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GLOBAL EXISTENCE AND BLOW-UP PHENOMENA FOR THE DOUBLY NONLINEAR DIFFUSION EQUATION WITH NONLINEAR NEUMANN BOUNDARY CONDITIONS

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Abstract Under the nonlinear Neumann boundary conditions, an initial boundary value problem for doubly nonlinear diffusion equation is considered in this paper. We establish the new sufficient conditions on nonlinear functions to guarantee that the positive solution $u(\boldsymbol{x},t)$ exists globally. Under the conditions to guarantee that the positive solution blows up, by establishing the Sobolev inequality in multidimensional space, we obtain upper and lower bounds of the blow-up time T.

Keywords Nonlinear porous medium equations, nonlinear Neumann boundary condition, global existence, blow up.

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1. Introduction

In this paper, we consider the problem

$$\begin{cases} u_t = \operatorname{div}(|\nabla u^m|^{p-2}\nabla u^m) + k(t)f(u), & (\boldsymbol{x},t) \in \Omega \times (0,T), \\ |\nabla u^m|^{p-2}\frac{\partial u}{\partial \boldsymbol{\nu}} = g(u), & (\boldsymbol{x},t) \in \partial\Omega \times (0,T), \\ u(\boldsymbol{x},0) = u_0(\boldsymbol{x}) > 0, & \boldsymbol{x} \in \Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded star-shaped domain with a smooth boundary $\partial\Omega$, $p \geq 2$ and $m \geq 1$, T is the blow-up time if blow-up occurs, or else $T = \infty$. We assume that $k(t)$ is a nonnegative differential function, f is locally Lipschitz continuous on $\mathbb{R}$, $f(0) = 0$, $g(u)$ is a nonnegative continuous function.

Definition 1.1. A function $u \in L^\infty(\Omega \times (0,T)) \cap L^p(0,T;\ W^{1,p}(\Omega))$ is said to be a weak solution for (1.1) if

$$\int_0^T \int_\Omega u\phi_\tau d\boldsymbol{x}d\tau + m\int_0^T \int_{\partial\Omega} \phi u^{m-1}g(u)dSd\tau - \int_0^T \int_\Omega |\nabla u^m|^{p-2}\nabla u^m \cdot \nabla\phi d\boldsymbol{x}d\tau$$

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$$
\begin{aligned}
&+\int_0^T \int_\Omega k(\tau) f \phi d\boldsymbol{x} d\tau \\
&= \int_\Omega u(\boldsymbol{x}, T)\phi(\boldsymbol{x}, T) d\boldsymbol{x} - \int_\Omega u(\boldsymbol{x}, 0)\phi(\boldsymbol{x}, 0) d\boldsymbol{x},
\end{aligned}
$$

for any $\phi \in C^\infty(\bar{\Omega}) \times (0, T)$.

One of the important examples of the doubly nonlinear diffusion equation is the porous medium equation, which describes widely processes involving fluid flow, heat transfer or diffusion, and its other applications in different fields such as mathematical biology, lubrication, boundary layer theory, and etc. At present, many articles are known for the study of blow-up phenomena for the parabolic equations (for instance, [4–7, 16–18]). Recently, some new developments have been made in the study of the blow-up time estimates for parabolic equations under nonlinear boundary conditions. We refer to [6, 19, 20, 25]. In order to investigate the blow-up problems of (1.1), we focus on the papers [6,19,20,25]. In [25], Zhang and Li studied the following problems

$$
\begin{cases}
u_t - \sum\limits_{i,j=1}^n (a_{ij}(\boldsymbol{x}) u_{x_i})_{x_j} = k(t) f(u), & (\boldsymbol{x}, t) \in \Omega \times (0, t^*), \\
\sum\limits_{i,j=1}^n a_{ij}(\boldsymbol{x}) u_{x_i} \nu_j = g(u), & (\boldsymbol{x}, t) \in \partial\Omega \times (0, t^*), \\
u(\boldsymbol{x}, 0) = u_0(\boldsymbol{x}) > 0, & \boldsymbol{x} \in \Omega,
\end{cases}
$$

where Ω is a bounded star-shaped domain in $\mathbb{R}^n (n \geq 3)$ with smooth boundary $\partial\Omega$. Under certain conditions on data, they showed that the solution blows up or remain global for $\Omega \in \mathbb{R}^n (n \geq 3)$, and by establishing the Sobolev inequality in multidimensional space and constructing the unified functionals, they obtain upper and lower bounds of the blow-up time t^*. Payne, Philippin and Vernier Piro [19] investigate the blow-up phenomenon of the classical solution $u(\boldsymbol{x}, t)$ of the following initial-boundary value problem

$$
\begin{cases}
u_t = \Delta u - f(u), \ (\boldsymbol{x}, t) \in \Omega \times (0, t^*), \\
\frac{\partial u}{\partial \boldsymbol{\nu}} = g(u), \qquad (\boldsymbol{x}, t) \in \partial\Omega \times (0, t^*), \\
u(\boldsymbol{x}, 0) = u_0(\boldsymbol{x}), \ \ \boldsymbol{x} \in \bar{\Omega},
\end{cases}
$$

where Ω is a bounded star-shaped domain in $\mathbb{R}^n (n \geq 2)$ with smooth boundary $\partial\Omega$. They establish conditions on nonlinearities sufficient to guarantee that $u(\boldsymbol{x}, t)$ exist for all time $t > 0$ as well as conditions on data forcing the solution $u(\boldsymbol{x}, t)$ to blow up at finite time t^*. Moreover, under somewhat more restrictive conditions, upper and lower bounds for t^* are derived. Ding and Shen [6] consider the blow-up problem

$$
\begin{cases}
u_t = \Delta(u^m) + k(t) f(u), & (\boldsymbol{x}, t) \in \Omega \times (0, t^*), \\
\frac{\partial u}{\partial \boldsymbol{\nu}} = g(u), & (\boldsymbol{x}, t) \in \partial\Omega \times (0, t^*), \\
u(\boldsymbol{x}, 0) = u_0(\boldsymbol{x}), & \boldsymbol{x} \in \bar{\Omega},
\end{cases}
$$

where $m > 1$, $\Omega \subset \mathbb{R}^n (n \geq 2)$ is a bounded convex domain with smooth boundary. Under appropriate assumptions, the sufficient condition is given to guarantee that solution u blows up at finite time. Moreover, upper and lower bounds for blow-up time are derived. Sabitbek and Torebek [20] investigate a global existence and blow-up of the positive solutions to a nonlinear porous medium problem

$$\begin{cases} u_t = \operatorname{div}(|\nabla u^m|^{p-2}\nabla u^m) + f(u), & (\boldsymbol{x},t) \in \Omega \times (0,+\infty), \\ u(\boldsymbol{x},t) = 0, & (\boldsymbol{x},t) \in \partial\Omega \times (0,+\infty), \\ u(\boldsymbol{x},0) = u_0(\boldsymbol{x}), & \boldsymbol{x} \in \overline{\Omega}. \end{cases}$$

By using the concavity method and constructing auxiliary functions, they obtain the sufficient conditions of the existence of global solution and blow up solution and the upper bound on blow-up time.

In this paper, the more generate problems (1.1) are studied. It seems that the auxiliary functions defined in [6,19,20,25] are not applicable for the problem (1.1). By defining completely different auxiliary functions from those in [6,19,20,25] and using a first-order differential inequality technical, we obtain conditions sufficient to ensure the solution exists for all time or blows up at some finite time. The upper and lower bounds on blow-up time are also given.

We process as follows. In section 2, we establish the conditions on the nonlinearities to guarantee that $u(\boldsymbol{x},t)$ exists globally. In section 3, we show the conditions on the nonlinearities which ensure that the solution blows up at some finite time and obtain an upper bound of the blow-up time. Section 4 is devoted to showing a lower bound of blow-up time under some assumptions.

2. Global existence

In this section, we show the global existence of the positive solution to problem (1.1). Firstly, we give the following general lemma [25].

Lemma 2.1. [25] *Let $\Omega \subset \mathbb{R}^n (n \geq 2)$ be a bounded star-shaped domain assumed to be convex in $n-1$ orthogonal directions. Then, for any nonnegative increasing C^1 function $h(w)$, we have*

$$\int_{\partial\Omega} h(w)dS \leq \frac{n}{\rho_0}\int_\Omega h(w)d\boldsymbol{x} + \frac{d}{\rho_0}\int_\Omega h'(w)|\nabla w|d\boldsymbol{x},$$

where

$$\rho_0 := \min_{\boldsymbol{x}\in\partial\Omega}(\boldsymbol{x}\cdot\boldsymbol{\nu}), \quad d := \max_{\boldsymbol{x}\in\partial\Omega}|\boldsymbol{x}|.$$

Theorem 2.1. *Let $\Omega \subset \mathbb{R}^n (n \geq 2)$ is a bounded star-shaped domain with a smooth boundary $\partial\Omega$, $p \geq 2$, $m \geq 1$. Let the functions f, g and $k(t)$ satisfy*

$$f(s) \leq -c_1 s^{r_1}, \quad g(s) \leq c_2 s^{r_2}, \quad k(t) \geq c_3, \tag{2.1}$$

where $c_1, c_2, c_3 > 0$, $r_2 > (p-2)m+1$, $r_1 > r_2 + m - 1$, $(p-1)r_1 - pr_2 > m - p$, $s > 0$. Then the positive solution $u(\boldsymbol{x},t)$ of problem (1.1) is bounded for all time $t > 0$.

Proof. Multiplying the equation of (1.1) by $u^m(\boldsymbol{x},t)$ and integrating by parts, we have

$$\begin{aligned}
&\int_\Omega u^m u_t d\boldsymbol{x} - \int_\Omega \left[u^m \mathrm{div}(|\nabla u^m|^{p-2}\nabla u^m) + k(t)u^m f(u)\right] d\boldsymbol{x}\\
=&\frac{1}{m+1}\frac{d}{dt}\int_\Omega u^{m+1}d\boldsymbol{x} - \int_{\partial\Omega} mu^{2m-1}g(u)dS + \int_\Omega |\nabla u^m|^p d\boldsymbol{x} - \int_\Omega k(t)u^m f(u)d\boldsymbol{x}\\
=&0. \qquad (2.2)
\end{aligned}$$

Define the auxiliary functional

$$\Phi(t) := \frac{1}{m+1}\int_\Omega u^{m+1}d\boldsymbol{x}, \qquad (2.3)$$

then we have

$$\begin{aligned}
\Phi'(t) =&\int_{\partial\Omega} mu^{2m-1}g(u)dS - \int_\Omega |\nabla u^m|^p d\boldsymbol{x} + \int_\Omega k(t)u^m f(u)d\boldsymbol{x}\\
\le& mc_2\int_{\partial\Omega} u^{2m+r_2-1}dS - \int_\Omega |\nabla u^m|^p d\boldsymbol{x} - c_1c_3\int_\Omega u^{m+r_1}d\boldsymbol{x}. \qquad (2.4)
\end{aligned}$$

Application of Lemma 2.1 leads to the inequality

$$\int_{\partial\Omega} u^{2m+r_2-1}dS \le \frac{n}{\rho_0}\int_\Omega u^{2m+r_2-1}d\boldsymbol{x} + \frac{d(2m+r_2-1)}{\rho_0}\int_\Omega u^{2m+r_2-2}|\nabla u|d\boldsymbol{x}. \qquad (2.5)$$

Using Young's inequality with $\epsilon > 0$, we have

$$\begin{aligned}
\int_\Omega u^{2m+r_2-2}|\nabla u|d\boldsymbol{x} =&\frac{1}{m}\int_\Omega u^{m+r_2-1}|\nabla u^m|d\boldsymbol{x}\\
\le&\frac{\epsilon}{m}\int_\Omega |\nabla u^m|^p d\boldsymbol{x} + \frac{c(\epsilon)}{m}\int_\Omega u^{\frac{p(m+r_2-1)}{p-1}}d\boldsymbol{x}, \qquad (2.6)
\end{aligned}$$

where $c(\epsilon) = \frac{p-1}{p}(\epsilon p)^{-\frac{1}{p-1}}$.

Inserting (2.5) and (2.6) into (2.4) and choosing $\epsilon = \frac{\rho_0}{c_2 d(2m+r_2-1)} > 0$, we have

$$\begin{aligned}
\Phi'(t) \le&\frac{nmc_2}{\rho_0}\int_\Omega u^{2m+r_2-1}d\boldsymbol{x} + \frac{c_2c(\epsilon)d(2m+r_2-1)}{\rho_0}\int_\Omega u^{\frac{p(m+r_2-1)}{p-1}}d\boldsymbol{x} \qquad (2.7)\\
&- c_1c_3\int_\Omega u^{m+r_1}d\boldsymbol{x}.
\end{aligned}$$

Now let's estimate $\int_\Omega u^{\frac{p(m+r_2-1)}{p-1}}d\boldsymbol{x}$. To this end we make use of Hölder's inequality to write

$$\int_\Omega u^{\frac{p(m+r_2-1)}{p-1}}d\boldsymbol{x} \le \left(\int_\Omega u^{2m+r_2-1}d\boldsymbol{x}\right)^{1-\sigma}\left(\int_\Omega u^{m+r_1}d\boldsymbol{x}\right)^{\sigma} \qquad (2.8)$$

with

$$\sigma := \frac{r_2-(p-2)m-1}{(p-1)(r_1-r_2-m+1)}. \qquad (2.9)$$

Note that $\sigma \in (0,1)$ in view of $r_1 > r_2 > (p-2)m+1$, $(p-1)r_1 - pr_2 > m-p$.

From (2.8), we obtain

$$\begin{aligned}\int_\Omega u^{\frac{p(m+r_2-1)}{p-1}} d\boldsymbol{x} \le & \left(\delta^{\frac{\sigma}{\sigma-1}} \int_\Omega u^{2m+r_2-1} d\boldsymbol{x}\right)^{1-\sigma} \left(\delta \int_\Omega u^{m+r_1} d\boldsymbol{x}\right)^{\sigma} \\ \le & (1-\sigma)\delta^{\frac{\sigma}{\sigma-1}} \int_\Omega u^{2m+r_2-1} d\boldsymbol{x} + \sigma\delta \int_\Omega u^{m+r_1} d\boldsymbol{x},\end{aligned} \tag{2.10}$$

for arbitrary $\delta > 0$. By inserting (2.10) in (2.7), we obtain

$$\Phi'(t) \le M_1 \int_\Omega u^{2m+r_2-1} d\boldsymbol{x} - M_2 \int_\Omega u^{m+r_1} d\boldsymbol{x} \tag{2.11}$$

with

$$\begin{aligned} M_1 :=& \frac{nmc_2}{\rho_0} + \frac{(1-\sigma)\delta^{\frac{\sigma}{\sigma-1}} c_2 c(\epsilon) d(2m+r_2-1)}{\rho_0} > 0, \\ M_2 :=& c_1 c_3 - \frac{\sigma\delta c_2 c(\epsilon) d(2m+r_2-1)}{\rho_0}, \end{aligned} \tag{2.12}$$

and choose δ small enough to have $M_2 > 0$. From Hölder's inequality, we have

$$\int_\Omega u^{2m+r_2-1} d\boldsymbol{x} \le \left(\int_\Omega u^{m+r_1} d\boldsymbol{x}\right)^{\frac{2m+r_2-1}{m+r_1}} |\Omega|^{\frac{r_1-r_2-m+1}{m+r_1}}, \tag{2.13}$$

where $|\Omega| := \int_\Omega d\boldsymbol{x}$. Combining (2.11) and (2.13), we obtain

$$\begin{aligned} &\Phi'(t) \\ \le & M_1 \left(\int_\Omega u^{m+r_1} d\boldsymbol{x}\right)^{\frac{2m+r_2-1}{m+r_1}} \left[|\Omega|^{\frac{r_1-r_2-m+1}{m+r_1}} - \frac{M_2}{M_1} \left(\int_\Omega u^{m+r_1} d\boldsymbol{x}\right)^{\frac{r_1-r_2-m+1}{m+r_1}} \right]. \end{aligned} \tag{2.14}$$

Using again Hölder inequality, we have

$$\Phi(t) := \frac{1}{m+1} \int_\Omega u^{m+1} d\boldsymbol{x} \le \frac{1}{m+1} \left(\int_\Omega u^{m+r_1} d\boldsymbol{x}\right)^{\frac{m+1}{m+r_1}} |\Omega|^{\frac{r_1-1}{m+r_1}}. \tag{2.15}$$

It follows from (2.14), (2.15) that

$$\Phi'(t) \tag{2.16}$$

$$\begin{aligned} \le & M_1 \left(\int_\Omega u^{m+r_1} d\boldsymbol{x}\right)^{\frac{2m+r_2-1}{m+r_1}} \\ & \times \left[|\Omega|^{\frac{r_1-r_2-m+1}{m+r_1}} - \frac{M_2}{M_1}(m+1)^{\frac{r_1-r_2-m+1}{m+1}} |\Omega|^{\frac{(1-r_1)(r_1-r_2-m+1)}{(m+1)(m+r_1)}} \Phi(t)^{\frac{r_1-r_2-m+1}{m+1}} \right]. \end{aligned} \tag{2.17}$$

From the inequality (2.16), we can conclude that $\Phi(t)$ remains bounded for all time under the conditions in Theorem 2.1. In fact, if $u(\boldsymbol{x}, t)$ blows up at finite time T, then $\Phi(t)$ is unbounded near T which forces $\Phi'(t) \le 0$ in some interval $[t_0, T)$. So we have $\Phi(t) \le \Phi(t_0)$ in $[t_0, T)$ which implies that $\Phi(t)$ is unbounded in $[t_0, T)$, this is a contradiction.

The proof of Theorem 2.1 is completed. □

Remark 2.1. If there is no heat source and the boundary is adiabatic, that is, $k(t) \equiv 0$ and $g \equiv 0$, from (2.2)-(2.4) in the proof process, we conclude that the nonnegative solution exists globally.

Remark 2.2. If $k(t) = 1, p = 2, m = 1$ is taken in this problem, then this problem is the model in reference [19], which is consistent with the results in reference [19]. In this case, $r_1 = 2$ and $r_2 = 1$ can satisfy all the conditions in Theorem 2.1.

Remark 2.3. If we choose $f(s) = -c_1 s^{r_1}$, $g(s) = c_2 s^{r_2}$, $k = c_3$, where $c_1, c_2, c_3 > 0$ and $r_1 = \frac{(p^2+1)m}{p-1}$, $r_2 = pm$, then all the conditions in Theorem 2.1 are satisfied.

3. Blow-up and upper bound estimation of T

In this section, we determine a condition sufficient to ensure the solution blows up at some finite time and obtain an upper bound on blow-up time. From the physical background and characteristics of the heat equation, we known that if the function f, g and $k(t)$ are nonnegative, then the solution to (1.1) is nonnegative and smooth.

Lemma 3.1. [13, 14] *If the function $\omega(t) > 0$ is twice-differentiable and for the constant $\theta > 0$ and any $t > 0$ which satisfies*

$$\omega''(t)\omega(t) - (1+\theta)\left(\omega'(t)\right)^2 \geq 0$$

and $\omega'(0) > 0$, then there exists $0 < t_1 \leq \frac{\omega(0)}{\theta\omega'(0)}$ such that $\omega(t)$ tends to infinity as $t \to t_1$.

Theorem 3.1. *Let $\Omega \subset \mathbb{R}^n (n \geq 1)$ is a bounded domain with a smooth boundary $\partial\Omega$. The heat source coefficient $k(t)$ is a nonnegative nondecreasing differentiable function, and the nonnegative integrable functions f, g satisfy the conditions for some constant $\alpha > m+1$:*

$$u^m f(u) \geq \alpha F(u), \ u^{2m-1} g(u) \geq \alpha G(u), u \geq 0 \tag{3.1}$$

with

$$F(u) = \frac{pm}{m+1}\int_0^u \xi^{m-1} f(\xi)d\xi, \ G(u) = \frac{pm^2}{m+1}\int_0^u \xi^{2(m-1)} g(\xi)d\xi, u \geq 0. \tag{3.2}$$

Moreover, we assume $\Theta(0) > 0$ with

$$\Theta(t) := \int_{\partial\Omega} G(u)dS - \frac{1}{m+1}\int_\Omega |\nabla u^m|^p d\boldsymbol{x} + k(t)\int_\Omega F(u)d\boldsymbol{x}. \tag{3.3}$$

Then the nonnegative classical solution $u(\boldsymbol{x}, t)$ blows up at some finite time $T \leq T_U$ with

$$T_U := \frac{(1+\frac{1}{\sigma})^2 \int_\Omega u_0^{m+1} d\boldsymbol{x}}{\alpha(m+1)\Theta(0)}, \tag{3.4}$$

where $\sigma = \frac{\sqrt{\alpha p m}}{m+1} - 1 > 0$.

Proof. By the definition of $\Theta(t)$ and the condition (3.1), and in view of $k'(t) \geq 0$, we deduce

$$\begin{aligned}
\Theta'(t) =&\frac{pm^2}{m+1}\int_{\partial\Omega} u^{2(m-1)}u_t g(u)dS - \frac{p}{m+1}\int_\Omega |\nabla u^m|^{p-2}\nabla u^m \cdot \nabla(u^m)_t d\boldsymbol{x} \\
&+ \frac{pm}{m+1}k(t)\int_\Omega u^{m-1}u_t f(u)d\boldsymbol{x} + \frac{pm}{m+1}k^{'}(t)\int_\Omega\int_0^u \xi^{m-1}f(\xi)d\xi \\
\geq&\frac{p}{m+1}\left[m\int_{\partial\Omega} u^{m-1}(u^m)_t g(u)dS - \int_\Omega |\nabla u^m|^{p-2}\nabla u^m \cdot \nabla(u^m)_t d\boldsymbol{x}\right. \\
&\left.+k(t)\int_\Omega (u^m)_t f(u)d\boldsymbol{x}\right] \\
=&\frac{p}{m+1}\left[\int_\Omega (u^m)_t|\nabla u^m|^{p-2}\Delta u^m d\boldsymbol{x} + \int_\Omega (u^m)_t\nabla(|\nabla u^m|^{p-2})\cdot\nabla u^m d\boldsymbol{x}\right. \\
&\left.+k(t)\int_\Omega (u^m)_t f(u)d\boldsymbol{x}\right] \\
=&\frac{p}{m+1}\left[\int_\Omega (u^m)_t \mathrm{div}(|\nabla u^m|^{p-2}\nabla u^m)d\boldsymbol{x} + k(t)\int_\Omega (u^m)_t f(u)d\boldsymbol{x}\right] \\
=&\frac{p}{m+1}\int_\Omega (u^m)_t u_t d\boldsymbol{x} \\
=&\frac{pm}{m+1}\int_\Omega u^{m-1}u_t^2 d\boldsymbol{x} \\
\geq&0,
\end{aligned} \tag{3.5}$$

which with $\Theta(0) > 0$ imply $\Theta(t) > 0$ for all $t \in (0, T)$. And we can see that

$$\Theta(t) = \Theta(0) + \int_0^t \frac{d}{d\tau}\Theta(\tau)d\tau \geq \Theta(0) + \int_0^t\int_\Omega \frac{pm}{m+1}u^{m-1}u_\tau^2 d\boldsymbol{x}d\tau. \tag{3.6}$$

Now, we introduce a new function

$$\Phi(t) = \int_0^t\int_\Omega u^{m+1}d\boldsymbol{x}dt + M, \tag{3.7}$$

where $M > 0$ is a constant to be determined later. Then it is easy to see that

$$\Phi'(t) = \int_\Omega u^{m+1}d\boldsymbol{x} = (m+1)\int_\Omega\int_0^t u^m u_\tau d\tau d\boldsymbol{x} + \int_\Omega u_0^{m+1}d\boldsymbol{x}. \tag{3.8}$$

Use integration by parts, the condition (3.1) and (3.6) in turn to obtain

$$\begin{aligned}
\Phi''(t) =&(m+1)\int_\Omega u^m u_t d\boldsymbol{x} \\
=&(m+1)\int_\Omega \left[u^m \mathrm{div}(|\nabla u^m|^{p-2}\nabla u^m) + k(t)u^m f(u)\right] d\boldsymbol{x} \\
=&(m+1)\int_{\partial\Omega} mu^{2m-1}g(u)dS - (m+1)\int_\Omega |\nabla u^m|^p d\boldsymbol{x} \\
&+ (m+1)\int_\Omega k(t)u^m f(u)d\boldsymbol{x}
\end{aligned}$$

$$\begin{aligned}
&\geq \alpha m(m+1)\int_{\partial\Omega} G(u)dS-(m+1)\int_\Omega |\nabla u^m|^p d\boldsymbol{x}\\
&\quad +\alpha(m+1)k(t)\int_\Omega F(u)d\boldsymbol{x}\\
&\geq \alpha(m+1)\int_{\partial\Omega} G(u)dS-(m+1)\int_\Omega |\nabla u^m|^p d\boldsymbol{x}+\alpha(m+1)k(t)\int_\Omega F(u)d\boldsymbol{x}\\
&=\alpha(m+1)\left[\int_{\partial\Omega} G(u)dS-\frac{1}{m+1}\int_\Omega |\nabla u^m|^p d\boldsymbol{x}+k(t)\int_\Omega F(u)d\boldsymbol{x}\right]\\
&\quad +(\alpha-m-1)\int_\Omega |\nabla u^m|^p d\boldsymbol{x}\\
&>\alpha(m+1)\left[\int_{\partial\Omega} G(u)dS-\frac{1}{m+1}\int_\Omega |\nabla u^m|^p d\boldsymbol{x}+k(t)\int_\Omega F(u)d\boldsymbol{x}\right]\\
&=\alpha(m+1)\Theta(t)\\
&\geq \alpha(m+1)\left[\Theta(0)+\int_0^t\int_\Omega \frac{pm}{m+1}u^{m-1}u_\tau^2 d\boldsymbol{x}d\tau\right]. \qquad (3.9)
\end{aligned}$$

Using the Schwarz inequality, we obtain

$$\begin{aligned}
\Phi'(t)^2 &= \left((m+1)\int_\Omega\int_0^t u^m u_\tau d\tau dx+\int_\Omega u_0^{m+1}dx\right)^2\\
&=\left((m+1)\int_\Omega\int_0^t u^m u_\tau d\tau dx\right)^2+\left(\int_\Omega u_0^{m+1}dx\right)^2\\
&\quad +2(m+1)(\int_\Omega\int_0^t u^m u_\tau d\tau dx)(\int_\Omega u_0^{m+1}dx)\\
&\leq (m+1)^2\left(\int_\Omega\int_0^t u^m u_\tau d\tau dx\right)^2+\left(\int_\Omega u_0^{m+1}dx\right)^2\\
&\quad +2\left[\frac{\epsilon(m+1)^2}{2}\left(\int_\Omega\int_0^t u^m u_\tau d\tau dx\right)^2+\frac{1}{2\epsilon}\left(\int_\Omega u_0^{m+1}dx\right)^2\right]\\
&=(m+1)^2(1+\epsilon)\left(\int_\Omega\int_0^t u^m u_\tau d\tau dx\right)^2+(1+\frac{1}{\epsilon})\left(\int_\Omega u_0^{m+1}dx\right)^2\\
&\leq (m+1)^2(1+\epsilon)\left[\int_\Omega(\int_0^t u^{m-1}u_\tau^2 d\tau)^{\frac{1}{2}}(\int_0^t u^{m+1}d\tau)^{\frac{1}{2}}dx\right]^2\\
&\quad +(1+\frac{1}{\epsilon})\left(\int_\Omega u_0^{m+1}dx\right)^2\\
&\leq (m+1)^2(1+\epsilon)\left(\int_\Omega\int_0^t u^{m-1}u_\tau^2 d\tau dx\right)\left(\int_\Omega\int_0^t u^{m+1}d\tau dx\right)\\
&\quad +(1+\frac{1}{\epsilon})\left(\int_\Omega u_0^{m+1}dx\right)^2, \qquad (3.10)
\end{aligned}$$

where $\epsilon > 0$ is arbitrary. Combining the above estimates (3.8), (3.9) and (3.10), we obtain that for $\sigma=\epsilon=\frac{\sqrt{\alpha pm}}{m+1}-1>0$,

$$\Phi''(t)\Phi(t)-(1+\sigma)\Phi'(t)^2$$

$$
\begin{aligned}
>&\alpha\left[(m+1)\Theta(0)+pm\int_0^t\int_\Omega u^{m-1}u_\tau^2dxd\tau\right]\left[\int_0^t\int_\Omega u^{m+1}dxd\tau+M\right]\\
&-(m+1)^2(1+\sigma)(1+\epsilon)\left(\int_\Omega\int_0^t u^{m-1}u_\tau^2d\tau dx\right)\left(\int_\Omega\int_0^t u^{m+1}d\tau dx\right)\\
&-(1+\sigma)(1+\frac{1}{\epsilon})\left(\int_\Omega u_0^{m+1}dx\right)^2\\
=&\alpha M(m+1)\Theta(0)+\alpha(m+1)\Theta(0)\int_0^t\int_\Omega u^{m+1}dxd\tau+\alpha Mpm\int_0^t\int_\Omega u^{m-1}u_\tau^2dxd\tau\\
&+\left[\alpha pm-(m+1)^2(1+\sigma)(1+\epsilon)\right]\left(\int_\Omega\int_0^t u^{m-1}u_\tau^2d\tau dx\right)\left(\int_\Omega\int_0^t u^{m+1}d\tau dx\right)\\
&-(1+\sigma)(1+\frac{1}{\epsilon})\left(\int_\Omega u_0^{m+1}dx\right)^2\\
\geq&\alpha M(m+1)\Theta(0)-(1+\sigma)(1+\frac{1}{\epsilon})\left(\int_\Omega u_0^{m+1}dx\right)^2. \qquad (3.11)
\end{aligned}
$$

Since $\Theta(0)>0$ by the assumption, we can choose

$$M=\frac{(1+\sigma)(1+\frac{1}{\sigma})\left(\int_\Omega u_0^{m+1}dx\right)^2}{\alpha(m+1)\Theta(0)}>0, \qquad (3.12)$$

that gives

$$\Phi''(t)\Phi(t)-(1+\sigma)\Phi'(t)^2>0. \qquad (3.13)$$

Then by the Lemma 3.1, we have derived the conclusion. □

Remark 3.1. From the above proof process, it is not difficult to find that when $\alpha=m+1$, $pm=m+1$, it holds that

$$\Phi(t)\geq\Phi(0)e^{\frac{\Phi'(0)}{\Phi(0)}t},$$

then we deduce

$$\lim_{t\to+\infty}\Phi(t)=+\infty.$$

Remark 3.2. If we choose $F(u)=u^{\frac{\alpha pm}{m+1}}f_1(u)$, $G(u)=u^{\frac{\alpha pm^2}{m+1}}g_1(u)$, where $f_1(u)$ and $g_1(u)$ is nonnegative nondecreasing function on $(0,+\infty)$, $u_0(\boldsymbol{x})=constant>0$, then all the conditions in Theorem 3.1 are satisfied.

Remark 3.3. The proof of Theorem 3.1 has not required that Ω is a star-shape domain.

4. Lower bound estimation of T

In this section, under assumption that $\Omega\subset\mathbb{R}^n(n\geq2)$ is a bounded star-sharped domain assumed to be convex in $n-1$ orthogonal directions, we give the lower bound estimation of blow-up time under occurrence of blow-up phenomena.

Lemma 4.1. [1] *For any function $u \in W_0^{1,p}(\Omega)$, we have the inequality*

$$\|u\|_r \leq C_* \|u\|_{W^{1,p}},$$

for all $1 \leq r \leq p^*$, where $p^* = \frac{np}{n-p}$ if $n > p$ and $p^* = \infty$ if $n \leq p$. The best constant C_* depends only on Ω, n, p and r.

We suppose that functions f, g and k satisfy

$$f(\xi) \leq a\xi^{r_1}, \quad g(\xi) \leq b\xi^{r_2}, \quad k(t) \leq M, \quad t > 0, \tag{4.1}$$

where a, b, r_1, r_2, M are some positive constants and

$$pr_2 - (p-1)r_1 > p - m, \quad r_2 > 1 + m(p-2). \tag{4.2}$$

The auxiliary function is defined as follows

$$\Phi(t) = \int_\Omega u^\beta d\boldsymbol{x}$$

with

$$\beta > \max\left\{1, \frac{p^* p\left[r_2 - m(p-2) - 1\right]}{(p^* - p)(p-1)}\right\}. \tag{4.3}$$

Theorem 4.1. *Let $\Omega \subset \mathbb{R}^n (n \geq 2)$ is a bounded star-sharped domain assumed to be convex in $n-1$ orthogonal directions. Assume that (4.1)-(4.3) hold. Then, the nonnegative classical solution $u(\boldsymbol{x}, t) \in L^\infty(\Omega \times (0,T)) \cap L^p(0,T;\ W_0^{1,p}(\Omega))$ to (1.1) blows up at finite time, and the blow-up time*

$$T \geq \int_{\Phi(0)}^{+\infty} \frac{d\tau}{C_1 + C_2 \tau^{\frac{p(1-\sigma_1)}{p - p^* \sigma_1}}},$$

where C_1, C_2 and σ_1 are certain positive constants, we will provide them later.

Proof. Using (4.1), (4.3), and the divergence theorem, we have

$$\begin{aligned}
\Phi'(t) =& \beta \int_\Omega u^{\beta-1} u_t d\boldsymbol{x} \\
=& \beta \int_\Omega u^{\beta-1} \left[\operatorname{div}(|\nabla u^m|^{p-2} \nabla u^m) + k(t) f(u)\right] d\boldsymbol{x} \\
=& \beta \int_\Omega u^{\beta-1} \nabla(|\nabla u^m|^{p-2}) \cdot \nabla u^m d\boldsymbol{x} + \beta \int_\Omega u^{\beta-1} |\nabla u^m|^{p-2} \Delta u^m d\boldsymbol{x} \\
&+ \beta \int_\Omega k(t) u^{\beta-1} f(u) d\boldsymbol{x} \\
=& \beta m \int_{\partial\Omega} u^{\beta+m-2} g(u) dS - \beta \int_\Omega |\nabla u^m|^{p-2} \nabla u^{\beta-1} \cdot \nabla u^m d\boldsymbol{x} \\
&+ \beta \int_\Omega k(t) u^{\beta-1} f(u) d\boldsymbol{x} \\
=& \beta m \int_{\partial\Omega} u^{\beta+m-2} g(u) dS - \beta(\beta-1)m \int_\Omega u^{\beta+m-3} |\nabla u^m|^{p-2} |\nabla u|^2 d\boldsymbol{x} \\
&+ \beta \int_\Omega k(t) u^{\beta-1} f(u) d\boldsymbol{x}.
\end{aligned} \tag{4.4}$$

We note

$$\begin{aligned} mu^{\beta+m-3}|\nabla u^m|^{p-2}|\nabla u|^2 =&\frac{1}{m}u^{\beta-m-1}|\nabla u^m|^{p-2}|\nabla u^m|^2 \\ =&\frac{1}{m}\left(\frac{mp}{\beta+(p-1)m-1}\right)^p|\nabla\left(u^{\frac{\beta+(p-1)m-1}{p}}\right)|^p. \end{aligned} \tag{4.5}$$

Inserting (4.5) into (4.4), we get

$$\begin{aligned} \Phi'(t) \leq&\beta mb\int_{\partial\Omega}u^{\beta+m+r_2-2}dS \\ &-\frac{\beta(\beta-1)}{m}\left(\frac{mp}{\beta+(p-1)m-1}\right)^p\int_\Omega|\nabla\left(u^{\frac{\beta+(p-1)m-1}{p}}\right)|^pd\boldsymbol{x} \\ &+\beta aM\int_\Omega u^{\beta+r_1-1}d\boldsymbol{x}. \end{aligned} \tag{4.6}$$

To the first term of right side of (4.6), we apply the Lemma 2.1 to obtain

$$\begin{aligned} &\int_{\partial\Omega}u^{\beta+m+r_2-2}dS \\ \leq&\frac{n}{\rho_0}\int_\Omega u^{\beta+m+r_2-2}d\boldsymbol{x}+\frac{d(\beta+m+r_2-2)}{\rho_0}\int_\Omega u^{\beta+m+r_2-3}|\nabla u|d\boldsymbol{x}, \end{aligned} \tag{4.7}$$

where $\rho_0 := \min\limits_{\boldsymbol{x}\in\partial\Omega}(\boldsymbol{x}\cdot\boldsymbol{\nu})$ and $d := \max\limits_{\boldsymbol{x}\in\partial\Omega}|\boldsymbol{x}|$. By (4.5), the Hölder inequality, and Young inequality, we derive

$$\begin{aligned} &\int_\Omega u^{\beta+m+r_2-3}|\nabla u|d\boldsymbol{x} \\ \leq&\left(\int_\Omega u^{\beta+(p-1)m-p-1}|\nabla u|^pd\boldsymbol{x}\right)^{\frac{1}{p}}\left(\int_\Omega u^{\beta+\frac{p(r_2-2)+m+1}{p-1}}d\boldsymbol{x}\right)^{\frac{p-1}{p}} \\ =&\left(\epsilon_1\int_\Omega u^{\beta+(p-1)m-p-1}|\nabla u|^pd\boldsymbol{x}\right)^{\frac{1}{p}}\left(\frac{1}{\epsilon_1^{p-1}}\int_\Omega u^{\beta+\frac{p(r_2-2)+m+1}{p-1}}d\boldsymbol{x}\right)^{\frac{p-1}{p}} \\ \leq&\frac{\epsilon_1}{p}\int_\Omega u^{\beta+(p-1)m-p-1}|\nabla u|^pd\boldsymbol{x}+\frac{p-1}{p\epsilon_1^{p-1}}\int_\Omega u^{\beta+\frac{p(r_2-2)+m+1}{p-1}}d\boldsymbol{x} \\ =&\frac{\epsilon_1}{p}\left(\frac{p}{\beta+(p-1)m-1}\right)^p\int_\Omega|\nabla\left(u^{\frac{\beta+(p-1)m-1}{p}}\right)|^pd\boldsymbol{x}+\frac{p-1}{p\epsilon_1^{p-1}}\int_\Omega u^{\beta+\frac{p(r_2-2)+m+1}{p-1}}d\boldsymbol{x}, \end{aligned} \tag{4.8}$$

where

$$\epsilon_1=\frac{(\beta-1)\rho_0m^{p-2}}{bd(\beta+m+r_2-2)}>0. \tag{4.9}$$

Substituting (4.7)-(4.9) into (4.6), we deduce

$$\begin{aligned} \Phi'(t) \leq&\beta mb\left(\frac{n}{\rho_0}\int_\Omega u^{\beta+m+r_2-2}d\boldsymbol{x}+\frac{d(\beta+m+r_2-2)}{\rho_0}\int_\Omega u^{\beta+m+r_2-3}|\nabla u|d\boldsymbol{x}\right) \\ &-\frac{\beta(\beta-1)}{m}\left(\frac{mp}{\beta+(p-1)m-1}\right)^p\int_\Omega|\nabla\left(u^{\frac{\beta+(p-1)m-1}{p}}\right)|^pd\boldsymbol{x} \end{aligned}$$

$$
\begin{aligned}
&+\beta aM\int_\Omega u^{\beta+r_1-1}d\boldsymbol{x}\\
\leq&\frac{\beta mbn}{\rho_0}\int_\Omega u^{\beta+m+r_2-2}d\boldsymbol{x}+\frac{d\beta mb(\beta+m+r_2-2)}{\rho_0}\\
&\times\left(\frac{\epsilon_1}{p}\left(\frac{p}{\beta+(p-1)m-1}\right)^p\int_\Omega|\nabla\left(u^{\frac{\beta+(p-1)m-1}{p}}\right)|^pd\boldsymbol{x}\right.\\
&\left.+\frac{p-1}{p\epsilon_1^{p-1}}\int_\Omega u^{\beta+\frac{p(r_2-2)+m+1}{p-1}}d\boldsymbol{x}\right)\\
&-\frac{\beta(\beta-1)}{m}\left(\frac{mp}{\beta+(p-1)m-1}\right)^p\int_\Omega|\nabla\left(u^{\frac{\beta+(p-1)m-1}{p}}\right)|^pd\boldsymbol{x}\\
&+\beta aM\int_\Omega u^{\beta+r_1-1}d\boldsymbol{x}\\
=&\frac{\beta mbn}{\rho_0}\int_\Omega u^{\beta+m+r_2-2}d\boldsymbol{x}\\
&+\frac{d\beta mb(p-1)(\beta+m+r_2-2)}{\rho_0p\epsilon_1^{p-1}}\int_\Omega u^{\beta+\frac{p(r_2-2)+m+1}{p-1}}d\boldsymbol{x}\\
&+\left(\frac{d\beta mb(\beta+m+r_2-2)}{\rho_0p}\left(\frac{p}{\beta+(p-1)m-1}\right)^p\epsilon_1\right.\\
&\left.-\frac{\beta(\beta-1)}{m}\left(\frac{mp}{\beta+(p-1)m-1}\right)^p\right)\int_\Omega|\nabla\left(u^{\frac{\beta+(p-1)m-1}{p}}\right)|^pd\boldsymbol{x}\\
&+\beta aM\int_\Omega u^{\beta+r_1-1}d\boldsymbol{x}\\
=&\frac{\beta mbn}{\rho_0}\int_\Omega u^{\beta+m+r_2-2}d\boldsymbol{x}\\
&+\frac{d\beta mb(p-1)(\beta+m+r_2-2)}{\rho_0p\epsilon_1^{p-1}}\int_\Omega u^{\beta+\frac{p(r_2-2)+m+1}{p-1}}d\boldsymbol{x}\\
&+\frac{\beta(\beta-1)(mp)^{p-1}(1-p)}{(\beta+(p-1)m-1)^p}\int_\Omega|\nabla\left(u^{\frac{\beta+(p-1)m-1}{p}}\right)|^pd\boldsymbol{x}+\beta aM\int_\Omega u^{\beta+r_1-1}d\boldsymbol{x}.
\end{aligned}
\tag{4.10}
$$

We use the Hölder inequality and the Young inequality to the first and fourth terms of right side of (4.10) to obtain

$$
\begin{aligned}
\int_\Omega u^{\beta+m+r_2-2}d\boldsymbol{x}\leq&\left(\int_\Omega u^{\beta+\frac{p(r_2-2)+m+1}{p-1}}d\boldsymbol{x}\right)^{\frac{(p-1)(\beta+m+r_2-2)}{(p-1)\beta+p(r_2-2)+m+1}}|\Omega|^{\frac{r_2-(p-2)m-1}{(p-1)\beta+p(r_2-2)+m+1}}\\
\leq&\frac{(p-1)(\beta+m+r_2-2)}{(p-1)\beta+p(r_2-2)+m+1}\int_\Omega u^{\beta+\frac{p(r_2-2)+m+1}{p-1}}d\boldsymbol{x}\\
&+\frac{r_2-(p-2)m-1}{(p-1)\beta+p(r_2-2)+m+1}|\Omega|
\end{aligned}
\tag{4.11}
$$

and

$$
\int_\Omega u^{\beta+r_1-1}d\boldsymbol{x}\leq\left(\int_\Omega u^{\beta+\frac{p(r_2-2)+m+1}{p-1}}d\boldsymbol{x}\right)^{\frac{(p-1)(\beta+r_1-1)}{(p-1)\beta+p(r_2-2)+m+1}}|\Omega|^{\frac{pr_2-(p-1)r_1-p+m}{(p-1)\beta+p(r_2-2)+m+1}}
$$

$$\leq \frac{(p-1)(\beta+r_1-1)}{(p-1)\beta+p(r_2-2)+m+1}\int_\Omega u^{\beta+\frac{p(r_2-2)+m+1}{p-1}}d\boldsymbol{x} + \frac{pr_2-(p-1)r_1-p+m}{(p-1)\beta+p(r_2-2)+m+1}|\Omega|, \tag{4.12}$$

where $0<\frac{(p-1)(\beta+m+r_2-2)}{(p-1)\beta+p(r_2-2)+m+1}<1$ and $0<\frac{(p-1)(\beta+r_1-1)}{(p-1)\beta+p(r_2-2)+m+1}<1$ in consideration of (4.2) and (4.3). and $|\Omega|$ is the measure of Ω. Inserting (4.11) and (4.12) into (4.10), we have

$$\Phi'(t) \leq A_1 + A_2\int_\Omega u^{\beta+\frac{p(r_2-2)+m+1}{p-1}}d\boldsymbol{x} + \frac{\beta(\beta-1)(mp)^{p-1}(1-p)}{(\beta+(p-1)m-1)^p}\int_\Omega |\nabla\left(u^{\frac{\beta+(p-1)m-1}{p}}\right)|^p d\boldsymbol{x}, \tag{4.13}$$

where

$$A_1 = \left[\frac{\beta bmn[r_2-(p-2)m-1]+\rho_0\beta aM[pr_2-(p-1)r_1-p+m]}{\rho_0[(p-1)\beta+p(r_2-2)+m+1]}\right]|\Omega|>0 \tag{4.14}$$

and

$$A_2 = \frac{\beta bmn(p-1)(\beta+m+r_2-2)+\rho_0\beta aM[(p-1)(\beta+r_1-1)]}{\rho_0[(p-1)\beta+p(r_2-2)+m+1]} + \frac{\beta mbd(p-1)(\beta+m+r_2-2)}{\rho_0 p\epsilon_1^{p-1}} > 0. \tag{4.15}$$

We use the Hölder inequality, the second term on the right-hand side of (4.13) can be estimated as follows

$$\int_\Omega u^{\beta+\frac{p(r_2-2)+m+1}{p-1}}d\boldsymbol{x} \leq \left(\int_\Omega u^\beta d\boldsymbol{x}\right)^{1-\sigma_1}\left[\int_\Omega \left(u^{\frac{\beta+(p-1)m-1}{p}}\right)^{p*}d\boldsymbol{x}\right]^{\sigma_1}, \tag{4.16}$$

where

$$\sigma_1 = \frac{p[p(r_2-2)+m+1]}{(p-1)[p^*(\beta+(p-1)m-1)-\beta p]},\ \ 0<\sigma_1<1$$

in view of (4.2). And by Lemma 4.1, we have

$$\int_\Omega \left(u^{\frac{\beta+(p-1)m-1}{p}}\right)^{p^*}d\boldsymbol{x} \leq C_*^{p*}\left[\int_\Omega u^{\beta+(p-1)m-1}d\boldsymbol{x} + \int_\Omega |\nabla\left(u^{\frac{\beta+(p-1)m-1}{p}}\right)|^p d\boldsymbol{x}\right]^{\frac{p^*}{p}}, \tag{4.17}$$

where $C_* = C_*(n,\Omega)$ is an embedding constant depending on n and Ω.

Inserting (4.17) into (4.16), we have

$$\int_\Omega u^{\beta+\frac{p(r_2-2)+m+1}{p-1}}d\boldsymbol{x}$$

$$\leq\left(\int_\Omega u^\beta d\boldsymbol{x}\right)^{1-\sigma_1}\left[C_*^{p*}\left[\int_\Omega u^{\beta+(p-1)m-1}d\boldsymbol{x}+\int_\Omega|\nabla\left(u^{\frac{\beta+(p-1)m-1}{p}}\right)|^p d\boldsymbol{x}\right]^{\frac{p^*}{p}}\right]^{\sigma_1}$$
$$=C_*^{p^*\sigma_1}\left(\int_\Omega u^\beta d\boldsymbol{x}\right)^{1-\sigma_1}\left(\int_\Omega u^{\beta+(p-1)m-1}d\boldsymbol{x}+\int_\Omega|\nabla\left(u^{\frac{\beta+(p-1)m-1}{p}}\right)|^p d\boldsymbol{x}\right)^{\frac{p^*\sigma_1}{p}}. \tag{4.18}$$

For (4.18), by using the following basic inequality

$$(j_1+j_2)^l\leq 2^l(j_1^l+j_2^l),\ \ j_1>0,\ \ j_2>0,\ \ l>0, \tag{4.19}$$

we deduce

$$\int_\Omega u^{\beta+\frac{p(r_2-2)+m+1}{p-1}}d\boldsymbol{x}$$
$$\leq(2C_*^p)^{\frac{p^*\sigma_1}{p}}\left(\int_\Omega u^\beta d\boldsymbol{x}\right)^{1-\sigma_1}\left(\int_\Omega u^{\beta+(p-1)m-1}d\boldsymbol{x}\right)^{\frac{p^*\sigma_1}{p}}$$
$$+(2C_*^p)^{\frac{p^*\sigma_1}{p}}\left(\int_\Omega u^\beta d\boldsymbol{x}\right)^{1-\sigma_1}\left(\int_\Omega|\nabla\left(u^{\frac{\beta+(p-1)m-1}{p}}\right)|^p d\boldsymbol{x}\right)^{\frac{p^*\sigma_1}{p}}. \tag{4.20}$$

Due to (4.3), we have

$$0<\frac{p^*\sigma_1}{p}<1. \tag{4.21}$$

By (4.21) and Young inequality, the first term of (4.20) can be rewritten as

$$(2C_*^p)^{\frac{p^*\sigma_1}{p}}\left(\int_\Omega u^\beta d\boldsymbol{x}\right)^{1-\sigma_1}\left(\int_\Omega u^{\beta+(p-1)m-1}d\boldsymbol{x}\right)^{\frac{p^*\sigma_1}{p}}$$
$$=\left[(2C_*^p)^{\frac{p^*\sigma_1}{p-p^*\sigma_1}}\left(\frac{p}{p^*\sigma_1}\right)^{-\frac{p^*\sigma_1}{p-p^*\sigma_1}}\left(\int_\Omega u^\beta d\boldsymbol{x}\right)^{\frac{p(1-\sigma_1)}{p-p^*\sigma_1}}\right]^{1-\frac{p^*\sigma_1}{p}}$$
$$\times\left(\frac{p}{p^*\sigma_1}\int_\Omega u^{\beta+(p-1)m-1}d\boldsymbol{x}\right)^{\frac{p^*\sigma_1}{p}}$$
$$\leq\left(1-\frac{p^*\sigma_1}{p}\right)(2C_*^p)^{\frac{p^*\sigma_1}{p-p^*\sigma_1}}\left(\frac{p}{p^*\sigma_1}\right)^{-\frac{p^*\sigma_1}{p-p^*\sigma_1}}\left(\int_\Omega u^\beta d\boldsymbol{x}\right)^{\frac{p(1-\sigma_1)}{p-p^*\sigma_1}}$$
$$+\int_\Omega u^{\beta+(p-1)m-1}d\boldsymbol{x}. \tag{4.22}$$

It follows from the Hölder inequality and the Young inequality that

$$\int_\Omega u^{\beta+(p-1)m-1}d\boldsymbol{x}$$
$$\leq\left(\frac{1}{2\sigma_2}\int_\Omega u^{\beta+\frac{p(r_2-2)+m+1}{p-1}}d\boldsymbol{x}\right)^{\sigma_2}\left(\left(\frac{1}{2\sigma_2}\right)^{-\frac{\sigma_2}{1-\sigma_2}}|\Omega|\right)^{1-\sigma_2}$$

$$\leq \frac{1}{2}\int_\Omega u^{\beta+\frac{p(r_2-2)+m+1}{p-1}}d\boldsymbol{x} + (1-\sigma_2)\left(\frac{1}{2\sigma_2}\right)^{-\frac{\sigma_2}{1-\sigma_2}}|\Omega|, \tag{4.23}$$

where

$$\sigma_2 = \frac{(p-1)[\beta+(p-1)m-1]}{\beta(p-1)+p(r_2-2)+m+1},\ \ 0<\sigma_2<1$$

in view of (4.2). For the second term of (4.20), we apply (4.21) and the Young inequality to obtain

$$\begin{aligned}
&(2C_*^p)^{\frac{p^*\sigma_1}{p}}\left(\int_\Omega u^\beta d\boldsymbol{x}\right)^{1-\sigma_1}\left(\int_\Omega |\nabla\left(u^{\frac{\beta+(p-1)m-1}{p}}\right)|^p d\boldsymbol{x}\right)^{\frac{p^*\sigma_1}{p}}\\
=&\left((2C_*^p)^{\frac{p^*\sigma_1}{p-p^*\sigma_1}}\epsilon_2^{-\frac{p^*\sigma_1}{p-p^*\sigma_1}}\left(\int_\Omega u^\beta d\boldsymbol{x}\right)^{\frac{p(1-\sigma_1)}{p-p^*\sigma_1}}\right)^{1-\frac{p^*\sigma_1}{p}}\\
&\times\left(\epsilon_2\int_\Omega |\nabla\left(u^{\frac{\beta+(p-1)m-1}{p}}\right)|^p d\boldsymbol{x}\right)^{\frac{p^*\sigma_1}{p}}\\
\leq&\left(1-\frac{p^*\sigma_1}{p}\right)(2C_*^p)^{\frac{p^*\sigma_1}{p-p^*\sigma_1}}\epsilon_2^{-\frac{p^*\sigma_1}{p-p^*\sigma_1}}\left(\int_\Omega u^\beta d\boldsymbol{x}\right)^{\frac{p(1-\sigma_1)}{p-p^*\sigma_1}}\\
&+\frac{p^*\sigma_1}{p}\epsilon_2\int_\Omega |\nabla u^{\frac{\beta+(p-1)m-1}{p}}|^p d\boldsymbol{x},
\end{aligned} \tag{4.24}$$

where

$$\epsilon_2 = \frac{\beta(\beta-1)(mp)^{p-1}p(p-1)}{2A_2p^*\sigma_1[\beta+(p-1)m-1]^p} > 0. \tag{4.25}$$

Now inserting (4.22)-(4.24) into (4.20), we get

$$\begin{aligned}
&\int_\Omega u^{\beta+\frac{p(r_2-2)+m+1}{p-1}}d\boldsymbol{x}\\
\leq&\left(1-\frac{p^*\sigma_1}{p}\right)(2C_*^p)^{\frac{p^*\sigma_1}{p-p^*\sigma_1}}\left(\frac{n-p}{n\sigma_1}\right)^{-\frac{p^*\sigma_1}{p-p^*\sigma_1}}\left(\int_\Omega u^\beta d\boldsymbol{x}\right)^{\frac{p(1-\sigma_1)}{p-p^*\sigma_1}}\\
&+\frac{1}{2}\int_\Omega u^{\beta+\frac{p(r_2-2)+m+1}{p-1}}d\boldsymbol{x}+(1-\sigma_2)\left(\frac{1}{2\sigma_2}\right)^{-\frac{\sigma_2}{1-\sigma_2}}|\Omega|\\
&+\left(1-\frac{p^*\sigma_1}{p}\right)(2C_*^p)^{\frac{p^*\sigma_1}{p-p^*\sigma_1}}\epsilon_2^{-\frac{p^*\sigma_1}{p-p^*\sigma_1}}\left(\int_\Omega u^\beta d\boldsymbol{x}\right)^{\frac{p(1-\sigma_1)}{p-p^*\sigma_1}}\\
&+\frac{p^*\sigma_1}{p}\epsilon_2\int_\Omega |\nabla u^{\frac{\beta+(p-1)m-1}{p}}|^p d\boldsymbol{x}.
\end{aligned}$$

That is

$$\begin{aligned}
&\int_\Omega u^{\beta+\frac{p(r_2-2)+m+1}{p-1}}d\boldsymbol{x}\\
\leq&2\left(1-\frac{p^*\sigma_1}{p}\right)(2C_*^p)^{\frac{p^*\sigma_1}{p-p^*\sigma_1}}\left(\frac{p}{p^*\sigma_1}\right)^{-\frac{p^*\sigma_1}{p-p^*\sigma_1}}\left(\int_\Omega u^\beta d\boldsymbol{x}\right)^{\frac{p(1-\sigma_1)}{p-p^*\sigma_1}}
\end{aligned}$$

$$
\begin{aligned}
&+2(1-\sigma_2)\left(\frac{1}{2\sigma_2}\right)^{-\frac{\sigma_2}{1-\sigma_2}}|\Omega| \\
&+2\left(1-\frac{p^*\sigma_1}{p}\right)(2C_*^p)^{\frac{p^*\sigma_1}{p-p^*\sigma_1}}\epsilon_2^{-\frac{p^*\sigma_1}{p-p^*\sigma_1}}\left(\int_\Omega u^\beta d\boldsymbol{x}\right)^{\frac{p(1-\sigma_1)}{p-p^*\sigma_1}} \\
&+\frac{2p^*\sigma_1}{p}\epsilon_2\int_\Omega |\nabla u^{\frac{\beta+(p-1)m-1}{p}}|^p d\boldsymbol{x}. \qquad (4.26)
\end{aligned}
$$

We substitute (4.26) into (4.13) to derive

$$
\begin{aligned}
\Phi'(t) \leq & A_1+2A_2(1-\sigma_2)(\frac{1}{2\sigma_2})^{-\frac{\sigma_2}{1-\sigma_2}}|\Omega| \\
&+\left\{2\left(1-\frac{p^*\sigma_1}{p}\right)(2C_*^p)^{\frac{p^*\sigma_1}{p-p^*\sigma_1}}A_2\left[\left(\frac{p}{p^*\sigma_1}\right)^{-\frac{p^*\sigma_1}{p-p^*\sigma_1}}+\epsilon_2^{-\frac{p^*\sigma_1}{p-p^*\sigma_1}}\right]\right\} \\
&\times\left(\int_\Omega u^\beta d\boldsymbol{x}\right)^{\frac{p(1-\sigma_1)}{p-p^*\sigma_1}} \\
&+\left[\frac{\beta(\beta-1)(mp)^{p-1}(1-p)}{(\beta+(p-1)m-1)^p}+\frac{2A_2p^*\sigma_1}{p}\epsilon_2\right]\int_\Omega |\nabla\left(u^{\frac{\beta+(p-1)m-1}{p}}\right)|^p d\boldsymbol{x} \\
=& C_1+C_2\Phi^{\frac{p(1-\sigma_1)}{p-p^*\sigma_1}}(t), \qquad (4.27)
\end{aligned}
$$

where

$$
C_1=A_1+2A_2(1-\sigma_2)(\frac{1}{2\sigma_2})^{-\frac{\sigma_2}{1-\sigma_2}}|\Omega| \qquad (4.28)
$$

and

$$
C_2=2\left(1-\frac{p^*\sigma_1}{p}\right)(2C_*^p)^{\frac{p^*\sigma_1}{p-p^*\sigma_1}}A_2\left[\left(\frac{p}{p^*\sigma_1}\right)^{-\frac{p^*\sigma_1}{p-p^*\sigma_1}}+\epsilon_2^{-\frac{p^*\sigma_1}{p-p^*\sigma_1}}\right]. \qquad (4.29)
$$

Integrating (4.27) from 0 to t, we have

$$
t \geq \int_{\Phi(0)}^{\Phi(t)} \frac{d\tau}{C_1+C_2\tau^{\frac{p(1-\sigma_1)}{p-p^*\sigma_1}}}. \qquad (4.30)
$$

Since u blows up in measure $\Phi(t)$ at T, we pass the limits as $t \to T^-$ to obtain a lower bound

$$
T \geq \int_{\Phi(0)}^{+\infty} \frac{d\tau}{C_1+C_2\tau^{\frac{p(1-\sigma_1)}{p-p^*\sigma_1}}}, \qquad (4.31)
$$

where $\frac{p(1-\sigma_1)}{p-p^*\sigma_1} > 1$ in view of (4.3), So the above integral converges. □

Remark 4.1. If we choose $p = 2$ in problem (1.1), then the results in Theorem 4.1 are identical with the results in literature [6].

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