Journal of Applied Analysis and Computation
Volume 14, Number 3, June 2024, 1508–1521
Website:http://www.jaac-online.com
DOI:10.11948/20230264

FURTHER STUDIES OF TOPOLOGICAL TRANSITIVITY IN NON-AUTONOMOUS DISCRETE DYNAMICAL SYSTEMS*

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Abstract In this paper, notions related to transitivity in autonomous discrete dynamical systems are generalized to non-autonomous discrete dynamical systems. Some sufficient conditions or necessary conditions of transitive were given. Then, it is obtained that the mapping sequence $f_{1,\infty} = (f_1, f_2, \cdots)$ is $\mathcal{P}$-chaotic if and only if the mapping sequence $f_{n,\infty} = (f_n, f_{n+1}, \cdots), \forall n \in \mathbb{N}$ ($\mathbb{N} = \{1, 2, \cdots\}$) would also be $\mathcal{P}$-chaotic. Where $\mathcal{P}$-chaos represents one of the following six properties: transitive, mixing, weakly mixing, syndetically transitive, strongly transitive, and $\mathbb{Z}$-transitive. Finally, an example is given to show that the condition 'the space has no isolated point' cannot be removed, and a $\mathcal{P}$-chaotic non-autonomous mapping sequence is provided.

Keywords Non-autonomous discrete dynamical systems, transitivity, mixing.

MSC(2010) 37B40, 37D45, 54H20.

1. Introduction

The application of dynamical systems in the cross field has achieved fruitful results, and gradually developed into an important branch of mathematics. One of the main contents of topological dynamical systems is the evolution process of the system. The topological dynamical system theory is used to deal with the complexity, instability, or chaoticity in the real world, such as in meteorology, ecology, celestial mechanics, and other natural sciences. In recent years, more and more scholars and experts have begun to devote themselves to the research in topological dynamical systems, and have achieved many significant results (see [1,9–11,18,20] and others).

As we all know, topological transitivity (shortly, transitivity) has been an eternal topic in the study of topological dynamical systems. The concept of transitivity can be traced back to Birkhoff [4]. Since then, many studies have been devoted

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* This work was funded by the Project of the Natural Science Foundation of Sichuan Province (Nos. 2023NSFSC0070, 2022NSFSC1821), the Scientific Research and Innovation Team Program of Sichuan University of Science and Engineering (No. SUSE652B002), and the Graduate Student Innovation Fundings (No. y2022189).
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to this topic (see [5, 7]). Transitivity is another crucial measure of system complexity. Liao [13] investigated transitivity, weak mixing, and mixing in hyperspace. Moothathu [10] shows that, for multiple transitive systems, minimal is equivalent to weak mixing. Kwietniak [8] raises some open questions about the connection between transitivity and weak mixing. Chen [6] discussed multi-transitivity with respect to a vector. It is shown that multi-transitivity can be characterized by the hitting time sets of open sets, and multi-transitive systems are Li-Yorke chaotic. Wu [19] proved that a dynamical system is weakly mixing if and only if its induced Zadeh's extension is transitive.

The above literatures study transitivity in autonomous discrete dynamical systems (ADDS). In this paper, non-autonomous discrete dynamical systems (NDDS) as follows are discussed.

Let $\mathbb{Z}$ and $\mathbb{N}$ denote the sets of integers and positive integers, respectively. $f_n : X \to X$ ($n \in \mathbb{N}$, (X, ρ) is a compact metric Hausdorff space) is a continuous mapping sequence, and denoted by $f_{1,\infty} = (f_1, f_2, \cdots) = (f_n)_{n=1}^{\infty}$. This mapping sequence defines an NDDS, denoted by $(X, f_{1,\infty})$. Under this sequence, the orbit of the point $x \in X$ is $Orb_{f_{1,\infty}}(x) = \{x, f_1(x), f_2 \circ f_1(x), \cdots, f_1^n(x), \cdots\}$ ($n \in \mathbb{N}$), where $f_1^n = f_n \circ \cdots \circ f_2 \circ f_1$. Similarly, $f_n^k = f_k \circ \cdots \circ f_{n+1} \circ f_n$ ($k \geq n$). Denoted $f_1^{-n} = (f_n \circ \cdots \circ f_2 \circ f_1)^{-1} = f_1^{-1} \circ f_2^{-1} \circ \cdots \circ f_n^{-1}$. And f_1^0 denotes the identity mapping.

If $f_i = f_j$ ($i, j \in \mathbb{N} : i \neq j$), $(X, f_{1,\infty})$ is called an ADDS. Generally, it is relatively more difficult to study the dynamic behavior of NDDSs than that in ADDSs. However, many complex systems in the real world, such as physics, biology, and economics, must be better described by NDDSs. Therefore, it is of great interest to study the dynamic behavior of NDDSs (see [2, 12, 14–17, 21, 22]).

This paper generalizes some notions of transitivity from ADDSs to NDDSs, and then gives some necessary and sufficient conditions for transitivity in NDDSs. Moreover, some examples are given to illustrate the conclusions.

2. Preliminaries

A subset T of $\mathbb{N} \cup \{0\}$ is said to be *syndetic* if, there exists an $M \in \mathbb{N}$, for every $m \in \mathbb{N}$ satisfying $\{m, m+1, \cdots, m+M\} \cap T \neq \emptyset$.

Let $(X, f_{1,\infty})$ be an NDDS. For any two nonempty open sets $U, V \subset X$, denote the recurrent time set of U and V by

$$N_{f_{1,\infty}}(U, V) = \{n \in \mathbb{N} : f_1^n(U) \cap V \neq \emptyset\}.$$

Definition 2.1. Let $(X, f_{1,\infty})$ be an NDDS. $f_{1,\infty}$ is said to be

(1) *transitive* if for any two nonempty open sets U and V in X, $N_{f_{1,\infty}}(U, V) \neq \emptyset$;

(2) *mixing* if for any two nonempty open sets U and V in X, there is an $N > 0$ such that $f_1^n(U) \cap V \neq \emptyset$ for all $n \geq N$;

(3) *weakly mixing* if $f_{1,\infty} \times f_{1,\infty}$ is transitive on $X \times X$, i.e., for any nonempty open sets U_i, V_i ($i = 1, 2$) in X, there is an integer $k \geq 1$ satisfying $f_1^k(U_1) \cap V_1 \neq \emptyset$ and $f_1^k(U_2) \cap V_2 \neq \emptyset$;

(4) *syndetically transitive* if for two nonempty open sets U and V in X, $N_{f_{1,\infty}}(U, V)$ is syndetic;

(5) *strongly transitive* if for any nonempty open set $U \subset X$, there exists a $k > 0$ such that $\cup_{i=1}^{k} f_1^i(U) = X$;

(6) *minimal* if there is no proper subset U of X, which is nonempty, closed, and invariant, i.e., $f_n(U) \subset U (n \in \mathbb{N})$;

(7) $\mathbb{Z}$-*transitive* if for any two nonempty open sets U and V in X, there is a $m \in \mathbb{Z}$ such that $f_1^m(U) \cap V \neq \emptyset$.

Remark 2.1. According to the definitions of $\mathbb{Z}$-transitive and transitive, transitive implies $\mathbb{Z}$-transitive. While $\mathbb{Z}$-transitive does not implies transitive. An example is given below to illustrate this conclusion.

Example 2.1. Let $X = \{0, 1, \frac{1}{2}, \frac{1}{3}, \cdots, \frac{1}{n}, \cdots\}$, defined

$$\omega(x) = \begin{cases} 0, & x = 0; \\ \frac{1}{\frac{1}{x}+1}, & x \in X \backslash \{0\}, \end{cases}$$

$$\phi(x) = \begin{cases} 1, & x = 0; \\ \frac{1}{\frac{1}{x}+1}, & x \in X \backslash \{0\}, \end{cases}$$

$g_{1,\infty} = \{\omega, \phi, \omega, \phi, \cdots\}$.

Obviously, for any $m_1, m_2 \in \mathbb{N}$, there is a $k \in \mathbb{Z}$ such that $g_1^k(\frac{1}{m_1}) = \frac{1}{m_2}$. Then, for any two nonempty open sets U and V in X, there is an $m \in \mathbb{Z}$ such that $g_1^m(U) \cap V \neq \emptyset$. Thus, $g_{1,\infty}$ is $\mathbb{Z}$-transitive. Let $U_0 = \{\frac{1}{3}, \frac{1}{4}\}$, $V_0 = \{\frac{1}{2}\}$. Then, $g_1^n(U_0) \cap V_0 = \emptyset$ for any $n \in \mathbb{N}$. Therefore, $g_{1,\infty}$ is not transitive.

3. Main results

Banks [3] had proved that if an ADDS (X, f) is totally transitive with a dense set of periodic points, then (X, f) is weakly mixing. Inspired by this, the following conclusion have been obtained.

Theorem 3.1. *Let $(X, f_{1,\infty})$ be an NDDS. If the following three conditions are held, then $f_{1,\infty}$ is minimal.*

(1) X has no isolated point;

(2) there exists an orbit of $f_{1,\infty}$, which is dense in X;

(3) for any $x, y \in X$ and any $k \in \mathbb{N}$,

$$\rho(f_1^{m_1}(x), f_1^{m_2}(y)) \leq \rho(f_1^{m_1-k}(x), f_1^{m_2-k}(y)) \ (m_1, m_2 \in \mathbb{N}).$$

Proof. By (1), there is a $a_0 \in X$ such that

$$Orb_{f_{1,\infty}}(a_0) = \{a_0, f_1(a_0), f_1^2(a_0), \cdots, f_1^n(a_0), \cdots\} \ (n \in \mathbb{N})$$

is dense in X. For any nonempty open set $U \subset X$, there is a $m \in \mathbb{N}$ such that $f_1^m(a_0) \in U$.

Claim 1. *$Q = \{n \in \mathbb{N} : f_1^n(a_0) \in U\}$ is an infinite set.*

Let $U_1 = \{a_0, f_1(a_0), f_1^2(a_0), \cdots, f_1^m(a_0)\}$, $W_1 = U \backslash U_1$. Since X has no isolated point, then W_1 is a nonempty open set. Since $Orb_{f_{1,\infty}}(a_0)$ is dense in X, then there is an $m_1 > m$ such that $f_1^{m_1}(a_0) \in W_1 \subset U$. In the same way, let $U_2 = \{a_0, f_1(a_0), f_1^2(a_0), \cdots, f_1^{m_1}(a_0)\}$, $W_2 = U \backslash U_2$. Then, there is an $m_2 > m_1$ such that $f_1^{m_2}(a_0) \in W_2 \subset U$. Carry on like this, it is not hard to get that $Q = \{n \in \mathbb{N} : f_1^n(a_0) \in U\}$ is an infinite set.

For any $a, b \in X$ and $\varepsilon > 0$, there exist $n_1, n_2 \in \mathbb{N}$ such that

$$f_1^{n_1}(a_0) \in B(a, \frac{\varepsilon}{2}) \quad \text{and} \quad f_1^{n_2}(a_0) \in B(b, \frac{\varepsilon}{2}).$$

Without loss of generality, assume that $n_1 \geq n_2$. According to condition (3),

$$\rho(f_1^{n_1}(a_0), f_1^{n_1-n_2}(a)) = \rho(f_1^{n_2+n_1-n_2}(a_0), f_1^{n_1-n_2}(a)) \leq \rho(f_1^{n_2}(a_0), a) < \frac{\varepsilon}{2}.$$

By the triangle inequality,

$$\rho(f_1^{n_1-n_2}(a), b) \leq \rho(f_1^{n_1}(a_0), f_1^{n_1-n_2}(a)) + \rho(f_1^{n_1}(a_0), b) < \varepsilon.$$

This implies that $f_1^{n_1-n_2}(a) \in B(b, \varepsilon)$. So, $Orb_{f_{1,\infty}}(a)$ is dense in X.

Claim 2. *$f_{1,\infty}$ is minimal.*

If not, there is a nonempty open set $U^* \subset X$ such that U^* is invariant. That is to say, $f_n(U^*) \subset U^*(n \in \mathbb{N})$. Then, $f_1^n(a^*) \in U^*$ for any $a^* \in U^*$ and $n \in \mathbb{N}$. Let $V = X \backslash U^*$, then $Orb_{f_{1,\infty}}(a^*) \cap V = \emptyset$. This contradicts that $Orb_{f_{1,\infty}}(a)$ is dense in X. So, $f_{1,\infty}$ is minimal. □

In [8], the author discusses the relationship between strong transitivity and Δ-transitivity in ADDSs. Next, we will discuss strong transitivity in NDDSs (Theorems 3.2-3.4).

Theorem 3.2. *$f_{1,\infty} = (f_1, f_2, \cdots, f_n, \cdots)$ is strongly transitive if and only if $\cup_{m=0}^{\infty} f_1^{-m}(\{a\})$ is dense in X for any $a \in X$.*

Proof. (Necessity) Assume that $f_{1,\infty}$ is strongly transitive, then for any nonempty open set $U \subset X$, there exists a $k > 0$ such that $\cup_{i=1}^{k} f_1^i(U) = X$. So, for any $a \in X$, there must be an $m > 0$ such that $a \in f_1^m(U)$. Then there is a $a^* \in U$, such that $a = f_1^m(a^*)$. Then, $a^* \in f_1^m(\{a\})$. Thus, $a^* \in \cup_{m=0}^{\infty} f_1^{-m}(\{a\}) \cap U \neq \emptyset$. Due to the arbitrariness of U, $\cup_{n=0}^{\infty} f_1^{-n}(\{a\})$ is dense in X for any $a \in X$.

(Sufficiency) Assume that $\cup_{m=0}^{\infty} f_1^{-m}(\{a\})$ is dense in X for any $a \in X$. Then, for any nonempty open set $U \subset X$, there exist an $M \in \mathbb{N} \cup \{0\}$ such that $a \in f_1^M(U)$. So, $\cup_{i=0}^{\infty} f_1^i(U) = X$. Due to X being a compact metric space, X is covered by finite subsets in $\{f_1^i(U)\}_{i=0}^{\infty}$. That is to say, there is an $M > 0$ such that $\cup_{i=0}^{M} f_1^i(U) = X$. So, $f_{1,\infty}$ is strongly transitive. □

Theorem 3.3. *Let $int(f_n(U)) \neq \emptyset$ for any nonempty open set $U \subset X$ and any $n \in \mathbb{N}$, where $int(U)$ is the interior of the set U. If $f_{1,\infty}$ is strongly transitive, then for any nonempty open set U, there exists an $M > 0$ such that $f_1(U) \cap (\cup_{j=2}^{M} f_1^j(U)) \neq \emptyset$.*

Proof. Assuming that there exists a nonempty open set U_0 in X such that $f_1(U_0) \cap (\cup_{j=2}^{M} f_1^j(U_0)) = \emptyset$ for any $M > 0$.

Claim. *For any nonempty open set U in X, there exist $u, v \in U$ and $\lambda > 0$ satisfying that*

$$B(u, \lambda) \cap B(v, \lambda) = \emptyset \quad \text{and} \quad B(u, \lambda) \cup B(v, \lambda) \subset U.$$

Since U is a nonempty open set, for any $u \in U$, there is a $\varepsilon > 0$ such that $B(u, \varepsilon) \subset U$. There exist $u_1, v_1 \in B(u, \varepsilon)$ with the distance $\rho(u_1, v_1) = l$, then

$$B(u_1, \frac{l}{4}) \cap B(v_1, \frac{l}{4}) = \emptyset.$$

If not, $B(u,\varepsilon)$ is a single-point set. This contradicts the condition that X has no isolated point. Therefore, for any nonempty open set U in X, there exist $u, v \in U$ and $\lambda > 0$ satisfying that

$$B(u,\lambda) \cap B(v,\lambda) = \emptyset \quad \text{and} \quad B(u,\lambda) \cup B(v,\lambda) \subset U.$$

Since $int(f_1(U_0))$ is a nonempty open set, then there exist $u^*, v^* \in int(f_1(U_0))$ and $\lambda^* > 0$ such that

$$B(u^*,\lambda^*) \cap B(v^*,\lambda^*) = \emptyset \quad \text{and} \quad B(u^*,\lambda^*) \cup B(v^*,\lambda^*) \subset int(f_1(U_0)).$$

According to the hypothesis, for any $M > 0$, $f_1(U_0) \cap (\cup_{j=2}^{M} f_1^j(U_0)) = \emptyset$. Since f_1 is continuous mapping, then $f_1^{-1}(B(u^*,\lambda^*))$ is a nonempty open set, and $f_1^{-1}(B(u^*,\lambda^*)) \subset U_0$. Due to $B(v^*,\lambda^*) \subset f_1(U_0)$, then for any $M > 0$, $B(v^*,\lambda^*) \cap (\cup_{j=1}^{M} f_1^j(B(u^*,\lambda^*))) = \emptyset$.

However, $f_{1,\infty}$ is strongly transitive, then for $f_1^{-1}(B(u^*,\lambda^*))$, there is an $M_1 > 0$ such that

$$\cup_{j=1}^{M_1} f_1^j(f_1^{-1}(B(u^*,\lambda^*))) = X.$$

So they contradict each other. Thus, for any nonempty open set U, there exists an $M > 0$ such that $f_1(U) \cap (\cup_{j=2}^{M} f_1^j(U)) \neq \emptyset$. □

Theorem 3.4. *Let $int(f_n(U)) \neq \emptyset$ for any nonempty open set $U \subset X$ and $n \in \mathbb{N}$. If $f_{1,\infty}$ is strongly transitive, then $\cup_{j=n+1}^{\infty} f_1^j(U) = X$ for every nonempty open set U in X and any $n \in \mathbb{N}$.*

Proof. Hypothesis that there exist an $x_0 \in X$, a nonempty open set U_1 in X, and an $n_0 \in \mathbb{N}$ such that $x_0 \notin \cup_{j=n_0+1}^{\infty} f_1^j(U_1)$. Since $f_{1,\infty}$ is strongly transitive, then for the above U_1, there is an $M > 0$ such that $\cup_{j=1}^{M} f_1^j(U_1) = X$. So, $\cup_{j=1}^{\infty} f_1^j(U_1) = X$. Then, $x_0 \in \cup_{j=1}^{n_0} f_1^j(U_1)$.

Let $U_2 = int(f_1^{n_0}(U_1))$. Due to $f_n (n \in \mathbb{N})$ being continuous mappings, then $f_{n_0}^{-1}(U_2)$ and $f_1^{-n_0}(U_2)$ are open sets. Since $f_n\,(n \in \mathbb{N})$ are surjections, then $f_{n_0}(f_{n_0}^{-1}(U_2)) = U_2$. Due to $f_1^{-n_0}(U_2) \subset U_1$, then $x_0 \notin \cup_{j=n_0+1}^{\infty} f_1^j(f_1^{-n_0}(U_2))$. And because $f_{1,\infty}$ is strongly transitive, then for $f_1^{-n_0}(U_2)$, there is an $M_1 > 0$ such that $\cup_{j=1}^{M_1} f_1^j(f_1^{-n_0}(U_2)) = X$. Then, $x_0 \in \cup_{j=1}^{n_0} f_1^j(f_1^{-n_0}(U_2))$.

Let $U_3 \subset U_2$ and U_3 be nonempty open set. By similar proof, $x_0 \in \cup_{j=1}^{n_0} f_1^j(f_1^{-n_0}(U_3))$. Continuing like this, the nonempty open sets $U_k\,(k \in \mathbb{N})$ can be found, and

$$x_0 \in \cup_{j=1}^{n_0} f_1^j(f_1^{-n_0}(U_k)) \quad \text{and} \quad \lim_{k\to+\infty} \rho \sum_{j=1}^{n_0} (f_1^j(f_1^{-n_0}(U_k))) = 0.$$

Since X has no isolated point, then $x_0 \in \cap_{k=2}^{\infty} \cup_{j=1}^{n_0} f_1^j(f_1^{-n_0}(U_k)) = \emptyset$. It is a contradiction. Thus, for every nonempty open set U in X and any $n \in \mathbb{N}$, $\cup_{j=n+1}^{\infty} f_1^j(U) = X$. □

In order to discuss the relationship between $(X, f_{1,\infty})$ and $(X, f_{n,\infty})$ $(n \in \mathbb{N})$ regarding various types of transitivity, the following lemma is necessary first.

Lemma 3.1. *If X has no isolated point. Then $f_{1,\infty}$ is transitive if and only if $N_{f_{1,\infty}}(U,V)$ is an infinite set for any nonempty open set $U, V \subset X$.*

Proof. (Necessity) Hypothesis that $N_{f_{1,\infty}}(U,V)$ is not an infinite set for some $U, V \subset X$, i.e., there exist an $m > 0$ and nonempty open sets $U_0, V_0 \subset X$ such that $f_1^n(U_0) \cap V_0 = \emptyset$ for any $n > m$. And there must be a $i_0 \leq m$ satisfying $f_1^{i_0}(U_0) \cap V_0 \neq \emptyset$ for the reason that $f_{1,\infty}$ is transitive.

Case 1. $f_1^{i_0}(U_0) \subset V_0$ and $\overline{f_1^{i_0}(U_0)} \neq \overline{V_0}$.

In this case, there exists a v_1 in V_0, but $v_1 \notin f_1^{i_0}(U_0)$. Then there is a $\varsigma_1 > 0$ such that

$$B(v_1, \varsigma_1) \subset V_0 \quad \text{and} \quad B(v_1, \varsigma_1) \cap f_1^{i_0}(U_0) = \emptyset.$$

So, one can find a point $u_1 \in U_0$ such that

$$B(v_1, \varsigma_1) \cap B(f_1^{i_0}(u_1), \varsigma_1) = \emptyset.$$

According to the continuity of $f_1^{i_0}$, there exists a $\delta_1 > 0$ such that $B(u_1, \delta_1)$ of u_1 such that

$$f_1^{i_0}(B(u_1, \delta_1)) \subset B(f_1^{i_0}(u_1), \varsigma_1).$$

Then, $f_1^{i_0}(B(u_1, \delta_1)) \cap B(v_1, \frac{\varsigma_1}{2}) = \emptyset$.

Case 2. $\overline{f_1^{i_0}(U_0)} = \overline{V_0}$.

In this case, one can take two points $u_1 \in U_0, v_1 \in V_0$ and a $\varsigma_1 > 0$ such that

$$B(v_1, \varsigma_1) \cap B(f_1^{i_0}(u_1), \varsigma_1) = \emptyset.$$

Similarly, it can be obtained that

$$f_1^{i_0}(B(u_1, \delta_1)) \cap B(v_1, \frac{\varsigma_1}{2}) = \emptyset$$

for some $\delta_1 > 0$.

Case 3. There exists an $x_1 \in f_1^{i_0}(U_0)$ but $x_1 \notin V_0$.

In this case, there exist a $u_1 \in U_0$ and a $\delta_1 > 0$ such that

$$B(u_1, \delta_1) \subset U_0 \quad \text{and} \quad f_1^{i_0}(B(u_1, \delta_1)) \cap V_0 = \emptyset.$$

Then, there exists a $v_1 \in V_0$ and a $\varsigma_1 > 0$ such that

$$f_1^{i_0}(B(u_1, \delta_1)) \cap B(v_1, \frac{\varsigma_1}{2}) = \emptyset.$$

In summary, whether case 1, case 2, or case 3, it can be obtained that $f_1^{i_0}(B(u_1, \delta_1)) \cap B(v_1, \frac{\varsigma_1}{2}) = \emptyset$, where $B(u_1, \delta_1) \subset U_0, B(v_1, \frac{\varsigma_1}{2}) \subset V_0$.

For any nonempty open sets $U, V \subset X$, denote

$$M(U,V) = \{i \leq m \mid f_1^i(U) \cap V \neq \emptyset\}.$$

Let

$$U_1 = B(u_1, \delta_1) \subset U_0, \; V_1 = B(v_1, \frac{\varsigma_1}{2}) \subset V_0,$$
$$Q_1(U_1, V_1) = \{i \in M(U_0, V_0) \mid f_1^i(U_1) \cap V_1 = \emptyset\}.$$

Obviously, $\#(Q_1(U_1, V_1)) \geq 1$.

Since $f_{1,\infty}$ is transitive, then $\#(M(U_1,V_1)) = \{i \leq m \mid f_1^i(U_1) \cap V_1 \neq \emptyset\} \geq 1$.

For U_1, V_1, similar to U_0, V_0, one can get that there exist $i_1 \in M(U_1,V_1), u_2 \in U_1, v_2 \in V_1, \varsigma_2 > 0$, and $\delta_2 > 0$ such that $f_1^{i_1}(B(u_2,\delta_2)) \cap B(v_2, \frac{\varsigma_2}{2}) = \emptyset$, where $B(u_2,\delta_2) \subset U_1$, $B(v_2, \frac{\varsigma_2}{2} \subset V_1)$.

Let

$$U_2 = B(u_2,\delta_2),\ V_2 = B(v_2, \frac{\varsigma_2}{2}),$$
$$Q_2(U_2,V_2) = \{i \in M(U_1,V_1) \mid f_1^i(U_2) \cap V_2 = \emptyset\}.$$

Obviously, $\#(Q_2(U_2,V_2)) \geq 1$.

And because $Q_2(U_2,V_2) \subset M(U_1,V_1)$, then $Q_2(U_2,V_2) \cap Q_1(U_1,V_1) = \emptyset$.

Repeat $m+2$ times like this, one can get that

$$\sum_{i=1}^{m+2} \#(Q_i(U_i,V_i)) > m \quad \text{and} \quad Q_i(U_i,V_i) \cap Q_j(U_j,V_j) = \emptyset\ (i \neq j),$$

where

$$Q_n(U_n,V_n) = \{i \in M(U_{n-1},V_{n-1}) \mid f_1^i(U_n) \cap V_n = \emptyset\}\ (n \in \mathbb{N}).$$

Therefore,

$$\#(M(U_0,V_0)) \geq \#(\cup_{j=1}^{m+2} Q_j(U_j,V_j)) = \sum_{i=1}^{m+2} \#(Q_i(U_i,V_i)) > m.$$

It contradicts that

$$\#(M(U_0,V_0)) = \#(\{i \leq m \mid f_1^i(U_0) \cap V_0 \neq \emptyset\}) \leq m.$$

So, $N_{f_{1,\infty}}(U,V)$ is an infinite set.

(Sufficiency) By the definition of transitivity, it is clear. □

In Lemma 3.1, the condition 'X has no isolated point' can not be cut. We give an example as follows.

Example 3.1. Let $X = \{1,2\}$. The open sets on X are $\emptyset, \{1\}, \{2\}, \{1,2\}$.

$\{\emptyset, \{1\}, \{2\}, \{1,2\}\}$ is induced by metric

$$\rho(x,y) = \mid x-y \mid, \forall x,y \in X.$$

Let $f_1(x) = 1, f_i(x) = 2, \forall i \geq 2, \forall x \in X$. Take $U = \{2\}, V = \{1\}$, then $N_{f_{1,\infty}}(U,V) = \{1\}$ is a finite set.

Now, we give the relationship between $(X, f_{1,\infty})$ and $(X, f_{n,\infty})$ $(n \in \mathbb{N})$ regarding various types of transitivity and mixing.

Theorem 3.5. *Let $f_n\,(n \in \mathbb{N})$ are surjections, and X has no isolated point. If $int(f_n(U)) \neq \emptyset$ for any nonempty open set $U \subset X$ and $n \in \mathbb{N}$, then*

(1) $f_{1,\infty}$ is transitive if and only if for any $n \geq 2$ $(n \in \mathbb{N})$, $f_{n,\infty}$ is transitive;

(2) $f_{1,\infty}$ is mixing if and only if for any $n \geq 2$ $(n \in \mathbb{N})$, $f_{n,\infty}$ is mixing;

(3) $f_{1,\infty}$ is weakly mixing if and only if for any $n \geq 2$ $(n \in \mathbb{N})$, $f_{n,\infty}$ is weakly mixing;

(4) $f_{1,\infty}$ is syndetically transitive if and only if for any $n \geq 2$ ($n \in \mathbb{N}$). $f_{n,\infty}$ is syndetically transitive;

(5) $f_{1,\infty}$ is strongly transitive if and only if for any $n \geq 2$ ($n \in \mathbb{N}$), $f_{n,\infty}$ is strongly transitive;

(6) $f_{1,\infty}$ is $\mathbb{Z}$-transitive if and only if for any $n \geq 2$ ($n \in \mathbb{N}$), $f_{n,\infty}$ is $\mathbb{Z}$-transitive.

Proof. Just prove the case $n = 2$. The other cases are similar.

(1) It is the special case of (2).

(2) (Necessity) For any two nonempty open sets $U, V \subset X$, let $U^* = f_1^{-(n-1)}(U)$. Since f_n is continuous, then U^* is an open set. And because f_n is a surjection, then $U^* \neq \emptyset$. Assume that $f_{1,\infty}$ is mixing, then there is an $M > 0$ such that $f_1^m(U^*) \cap V \neq \emptyset$ for all $m \geq M$. Due to

$$f_1^m(U^*) = f_1^m(f_1^{-(n-1)}(U)) = f_n^m \circ f_1^{n-1}(f_1^{-(n-1)}(U))$$

$$= f_n^m(f_1^{n-1} \circ f_1^{-(n-1)}(U)) = f_n^m(U),$$

then, for any two nonempty open sets $U, V \subset X$, there exists an $M^* = max\{n + 1, M\}$ satisfying that $f_n^{m^*}(U) \cap V \neq \emptyset$ for all $m^* > M^*$. This means $f_{n,\infty}$ is mixing.

(Sufficiency) For any two nonempty open sets $U, V \subset X$, since $int(f_n(U')) \neq \emptyset$ for any nonempty open set $U' \subset X$, then there exists a nonempty open set U^* such that $U^* \subset f_1^n(U)$. Since $f_{n,\infty}$ is mixing, there is an $M > 0$ satisfying that $f_n^m(U^*) \cap V \neq \emptyset$ for any $m > M$. While, $(f_1^m(U) \cap V) \supset (f_n^m(U^*) \cap V)$, then $f_1^m(U) \cap V \neq \emptyset$ for any two nonempty open sets $U, V \subset X$ and all $m \geq M$. This means $f_{1,\infty}$ is mixing.

(3) (Necessity) For any four nonempty open subsets U_i, V_i ($i = 1, 2$) of X, let $U_i^* = f_1^{-n+1}(U_i) (i = 1, 2)$. Similar to the above proof, U_i^* is an open set. By Lemma 3.1, for any $n \in \mathbb{N}$, there is an $m > n$, such that $f_n^m(U_i) \cap V_i \neq \emptyset \, (i = 1, 2)$. Thus, $f_{n,\infty}$ is weakly mixing.

(Sufficiency) For any four nonempty open subsets U_i, V_i ($i = 1, 2$) of X. There exist $U_i^* \subset int(f_1^n(U_i))$ ($i = 1, 2$). So, one can find $a_i \in U_i^*$ and some $m \in \mathbb{N} : m > n$ satisfies $f_1^m(a_i) \in V_i$. Thus, $f_1^m(a_i) \in f_n^m(U_i) \cap V_i$. This means that $f_{1,\infty}$ is weakly mixing.

(4) (Necessity) Assume that $f_{1,\infty}$ is syndetically transitive. Then for any two nonempty open sets U and V in X, $N_{f_{1,\infty}}(U, V)$ is syndetic. That is to say, there exists an $M \in \mathbb{N}$, for every $m \in \mathbb{N}$ such that $\{m, m+1, \cdots, m+M\} \cap N_{f_{1,\infty}}(U, V) \neq \emptyset$, where $N_{f_{1,\infty}}(U, V) = \{n \in \mathbb{N} : f_1^n(U) \cap V \neq \emptyset\}$.

For any nonempty open set $U \in X$, due to f_1 being a continuous map, $f_1^{-1}(U)$ is an open set. Then,

$$N_{f_{2,\infty}}(U, V) = N_{f_{1,\infty}}(f_1^{-1}(U), V) \backslash \{1\}.$$

Since $N_{f_{1,\infty}}(f_1^{-1}(U), V)$ is syndetic, then $N_{f_{2,\infty}}(U, V)$ is syndetic. Therefore, $f_{2,\infty}$ is syndetically transitive.

(Sufficiency) For any two nonempty open sets U and V in X, take $U^* = int(f_n(U))$. Then,

$$N_{f_{1,\infty}}(U, V) \backslash \{1\} = N_{f_{2,\infty}}(f_1(U), V) \supset N_{f_{2,\infty}}(U^*, V).$$

Since $f_{2,\infty}$ is syndetically transitive, then for any two nonempty open sets U and V in X, $N_{f_{2,\infty}}(U,V)$ is syndetic. Then $N_{f_{2,\infty}}(U^*,V)$ is syndetic. So $N_{f_{1,\infty}}(U,V)$ is syndetic. Therefore, $f_{1,\infty}$ is syndetically transitive.

(5) (Necessity) Since $int(f_1(U)) \neq \emptyset$ for any nonempty open set $U \subset X$, then there exist $u, v \in int(f_1(U))$ and $\lambda > 0$ such that

$$B(u,\lambda), B(v,\lambda) \subset U^* \quad \text{and} \quad B(u,\lambda) \cap B(v,\lambda) = \emptyset,$$

where $U^* = int(f_1(U))$.

Since f_1 is a continuous map, then $f_1^{-1}(B(u,\lambda)), f_1^{-1}(B(v,\lambda))$ are open sets. Due to $f_{1,\infty}$ being strongly transitive, then for $f_1^{-1}(B(u,\lambda))$, there is a $M_u > 0$ satisfying that

$$\cup_{j=1}^{M_u} f_1^j(f_1^{-1}(B(u,\lambda))) = \cup_{j=1}^{M_u} f_2^j(B(u,\lambda)) = X.$$

And for $f_1^{-1}(B(v,\lambda))$, there is a $M_v > 0$ satisfying that

$$\cup_{j=1}^{M_v} f_1^j(f_1^{-1}(B(v,\lambda))) = \cup_{j=1}^{M_v} f_2^j(B(v,\lambda)) = X.$$

Since

$$f_1(U \backslash f_1^{-1}(B(u,\lambda))) \cap B(u,\lambda) = \emptyset \quad \text{and} \quad f_1(U \backslash f_1^{-1}(B(v,\lambda))) \cap B(v,\lambda) = \emptyset,$$

then

$$f_1(U \backslash f_1^{-1}(B(u,\lambda))) \subset \cup_{j=2}^{M_u} f_2^j(B(u,\lambda))$$

and

$$f_1(U \backslash f_1^{-1}(B(v,\lambda))) \subset \cup_{j=2}^{M_v} f_2^j(B(v,\lambda)).$$

Take $M = \max\{M_u, M_v\}$, then

$$\cup_{j=2}^{M_u} f_2^j(B(u,\lambda)) \subset \cup_{j=2}^{M} f_2^j(U^*) \subset \cup_{j=2}^{M} f_2^j(f_1(U)) = \cup_{j=2}^{M} f_1^j(U);$$

$$\cup_{j=2}^{M_v} f_2^j(B(v,\lambda)) \subset \cup_{j=2}^{M} f_2^j(U^*) \subset \cup_{j=2}^{M} f_2^j(f_1(U)) = \cup_{j=2}^{M} f_1^j(U).$$

So,

$$f_1(U) = (f_1(U \backslash f_1^{-1}(B(u,\lambda))) \cup f_1(U \backslash f_1^{-1}(B(v,\lambda)))) \subset \cup_{j=2}^{M} f_1^j(U).$$

Thus,

$$X = \cup_{j=1}^{M_v} f_2^j(B(v,\lambda)) \subset \cup_{j=1}^{M} f_2^j(U^*) \subset \cup_{j=1}^{M} f_1^j(U) = \cup_{j=2}^{M} f_1^j(U).$$

For any $U' \in X$, due to f_1 being a continuous mapping, then $f_1^{-1}(U')$ is an open set. Take $f_1^{-1}(U') = U$, then, $\cup_{j=2}^{M} f_2^j(U') = X$. Thus, $f_{2,\infty}$ is strongly transitive.

(Sufficiency) Since $int(f_1(U)) \neq \emptyset$ for any nonempty open set $U \subset X$, then, one can take a nonempty open subset $U^* \subset X$. Due to $f_{2,\infty}$ being strongly transitive, then for the above U^* of X, there is an $M > 0$ satisfying that $\cup_{j=2}^{M} f_2^j(U^*) = X$. Thus,

$$\begin{aligned}
\cup_{j=1}^{M} f_1^j(U) = \cup_{j=1}^{M}\ & f_2^j(f_1(U)) \\
= & f_1(U) \cup f_1^2(U) \cup f_1^3(U) \cup \cdots \cup f_1^M(U) \\
\supset & U^* \cup f_2(U^*) \cup f_2^3(U^*) \cup \cdots \cup f_2^M(U^*) \\
\supset & f_2(U^*) \cup f_2^3(U^*) \cup \cdots \cup f_2^M(U^*) \\
= & \cup_{j=2}^{M}\ f_2^j(U^*).
\end{aligned}$$

So,

$$\cup_{j=1}^{M} f_1^j(U) \supset \cup_{j=2}^{M} f_2^j(U^*) = X.$$

Therefore, $f_{1,\infty}$ is strongly transitive.

(6) Similar to the proof of weakly mixing. □

Remark 3.1. The necessity in the proof for strongly transitive can also be proved as follows.

Proof. It follows from Theorem 3.4 that, for any nonempty open sets U in X and any $n \in \mathbb{N}$, $\cup_{j=n+1}^{\infty} f_1^j(U) = X$. Since X is a compact metric space, then X is covered by finite sets in $\{f_1^j(U)\}_{j=n+1}^{\infty}$. That is to say, there is an $M > 0$ such that $\cup_{j=n+1}^{M} f_1^j(U) = X$. Obviously, for any nonempty open sets $U \in X$, $\cup_{j=1}^{M} f_1^j(U) = X$. So, $f_{n,\infty}$ is strongly transitive. □

The following constructs an example which satisfies Theorem 3.5.

Example 3.2. Let $X = [0, 1]$,

$$f_i(x) = \begin{cases} ix & \text{for } x \in \left[0, \frac{1}{i}\right]; \\ 2 - ix & \text{for } x \in \left[\frac{1}{i}, \frac{2}{i}\right]; \\ i(x - \frac{2}{i}) & \text{for } x \in \left[\frac{2}{i}, \frac{3}{i}\right]; \\ 2 - i(x - \frac{2}{i}) & \text{for } x \in \left[\frac{3}{i}, \frac{4}{i}\right]; \\ ix - 4 & \text{for } x \in \left[\frac{4}{i}, \frac{5}{i}\right]; \\ 4 - i(x - \frac{2}{i}) & \text{for } x \in \left[\frac{5}{i}, \frac{6}{i}\right]; \\ \vdots & \\ ix - (i-3) & \text{for } x \in \left[\frac{i-3}{i}, \frac{i-2}{i}\right]; \\ (i-3) - i(x - \frac{2}{i}) & \text{for } x \in \left[\frac{i-2}{i}, \frac{i-1}{i}\right]; \\ ix - (i-1) & \text{for } x \in \left[\frac{i-1}{i}, 1\right], \end{cases} \quad (i \text{ is a positive odd number}),$$

$$f_i(x) = \begin{cases} ix & \text{for } x \in \left[0, \frac{1}{i}\right]; \\ 2 - ix & \text{for } x \in \left[\frac{1}{i}, \frac{2}{i}\right]; \\ i(x - \frac{2}{i}) & \text{for } x \in \left[\frac{2}{i}, \frac{3}{i}\right]; \\ 2 - i(x - \frac{2}{i}) & \text{for } x \in \left[\frac{3}{i}, \frac{4}{i}\right]; \\ ix - 4 & \text{for } x \in \left[\frac{4}{i}, \frac{5}{i}\right]; \\ 4 - i(x - \frac{2}{i}) & \text{for } x \in \left[\frac{5}{i}, \frac{6}{i}\right]; \\ \vdots & \\ ix - (i-2) & \text{for } x \in \left[\frac{i-2}{i}, \frac{i-1}{i}\right]; \\ (i-2) - i(x - \frac{2}{i}) & \text{for } x \in \left[\frac{i-1}{i}, 1\right]. \end{cases} \quad (i \text{ is a positive even number}).$$

For the mapping sequence $f_{1,\infty} = \{f_1, f_2, \cdots\}$, the function images of f_1^2, f_1^3 and f_1^4 are shown in Figure 1. The function image of f_1^i can be inferred. Then the following conclusions (Propositions 3.1-3.6) are obtained.

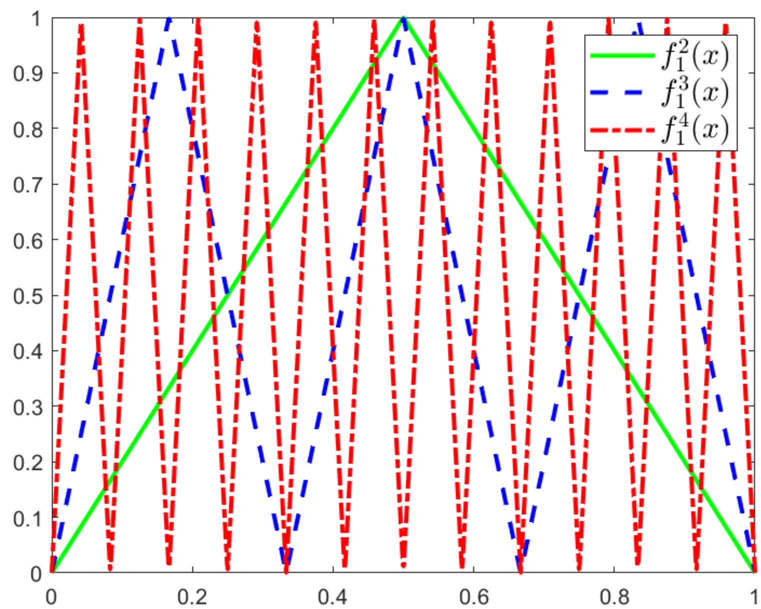

Figure 1. The function images of f_1^2, f_1^3 and f_1^4

Proposition 3.1. *The mapping sequence $f_{n,\infty}$ is transitive for any $n \in \mathbb{N}$.*

Proof. From Figure 1, it can be seen that, if x is in $[0, \frac{1}{i!}]$, $[\frac{2}{i!}, \frac{3}{i!}]$, $[\frac{4}{i!}, \frac{5}{i!}]$, $\cdots$, or $[\frac{i!-2}{i!}, \frac{i!-1}{i!}]$, the function values of $f_1^i(x)$ rise monotonically from 0 to 1; if x is in $[\frac{1}{i!}, \frac{2}{i!}]$, $[\frac{3}{i!}, \frac{4}{i!}]$, $\cdots$, or $[\frac{i!-1}{i!}, 1]$, the function values of $f_1^i(x)$ decrease monotonically from 1 to 0. As $i \in \mathbb{N}$ increases, $[0,1]$ is infinitely subdivided. Then for any $U, V \subset X$, there always exist an $i^* \in \mathbb{N}$ and a $k \in \mathbb{N}$ such that $w = [\frac{2k}{(i^*)!}, \frac{2k+1}{(i^*)!}] \subset U$ (or $w = [\frac{2k-1}{(i^*)!}, \frac{2k}{(i^*)!}] \subset U$). So

$$[0,1] = f_1^{i^*}(w) \subset f_1^{i^*}(U) \subset [0,1].$$

Thus, $f_1^{i^*}(U) \cap V \neq \emptyset$. Therefore, $f_{1,\infty}$ is transitive.

Similar to the above proof, $f_{n,\infty}$ is transitive for any $n \in \mathbb{N}$. □

Proposition 3.2. *The mapping sequence $f_{n,\infty}$ is $\mathbb{Z}$-transitive for any $n \in \mathbb{N}$.*

Proof. By the proof of Proposition 3.1, it is obvious. □

Proposition 3.3. *The mapping sequence $f_{n,\infty}$ is mixing for any $n \in \mathbb{N}$.*

Proof. Since $[0,1]$ is infinitely subdivided along with the increase of $i \in \mathbb{N}$, and each cell is mapped to $[0,1]$, then, for any $n \in \mathbb{N}$ and any $U, V \subset X$, there exists an $i^* \in \mathbb{N}$ such that $f_n^p(U) \cap V \neq \emptyset$ for all $p > i^*$. So, $f_{n,\infty}$ is mixing. □

Proposition 3.4. *The mapping sequence $f_{n,\infty}$ is weakly mixing for any $n \in \mathbb{N}$.*

Proof. For any nonempty open sets U_i, V_i $(i = 1, 2)$ in X, by Proposition 3.3, there exist $N_1 \geq 1, N_2 \geq 1$ such that $f_n^i(U_1) \cap V_1 \neq \emptyset$ for any $i \geq N_1$ and $f_n^i(U_2) \cap V_2 \neq \emptyset$ for any $i \geq N_2$. Take $N = \max\{N_1, N_2\}$, then $f_n^i(U_1) \cap V_1 \neq \emptyset$ and $f_n^i(U_2) \cap V_2 \neq \emptyset$ for any $i \geq N$. So, $f_{n,\infty}$ is weakly mixing. □

Proposition 3.5. *The mapping sequence $f_{n,\infty}$ is syndetically transitive for any $n \in \mathbb{N}$.*

Proof. For any nonempty open sets $U, V \subset X$, by Proposition 3.3, there exist an $i^* \in \mathbb{N}$ such that $f_n^i(U) \cap V \neq \emptyset$ for any $i > i^*$. Then, $N_{f_{n,\infty}}(U, V) = \{i^*, i^* + 1, \cdots\}$. So, there exists an $M \in \mathbb{N}$, for every $m \in \mathbb{N}$ satisfying $\{m, m+1, \cdots, m+M\} \cap N_{f_{n,\infty}}(U, V) \neq \emptyset$. Thus, $N_{f_{n,\infty}}(U, V)$ is a syndetic set, i.e. $f_{n,\infty}$ is syndetically transitive. □

Proposition 3.6. *The mapping sequence $f_{n,\infty}$ is strongly transitive for any $n \in \mathbb{N}$.*

Proof. By the proof of Proposition 3.1, for any nonempty open set $U \subset X$, there exists an $i^* \in \mathbb{N}$ such that $f_n^{i^*}(U) = [0, 1]$. Then, $\cup_{i=1}^{i^*} f_n^i(U) = [0, 1]$. This means that $f_{n,\infty}$ is strongly transitive. □

According to the proof of Theorem 3.5, the following results can be obtained.

Theorem 3.6. *Assumed that f_n $(n \in \mathbb{N})$ are surjections and X has no isolated point. If $int(f_n(U)) \neq \emptyset$ for any nonempty open set $U \subset X$ and $n \in \mathbb{N}$, then*

(1) $f_{1,\infty}$ is transitive if and only if there exists an $n \in \mathbb{N}$, $f_{n,\infty}$ is transitive;

(2) $f_{1,\infty}$ is mixing if and only if there exists an $n \in \mathbb{N}$, $f_{n,\infty}$ is mixing;

(3) $f_{1,\infty}$ is weakly mixing if and only if there exists an $n \in \mathbb{N}$, $f_{n,\infty}$ is weakly mixing;

(4) $f_{1,\infty}$ is syndetically transitive if and only if there exists an $n \in \mathbb{N}$, $f_{n,\infty}$ is syndetically transitive;

(5) $f_{1,\infty}$ is strongly transitive if and only if there exists an $n \in \mathbb{N}$, $f_{n,\infty}$ is strongly transitive;

(6) $f_{1,\infty}$ is $\mathbb{Z}$-transitive if and only if there exist an $n \in \mathbb{N}$, $f_{n,\infty}$ is $\mathbb{Z}$-transitive.

Remark 3.2. In Theorems 3.5 and 3.6, the condition 'X has no isolated point' can not be removed. The following gives an example for strongly transitive.

Example 3.3. In Example 3.1, obviously, for any nonempty open set $U \subset X$,

$$\cup_{j=1}^{3} f_1^j(U) = X, \quad \cup_{j=2}^{\infty} f_2^j(U) = \{2\}.$$

So, $f_{1,\infty}$ is strongly transitive, but $f_{2,\infty}$ is not strongly transitive.

Remark 3.3. In Theorems 3.5 and 3.6, the condition '$int(f_n(U)) \neq \emptyset$ for any nonempty open set $U \subset X$ and $n \in \mathbb{N}$' can not be removed. The following gives an example for weakly mixing.

Example 3.4. Let $X = [0, 1]$. Hypothesis that there is a nonempty open set $U_1 \subset X$ such that $int(f_1(U_1)) = \emptyset$. Assume that $f_{2,\infty}$ is weakly mixing. Since U_1 is a nonempty open set, then there exist $a_1, b_1 \in [0, 1] : (a_1, b_1) \subset U_1$. So, $int(f_1(a_1, b_1)) = \emptyset$. That is to say, there is a $\alpha \in [0, 1]$ such that $f_1(a_1, b_1) = \{\alpha\}$.

The following proves that $f_{1,\infty}$ is not weakly mixing by using the counterproof method.

If $f_{1,\infty}$ is weakly mixing. Then for nonempty open sets $U_1 = U_2 = (a_1, b_1), V_1 = (0, \frac{1}{3}), V_2 = (\frac{2}{3}, 1)$, there is a $n_0 \geq 1$ such that

$$f_1^{n_0}(a_1, b_1) \cap V_1 \neq \emptyset \quad \text{and} \quad f_1^{n_0}(a_1, b_1) \cap V_2 \neq \emptyset.$$

Since $f_1^{n_0}(a_1, b_1) = f_2^{n_0}(\alpha)$ is a single point set, then,

$$f_2^{n_0}(\alpha) \in V_1 \quad \text{and} \quad f_2^{n_0}(\alpha) \in V_2.$$

This contradicts to $V_1 \cap V_2 = \emptyset$. Therefore, $f_{1,\infty}$ is not weakly mixing.

Statements and declarations

All authors agree to publish this paper. There is no conflict of interest.

Acknowledgements

Many thanks to the referees for the careful reading of the manuscript and valuable suggestions.

References

[1] S. Ahmadi, X. Wu and G. Chen, *Topological chain and shadowing properties of dynamical systems on uniform spaces*, Topol. Appl., 2020, 275, 107153.

[2] W. Anwar, T. Lu and X. Yang, *Sensitivity of iterated function systems under the product operation*, Results Math., 2022, 77, 185.

[3] J. Banks, *Regular periodic decompositions for topologically transitive maps*, Ergod. Theor. Dyn. Syst., 1997, 17, 505–529.

[4] G. Birkhoff, *Dynamical Systems*, American Mathematical Society, Washington, 2008.

[5] L. Block and D. Ledis, *Topological conjugacy, transitivity and patterns*, Topol. Appl., 2014, 167, 53–61.

[6] Z. Chen, J. Li and L. Jie, *On multi-transitivity with respect to a vector*, Sci. China Math., 2014, 57, 1639–1648.

[7] M. Javaheri, *Hypercyclic and topologically transitive semigroups of composition operators*, Topol. Appl., 2014, 164, 105–112.

[8] D. Kwietniak and P. Oprocha, *On weak mixing, minimality and weak disjointness of all iterates*, Ergod. Theor. Dyn. Syst., 2012, 32, 1661–1672.

[9] J. Li, C. Liu , S. Tu and T. Yu, *Sequence entropy tuples and mean sensitive tuples*, Ergod. Theor. Dyn. Syst., 2024, 44, 184–203.

[10] J. Li, X. Ye and T. Yu, *Equicontinuity and sensitivity in mean forms*, J. Dyn. Differ. Equ., 2022, 34, 133–154.

[11] J. Li, X. Ye and T. Yu, *Mean equicontinuity, complexity and applications*, Discrete contin. dyn. syst., 2021, 41, 359–393.

[12] R. Li, Y. Zhao and H. Wang, *Stronger forms of transitivity and sensitivity for nonautonomous discrete dynamical systems and Furstenberg families*, J. Dyn. Control Syst., 2020, 26, 109–126.

[13] G. Liao, L. Wang and Y. Zhang, *Transitivity, mixing and chaos for a class of set-valued mappings*, Sci. China Math., 2006, 49, 1–8.

[14] J. Pi, T. Lu and Y. Chen, *Collective sensitivity and collective accessibility of non-autonomous discrete dynamical systems*, Fractal. Fract., 2022, 6, 535.

[15] J. Pi, T. Lu and Y. Xue, *Transitivity and shadowing properties of non-autonomous discrete dynamical systems*, Int. J. Bifurcat. Chaos, 2022, 32, 2250246.

[16] H. Shao, Y. Shi and H. Zhu, *Estimations of topological entropy for nonautonomous discrete systems*, J. Differ. Equ. Appl., 2016, 22, 474–484.

[17] R. Thakur and R. Das, *Transitivity and sensitivity of iterated function systems via Furstenberg families*, Aequationes math., 2020, 94, 1123–1140.

[18] R. Vasisht and R. Das, *Generalizations of expansiveness in non-autonomous discrete systems*, B. Iran. Math. Soc., 2022, 48, 417–433.

[19] X. Wu, X. Ding, T. Lu and J. Wang, *Topological dynamics of Zadeh's extension on the space of upper semi-continuous fuzzy sets*, Int. J. Bifurcat. Chaos, 2017, 27, 1750165.

[20] X. Xie and J. Yin, *On the eventual shadowing property and eventually shadowable point of set-valued dynamical systems*, Acta. Math. Sin., 2022, 38, 1105–1115.

[21] X. Yang, T. Lu and W. Anwar, *Chaotic properties of a class of coupled mapping lattice induced by fuzzy mapping in non-autonomous discrete systems*, Chaos Soliton Fract., 2021, 148, 110979.

[22] X. Yang, T. Lu, J. Pi and Y. Jiang, *On shadowing system generated by a uniformly convergent mappings sequence*, J. Dyn. Control Syst., 2022, 28, 238.