_i(t, x-\gamma_i(t)) - \sum_{i=1}^{N} p_i(x)\Big)\Big| \le N\epsilon.$$

That gives the proof of statement (ii). Thus we complete the proof. $\square$

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