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SOME NOVEL INERTIAL BALL-RELAXED CQ ALGORITHMS FOR SOLVING THE SPLIT FEASIBILITY PROBLEM WITH MULTIPLE OUTPUT SETS

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Abstract The split feasibility problem with multiple output sets (SFPMOS) is a generalization of the well-known split feasibility problem (SFP), which has gained significant research attention due to its applications in theoretical and practical problems. However, the original CQ method for solving the SFP seems less efficient when the involved subsets are general convex sets since the method requires calculating projection onto the given sets directly. The relaxed CQ method was introduced to overcome this difficulty when the subsets are level sets of convex functions, where the projections onto the constructed half-spaces were used instead of the projections onto the original subsets. In this paper, we propose and investigate new algorithms for solving the SFPMOS when the involved subsets are given as the level sets of strongly convex functions. In this situation, we replace the half-spaces in the relaxed CQ method with balls constructed in each iteration. The algorithms are accelerated using the inertial technique and eliminate the need for calculating or estimating the norms of linear operators by employing self-adaptive step size criteria. We then analyze the strong convergence of the algorithms under some mild conditions. Some applications to the split feasibility problem are also reported. Finally, we present three numerical results, including an application to the LASSO problem with elastic net regularization, illustrating the better performance of our algorithms compared to the relevant ones.

Keywords Split feasibility problems, CQ algorithm, inertial technique, self-adaptive step size, metric projection.

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1. Introduction

Let $\mathcal{C}$ and $\mathcal{Q}_j$ be nonempty closed convex subsets of real Hilbert spaces $\mathcal{H}$ and $\mathcal{H}_j$, $j = 1, \dots, N$, respectively, and let $\mathcal{F}_j : \mathcal{H} \to \mathcal{H}_j$, $j = 1, \dots, N$, be a bounded linear operator. The split feasibility problem with multiple output sets (SFPMOS)

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[1, 8, 9, 18–23] can be formulated as follows:

$$\text{Find } u^* \in \mathcal{C} \text{ such that } \mathcal{F}_j u^* \in \mathcal{Q}_j \ \forall j = 1, \ldots, N. \tag{1.1}$$

Throughout this paper, we denote by I the identity operator in $\mathcal{H}$ or $\mathcal{H}_j$, $j = 1, \ldots, N$, and denote by $\Omega = \{u^* \in \mathcal{C} \mid \mathcal{F}_j u^* \in \mathcal{Q}_j, \forall j = 1, \ldots, N\}$ the solution set of the SFPMOS. In [18], Reich et al. proposed the following algorithm

$$x^0 \in \mathcal{C},\ x^{k+1} = \alpha_k f(x^k) + (1-\alpha_k) P_{\mathcal{C}}\Big[x^k - \gamma_k \sum_{j=1}^{N} \mathcal{F}_j^*(I - P_{\mathcal{Q}_j})\mathcal{F}_j x^k\Big],\ k \geq 0, \tag{1.2}$$

that can be used to find $u^* \in \Omega$, which is the unique solution to the following variational inequality,

$$\langle (I - f)u^*, u - u^* \rangle \geq 0 \ \forall u \in \Omega, \tag{1.3}$$

where f is a contraction mapping, $\{\alpha_k\} \subset (0,1)$ and $\{\gamma_k\}$ satisfy the following conditions

$$\lim_{k\to\infty} \alpha_k = 0,\ \sum_{k=0}^{\infty} \alpha_k = \infty, \tag{α}$$

$$0 < \gamma_k < \frac{2}{N \max_{j=1,\ldots,N}\{\|\mathcal{F}_j\|^2\}}. \tag{γR}$$

Wang [23] introduced a stepsize sequence $\{\gamma_k\}$ for (1.2) with $x^0 \in \mathcal{H}$ as follows

$$\gamma_k = \begin{cases} \frac{\sum_{j=1}^{N} \|(I - P_{\mathcal{Q}_j})\mathcal{F}_j x^k\|^2}{\|\sum_{j=1}^{N} \mathcal{F}_j^*(I - P_{\mathcal{Q}_j})\mathcal{F}_j x^k\|^2}, & \text{if } \|\sum_{j=1}^{N} \mathcal{F}_j^*(I - P_{\mathcal{Q}_j})\mathcal{F}_j x^k\| \neq 0, \\ 0, & \text{otherwise.} \end{cases} \tag{γW}$$

In [9], given an arbitrary initial point $x^0 \in \mathcal{H}$, Cuong et al. proposed the following iterative scheme

$$x^{k+1} = P_{\mathcal{C}} y^k - \alpha_k A\Big[P_{\mathcal{C}}\big(x^k - \gamma_k \sum_{j=1}^{N} \mathcal{F}_j^*(I - P_{\mathcal{Q}_j})\mathcal{F}_j x^k\big)\Big],\ k \geq 0, \tag{1.4}$$

where $A : \mathcal{H} \to \mathcal{H}$ be η-strongly monotone and ℓ-Lipschitz continuous on $\mathcal{H}$. They proved the strong convergence of the iterative sequence $\{x^k\}$ generated by (1.4) to $u^* \in \Omega$, which is the unique solution to the VIP (1.3), with $I - f$ replaced by A, under the condition (α) and

$$\gamma_k = \begin{cases} \frac{\sum_{j=1}^{N} \|(I - P_{\mathcal{Q}_j})\mathcal{F}_j x^k\|^2}{2\big(\sum_{j=1}^{N} \|\mathcal{F}_j^*(I - P_{\mathcal{Q}_j})\mathcal{F}_j x^k\|\big)^2}, & \text{if } \sum_{j=1}^{N} \|\mathcal{F}_j^*(I - P_{\mathcal{Q}_j})\mathcal{F}_j x^k\| \neq 0, \\ 0, & \text{otherwise.} \end{cases} \tag{γC}$$

When $N = 1$, the SFPMOS (1.1) becomes the split feasibility problem (SFP). The SFP introduced by Censor and Elfving [7] has been receiving much attention due to its applications in signal processing and image reconstruction [16]. The SFP requires to find a point $u^* \in \mathcal{H}$ satisfying the property:

$$u^* \in \mathcal{C} \text{ and } \mathcal{F}u^* \in \mathcal{Q}, \tag{1.5}$$

where $\mathcal{C}$ and $\mathcal{Q}$ are nonempty closed convex subsets of real Hilbert spaces $\mathcal{H}$ and $\mathcal{H}_1$, respectively, and $\mathcal{F} : \mathcal{H} \to \mathcal{H}_1$ is a bounded linear operator. To solve the SFP, Censor and Elfving [7] proposed an iterative algorithm based on the multidistance idea. But their algorithm involves matrix inverses at each iteration. Later, in [5,6], Byrne introduced a projection method called the CQ algorithm which does not involve matrix inverses. Denoting by $P_{\mathcal{C}}$ and $P_{\mathcal{Q}}$ the metric projections onto $\mathcal{C}$ and $\mathcal{Q}$, respectively. Then the CQ algorithm is formulated as follows:

$$x^{k+1} = P_{\mathcal{C}}\big(x^k - \gamma \mathcal{F}^*(I - P_{\mathcal{Q}})\mathcal{F}x^k\big), \ k \geq 0, \tag{1.6}$$

where $\mathcal{F}^*$ is the adjoint operator of $\mathcal{F}$ and $\gamma \in \big(0, 2/\|\mathcal{F}\|^2\big)$. It is worth noting that Byrne's CQ algorithm is a special case of the gradient-projection method in constrained convex minimization problems. The CQ algorithm (1.6) has been now widely studied since it is more easily performed. However, the computation of a projection onto a closed convex subset is generally difficult. To overcome this difficulty, Fukushima [12] suggested a way to calculate the projection onto a level set of a convex function by computing a sequence of projections onto half-spaces containing the original level set. This idea is followed by Yang [24], who introduced a relaxed CQ algorithm in a finite-dimensional Hilbert space, in which $\mathcal{C}$ and $\mathcal{Q}$ are level sets of convex functions $g : \mathcal{H} \to (-\infty, \infty]$ and $h : \mathcal{H}_1 \to (-\infty, \infty]$, respectively. The relaxed CQ algorithm is given as follows:

$$x^{k+1} = P_{\mathcal{C}_k}\big(x^k - \gamma \mathcal{F}^*(I - P_{\mathcal{Q}_k})\mathcal{F}x^k\big), \ k \geq 0, \tag{1.7}$$

where $\gamma \in \big(0, 2/\|\mathcal{F}\|^2\big)$ and

$$\mathcal{C}_k = \{x \in \mathcal{H} \mid g(x^k) + \langle \xi^k, x - x^k\rangle \leq 0\}, \ \xi^k \in \partial g(x^k), \tag{$\mathcal{C}_k$}$$

$$\mathcal{Q}_k = \{y \in \mathcal{H}_1 \mid h(\mathcal{F}x^k) + \langle \eta^k, y - \mathcal{F}x^k\rangle \leq 0\}, \ \eta^k \in \partial h(\mathcal{F}x^k). \tag{$\mathcal{Q}_k$}$$

In (1.7), since $\mathcal{C}_k$ and $\mathcal{Q}_k$ are both halfspaces, then the projections $P_{\mathcal{C}_k}$ and $P_{\mathcal{Q}_k}$ have closed-form expressions. Thus they are easily to be computed. However, to implement the relaxed CQ algorithm, one has to calculate or estimate the operator norm $\|\mathcal{F}\|$, which is generally not an easy task in practice. To overcome this difficulty, López et al. [16] improved Yang's relaxed CQ algorithm as follows:

$$x^{k+1} = P_{\mathcal{C}_k}\big(x^k - \gamma_k \mathcal{F}^*(I - P_{\mathcal{Q}_k})\mathcal{F}x^k\big), \ k \geq 0, \tag{1.8}$$

where

$$\gamma_k = \rho_k \frac{\|(I - P_{\mathcal{Q}_k})\mathcal{F}x^k\|^2}{\|\mathcal{F}^*(I - P_{\mathcal{Q}_k})\mathcal{F}x^k\|^2}, \ 0 < \rho_k < 2. \tag{γL}$$

Li et al. [15] and Yu and Wang [25] proposed some new relaxed CQ algorithms. The main idea of the algorithms is to replace the projections to the half-spaces $\mathcal{C}_k$ and $\mathcal{Q}_k$ with the projections to the intersection of $\mathcal{C}_k$ and $\mathcal{C}_{k-1}$, and the intersection of $\mathcal{Q}_k$ and $\mathcal{Q}_{k-1}$, respectively.

Yu et al. [26] introduced another ball-relaxed CQ method for solving the SFP under the condition that the functions g and h are β and β_1-strongly convex functions, respectively. The ball-relaxed CQ algorithm is formulated as follows:

$$x^{k+1} = P_{\mathcal{C}_k^b}\big(x^k - \gamma_k^b \mathcal{F}^*(I - P_{\mathcal{Q}_k^b})\mathcal{F}x^k\big), \ k \geq 0, \tag{1.9}$$

where

$$\gamma_k^b = \rho_k \frac{\|(I - P_{\mathcal{Q}_k^b})\mathcal{F}x^k\|^2}{\|\mathcal{F}^*(I - P_{\mathcal{Q}_k^b})\mathcal{F}x^k\|^2},\ 0 < \rho_k < 2 \tag{γY}$$

and

$$\mathcal{C}_k^b = \Big\{x \in \mathcal{H} \mid g(x^k) + \langle \xi^k, x - x^k\rangle + \frac{\beta}{2}\|x - x^k\|^2 \le 0\Big\},\ \xi^k \in \partial g(x^k), \tag{$\mathcal{C}_k^b$}$$

$$\mathcal{Q}_k^b = \Big\{y \in \mathcal{H}_1 \mid h(\mathcal{F}x^k) + \langle \eta^k, y - \mathcal{F}x^k\rangle + \frac{\beta_1}{2}\|y - \mathcal{F}x^k\|^2 \le 0\Big\},\ \eta^k \in \partial h(\mathcal{F}x^k). \tag{$\mathcal{Q}_k^b$}$$

In order to improve the convergence rate of the algorithms, the idea of inertial acceleration is widely applied. It was first proposed by Polyak in 1964 [17] for solving the smooth convex minimization problems. Based on the heavy ball methods of the two-order time dynamical system, the inertial algorithm is a two-step iterative method and the next iterative is defined by utilizing the previous two iterates. In [2], Alvarez and Attouch employed the inertial extrapolation technique for improving the performance of the proximal point algorithm. In [11], Dang et al. proposed an inertial relaxed CQ algorithm by applying the inertial extrapolation technique in (1.7):

$$\begin{cases} w^k = x^k + \theta_k(x^k - x^{k-1}), \\ x^{k+1} = P_{\mathcal{C}_k}\big(w^k - \gamma\mathcal{F}^*(I - P_{\mathcal{Q}_k})\mathcal{F}w^k\big), \end{cases} \tag{1.10}$$

where $\gamma \in (0, 2/\|\mathcal{F}\|^2)$ and $\theta \in [0, \overline{\theta}_k]$, $\overline{\theta}_k = \min\big\{\theta, (\max\{k^2\|x^k - x^{k-1}\|^2, k^2\|x^k - x^{k-1}\|\})^{-1}\big\}$, $\theta \in [0, 1)$. It is proved that the iterative sequence generated by (1.9) is weakly convergent to a solution of the SFP.

Motivated and inspired by the aforementioned works, in the present manuscript, we consider three ball-relaxed CQ algorithms for solving the SFPMOS (1.1) in real Hilbert spaces. In our algorithms, under the condition that $g : \mathcal{H} \to (-\infty, \infty]$ or $h_j : \mathcal{H}_j \to (-\infty, \infty]$, $j = 1, \ldots, N$, is strongly convex function, we replace the halfspace by a ball at the k-th iterate and the projections onto ball is easily executed. Meanwhile, to speed up convergence, we still use the variable stepsize and inertial acceleration in our algorithms.

The remaining part of this paper is organized as follows: Section 2 displays some lemmas that will be used for the validity and convergence of the algorithms. Section 3 gives three ball-relaxed CQ algorithms with inertial procedure and variable stepsize, and then proves the convergence of our algorithms. Section 4 gives some applications in the split feasibility problem. Finally, we present some numerical experiments to testify to the better performance of our algorithms in Section 5.

2. Preliminaries

In this section, we introduce some mathematical symbols, definitions, and lemmas which can be used in the proof of our main result. Let $\mathcal{H}$ be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$. In what follows, we write $x^k \rightharpoonup x$ to indicate that the sequence $\{x^k\}$ converges weakly to x, while $x^k \to x$ indicates that the sequence $\{x^k\}$ converges strongly to x. Let $\omega_w(x^k)$ denotes the weak ω-limit set of $\{x^k\}$, that is, the set of all those points x such that $x^{k_l} \rightharpoonup x$ as $l \to \infty$ for some subsequence $\{x^{k_l}\}$ of $\{x^k\}$.

Definition 2.1 (see [4], [13] Section 1.11)**.** Let $\mathcal{C}$ be a nonempty closed convex subset of a real Hilbert space $\mathcal{H}$. An operator $T : \mathcal{C} \to \mathcal{H}$ is called

(1) nonexpansive if $\|Tx - Ty\| \le \|x - y\|$ for all $x, y \in \mathcal{C}$;

(2) firmly nonexpansive if $\|Tx - Ty\|^2 \le \|x - y\|^2 - \|(I - T)x - (I - T)y\|^2$ for all $x, y \in \mathcal{C}$;

(3) η-inverse strongly monotone if exists $\eta > 0$ such that $\langle Tx - Ty, x - y\rangle \ge \eta\|Tx - Ty\|^2$ for all $x, y \in \mathcal{C}$.

Readers can refer to [13, Section 11] for more information about firmly nonexpansive mappings. For any $x \in \mathcal{H}$, the projection onto a nonempty closed convex subset $\mathcal{C}$ is defined as $P_{\mathcal{C}}x = \arg\min\{\|u - x\| \mid u \in \mathcal{C}\}$. The projection $P_{\mathcal{C}}$ has the following well-known properties.

Lemma 2.1 (see [4])**.** *Let $\mathcal{C} \subseteq \mathcal{H}$ be a nonempty closed convex subset. Then for all $x \in \mathcal{H}$ and $u \in \mathcal{C}$,*

(1) $\langle x - P_{\mathcal{C}}x, u - P_{\mathcal{C}}x\rangle \le 0$*;*

(2) $P_{\mathcal{C}}$ *and* $I - P_{\mathcal{C}}$ *are both nonexpansive;*

(3) $P_{\mathcal{C}}$ *and* $I - P_{\mathcal{C}}$ *are both firmly nonexpansive;*

(4) $P_{\mathcal{C}}$ *and* $I - P_{\mathcal{C}}$ *are both* 1*-inverse strongly monotone.*

Definition 2.2. Let $\lambda \in (0, 1)$ and $f : \mathcal{H} \to (-\infty, \infty]$ be a proper function.

(1) f is convex if $f[\lambda x + (1 - \lambda y)] \le \lambda f(x) + (1 - \lambda)f(y), \ \forall x, y \in \mathcal{H}$.

(2) f is strongly convex with constant β, where $\beta > 0$, if

$$f[\lambda x + (1 - \lambda y)] + \frac{\beta}{2}\lambda(1 - \lambda)\|x - y\|^2 \le \lambda f(x) + (1 - \lambda)f(y), \ \forall x, y \in \mathcal{H}.$$

(3) A vector $\xi \in \mathcal{H}$ is called a subgradient of f at a point $x \in \mathcal{H}$ if

$$f(y) \ge f(x) + \langle \xi, y - x\rangle \quad \forall y \in \mathcal{H}. \tag{2.1}$$

The set of all subgradients of f at x, denoted by $\partial f(x)$, is called the subdifferential of f at x. If $\partial f(x)$ is nonempty, then we say that f is subdifferentiable at x.

(4) f is said to be weakly lower semi-continuous (w-lsc) at a point x if $x^k \rightharpoonup x$ implies

$$f(x) \le \liminf_{k\to\infty} f(x^k).$$

Lemma 2.2. *Let $f : \mathcal{H} \to (-\infty, \infty]$ be a strongly convex function with constant β. Then for all $x, y \in \mathcal{H}$,*

$$f(y) \ge f(x) + \langle \xi, y - x\rangle + \frac{\beta}{2}\|y - x\|^2, \ \xi \in \partial f(x).$$

Lemma 2.3 (see [14, Lemma 8]). *Assume $\{s_k\}$ is a sequence of nonnegative real numbers such that*

$$s_{k+1} \leq (1-a_k)s_k + a_k b_k \quad \textit{and} \quad s_{k+1} \leq s_k - c_k + d_k,\ k \geq 1,$$

where $\{a_k\}$ is a sequence in $(0,1)$, $\{c_k\}$ is a sequence of nonnegative real numbers, and $\{b_k\}$ and $\{d_k\}$ are two real sequences in $\mathbb{R}$ such that

(1) $\sum_{k=0}^{\infty} a_k = \infty$;

(2) $\lim_{k\to\infty} d_k = 0$;

(3) $\lim_{l\to\infty} c_{k_l} = 0$ *implies* $\limsup_{l\to\infty} b_{k_l} \leq 0$ *for any subsequence* $\{k_l\} \subset \{k\}$.

Then $\lim_{k\to\infty} s_k = 0$.

3. Main results

3.1. The proposed algorithms

In this section, we first introduce a ball-relaxed CQ algorithm for the SFPMOS (1.1), in which the closed convex subsets $\mathcal{C}$ and $\mathcal{Q}_j$, $j = 1, \ldots, N$, satisfy the following assumptions.

(A1) The solution set $\Omega = \{x \in \mathcal{C} \mid \mathcal{F}_j x \in \mathcal{Q}_j,\ \forall j = 1, \ldots, N\}$ is nonempty.

(A2) The sets $\mathcal{C}$ and $\mathcal{Q}_j$, $j = 1, \ldots, N$, are given by

$$\mathcal{C} = \{x \in \mathcal{H} \mid g(x) \leq 0\} \text{ and } \mathcal{Q}_j = \{y_j \in \mathcal{H}_j \mid h_j(y_j) \leq 0\},$$

where $g : \mathcal{H} \to (-\infty, \infty]$ and $h_j : \mathcal{H}_j \to (-\infty, \infty]$ are β and β_j-strongly convex, subdifferentiable, and w-lsc functions on $\mathcal{H}$ and $\mathcal{H}_j$, respectively.

(A3) For any $x \in \mathcal{H}$, at least one subgradient $\xi \in \partial g(x)$ can be calculated. For any $y_j \in \mathcal{H}_j$, at least one subgradient $\eta_j \in \partial h_j(y_j)$, $j = 1, \ldots, N$, can be calculated. We assume also that the subdifferential operators ∂g and ∂h_j are bounded on bounded sets.

Remark 3.1. It is worth noting that every convex function defined on a finite-dimensional Hilbert space automatically satisfies assumption (A3) (see [3], Corollary 7.9).

In our algorithm, given the k-th iterative point x^k, we construct $N+1$ closed balls $\mathcal{C}_k^b$ and $\mathcal{Q}_{j,k}^b$, $j = 1, \ldots, N$, which contain $\mathcal{C}$ and $\mathcal{Q}_j$, respectively. Since g and h_j are strongly convex functions with constants β and β_j, respectively, it follows from Lemma 2.2 that

$$g(x) \geq g(x^k) + \langle \xi^k, x - x^k \rangle + \frac{\beta}{2}\|x - x^k\|^2,\ \xi^k \in \partial g(x^k), \tag{3.1}$$

$$h_j(y_j) \geq h_j(\mathcal{F}_j x^k) + \langle \eta_j^k, y_j - \mathcal{F}_j x^k \rangle + \frac{\beta_j}{2}\|y_j - \mathcal{F}_j x^k\|^2,\ \eta_j^k \in \partial h_j(\mathcal{F}_j x^k). \tag{3.2}$$

Define the set $\mathcal{C}_k^b$ at point x^k by ($\mathcal{C}_k^b$) and the sets $\mathcal{Q}_{j,k}^b$, $j = 1, \ldots, N$, at point x^k by

$$\mathcal{Q}_{j,k}^b = \Big\{ y_j \in \mathcal{H}_j \mid h_j(\mathcal{F}_j x^k) + \langle \eta_j^k, y_j - \mathcal{F}_j x^k \rangle + \frac{\beta_j}{2}\|y_j - \mathcal{F}_j x^k\|^2 \leq 0 \Big\}. \tag{$\mathcal{Q}_{j,k}^b$}$$

From $(\mathcal{C}_k^b)$ and $(\mathcal{Q}_{j,k}^b)$, we have $\mathcal{C}_k^b$ is a ball whose centre and radius are $x^k - (1/\beta)\xi^k$ and $\sqrt{(1/\beta^2)\|\xi^k\|^2 - (2/\beta)g(x^k)}$, respectively and $\mathcal{Q}_{j,k}^b$ is a ball whose centre and radius are $\mathcal{F}_j x^k - (1/\beta_j)\eta_j^k$ and $\sqrt{(1/\beta_j^2)\|\eta_j^k\|^2 - (2/\beta_j)h_j(\mathcal{F}_j x^k)}$, $j = 1, \dots, N$, respectively (see [26]).

Now we give our ball-relaxed CQ algorithm for solving the SFPMOS (1.1), where $\mathcal{C}$ and $\mathcal{Q}_j$, $j = 1, \dots, N$, are given in (A2).

Algorithm 3.1.

Initial Step. Select a sequence $\{\alpha_k\} \subset (0,1)$ satisfies the condition (α), a sequence $\{\eta_k\}$ such that $\lim_{k\to\infty} \frac{\eta_k}{\alpha_k} = 0$, $\{\rho_k\} \subset (0,2)$, $\{e_k\} \subset (0,c]$, where $c > 0$ is a constant, and $\theta \in (0,1)$. Let $x^0, x^1 \in \mathcal{H}$ be arbitrary. Input $k := 1$.

Iterative Step. Given the iterates x^{k-1} and x^k $(k \geq 1)$, calculate x^{k+1} as follows.

Step 1. Compute $w^k = x^k + \theta_k \left(x^k - x^{k-1}\right)$, where

$$\theta_k = \begin{cases} \min\left\{\frac{\eta_k}{\|x^k - x^{k-1}\|}, \theta\right\}, & \text{if } x^k \neq x^{k-1}, \\ \theta, & \text{otherwise.} \end{cases} \tag{θ}$$

Step 2. Compute $y^k = P_{\mathcal{C}_k^b} w^k$. Let $\ell_k = \|y^k - w^k\|$.

Step 3. Compute $v_j^k = P_{\mathcal{Q}_{j,k}^b} \mathcal{F}_j w^k$, $j = 1, \dots, N$.
Let $j_k \in \arg\max\left\{\|v_j^k - \mathcal{F}_j w^k\| \;\middle|\; j = 1, 2, \dots, N\right\}$, $\ell_{j_k} := \|v_{j_k}^k - \mathcal{F}_{j_k} w^k\|$.

Step 4. Let $L_k = \max\left\{\ell_k, \ell_{j_k}\right\}$.
If $L_k = \ell_k$, then compute $z^k = w^k - \gamma_k (I - P_{\mathcal{C}_k^b}) w^k$, where

$$\gamma_k = \rho_k \frac{\ell_k^2}{\|(I - P_{\mathcal{C}_k^b}) w^k\|^2 + e_k}. \tag{$\gamma_{\mathcal{C}}$T}$$

If $L_k = \ell_{j_k}$, then compute $z^k = w^k - \gamma_k \mathcal{F}_{j_k}^* (I - P_{\mathcal{Q}_{j_k,k}^b}) \mathcal{F}_{j_k} w^k$, where

$$\gamma_k = \rho_k \frac{\ell_{j_k}^2}{\|\mathcal{F}_{j_k}^* (I - P_{\mathcal{Q}_{j_k,k}^b}) \mathcal{F}_{j_k} w^k\|^2 + e_k}. \tag{$\gamma_{\mathcal{Q}}$T}$$

Step 5. Compute $x^{k+1} = \alpha_k f(x^k) + (1 - \alpha_k) P_{\mathcal{C}_k^b} z^k$. Set $k := k + 1$ and go to Step 1.

Remark 3.2. 1. From (3.1) and (3.2), we get $\mathcal{C} \subseteq \mathcal{C}_k^b$ and $\mathcal{Q}_j \subseteq \mathcal{Q}_{j,k}^b$ for all $j = 1, \dots, N$ and $k \geq 1$. Since $\mathcal{C}_k^b$ and $\mathcal{Q}_{j,k}^b$, $j = 1, \dots, N$, are balls, then the orthogonal projections onto $\mathcal{C}_k^b$ and $\mathcal{Q}_{j,k}^b$ can be directly calculated. Thus the proposed algorithm is easily implemented.

2. The inertial calculation criterion (θ) is easy to implement since the term $\left\|x^k - x^{k-1}\right\|$ is known before calculating θ_k. Moreover, it follows from (α) and (θ) that $\lim_{k\to\infty} \frac{\theta_k}{\alpha_k} \left\|x^k - x^{k-1}\right\| = 0$. Indeed, we obtain $\theta_k \left\|x^k - x^{k-1}\right\| \leq \eta_k$ for all $k \geq 1$ which together with $\lim_{k\to\infty} \frac{\eta_k}{\alpha_k} = 0$ implies that $\lim_{k\to\infty} \frac{\theta_k}{\alpha_k} \left\|x^k - x^{k-1}\right\| \leq \lim_{k\to\infty} \frac{\eta_k}{\alpha_k} = 0$.

In Algorithm 3.1, we assume that g and h_j, $j = 1, \ldots, N$, are strongly convex functions. When g is strongly convex function and h_j is only convex function, we modify Algorithm 3.1 to the following algorithm.

Algorithm 3.2.

Initial Step. Select a sequence $\{\alpha_k\} \subset (0,1)$ satisfies the condition (α), a sequence $\{\eta_k\}$ such that $\lim_{k\to\infty} \frac{\eta_k}{\alpha_k} = 0$, $\{\rho_k\} \subset (0,2)$, $\{e_k\} \subset (0,c]$, where $c > 0$ is a constant, and $\theta \in (0,1)$. Let $x^0, x^1 \in \mathcal{H}$ be arbitrary. Input $k := 1$.

Iterative Step. Given the iterates x^{k-1} and x^k $(k \geq 1)$, calculate x^{k+1} as follows.

Step 1. Compute $w^k = x^k + \theta_k \left(x^k - x^{k-1}\right)$, where θ_k is defined by (θ).

Step 2. Compute $y^k = P_{\mathcal{C}_k^b} w^k$. Let $\ell_k = \|y^k - w^k\|$.

Step 3. Compute $v_j^k = P_{\mathcal{Q}_{j,k}} \mathcal{F}_j w^k$, $j = 1, \ldots, N$, where $\mathcal{Q}_{j,k}$ is defined as

$$\mathcal{Q}_{j,k} = \Big\{ y_j \in \mathcal{H}_j \mid h_j(\mathcal{F}_j x^k) + \langle \eta_j^k, y_j - \mathcal{F}_j x^k \rangle \leq 0 \Big\},\ j = 1, \ldots, N. \qquad (\mathcal{Q}_{j,k})$$

Let $j_k \in \arg\max \left\{ \|v_j^k - \mathcal{F}_j w^k\| \;\middle|\; j = 1, 2, \ldots, N \right\}$, $\ell_{j_k} := \|v_{j_k}^k - \mathcal{F}_{j_k} w^k\|$.

Step 4. Let $L_k = \max\left\{\ell_k, \ell_{j_k}\right\}$.
If $L_k = \ell_k$, then compute $z^k = w^k - \gamma_k (I - P_{\mathcal{C}_k^b}) w^k$, where γ_k is defined by $(\gamma_{\mathcal{C}}\mathrm{T})$.
If $L_k = \ell_{j_k}$, then compute $z^k = w^k - \gamma_k \mathcal{F}_{j_k}^* (I - P_{\mathcal{Q}_{j_k,k}}) \mathcal{F}_{j_k} w^k$, where γ_k is defined by

$$\gamma_k = \rho_k \frac{\ell_{j_k}^2}{\|\mathcal{F}_{j_k}^* (I - P_{\mathcal{Q}_{j_k,k}}) \mathcal{F}_{j_k} w^k\|^2 + e_k}. \qquad (\gamma_{\mathcal{Q}2}\mathrm{T})$$

Step 5. Compute $x^{k+1} = \alpha_k f(x^k) + (1 - \alpha_k) P_{\mathcal{C}_k^b} z^k$. Set $k := k+1$ and go to Step 1.

If g is convex function and h_j is strongly convex function, then we use the following algorithm to solve the SFPMOS.

Algorithm 3.3.

Initial Step. Select a sequence $\{\alpha_k\} \subset (0,1)$ satisfies the condition (α), a sequence $\{\eta_k\}$ such that $\lim_{k\to\infty} \frac{\eta_k}{\alpha_k} = 0$, $\{\rho_k\} \subset (0,2)$, $\{e_k\} \subset (0,c]$, where $c > 0$ is a constant, and $\theta \in (0,1)$. Let $x^0, x^1 \in \mathcal{H}$ be arbitrary. Input $k := 1$.

Iterative Step. Given the iterates x^{k-1} and x^k $(k \geq 1)$, calculate x^{k+1} as follows.

Step 1. Compute $w^k = x^k + \theta_k \left(x^k - x^{k-1}\right)$, where θ_k is defimed by (θ).

Step 2. Compute $y^k = P_{\mathcal{C}_k} w^k$, where $\mathcal{C}_k$ is defined by $(\mathcal{C}_k)$. Let $\ell_k = \|y^k - w^k\|$.

Step 3. Compute $v_j^k = P_{\mathcal{Q}_{j,k}^b} \mathcal{F}_j w^k$, $j = 1, \ldots, N$.
Let $j_k \in \arg\max \left\{ \|v_j^k - \mathcal{F}_j w^k\| \;\middle|\; j = 1, 2, \ldots, N \right\}$, $\ell_{j_k} := \|v_{j_k}^k - \mathcal{F}_{j_k} w^k\|$.

Step 4. Let $L_k = \max\left\{\ell_k, \ell_{j_k}\right\}$.

If $L_k = \ell_k$, then compute $z^k = w^k - \gamma_k(I - P_{\mathcal{C}_k})w^k$, where γ_k is defined by

$$\gamma_k = \rho_k \frac{\ell_k^2}{\|(I - P_{\mathcal{C}_k})w^k\|^2 + e_k}. \tag{$\gamma_{\mathcal{C}2}$T}$$

If $L_k = \ell_{j_k}$, then compute $z^k = w^k - \gamma_k \mathcal{F}_{j_k}^*(I - P_{\mathcal{Q}^b_{j_k,k}})\mathcal{F}_{j_k} w^k$, where γ_k is defined by ($\gamma_{\mathcal{Q}}$T).

Step 5. Compute $x^{k+1} = \alpha_k f(x^k) + (1 - \alpha_k)P_{\mathcal{C}_k} z^k$. Set $k := k + 1$ and go to Step 1.

3.2. Convergence analysis

In this section, we will establish the convergence of the proposed algorithms. We only give the convergence proof of Algorithm 3.1, since the convergence of Algorithms 3.2 and 3.3 can be obtained by the same method.

Theorem 3.1. *Assume that the conditions* (A1), (A2), *and* (A3) *are satisfied,* $\inf_k \rho_k(2 - \rho_k) > 0$, *and* $f : \mathcal{H} \to \mathcal{H}$ *is a contraction mapping with the contraction coefficient* $\tau \in [0, 1)$. *Then the sequence* $\{x^k\}$ *generated by Algorithm 3.1 converges strongly to a point* $u^* \in \Omega$, *which is the unique solution to the* VIP (1.3).

Proof. Since $P_\Omega f$ is a contraction mapping, Banach's fixed point theorem guarantees that $P_\Omega f$ has a unique fixed point u^*, which is the unique solution to the variational inequality problem (1.3). In particular $u^* \in \Omega$, i.e., $u^* \in \mathcal{C}$ and $\mathcal{F}_j u^* \in \mathcal{Q}_j$ for all $j = 1, \ldots, N$. We divide the proof into two claims.

Claim 1. *The sequences* $\{x^k\}$ *is bounded.*

Let us consider two cases.

Case 1 ($L_k = \ell_k$)**.** Since $u^* \in \mathcal{C}$ and $\mathcal{C} \subseteq \mathcal{C}_k^b$, then $u^* = P_{\mathcal{C}_k^b} u^*$. It follows from the 1-inverse strong monotonicity of $I - P_{\mathcal{C}_k^b}$ that

$$\langle (I - P_{\mathcal{C}_k^b})w^k, w^k - u^* \rangle = \langle (I - P_{\mathcal{C}_k^b})w^k - (I - P_{\mathcal{C}_k^b})u^*, w^k - u^* \rangle \geq \|(I - P_{\mathcal{C}_k^b})w^k\|^2.$$

From this inequality and Step 4 in Algorithm 3.1, we have

$$\begin{aligned}
\|z^k - u^*\|^2 &= \|w^k - u^*\|^2 + \gamma_k^2 \|(I - P_{\mathcal{C}_k^b})w^k\|^2 - 2\gamma_k \langle (I - P_{\mathcal{C}_k^b})w^k, w^k - u^* \rangle \\
&\leq \|w^k - u^*\|^2 + \gamma_k^2 \|(I - P_{\mathcal{C}_k^b})w^k\|^2 - 2\gamma_k \|(I - P_{\mathcal{C}_k^b})w^k\|^2 \\
&\leq \|w^k - u^*\|^2 - \rho_k(2 - \rho_k) \frac{\ell_k^4}{\|(I - P_{\mathcal{C}_k^b})w^k\|^2 + e_k}.
\end{aligned} \tag{3.3}$$

Case 2 ($L_k = \ell_{j_k}$)**.** From $\mathcal{F}_j u^* \in \mathcal{Q}_j$ and $\mathcal{Q}_j \subseteq \mathcal{Q}^b_{j,k}$ for all $j = 1, \ldots, N$, we get $\mathcal{Q}_{j_k} \subseteq \mathcal{Q}^b_{j_k,k}$ and $\mathcal{F}_{j_k} u^* = P_{\mathcal{Q}^b_{j_k,k}} \mathcal{F}_{j_k} u^*$. It follows from the 1-inverse strong monotonicity of $I - P_{\mathcal{Q}^b_{j_k,k}}$ that

$$\begin{aligned}
&\langle (I - P_{\mathcal{Q}^b_{j_k,k}})\mathcal{F}_{j_k} w^k, \mathcal{F}_{j_k} w^k - \mathcal{F}_{j_k} u^* \rangle \\
=&\left\langle (I - P_{\mathcal{Q}^b_{j_k,k}})\mathcal{F}_{j_k} w^k - (I - P_{\mathcal{Q}^b_{j_k,k}})\mathcal{F}_{j_k} u^*, \mathcal{F}_{j_k} w^k - \mathcal{F}_{j_k} u^* \right\rangle
\end{aligned}$$

$$\geq \|(I - P_{\mathcal{Q}^b_{j_k,k}})\mathcal{F}_{j_k} w^k\|^2.$$

From this inequality, Step 4 in Algorithm 3.1, and the property of the adjoint operator $\mathcal{F}^*_{j_k}$, we get

$$\begin{aligned}
\|z^k - u^*\|^2 = {} & \|w^k - u^*\|^2 + \gamma_k^2 \|\mathcal{F}^*_{j_k}(I - P_{\mathcal{Q}^b_{j_k,k}})\mathcal{F}_{j_k} w^k\|^2 \\
& - 2\gamma_k \big\langle (I - P_{\mathcal{Q}^b_{j_k,k}})\mathcal{F}_{j_k} w^k, \mathcal{F}_{j_k} w^k - \mathcal{F}_{j_k} u^* \big\rangle \\
\leq {} & \|w^k - u^*\|^2 + \gamma_k^2 \|\mathcal{F}^*_{j_k}(I - P_{\mathcal{Q}^b_{j_k,k}})\mathcal{F}_{j_k} w^k\|^2 \qquad (3.4) \\
& - 2\gamma_k \|(I - P_{\mathcal{Q}^b_{j_k,k}})\mathcal{F}_{j_k} w^k\|^2 \\
\leq {} & \|w^k - u^*\|^2 - \rho_k(2 - \rho_k) \frac{\ell^4_{j_k}}{\|\mathcal{F}^*_{j_k}(I - P_{\mathcal{Q}^b_{j_k,k}})\mathcal{F}_{j_k} w^k\|^2 + e_k}. \qquad (3.5)
\end{aligned}$$

It follows from $\rho_k \in (0, 2)$, (3.3) and (3.4) that

$$\|z^k - u^*\|^2 \leq \|w^k - u^*\|^2. \qquad (3.6)$$

Now, from Step 1 in Algorithm 3.1, we get

$$\|w^k - u^*\| \leq \|x^k - u^*\| + \theta_k \|x^k - x^{k-1}\| \leq \|x^k - u^*\| + \alpha_k M_1, \qquad (3.7)$$

where M_1 is a positive number such that $\frac{\theta_k}{\alpha_k}\|x^k - x^{k-1}\| \leq M_1$ for all $k \geq 1$.

Since f is a contraction with a contraction coefficient $\tau \in [0, 1)$, $P_{\mathcal{C}^b_k}$ is a nonexpansive mapping, it follows from Step 5 in Algorithm 3.1, the convexity of the $\|\cdot\|$, $\{\alpha_k\} \subset (0, 1)$, (3.6), and (3.7) that

$$\begin{aligned}
\|x^{k+1} - u^*\| = {} & \big\|\alpha_k (f(x^k) - u^*) + (1 - \alpha_k)\big(P_{\mathcal{C}^b_k} z^k - P_{\mathcal{C}^b_k} u^*\big)\big\| \\
\leq {} & \alpha_k \|f(x^k) - f(u^*) + f(u^*) - u^*\| + (1 - \alpha_k)\|z^k - u^*\| \\
\leq {} & \alpha_k \tau \|x^k - u^*\| + \alpha_k \|f(u^*) - u^*\| + (1 - \alpha_k)\big(\|x^k - u^*\| + \alpha_k M_1\big) \\
\leq {} & \big[1 - \alpha_k(1 - \tau)\big]\|x^k - u^*\| + \alpha_k(1 - \tau)\frac{\|f(u^*) - u^*\| + (1 - \alpha_k)M_1}{1 - \tau} \\
\leq {} & \max\Big\{\|x^k - u^*\|, \frac{\|f(u^*) - u^*\| + (1 - \alpha_k)M_1}{1 - \tau}\Big\} \\
& \vdots \\
\leq {} & \max\Big\{\|x^1 - u^*\|, \frac{\|f(u^*) - u^*\| + (1 - \alpha_1)M_1}{1 - \tau}\Big\}, \ k \geq 1. \qquad (3.8)
\end{aligned}$$

Hence, the sequence $\{x^k\}$ is bounded and so are the sequences $\{w^k\}$, $\{\mathcal{F}_j w^k\}$, and $\{f(x^k)\}$ thanks to (3.7), the boundedness of $\mathcal{F}_j$, and the contraction property of f.

Claim 2. *The sequence $\{x^k\}$ converges strongly to u^*.*

It follows from Step 1 in Algorithm 3.1 that

$$\begin{aligned}\|w^k - u^*\|^2 &\le \|x^k - u^*\|^2 + \theta_k^2\|x^k - x^{k-1}\|^2 + 2\theta_k\|x^k - u^*\|\|x^k - x^{k-1}\| \\ &\le \|x^k - u^*\|^2 + \eta_k M_2, \qquad (3.9)\end{aligned}$$

where $M_2 = \sup_{k\ge1}\{2\|x^k - u^*\| + \alpha_k M_1\}$. From Step 5 in Algorithm 3.1 and the nonexpansiveness of $P_{\mathcal{C}_k^b}$, we obtain

$$\begin{aligned}&\|x^{k+1} - u^*\|^2 \\ =&\langle \alpha_k(f(x^k) - u^*) + (1-\alpha_k)(P_{\mathcal{C}_k^b} z^k - P_{\mathcal{C}_k^b} u^*), x^{k+1} - u^*\rangle \\ =&\alpha_k\langle f(x^k) - f(u^*), x^{k+1} - u^*\rangle + \alpha_k\langle f(u^*) - u^*, x^{k+1} - u^*\rangle \\ &+ (1-\alpha_k)\langle P_{\mathcal{C}_k^b} z^k - P_{\mathcal{C}_k^b} u^*, x^{k+1} - u^*\rangle \\ \le&\frac{\alpha_k}{2}\big(\|f(x^k) - f(u^*)\|^2 + \|x^{k+1} - u^*\|^2\big) + \alpha_k\langle f(u^*) - u^*, x^{k+1} - u^*\rangle \\ &+ \frac{1-\alpha_k}{2}\big(\|z^k - u^*\|^2 + \|x^{k+1} - u^*\|^2\big).\end{aligned}$$

Since f is a contraction with a contraction coefficient $\tau \in [0,1)$, it follows from this inequality, (3.6) and (3.9) that

$$\begin{aligned}\|x^{k+1} - u^*\|^2 &\le \alpha_k\tau^2\|x^k - u^*\|^2 + 2\alpha_k\langle f(u^*) - u^*, x^{k+1} - u^*\rangle \\ &\quad + (1-\alpha_k)\|x^k - u^*\|^2 + (1-\alpha_k)\eta_k M_2 \\ &= \big[1 - \alpha_k(1-\tau^2)\big]\|x^k - u^*\|^2 + 2\alpha_k\langle f(u^*) - u^*, x^{k+1} - u^*\rangle \\ &\quad + (1-\alpha_k)\eta_k M_2 \\ &= (1-a_k)\|x^k - u^*\|^2 + a_k b_k, \qquad (3.10)\end{aligned}$$

where,

$$a_k = \alpha_k(1-\tau^2) \text{ and } b_k = \frac{1}{1-\tau^2}\Big[(1-\alpha_k)\frac{\eta_k}{\alpha_k}M_2 + 2\langle f(u^*) - u^*, x^{k+1} - u^*\rangle\Big].$$

Consider two possible cases.

Case 1 ($L_k = \ell_k$)**.** Since $P_{\mathcal{C}_k^b}$ is firmly nonexpansive, from Step 5 in Algorithm 3.1, the convexity of the $\|\cdot\|^2$, $\{\alpha_k\} \subset (0,1)$, (3.3), and (3.9), we have

$$\begin{aligned}\|x^{k+1} - u^*\|^2 &\le \alpha_k\|f(x^k) - u^*\|^2 + (1-\alpha_k)\|P_{\mathcal{C}_k^b} z^k - P_{\mathcal{C}_k^b} u^*\|^2 \\ &\le \alpha_k\|f(x^k) - u^*\|^2 + (1-\alpha_k)\big[\|z^k - u^*\|^2 - \|(I - P_{\mathcal{C}_k^b})z^k\|^2\big] \\ &\le \|x^k - u^*\|^2 - \rho_k(2-\rho_k)\frac{\ell_k^4}{\|(I - P_{\mathcal{C}_k^b})w^k\|^2 + e_k} - \|(I - P_{\mathcal{C}_k^b})z^k\|^2 \\ &\quad + \alpha_k\|f(x^k) - u^*\|^2 + \eta_k M_2\end{aligned}$$

$$= \|x^k - u^*\|^2 - c_k + d_k, \tag{3.11}$$

where

$$c_k = \rho_k(2-\rho_k)\frac{\ell_k^4}{\|(I-P_{\mathcal{C}_k^b})w^k\|^2 + e_k} + \|(I-P_{\mathcal{C}_k^b})z^k\|^2$$

and

$$d_k = \alpha_k\|f(x^k) - u^*\|^2 + \eta_k M_2.$$

Putting $s_k = \|x^k - u^*\|^2$, then (3.10) and (3.11) can be rewritten as

$$s_{k+1} \le (1-a_k)s_k + a_k b_k \text{ and } s_{k+1} \le s_k - c_k + d_k. \tag{3.12}$$

From (α) and $\tau \in [0,1)$, it is easy to know that $\{a_k\} \subset (0,1)$ and $\sum_{k=0}^{\infty} a_k = \infty$. From the boundedness of sequence $\{f(x^k)\}$, $\eta_k/\alpha_k \to 0$ as $k \to \infty$, and $\alpha_k \to 0$, we get $\lim_{k\to\infty} d_k = 0$. So, from Lemma 2.3, $\lim_{k\to\infty} s_k = 0$ if we can show that $\limsup_{l\to\infty} b_{k_l} \le 0$ whenever $\lim_{l\to\infty} c_{k_l} = 0$ for any $\{k_l\} \subset \{k\}$. Indeed, for any $\{k_l\} \subset \{k\}$, by the boundedness of $\{w^{k_l}\}$ and $e_k \in (0,c]$, we have the sequence $\{\|(I-P_{\mathcal{C}_{k_l}^b})w^{k_l}\|^2 + e_{k_l}\}$ is bounded. It then follows from $\lim_{l\to\infty} c_{k_l} = 0$ and $\rho_k \in (0,2)$ that

$$\lim_{l\to\infty} \|(I-P_{\mathcal{C}_{k_l}^b})w^{k_l}\| = 0, \tag{3.13}$$

$$\lim_{l\to\infty} \|(I-P_{\mathcal{C}_{k_l}^b})z^{k_l}\| = 0. \tag{3.14}$$

Since the sequence $\{x^{k_l}\}$ is bounded, $\omega_w(x^{k_l}) \neq \emptyset$. Taking $\widehat{u} \in \omega_w(x^{k_l})$, there exist subsequences $\{x^{k_{l_m}}\}$ of $\{x^{k_l}\}$ such that it is weakly convergent to $\widehat{u}$ as $m \to \infty$. Since each $\mathcal{F}_j$ is a bounded linear operator, it follows that $\mathcal{F}_j x^{k_{l_m}} \rightharpoonup \mathcal{F}_j\widehat{u}$ for all $j = 1,\ldots,N$. The main purpose of the remaining proof is to show that $\widehat{u}$ is a solution of the SFPMOS (1.1).

First we show $\widehat{u} \in \mathcal{C}$. Indeed, from Step 1 in Algorithm 3.1, $\alpha_k \to 0$, and $\eta_k/\alpha_k \to 0$ as $k \to \infty$, we get

$$\|w^k - x^k\| = \theta_k\|x^k - x^{k-1}\| \le \eta_k \to 0 \text{ as } k \to \infty. \tag{3.15}$$

By the definition of $\mathcal{C}_k^b$ and the fact that $P_{\mathcal{C}_{k_{l_m}}^b} w^{k_{l_m}} \in \mathcal{C}_{k_{l_m}}^b$, we obtain

$$\begin{aligned} g(x^{k_{l_m}}) &\le \langle \xi^{k_{l_m}}, x^{k_{l_m}} - P_{\mathcal{C}_{k_{l_m}}^b} w^{k_{l_m}}\rangle - \frac{\beta}{2}\|P_{\mathcal{C}_{k_{l_m}}^b} w^{k_{l_m}} - x^{k_{l_m}}\|^2 \\ &\le \|\xi^{k_{l_m}}\|\big(\|x^{k_{l_m}} - w^{k_{l_m}}\| + \|(I-P_{\mathcal{C}_{k_{l_m}}^b})w^{k_{l_m}}\|\big). \end{aligned}$$

This together with g is ω-lsc, ∂g is bounded on bounded sets, (3.13), and (3.15), implies that

$$g(\widehat{u}) \le \liminf_{m\to\infty} g(x^{k_{l_m}}) \le \lim_{m\to\infty} \|\xi^{k_{l_m}}\|\big(\|x^{k_{l_m}} - w^{k_{l_m}}\| + \|(I-P_{\mathcal{C}_{k_{l_m}}^b})w^{k_{l_m}}\|\big) = 0,$$

which means that $\widehat{u} \in \mathcal{C}$.

We next turn to prove $\mathcal{F}_j\widehat{u} \in \mathcal{Q}_j$ for all $j = 1, \ldots, N$. Indeed, since $L_k = \ell_k$, (3.13), and Steps 2–4 in Algorithm 3.1, we get

$$\lim_{l\to\infty} \|(I - P_{\mathcal{Q}^b_{j_k,k_l}})\mathcal{F}_{j_k} w^{k_l}\| = 0$$

and then

$$\lim_{m\to\infty} \|(I - P_{\mathcal{Q}^b_{j,k_{l_m}}})\mathcal{F}_j w^{k_{l_m}}\| = 0 \ \text{ for all } j = 1, \ldots, N. \tag{3.16}$$

From the definition of $\mathcal{Q}^b_{j,k_{l_m}}$ and the fact that $P_{\mathcal{Q}^b_{j,k_{l_m}}} \mathcal{F}_j w^{k_{l_m}} \in \mathcal{Q}^b_{j,k_{l_m}}$, we get

$$\begin{aligned}
&h_j(\mathcal{F}_j x^{k_{l_m}})\\
\le&\langle \eta_j^{k_{l_m}}, \mathcal{F}_j x^{k_{l_m}} - P_{\mathcal{Q}^b_{j,k_{l_m}}} \mathcal{F}_j w^{k_{l_m}}\rangle - \frac{\beta_j}{2}\|P_{\mathcal{Q}^b_{j,k_{l_m}}} \mathcal{F}_j w^{k_{l_m}} - \mathcal{F}_j x^{k_{l_m}}\|^2\\
\le&\|\eta_j^{k_{l_m}}\|\big(\|\mathcal{F}_j\|\|x^{k_{l_m}} - w^{k_{l_m}}\| + \|(I - P_{\mathcal{Q}^b_{j,k_{l_m}}})\mathcal{F}_j w^{k_{l_m}}\|\big) \ \forall j = 1, \ldots, N.
\end{aligned}$$

Combining this inequality with $\mathcal{F}_j$ is a bounded linear operator, ∂h_j is bounded on bounded sets, h_j is ω-lsc, (3.15), and (3.16), we obtain

$$\begin{aligned}
h_j(\mathcal{F}_j\widehat{u}) &\le \liminf_{m\to\infty} h_j(\mathcal{F}_j x^{k_{l_m}})\\
&\le \lim_{m\to\infty} \|\eta_j^{k_{l_m}}\|\big(\|\mathcal{F}_j\|\|x^{k_{l_m}} - w^{k_{l_m}}\| + \|(I - P_{\mathcal{Q}^b_{j,k_{l_m}}})\mathcal{F}_j w^{k_{l_m}}\|\big)\\
&= 0,
\end{aligned}$$

for all $j = 1, \ldots, N$, which means that $\mathcal{F}_j\widehat{u} \in \mathcal{Q}_j$ for all $j = 1, \ldots, N$. Hence, we conclude that $\widehat{u} \in \Omega$.

Now, we prove that $\limsup_{l\to\infty} b_{k_l} \le 0$. Indeed, let $\{x^{k_{l'}}\}$ be a subsequence of $\{x^{k_l}\}$ such that

$$\limsup_{l\to\infty}\langle f(u^*) - u^*, x^{k_l} - u^*\rangle = \lim_{l'\to\infty}\langle f(u^*) - u^*, x^{k_{l'}} - u^*\rangle. \tag{3.17}$$

Since $\{x^{k_{l'}}\}$ is bounded, there exists a subsequence of $\{x^{k_{l'}}\}$ which converges weakly to $\widetilde{u}$. We may assume, without any loss of generality, that $x^{k_{l'}} \rightharpoonup \widetilde{u}$. Similar to the proof of $\widehat{u} \in \Omega$, we have $\widetilde{u} \in \Omega$. Thus, by (1.3) and (3.17), we obtain

$$\limsup_{l\to\infty}\langle f(u^*) - u^*, x^{k_l} - u^*\rangle = \langle (f - I)u^*, \widetilde{u} - u^*\rangle \le 0. \tag{3.18}$$

In order to show that $\limsup_{l\to\infty} b_{k_l} \le 0$, we need to prove that

$$\lim_{l\to\infty} \|x^{k_l+1} - x^{k_l}\| = 0.$$

Indeed, from the boundedness of the sequences $\{w^k\}$, $\{f(x^k)\}$ and $\{\|(I-P_{\mathcal{C}^b_k} w^k)\|^2 + e_k\}$, $\alpha_k \to 0$, $\rho_k \in (0, 2)$, (3.13), and (3.14), it follows from Steps 4 and 5 in Algorithm 3.1 that

$$\|x^{k+1} - z^k\| = \|\alpha_k(f(x^k) - w^k + w^k - z^k) + (1 - \alpha_k)(P_{\mathcal{C}^b_k} z^k - z^k)\|$$

$$\begin{aligned}&\le \alpha_k\big(\|f(x^k)\| + \|w^k\| + \rho_k\|(I - P_{\mathcal{C}_k^b})w^k\|\big)\\&\quad + (1-\alpha_k)\|(I - P_{\mathcal{C}_k^b})z^k\| \to 0 \text{ as } k \to \infty. \qquad (3.19)\end{aligned}$$

From Step 4 in Algorithm 3.1, $\rho_k \in (0,2)$, and (3.13), we also have

$$\|z^k - w^k\| \le \rho_k\|(I - P_{\mathcal{C}_k^b})w^k\| \to 0 \text{ as } k \to \infty. \qquad (3.20)$$

It follows from (3.15), (3.19), and (3.20) that

$$\|x^{k_l+1} - x^{k_l}\| \le \|x^{k_l+1} - z^{k_l}\| + \|z^{k_l} - w^{k_l}\| + \|w^{k_l} - x^{k_l}\| \to 0 \text{ as } l \to \infty. \qquad (3.21)$$

From (3.18) and (3.21), we get

$$\begin{aligned}\limsup_{l\to\infty} b_{k_l} &= \frac{2}{1-\tau^2}\limsup_{l\to\infty}\langle f(u^*) - u^*, x^{k_l+1} - u^*\rangle\\&= \frac{2}{1-\tau^2}\limsup_{l\to\infty}\big[\langle f(u^*) - u^*, x^{k_l+1} - x^{k_l}\rangle + \langle f(u^*) - u^*, x^{k_l} - u^*\rangle\big]\\&= \limsup_{l\to\infty}\langle f(u^*) - u^*, x^{k_l} - u^*\rangle\\&\le 0.\end{aligned}$$

Case 2 ($L_k = \ell_{j_k}$)**.** Similarly as (3.11), since $P_{\mathcal{C}_k^b}$ is firmly nonexpansive, from Step 5 in Algorithm 3.1, the convexity of $\|\cdot\|^2$, and (3.4), we have

$$\begin{aligned}\|x^{k+1} - u^*\|^2 &\le \|x^k - u^*\|^2 - \rho_k(2-\rho_k)\frac{\ell_{j_k}^4}{\|\mathcal{F}_{j_k}^*(I - P_{\mathcal{Q}_{j_k,k}^b})\mathcal{F}_{j_k}w^k\|^2 + e_k}\\&\quad - \|(I - P_{\mathcal{C}_k^b})z^k\|^2 + \alpha_k\|f(x^k) - u^*\|^2 + \eta_k M_2\\&= \|x^k - u^*\|^2 - \widehat{c}_k + d_k, \qquad (3.22)\end{aligned}$$

where

$$\widehat{c}_k = \rho_k(2-\rho_k)\frac{\ell_{j_k}^4}{\|\mathcal{F}_{j_k}^*(I - P_{\mathcal{Q}_{j_k,k}^b})\mathcal{F}_{j_k}w^k\|^2 + e_k} + \|(I - P_{\mathcal{C}_k^b})z^k\|^2,$$

and then (3.10) and (3.22) can also be rewritten as

$$s_{k+1} \le (1-a_k)s_k + a_k b_k \text{ and } s_{k+1} \le s_k - \widehat{c}_k + b_k. \qquad (3.23)$$

Arguing similarly as in the first case, for any $\{k_l\} \subset \{k\}$ and $\lim_{l\to\infty}\widehat{c}_{k_l} = 0$, we have

$$\begin{aligned}&\lim_{l\to\infty}\|(I - P_{\mathcal{Q}_{j,k_l}^b})\mathcal{F}_j w^{k_l}\| = 0 \text{ for all } j = 1,\dots,N,\\&\lim_{l\to\infty}\|(I - P_{\mathcal{C}_{k_l}^b})z^{k_l}\| = 0,\end{aligned}$$

and $\limsup_{l\to\infty} b_{k_l} \le 0$.

Therefore, it follows from (3.12), (3.23), and Lemma 2.3 that $\lim_{k\to\infty} s_k = 0$, that is

$$\lim_{k\to\infty} \|x^k - u^*\| = 0.$$

Thus, the sequence $\{x^k\}$ generated by Algorithm 3.1 converges strongly to $u^* \in \Omega$ which is the unique solution of the VIP (1.3).

□

Theorem 3.2. *Assume that all the conditions in Theorem 3.1 are satisfied. Then the sequence $\{x^k\}$ generated by Algorithm 3.2 converges strongly to a point $u^* \in \Omega$, which is the unique solution to the variational inequality* (1.3)*.*

Theorem 3.3. *Assume that all the conditions in Theorem 3.1 are satisfied. Then the sequence $\{x^k\}$ generated by Algorithm 3.3 converges strongly to a point $u^* \in \Omega$, which is the unique solution to the variational inequality* (1.3)*.*

4. Corollaries

If $N = 1$, then the SFPMOS reduces to the SFP. In what follows, from Algorithms 3.1, 3.2 and 3.3, we get some methods to SFP, where $\mathcal{C}_k^b$ and $\mathcal{Q}_k^b$ are defined by $(\mathcal{C}_k^b)$ and $(\mathcal{Q}_k^b)$, respectively.

Corollary 4.1. *Assume that all the conditions in Theorem 3.1 are satisfied when $N = 1$. Then the sequence $\{x^k\}$ generated by*

$$\begin{cases} w^k = x^k + \theta_k\left(x^k - x^{k-1}\right), \\ y^k = P_{\mathcal{C}_k^b} w^k,\ \ell_k = \|y^k - w^k\|, \\ v^k = P_{\mathcal{Q}_k^b}\mathcal{F}w^k,\ \widetilde{\ell}_k = \|v^k - \mathcal{F}w^k\|, \\ \text{if } \ell_k \geq \widetilde{\ell}_k, \text{ then } z^k = w^k - \rho_k \frac{\ell_k^2}{\|(I-P_{\mathcal{C}_k^b})w^k\|^2 + e_k}(I - P_{\mathcal{C}_k^b})w^k, \\ \text{if } \widetilde{\ell}_k > \ell_k, \text{ then } z^k = w^k - \rho_k \frac{\widetilde{\ell}_k^2}{\|\mathcal{F}^*(I-P_{\mathcal{Q}_k^b})\mathcal{F}w^k\|^2 + e_k}\mathcal{F}^*(I - P_{\mathcal{Q}_k^b})\mathcal{F}w^k, \\ x^{k+1} = \alpha_k f(x^k) + (1-\alpha_k)P_{\mathcal{C}_k^b} z^k, \end{cases} \tag{4.1}$$

strongly converges to $u^* \in \Omega_{\mathrm{SFP}}$, the unique solution to the VIP (1.3) *with Ω replaced by Ω_{SFP}, provided that the solution set $\Omega_{\mathrm{SFP}} = \{x \in \mathcal{C} \mid \mathcal{F}x \in \mathcal{Q}\}$ of the* SFP *is nonempty.*

When $\mathcal{Q}_k^b$ replaced by $\mathcal{Q}_k$ in $(\mathcal{Q}_k)$, we obtain the following result.

Corollary 4.2. *Assume that all the conditions in Theorem 3.1 are satisfied when $N = 1$. Then the sequence $\{x^k\}$ generated by*

$$\begin{cases} w^k = x^k + \theta_k\left(x^k - x^{k-1}\right), \\ y^k = P_{\mathcal{C}_k^b} w^k,\ \ell_k = \|y^k - w^k\|, \\ v^k = P_{\mathcal{Q}_k}\mathcal{F}w^k,\ \widetilde{\ell}_k = \|v^k - \mathcal{F}w^k\|, \\ \text{if } \ell_k \geq \widetilde{\ell}_k, \text{ then } z^k = w^k - \rho_k \frac{\ell_k^2}{\|(I-P_{\mathcal{C}_k^b})w^k\|^2 + e_k}(I - P_{\mathcal{C}_k^b})w^k, \\ \text{if } \widetilde{\ell}_k > \ell_k, \text{ then } z^k = w^k - \rho_k \frac{\widetilde{\ell}_k^2}{\|\mathcal{F}^*(I-P_{\mathcal{Q}_k})\mathcal{F}w^k\|^2 + e_k}\mathcal{F}^*(I - P_{\mathcal{Q}_k})\mathcal{F}w^k, \\ x^{k+1} = \alpha_k f(x^k) + (1-\alpha_k)P_{\mathcal{C}_k^b} z^k, \end{cases} \tag{4.2}$$

strongly converges to $u^* \in \Omega_{\mathrm{SFP}}$*, the unique solution to the* VIP (1.3) *with* Ω *replaced by* Ω_{SFP}*, provided that the solution set* Ω_{SFP} *of the* SFP *is nonempty.*

When $\mathcal{C}_k^b$ replaced by $\mathcal{C}_k$ in $(\mathcal{C}_k)$, we obtain the following result.

Corollary 4.3. *Assume that all the conditions in Theorem 3.1 are satisfied when* $N = 1$*. Then the sequence* $\{x^k\}$ *generated by*

$$\begin{cases} w^k = x^k + \theta_k \left(x^k - x^{k-1}\right), \\ y^k = P_{\mathcal{C}_k} w^k, \ \ell_k = \|y^k - w^k\|, \\ v^k = P_{\mathcal{Q}_k^b} \mathcal{F} w^k, \ \widetilde{\ell}_k = \|v^k - \mathcal{F} w^k\|, \\ \textit{if } \ell_k \geq \widetilde{\ell}_k, \textit{ then } z^k = w^k - \rho_k \frac{\ell_k^2}{\|(I-P_{\mathcal{C}_k})w^k\|^2 + e_k}(I - P_{\mathcal{C}_k}) w^k, \\ \textit{if } \widetilde{\ell}_k > \ell_k, \textit{ then } z^k = w^k - \rho_k \frac{\widetilde{\ell}_k^2}{\|\mathcal{F}^*(I-P_{\mathcal{Q}_k^b})\mathcal{F} w^k\|^2 + e_k} \mathcal{F}^*(I - P_{\mathcal{Q}_k^b}) \mathcal{F} w^k, \\ x^{k+1} = \alpha_k f(x^k) + (1-\alpha_k) P_{\mathcal{C}_k} z^k, \end{cases} \tag{4.3}$$

strongly converges to $u^* \in \Omega_{\mathrm{SFP}}$*, the unique solution to the* VIP (1.3) *with* Ω *replaced by* Ω_{SFP}*, provided that the solution set* Ω_{SFP} *of the* SFP *is nonempty.*

5. Numerical experiments

This section presents three numerical experiments, where all the iterative schemes are implemented in Python 3.7 running on a laptop with Intel(R) Core(TM) i5-5200U CPU @ 2.20GHz, 12 GB RAM. In the following tables, we denote by Iter. (k) and CPU (s) the number of iterations and the computational time needed to reach the stopping condition of each algorithm, respectively.

Example 5.1. Firstly, we study the following instance of the split feasibility problem with multiple output sets. Let $\mathcal{H} = \mathbb{R}^2$, $\mathcal{H}_j = \mathbb{R}^{j+2}$, $j = 1, 2, 3$. The subsets $\mathcal{C}$ and $\mathcal{Q}_j$ are determined as follows

$$\begin{aligned} \mathcal{C} &= \left\{x \in \mathbb{R}^2 \mid \|x - (1,1)^\top\|^2 \leq 1\right\}, \\ \mathcal{Q}_1 &= \left\{y \in \mathbb{R}^3 \mid \|y - (2, 2, \sqrt{2})^\top\|^2 \leq 2\right\}, \\ \mathcal{Q}_2 &= \left\{y \in \mathbb{R}^4 \mid \|y - (\sqrt{3}, 2, 2, 2)^\top\|^2 \leq 3\right\}, \\ \mathcal{Q}_3 &= \left\{y \in \mathbb{R}^5 \mid \|y - (2, 2, 4, 2, 2)^\top\|^2 \leq 8\right\}. \end{aligned}$$

The bounded linear operators $\mathcal{F}_j : \mathbb{R}^2 \to \mathbb{R}^{j+2}$, $j = 1, 2, 3$, are determined as

$$\begin{aligned} &\mathcal{F}_1 : \mathbb{R}^2 \to \mathbb{R}^3, \quad \mathcal{F}_1(x) = (x_1, x_1, \sqrt{2} x_2)^\top, \\ &\mathcal{F}_2 : \mathbb{R}^2 \to \mathbb{R}^4, \quad \mathcal{F}_2(x) = (\sqrt{3} x_1, x_2, x_2, x_2)^\top, \\ &\mathcal{F}_3 : \mathbb{R}^2 \to \mathbb{R}^5, \quad \mathcal{F}_3(x) = (x_2, x_2, 2x_1, x_2, x_2)^\top. \end{aligned}$$

Then, the solution set of this (1.1) is given by

$$\Omega = \mathcal{C} \cap \left\{ x \in \mathbb{R}^2 \,\middle|\, \begin{cases} (x_1 - 2)^2 + (x_2 - 1)^2 \leq 1 \\ (x_1 - 1)^2 + (x_2 - 2)^2 \leq 1 \\ (x_1 - 2)^2 + (x_2 - 2)^2 \leq 2 \end{cases} \right\}.$$

Table 1. Numerical results of Algorithm 3.1 in Example 5.1 with different initial guesses

$x^\star$	x_1^k	x_2^k	$\|x^k - u^*\|$	Iter. (k)	CPU (s)
$(-15,-20)^\top$	1.0000069	0.9999008	9.9470756×10^{-5}	45	0.03398
$(0,0)^\top$	1.0000705	1.0000705	9.9766478×10^{-5}	75	0.04497
$(5,5)^\top$	0.9999171	0.9999820	8.4885202×10^{-5}	589	0.32381

Choosing the contraction mapping $f(x) = \tau x$ with $\tau \in [0,1)$, it is not difficult to see that $u^* = (1,1)^\top$ is the unique solution to the problem VIP (1.3).

Now, we consider Algorithm 3.1 with $\alpha_k = 1/(k+1)$, $\eta_k = 1/(k^2+1)$, $\rho_k = 1.99$, $e_k = 10^{-7}$ for all $k \geq 1$, $f(x) = 0.95x$ for all $x \in \mathbb{R}^2$, and $\theta = 0.5$. Let $x^\star$ be a given point in $\mathcal{C}$, and let $x^0 = x^\star/2$, $x^1 = x^\star$. Then, under the stopping criterion $\|x^k - u^*\| \leq 10^{-4}$, we obtain Table 1 of numerical results. It can be seen that the iterative sequences $\{x^k\}$ converge to the unique solution $u^* = (1,1)^\top$ of the VIP (1.3).

In what follows, we demonstrate the efficiency of our algorithm through comparing it with (1.4) denoted by Cuong's Alg. [9], and the two versions of the relevant iterative scheme (1.2): (1.2) with the step size criterion (γR) and (1.2) with the step size criterion (γW), which are denoted by Reich's Alg. [18] and Wang's Alg. [23], respectively. The parameters and mappings are chosen as follows:

- In our algorithm: $\alpha_k = \frac{1}{k+1}$, $\eta_k = \frac{1}{k^2+1}$, $\rho_k = 1.99$, $e_k = 10^{-7}$ for all $k \geq 1$, $f(x) = 0.95x$ for all $x \in \mathbb{R}^2$, $\theta = 0.5$, and $x^0 = \frac{1}{2}x^\star$, $x^1 = x^\star$, where $x^\star \in \mathbb{R}^2$ is a given initial point.
- In Reich's Alg.: $\alpha_k = \frac{1}{k+1}$, $\gamma_k = \frac{1.5}{3\max_{j=1,2,3}\|\mathcal{F}_j\|^2}$ for all $k \geq 1$, $f(x) = 0.95x$ for all $x \in \mathbb{R}^2$, and $x^1 = x^\star$, where $x^\star \in \mathbb{R}^2$ is a given initial point.
- In Wang's Alg.: $\alpha_k = \frac{1}{k+1}$ for all $k \geq 1$, $f(x) = 0.95x$ for all $x \in \mathbb{R}^2$, and $x^1 = x^\star$, where $x^\star \in \mathbb{R}^2$ is a given initial point.
- In Cuong's Alg.: $\alpha_k = \frac{1}{k+1}$ for all $k \geq 1$, $A = I - f$ where $f(x) = 0.95x$ for all $x \in \mathbb{R}^2$, and $x^1 = x^\star$, where $x^\star \in \mathbb{R}^2$ is a given initial point.

Then, using the stopping criterion $\|x^k - u^*\| \leq 10^{-4}$, we obtain Table 2 and Figure 1 of numerical results. From Table 2 and Figure 1, our algorithm presents better performance than the three related ones.

Example 5.2. It is especially significant when $\mathcal{C}$ and $\mathcal{Q}_j$, $j = 1,\dots,N$, are general closed convex sets, where the metric projection onto these sets might not be computed efficiently. The Example 5.2 demonstrates one such case.

Consider the following split feasibility problem with multiple output sets. Let $\mathcal{H} = \mathbb{R}^L$, $\mathcal{H}_j = \mathbb{R}^{L(j+1)}$, $j = 1,\dots,N$. Let

$$\mathcal{C} = \Big\{x \in \mathbb{R}^L \;\Big|\; \sum_{i=1}^{L} 10^{(i-1)/(L-1)} x_i^2 \leq 1\Big\},$$

$$\mathcal{Q}_j = \Big\{y \in \mathbb{R}^{L(j+1)} \;\Big|\; \sum_{i=1}^{L(j+1)} 10^{(i-1)/(L(j+1)-1)} y_i^2 \leq 1\Big\},\ j = 1,\dots,N.$$

Table 2. Numerical results of the algorithm (3.1), Reich's Alg [18], Wang's Alg. [23], and Cuong's Alg. [9] in Example 5.1 with different initial guesses

$x^\star$	Algorithm	Iter. (k)	CPU (s)
$x^\star = (0.5, 0.5)^\top$	Our algorithm	49	0.03298
	Reich's Alg.	2709	0.52870
	Wang's Alg.	723	0.19791
	Cuong's Alg.	1927	0.60965
$x^\star = (1.8, 0.8)^\top$	Our algorithm	482	0.23686
	Reich's Alg.	8103	1.39522
	Wang's Alg.	5267	1.33923
	Cuong's Alg.	5454	1.50514
$x^\star = \left(\frac{2-\sqrt{2}}{2}, \frac{2-\sqrt{2}}{2}\right)^\top$	Our algorithm	15	0.01299
	Reich's Alg.	2709	0.51572
	Wang's Alg.	723	0.18492
	Cuong's Alg.	1927	0.59264

The matrix $\mathcal{F}_j = (a_{pq})_{L(j+1)\times L}$, $a_{pq} \in [0, 5]$ are generated randomly. It is obvious that $\mathcal{C}$ and $\mathcal{Q}_j$ are ellipsoids [10]. Furthermore, $g(x) = \sum_{i=1}^{L} 10^{(i-1)/(L-1)} x_i^2 - 1$ is a strongly convex function with constant 2, and $h_j(y) = \sum_{i=1}^{L} 10^{(i-1)/(L(j+1)-1)} y_i^2 - 1$ is also a strongly convex function with constant 2. So such an SFPMOS can be solved by the proposed algorithms.

Now, we consider the convergence of Algorithm 3.1 with $L = 100$, $N = 10$. The parameters and mappings are chosen as follows: $\alpha_k = \frac{1}{k+1}$, $\eta_k = \frac{1}{k^2+1}$, $e_k = 10^{-5}$ for all $k \geq 1$, $\theta = 0.5$, $f(x) = 0.1x$ for all $x \in \mathbb{R}^2$. Let $x^\star \in \mathbb{R}^{100}$ be a given point, and let $x^0 = \frac{1}{2}x^\star$, $x^1 = x^\star$. Then, using the stopping condition $\|x^k - x^{k-1}\| \leq 10^{-6}$, we obtain Table 3 of numerical results.

Remark 5.1. In this example, the sets $\mathcal{C}$ and $\mathcal{Q}_j$, $j = 1, \ldots, N$ are ellipsoids, where an explicit form for the metric projection onto these sets as in Example 5.1 does not exist anymore. Although there exist some methods for approximating the metric projection onto ellipsoids, implementing algorithms that directly rely on projecting onto the original subsets, like Reich's Alg., Wang's Alg., and Cuong's Alg., remains challenging due to the potential errors associated with projection approximation.

Example 5.3 (see [26])**.** In statistics and, in particular, in the fitting of linear or logistic regression models, the elastic net is a regularized regression method that linearly combines the L_1 and L_2 penalties of the LASSO (least absolute shrinkage and selection operator) and ridge methods [27]. The elastic net method overcomes the limitations of the LASSO method which uses a penalty function based on $\|x\|_1 = \sum_{i=1}^{L} |x_i|$. The estimates from the elastic net method are defined by

$$x^* = \arg\min_x \left\{ \|y - \mathcal{F}x\|^2 + \lambda_1\|x\|_1 + \lambda_2\|x\|^2 \right\}, \tag{5.1}$$

where $\mathcal{F}$ is an $K \times L$ real matrix, $y \in \mathbb{R}^K$ is a vector of observation, and $x \in \mathbb{R}^L$ is a vector of estimating parameters. Let $\lambda = \lambda_2/(\lambda_1 + \lambda_2)$, then solving x^* in Equation

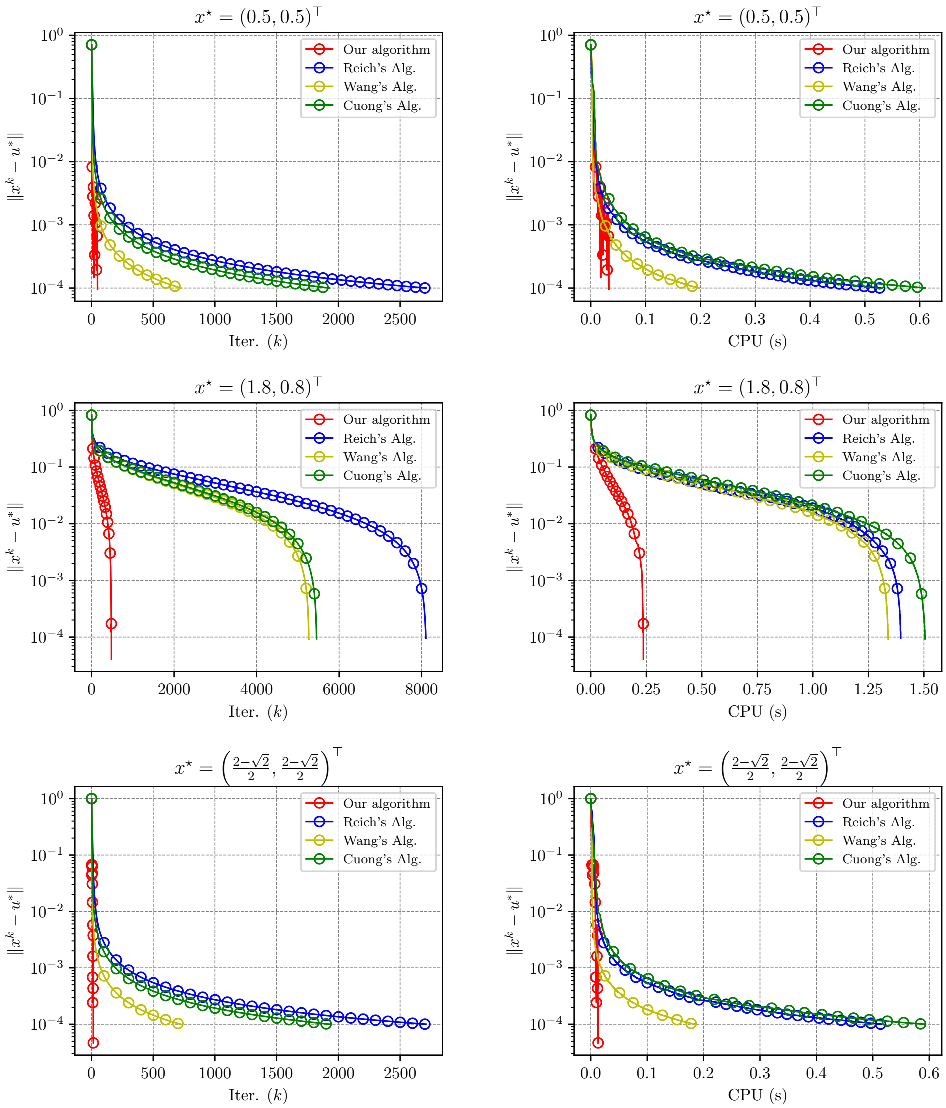

Figure 1. The behaviors of the algorithm (3.1), Reich's Alg [18], Wang's Alg. [23], and Cuong's Alg. [9] in Example 5.1 with different initial guesses

(5.1) is equivalent to the optimization problem

$$x^* = \arg\min_x \left\{\|y - \mathcal{F}x\|^2\right\} \text{ such that } (1-\lambda)\|x\|_1 + \lambda\|x\|^2 \leq t \quad \text{for some } t > 0. \tag{5.2}$$

The function $(1-\lambda)\|x\|_1 + \lambda\|x\|^2$ is called the elastic net penalty, which is a convex combination of the LASSO and ridge penalty. Let $g(x) = (1-\lambda)\|x\|_1 + \lambda\|x\|^2 - t$, then (5.2) is a particular case of the SFP, where $\mathcal{C} = \{x \in \mathbb{R}^L \mid g(x) \leq 0\}$ and $\mathcal{Q} = \{y\}$, i.e. find $x^* \in \mathcal{C}$ such that $\mathcal{F}x^* = y$. Moreover, when $\lambda > 0$, $g(x)$ is a strongly convex function with constant 2λ, since $g(x) - (2\lambda/2)\|x\|^2 = (1-\lambda)\|x\|_1 - t$ is a convex function. Therefore, our algorithm can be applied to solve (5.2).

Table 3. Numerical results of Algorithm 3.1 in Example 5.2 with different choices of ρ_k and initial guesses

$x^\star$	ρ_k	Iter. (k)	CPU (s)
$x^\star = (1, 1, \ldots, 1)^\top$	$\rho_k = 0.1$	474	6.69064
	$\rho_k = 0.5$	349	5.00825
	$\rho_k = 1.0$	133	1.87854
	$\rho_k = 1.5$	93	1.35527
	$\rho_k = 1.9$	78	1.10713
$x^\star = (10, 10, \ldots, 10)^\top$	$\rho_k = 0.1$	941	13.17145
	$\rho_k = 0.5$	693	9.89138
	$\rho_k = 1.0$	371	5.12882
	$\rho_k = 1.5$	401	5.56881
	$\rho_k = 1.9$	340	4.86921
$x^\star = (100, 100, \ldots, 100)^\top$	$\rho_k = 0.1$	1467	21.38174
	$\rho_k = 0.5$	867	12.36122
	$\rho_k = 1.0$	467	6.46011
	$\rho_k = 1.5$	600	8.44477
	$\rho_k = 1.9$	1092	15.12869

In this experiment, we set $K = 1500$, $L = 2000$. The matrix $\mathcal{F}$ is randomly generated by a standardized normal distribution, and the vector $x \in \mathbb{R}^L$ is also randomly generated with n nonzero coordinates taking values randomly in $[-2, 2]$ for some $n > 0$. Then, the observation vector y is generated as $y = \mathcal{F}x$. We provided some comparisons to the algorithm (1.7) of Yang [24], the algorithm (1.8) of López et al. [16], the algorithm (1.9) of Yu et al. [26], and the algorithm (1.10) of Dang et al. [10]. The involving parameters and operators are chosen as follows:

- In our algorithm: $\rho_k = 0.5$, $\alpha_k = \frac{1}{k+1}$, $\eta_k = \frac{1}{k^{1.1}+1}$, $e_k = 10^{-3}$ for all $k \geq 1$, $\theta = 0.1$, $f(x) = 0.9999x$.
- In Yu's Alg.: $\rho_k = 0.5$ for all $k \geq 1$.
- In López's Alg: $\rho_k = 0.5$ for all $k \geq 1$.
- In Yang's Alg.: $\gamma = \frac{1.95}{\|\mathcal{F}\|^2}$.
- In Dang's Alg.: $\gamma = \frac{1.95}{\|\mathcal{F}\|^2}$, $\theta = 0.1$, $\theta_k = \frac{1}{2}\bar{\theta}_k$ for each $k \geq 1$.

In all algorithms, the initial points are $x^0 = \frac{1}{2}x^\star$ (if needed), $x^1 = x^\star$, where $x^\star = (1, 1, \ldots, 1)^\top$, and the stopping criteria is $\varepsilon_k \leq 10^{-6}$, where $\varepsilon_k := \|x^k - x^{k-1}\|$. We tested the algorithms with different values of t and n. The numerical results are listed in Table 4.

From Table 4, it can be seen that our algorithm has better performance than the relevant methods, in terms of number of iteration (k) and the CPU time (s). This advantage of our method is especially significant when we are working in large-scale problems. In that case, our algorithm is highly faster than Yang's Alg. and Dang's Alg.

Table 4. Numerical results of the algorithm (4.1), Yu's Alg. [26], López's Alg. [16], Yang's Alg. [24], and Dang's Alg. [10] for Example 5.3 with different values of t and n

		$t = 50$		$t = 75$	
n	Algorithm	Iter. (k)	CPU (s)	Iter. (k)	CPU (s)
$n = 20$	Our Alg.	280	4.17062	299	5.35194
	Yu's Alg.	288	8.28327	300	14.58467
	López's Alg.	350	9.54555	347	13.13750
	Yang's Alg.	153641	1641.72481	170096	1991.93037
	Dang's Alg.	153628	1709.61426	170054	1973.73857
$n = 30$	Our Alg.	330	5.38993	313	5.45788
	Yu's Alg.	341	10.21217	332	10.91377
	López's Alg.	419	12.89763	383	11.67734
	Yang's Alg.	142416	1506.89022	156907	1804.71479
	Dang's Alg.	142372	1551.33575	156905	1907.00519
$n = 40$	Our Alg.	515	8.7879	304	5.68875
	Yu's Alg.	534	16.30069	307	12.83835
	López's Alg.	611	19.85566	345	12.80099
	Yang's Alg.	142097	1582.04282	151415	1824.85440
	Dang's Alg.	142005	1599.88914	151414	1760.53157

6. Conclusion

This paper has presented novel ball-relaxed CQ algorithms for efficiently solving the split feasibility problem with multiple output sets. The algorithms are enhanced by incorporating an inertial term within the iterative scheme to accelerate the convergence. Furthermore, they eliminate the need for computing norms of the involved bounded linear operators, thanks to the utilization of self-adaptive step size criteria. Strong convergence theorems of the proposed methods have been proved under some feasible assumptions widely used in the optimization theory. Some applications relating to the solution of the split feasibility problem were also reported. The efficiency and advantages of our algorithms were confirmed by three numerical experiments, including an application to the LASSO problem with elastic net regularization.

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