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OSCILLATION AND SURVIVAL ANALYSIS OF GENERALIZED STOCHASTIC LOGISTIC MODELS WITH PIECEWISE CONSTANT ARGUMENT

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Abstract The paper focuses on oscillation and survival analysis for a class of generalized stochastic logistic equations with piecewise constant argument. The existence of global positive solution is proved firstly. Then the necessary and sufficient conditions under which the population will be almost surely extinct and persistent are investigated. Furthermore, we study the condition for oscillation of the equation with constant coefficients, and the result shows that the solution oscillates around a new positive point induced by the noise. Finally, numerical experiments are given for several examples to support the results.

Keywords Stochastic logistic model, piecewise constant argument, global positive solution, extinction and persistence, oscillation.

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1. Introduction

Differential equations with piecewise constant argument were firstly considered by Cooke and Wiener [7], and Shah and Wiener [30]. These equations combine the properties of differential equations and difference equations and have the structure of continuous dynamical systems within intervals of unit length; see [1–5, 14–16, 27, 28] and the references cited therein. Amongst them most papers focus on the deterministic logistic equation with piecewise constant argument as a simpler special model with harvesting from the viewpoint of application.

For the autonomous and nonautonomous differential equations, researchers mainly studied the persistence and global stability of solutions. A classical logistic equation with piecewise constant argument is like

$$x'(t) = rx(t)\left(1 - ax(t) - bx\left([t]\right)\right), \tag{1.1}$$

where $r, a, b \in R^+$. Gopalsamy and Liu [9] studied equation (1.1) and gave a sufficient condition established for all positive solutions of the corresponding discrete dynamic system to converge eventually to a positive equilibrium.

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Now stochastic noise has been considered as an important factor in modelling population dynamics for the existence of random perturbation in almost all fields. For example, in [23], Mao et al. considered an autonomous system driven by stochastically environmental noise

$$dx\left(t\right)=x\left(t\right)\left\{\left(r+ax(t)\right)dt+\sigma dB\left(t\right)\right\}, \tag{1.2}$$

where $r, a, \sigma > 0$ are all constants. They revealed that the environmental noise may suppress the explosion, no matter how small the parameter $\sigma > 0$ is. While [6, 8, 12, 13, 17–21, 24, 26, 29, 32, 33] considered the persistence and stability of similar population models with stochastic perturbations. To be a little concrete, [17, 21] studied the population model perturbed by the white noises and regime switching which was modeled by the finite states Markov chains. The existence of stationary distribution and the ergodicity were also discussed for stochastic population model (see [13] and references therein). In [33], Zu and Jiang studied the extinction and strong persistence of a stochastic predator-prey system with Holling II functional response. [29] investigated the threshold problem for stochastic SIR model with saturated incidence rate and saturated treatment function.

Motivated by the above, an interesting and important question arises naturally, that is, how will the population tend to be when the population is modeled by (1.1) together with the stochastic perturbation. To the best of our knowledge, the stochastic population model with piecewise constant argument has not been investigated up to now. To fill this gap, we make the first attempt to study a generalized logistic population model (1.3) in below for the existence, extinction, persistence and oscillation of its solutions.

In this paper, we concern the following generalized stochastic logistic equations with piecewise constant argument of the form

$$dx\left(t\right)=x\left(t\right)\left\{\left(r\left(t\right)-a(t)x^{\theta}(t)-b(t)x^{\gamma}\left([t]\right)\right)dt+\sigma\left(t\right)dB\left(t\right)\right\} \tag{1.3}$$

and with initial condition $x(0) = x_0 > 0$. $x(t)$ means the population size at time t, $r(t)$ is called the intrinsic rate of growth, $a(t)$ is about the carrying capacity of environment, $b(t)x^{\gamma}([t])$ represents the control strategy made depending on the size of its population at time $[t]$ like the harvesting and so on, $\theta, \gamma \in R^+$, $B(t)$ is a standard Brown motion representing the effects induced by environmental noise on the natural growth, $\sigma(t)$ is the intensity of noise, and $[\cdot]$ denotes the greatest-integer function.

In fact, equation (1.3) stems from the famous logistic equation

$$x'(t)=rx(t)\left(1-\frac{x(t)}{K}\right), \tag{1.4}$$

where $r, K \in R^+$ denotes the growth rate and the carrying capacity of the environment, respectively. It is known that if $r < 0$, then $x(t) \to 0$ as $t \to \infty$; if $r > 0$, then $x(t)$ converges to K (see e.g. Murray [25]).

The remain part of this paper is organized as follows. In Section 2, we show the existence of the global positive solution of (1.3). In Section 3, we discuss the necessary and sufficient conditions for extinction and persistence of the population modeled by (1.3). In Section 4, the oscillation of solution is studied under assumption that all the coefficients are constants. Finally, examples with computer simulations are given to illustrate the obtained results.

For convenience, we introduce some notations.

$$R^+ = [0,+\infty); \quad \mathbf{N} = \{0,1,2,3,\cdots\cdots\},$$

$$x^* = \sup_{t\in R^+}\{x(t)\}, \quad x_* = \inf_{t\in R^+}\{x(t)\},$$

$$r_- = \liminf_{t\to\infty}\frac{1}{t}\int_0^t\left(r(s)-\frac{1}{2}\sigma^2(s)\right)ds,$$

$$r^+ = \limsup_{t\to\infty}\frac{1}{t}\int_0^t\left(r(s)-\frac{1}{2}\sigma^2(s)\right)ds,$$

$$x\vee y = \max\{x,y\}, \ x\wedge y = \min\{x,y\},$$

where $x, y \in R$; I_A is an indicator function.

Besides this, throughout this paper, we assume that:

$$\begin{aligned}&r(t)\in C(R,R) \text{ and } a(t), b(t), \sigma(t)\in C(R,R^+)\\ &\text{are all bounded functions with } b_*>0,\\ &\gamma>0.\end{aligned} \tag{1.5}$$

2. Global positive solution

Let $C(S_1, S_2)$ denote the set of all continuous functions $\varphi : S_1 \to S_2$. $B(t)$ is a standard Brownian motion on $\left(\Omega, F, \left(F^B(t)\right)_{t\geq 0}, P\right)$, where $F^B(t) = \sigma(B(s) : 0 \leq s \leq t)$. E(X) is the expectation of X. For each $x_0 \in R^+$, a solution of (1.3) with initial value x_0 is denoted by $x(t) = x(t, 0, x_0)$.

Theorem 2.1. *Suppose (1.5) holds and $x(0) = x_0 > 0$, then there exists a unique continuous positive solution of equation (1.3) on $t \geq 0$ with probability 1. Furthermore, for any $t \in R^+$ and $k \in [0, [t]]$, the solution can be denoted by*

$$x(t) = x(k)\,e^{\int_k^t\left(r(s)-a(s)x^\theta(s)-b(s)x^\gamma([s])-\frac{\sigma^2(s)}{2}\right)ds+\int_k^t\sigma(s)dB(s)}. \tag{2.1}$$

Proof. Let τ_e be the explosion time (see e.g., [22, 31]), then $\tau_e > 0$ *a.s.* since $x_0 > 0$. Denote

$$dx(t) = x(0)\,e^{\int_0^t\left(r(s)-a(s)x^\theta(s)-b(s)x^\gamma(0)-\frac{\sigma^2(s)}{2}\right)ds+\int_0^t\sigma(s)dB(s)}. \tag{2.2}$$

One can check by the Itô formula that (2.2) is a continuous positive solution of (1.3) for $t \in [0,1]\bigcap[0,\tau_e)$. Note that the coefficients of the equation are local Lipschitz continuous, thus there exists a unique continuous positive solution of (1.3) for $t \in [0,1]\bigcap[0,\tau_e)$ (see [22]), which can be represented by (2.2). Furthermore, $0 < x(1,\omega) < \infty$ for all $\omega \in \Omega_0 = \{\omega|\,\tau_e(\omega) > 1\}$.

For all $k \in \mathbf{N}$, similarly to the above, one can verify that there is a unique continuous positive solution of (1.3) for $t \in [k, k+1]\bigcap[0,\tau_e)$ by using the mathematical induction method, and the solution can be expressed as

$$x(t) = x(k)\,e^{\int_k^t\left(r(s)-a(s)x^\theta(s)-b(s)x^\gamma(k)-\frac{\sigma^2(s)}{2}\right)ds+\int_k^t\sigma(s)dB(s)}. \tag{2.3}$$

Thus (2.1) holds for $t \in [0, \tau_e)$.

To show that the solution is global, we need to prove that $\tau_e = \infty$ *a.s.*. Let $m_0 \in \mathbf{N}$ be sufficiently large such that $\frac{1}{m_0} \leq x_0 \leq m_0$. For any integer $m > m_0$, define the stopping time

$$\tau_m = \inf\left\{t \in [0, \tau_e) : x\left(t\right) \notin \left(\frac{1}{m}, m\right)\right\},$$

where throughout this paper we set $inf\emptyset = \infty$ (as usual $\emptyset$ denotes the empty set). Clearly, τ_m is increasing as $m \to \infty$. Set $\tau_\infty = \lim_{m\to\infty} \tau_m$, then $\tau_\infty \leq \tau_e$ *a.s.*. If $\tau_\infty = \infty$ is true, then $\tau_e = \infty$ *a.s.*. If $\tau_\infty = \infty$ *a.s.* is false, there are constants $T > 0$ and $\varepsilon \in (0, 1)$ such that $P\left(\tau_m \leq T\right) \geq \varepsilon$. For all $0 \leq t \leq \tau_m \wedge T$, define a C^2-function $V : R^+ \to R^+$ by

$$V\left(x\right) = 4\sqrt{x} - 4 - 2\ln\left(x\right),$$

which is not negative for $x > 0$.

Let $k > k_0$ and $T > 0$ be arbitrary. For $0 \leq t \leq \tau_k \wedge T$, using the Itô formula to (1.3), we get

$$\begin{aligned}
dV\left(x\right) =& 2\left(\frac{1}{\sqrt{x}} - \frac{1}{x}\right)dx - \frac{1}{2}\left(\frac{1}{\sqrt{x^3}} - \frac{2}{x^2}\right)\left(dx\right)^2 \\
=& \left\{2\left(\sqrt{x} - 1\right)\left(r\left(t\right) - a(t)x^\theta\left(t\right) - b(t)x^\gamma\left([t]\right)\right) + \frac{1}{2}\left(2 - \sqrt{x(t)}\right)\sigma^2\left(t\right)\right\}dt \\
&+ 2\left(\sqrt{x(t)} - 1\right)\sigma\left(t\right)dB\left(t\right) \\
\leq& \left\{-2a_* x^{\theta+1/2}\left(t\right) + 2r^*\sqrt{x\left(t\right)} + 2a^* x^\theta\left(t\right) + 2b^* x^\gamma\left([t]\right) + \left(\sigma^*\right)^2\right\}dt \\
&+ 2\left(\sqrt{x\left(t\right)} - 1\right)\sigma\left(t\right)dB\left(t\right).
\end{aligned}$$

Note that $a_* \geq 0$, it follows that

$$\begin{aligned}
dV\left(x\right) \leq& \left\{2r^*\sqrt{x\left(t\right)} + 2a^* x^\theta\left(t\right) + 2b^* x^\gamma\left([t]\right) + \left(\sigma^*\right)^2\right\}dt \\
&+ 2\left(\sqrt{x\left(t\right)} - 1\right)\sigma\left(t\right)dB\left(t\right).
\end{aligned}$$

Integrating both sides from 0 to $\tau_m \wedge T$, and taking expectations, it yields

$$\begin{aligned}
& E\left(V\left(x\left(\tau_m \wedge T\right)\right)\right) \\
\leq& V\left(x\left(0\right)\right) + E\left(\int_0^{\tau_m \wedge T}\left\{2r^*\sqrt{x\left(s\right)} + 2a^* x^\theta\left(s\right) + 2b^* x^\gamma\left([s]\right) + \left(\sigma^*\right)^2\right\}ds\right) \\
\leq& V\left(x\left(0\right)\right) + 2r^* E\left(\int_0^T \sqrt{x\left(s\right)} \cdot I_{\{s<\tau_m \wedge T\}}ds\right) + \left(\sigma^*\right)^2 T \\
&+ 2a^* E\left(\int_0^T x^\theta\left(s\right) \cdot I_{\{s<\tau_m \wedge T\}}ds\right) + 2b^* E\left(\int_0^T x^\gamma\left([s]\right) \cdot I_{\{s<\tau_m \wedge T\}}ds\right).
\end{aligned} \tag{2.4}$$

The following estimations are useful for our proof. By (2.1), for $s \in (0,1]$ we get that

$$
\begin{aligned}
& E\left(x^{\theta}(s)\cdot I_{\{s<\tau_m\wedge T\}}\right)\\
=& E\left(x^{\theta}(0)\, e^{\theta\int_0^s\left(r(u)-a(u)x^{\theta}(u)-b(u)x^{\gamma}(0)-\frac{\sigma^2(u)}{2}\right)du+\theta\int_0^s\sigma(u)dB(u)}\cdot I_{\{s<\tau_m\wedge T\}}\right)\\
\leq& E\left(x^{\theta}(0)\, e^{\theta\int_0^1 r(u)du+\theta^2\int_0^s\frac{\sigma^2(u)}{2}du-\theta^2\int_0^s\frac{\sigma^2(u)}{2}du+\theta\int_0^s\sigma(u)dB(u)}\cdot I_{\{s<\tau_m\wedge T\}}\right)\\
\leq& x^{\theta}(0)\, e^{\theta r^*+\theta^2\frac{(\sigma^*)^2}{2}} E\left(e^{-\theta^2\int_0^s\frac{\sigma^2(u)}{2}du+\theta\int_0^s\sigma(u)dB(u)}\cdot I_{\{s<\tau_m\wedge T\}}\right)\\
\leq& x^{\theta}(0)\, e^{\theta r^*+\theta^2\frac{(\sigma^*)^2}{2}} E\left(e^{-\theta^2\int_0^s\frac{\sigma^2(u)}{2}du+\theta\int_0^s\sigma(u)dB(u)}\right)\\
\leq& x^{\theta}(0)\, e^{\theta r^*+\theta^2\frac{(\sigma^*)^2}{2}},
\end{aligned}
\tag{2.5}
$$

which implies $E\left(x^{\frac{1}{2}}(s)\cdot I_{\{s<\tau_m\wedge T\}}\right)\leq x^{\frac{1}{2}}(0)\, e^{\frac{1}{2}r^*+\frac{1}{4}\frac{(\sigma^*)^2}{2}}$. As $s\in(1,T]$, there is

$$
\begin{aligned}
& E\left(x^{\theta}(s)\cdot I_{\{s<\tau_m\wedge T\}}\right)\\
=& E\Bigg(x^{\theta}([s]-1)\, e^{\theta\int_{[s]-1}^s\left(r(u)-a(u)x^{\theta}(u)-b(u)x^{\gamma}([u])-\frac{\sigma^2(u)}{2}\right)du+\theta\int_{[s]-1}^s\sigma(u)dB(u)}\\
& \times I_{\{s<\tau_m\wedge T\}}\Bigg)\\
\leq& E\left(x^{\theta}([s]-1)\, e^{\theta\int_{[s]-1}^s r(u)du-\theta x^{\gamma}([s]-1)\int_{[s]-1}^s b(u)du+\theta\int_{[s]-1}^s\sigma(u)dB(u)}\cdot I_{\{s<\tau_m\wedge T\}}\right)\\
\leq& E\Bigg(x^{\theta}([s]-1)\, e^{2\theta r^*-b_*\theta x^{\gamma}([s]-1)+\theta^2(\sigma^*)^2-\theta^2\int_{[s]-1}^s\frac{\sigma^2(u)}{2}du+\theta\int_{[s]-1}^s\sigma(u)dB(u)}\\
& \times I_{\{s<\tau_m\wedge T\}}\Bigg).
\end{aligned}
\tag{2.6}
$$

Let $f(z)=ze^{-b_*z^{\gamma}}$, then $f'(z)=e^{-b_*z^{\gamma}}\left(1-b_*\gamma z^{\gamma}\right)$. Clearly, it follows that

$$
\max_{z\geq 0}\{f(z)\}=f\left(\left(\frac{1}{b_*\gamma}\right)^{1/\gamma}\right)=\left(\frac{1}{b_*\gamma}\right)^{1/\gamma}e^{-\frac{1}{\gamma}}.
$$

By (2.6), we obtain

$$
\begin{aligned}
& E\left(x^{\theta}(s)\cdot I_{\{s<\tau_m\wedge T\}}\right)\\
\leq& E\left(f^{\theta}\left[x([s]-1)\right]e^{2\theta r^*+\theta^2(\sigma^*)^2-\theta^2\int_{[s]-1}^s\frac{\sigma^2(u)}{2}du+\theta\int_{[s]-1}^s\sigma(u)dB(u)}\cdot I_{\{s<\tau_m\wedge T\}}\right)
\end{aligned}
$$

$$
\begin{aligned}
&\leq\left(\frac{1}{b_*\gamma}\right)^{\theta/\gamma} e^{2\theta r^*-\frac{\theta}{\gamma}+\theta^2(\sigma^*)^2} E\left(e^{-\theta^2\int_{[s]-1}^{s}\frac{\sigma^2(u)}{2}du+\theta\int_{[s]-1}^{s}\sigma(u)dB(u)}\cdot I_{\{s<\tau_m\wedge T\}}\right)\\
&\leq\left(\frac{1}{b_*\gamma}\right)^{\theta/\gamma} e^{2\theta r^*-\frac{\theta}{\gamma}+\theta^2(\sigma^*)^2} E\left(e^{-\theta^2\int_{[s]-1}^{s}\frac{\sigma^2(u)}{2}du+\theta\int_{[s]-1}^{s}\sigma(u)dB(u)}\right)\\
&=\left(\frac{1}{b_*\gamma}\right)^{\theta/\gamma} e^{2\theta r^*-\frac{\theta}{\gamma}+\theta^2(\sigma^*)^2},
\end{aligned}
$$

where the last term is derived by using the property of the exponential martingale. Denote

$$
H_\theta=\left(x^\theta(0)\,e^{\theta r^*+\theta^2\frac{(\sigma^*)^2}{2}}\right)\vee\left(\left(\frac{1}{b_*\gamma}\right)^{\theta/\gamma} e^{2\theta r^*-\frac{\theta}{\gamma}+\theta^2(\sigma^*)^2}\right), \tag{2.7}
$$

then for $s\in[0,T]$,

$$
E\left(\sqrt{x(s)}\cdot I_{\{s<\tau_m\wedge T\}}\right)\leq H_{1/2}. \tag{2.8}
$$

Now we consider the last formula in (2.4).

For $s\in(0,1)$,

$$
E\left(x^\gamma([s])\cdot I_{\{s<\tau_m\wedge T\}}\right)=E\left(x^\gamma(0)\cdot I_{\{s<\tau_m\wedge T\}}\right)\leq x^\gamma(0). \tag{2.9}
$$

When $s\in[1,T)$, similarly to (2.6) we have

$$
\begin{aligned}
&E\left(x^\gamma([s])\cdot I_{\{s<\tau_m\wedge T\}}\right)\\
&\leq E\left(x^\gamma([s]-1)\,e^{\gamma\int_{[s]-1}^{[s]}\left(r(u)-b(u)x^\gamma([u]-1)-\frac{\sigma^2(u)}{2}\right)du+\gamma\int_{[s]-1}^{[s]}\sigma(u)dB(u)}\right.\\
&\quad\left.\times I_{\{s<\tau_m\wedge T\}}\right)\\
&\leq E\left(x^\gamma([s]-1)\,e^{\gamma\int_{[s]-1}^{[s]}r(u)du-\gamma x^\gamma([s]-1)\int_{[s]-1}^{[s]}b(u)du+\gamma^2\int_{[s]-1}^{[s]}\frac{\sigma^2(u)}{2}du}\right.\\
&\quad\left.\times e^{-\gamma^2\int_{[s]-1}^{[s]}\frac{\sigma^2(u)}{2}du+\gamma\int_{[s]-1}^{[s]}\sigma(u)dB(u)\cdot I_{\{s<\tau_m\wedge T\}}}\right)\\
&\leq E\left(x^\gamma([s]-1)\,e^{\gamma r^*-b_*\gamma x^\gamma([s]-1)+\frac{1}{2}\gamma^2(\sigma^*)^2-\gamma^2\int_{[s]-1}^{[s]}\frac{\sigma^2(u)}{2}du+\gamma\int_{[s]-1}^{[s]}\sigma(u)dB(u)}\right.\\
&\quad\left.\times I_{\{s<\tau_m\wedge T\}}\right)\\
&\leq\frac{1}{b_*\gamma}e^{\gamma r^*-1+\frac{1}{2}\gamma^2(\sigma^*)^2}E\left(e^{-\gamma^2\int_{[s]-1}^{[s]}\frac{\sigma^2(u)}{2}du+\gamma\int_{[s]-1}^{[s]}\sigma(u)dB(u)}\cdot I_{\{s<\tau_m\wedge T\}}\right)\\
&\leq\frac{1}{b_*\gamma}e^{\gamma r^*-1+\frac{1}{2}\gamma^2(\sigma^*)^2}.
\end{aligned} \tag{2.10}
$$

Denote

$$G_\gamma = x^\gamma(0) \vee \left(\frac{1}{b_*\gamma} e^{\gamma r^* - 1 + \frac{1}{2}\gamma^2(\sigma^*)^2}\right). \tag{2.11}$$

Substituting (2.5)-(2.11) into (2.4), we obtain

$$E(V(x(\tau_m \wedge T))) \le V(x(0)) + (\sigma^*)^2 T + 2\left\{r^* H_{1/2} + a^* H_\theta + b^* G_\gamma\right\} T. \tag{2.12}$$

Note that for every $\omega \in \{\omega : \tau_m(\omega) \le T\}$, $x(\tau_m,\ \omega) = m$ or $\frac{1}{m}$, then

$$\begin{aligned} E(V(x(\tau_m \le T))) =& E\left(I_{\{\tau_m \le T\}} V(x(\tau_m))\right) \\ =& \left\{(4m - 4 - 2\ln m) \wedge \left(4\frac{1}{m} - 4 + 2\ln m\right)\right\} P(\tau_m \le T) \\ \ge& \left\{(4m - 4 - 2\ln m) \wedge \left(4\frac{1}{m} - 4 + 2\ln m\right)\right\} \varepsilon \to \infty \end{aligned}$$

as $k \to \infty$, which contradicts with (2.12). Therefore, $\tau_\infty = \infty$ $a.s.$ holds, and hence $\tau_e = \infty$ $a.s.$. The proof is complete. □

From (2.5)-(2.11), one can also obtain the boundness result of the solution.

Theorem 2.2. *Suppose that (1.5) holds and $x(t)$ is a solution of equation (1.3) with $x(0) = x_0 > 0$.*

(i) If $p \in (0,1)$, then $E(x^p(t)) \le \max\left\{x_0^p e^{pr^*}, \left(\frac{1}{b_*\gamma}\right)^{p/\gamma} e^{2pr^* - \frac{p}{\gamma} - \frac{1}{2}(p - p^2)(\sigma_*)^2}\right\}$.

(ii) If $p \ge 1$, then

$E(x^p(t)) \le \max\left\{x_0^p e^{pr^* + (p^2 - p)(\sigma^*)^2}, \left(\frac{1}{b_*\gamma}\right)^{p/\gamma} e^{2pr^* - \frac{p}{\gamma} + (p^2 - p)(\sigma^*)^2}\right\}$.

In particular, $E(x(k)) \le \left(\frac{1}{b_*\gamma}\right)^{1/\gamma} e^{r^* - \frac{1}{\gamma}}$ *for all integer number $k \ge 1$.*

Proof. It can be proved from (2.5)-(2.11), we omit the details here. □

3. Survival analysis

In this section, we discuss the conditions for the extinction and persistence of the population. We first list some definitions used below.

Definition 3.1.

(1) The population $x(t)$ modeled by (1.3) is said to be extinct a.s. if $\lim\limits_{t\to\infty} x(t) = 0$ a.s. (almost surely).

(2) The population $x(t)$ modeled by (1.3) is said to be persistent a.s. if there exist constants Λ_0 and Λ_1 such that $0 < \Lambda_0 \le \liminf\limits_{t\to\infty} x(t) \le \limsup\limits_{t\to\infty} x(t) \le \Lambda_1 < \infty$ $a.s.$.

(3) Let $p > 0$ be a constant, the population $x(t)$ modeled by (1.3) is said to be p-persistent a.s. by time average if there exist constants H_0 and H_1 such that $0 < \mathrm{H}_0 \le \liminf\limits_{t\to\infty} \frac{1}{t}\int_0^t x^p(s)\,ds \le \limsup\limits_{t\to\infty} \frac{1}{t}\int_0^t x^p(s)\,ds \le \mathrm{H}_1 < \infty$ $a.s.$.

Remark 3.1. In (3) of Definition 3.1, we call $x(t)$ to be persistent a.s. by time average if $p = 1$.

3.1. Extinction

Theorem 3.1. *Suppose that (1.5) holds and $x(t)$ is a solution of equation (1.3) with $x(0) = x_0 > 0$. If $r^+ < 0$, the population $x(t)$ will be almost surely extinct, i.e. $\lim_{t\to\infty} x(t) = 0$ a.s.. Moreover, $x(t)$ converges exponentially to 0.*

Proof. By using the *Itô* formula, we have from (1.3)

$$\begin{aligned} d\ln x =& \frac{1}{x}dx - \frac{1}{2x^2}(dx)^2 \\ =& \left\{ r(t) - a(t)x^\theta(t) - b(t)x^\gamma([t]) - \frac{1}{2}\sigma^2(t) \right\} dt + \sigma(t)\, dB(t) \\ \le& \left\{ r(t) - \frac{1}{2}\sigma^2(t) \right\} dt + \sigma(t)\, dB(t). \end{aligned}$$

It follows that

$$\ln x(t) \le \ln x(0) + \int_0^t \left(r(s) - \frac{1}{2}\sigma^2(s) \right) ds + \int_0^t \sigma(s)\, dB(s), \tag{3.1}$$

where $M(t) = \int_0^t \sigma(s)\, dB(s)$ is continuously local martingale with

$$\langle M(t), M(t) \rangle = \int_0^t \sigma^2(s)\, ds \le (\sigma^*)^2 t.$$

An application of the strong law of large numbers for Brownian motion gives that

$$\lim_{t\to\infty} \frac{M(t)}{t} = 0 \ a.s. \tag{3.2}$$

Then from (3.1), $r^+ < 0$ implies that the population $x(t)$ will almost surely die out with exponential rate r^+. □

Theorem 3.2. *Suppose that (1.5) holds and $x(t)$ is a solution of equation (1.3) with $x(0) = x_0 > 0$. If $r^+ = 0$, and one of the following two conditions is satisfied*
(1) $\lim_{t\to\infty} \int_0^t \sigma^2(u)\, du = \infty$ and $\limsup_{t\to\infty} \left(\int_0^t \left(r(u) - \frac{\sigma^2(u)}{8} \right) du \right) \le 0$;
(2) $\limsup_{t\to\infty} \int_0^t r(u)\, du = -\infty$;
the population will be almost surely extinct.

Proof. Under the condition (1), the solution (2.1) shows that

$$\begin{aligned} \sqrt{x(t)} =& \sqrt{x_0} e^{\frac{1}{2}\int_0^t \left(r(u) - a(u)x^\theta(u) - b(u)x^\gamma([u]) - \frac{\sigma^2(u)}{2} \right) du + \frac{1}{2}\int_0^t \sigma(u) dB(u)} \\ \le& \sqrt{x_0} e^{\frac{1}{2}\int_0^t \left(r(u) - \frac{\sigma^2(u)}{8} \right) du - \frac{1}{2}\int_0^t \frac{\sigma^2(u)}{8} du + \left\{ -\frac{1}{4}\int_0^t \frac{\sigma^2(u)}{2} du + \frac{1}{2}\int_0^t \sigma(u) dB(u) \right\}}. \end{aligned} \tag{3.3}$$

Denote $\tilde{M}(t) = e^{-\frac{1}{4}\int_0^t \frac{\sigma^2(u)}{2} du + \frac{1}{2}\int_0^t \sigma(u) dB(u)}$ for $t > 0$, then $\tilde{M}(t)$ is a nonnegative martingale and converges to some finite random variable $\tilde{M}_\infty$ as $t \to \infty$ due to the Doob's martingale convergence theorem (see [11]). Therefore there exists an a.s. finite random variable $L = L(\omega) > 0$ such that

$$\tilde{M}(t)(\omega) < L(\omega) \text{ for all } t > 0. \tag{3.4}$$

This, together with (1.1) and (3.3), gives that

$$\sqrt{x\left(t\right)} \leq \sqrt{x_0} L e^{-\frac{1}{2}\int_0^t \frac{\sigma^2(u)}{8} du} \to 0 \tag{3.5}$$

as $t \to \infty$. Notice $f\left(x\right) = \sqrt{x}$ is a continuous and increasing function for $x \geq 0$, (3.5) implies that

$$\lim_{t\to\infty} x\left(t\right) = 0 \ a.s.$$

Under the condition (2), now the solution (2.1) indicates

$$\begin{aligned} x\left(t\right) =& x_0 e^{\int_0^t \left(r(u)-a(u)x^\theta(u)-b(u)x^\gamma([u])-\frac{\sigma^2(u)}{2}\right)du+\int_0^t \sigma(u)dB(u)} \\ \leq& x_0 e^{\int_0^t r(u)du-\int_0^t \frac{\sigma^2(u)}{2}du+\int_0^t \sigma(u)dB(u)}. \end{aligned}$$

The left proof is similar to that of (1), we omit it. □

3.2. Persistence

Theorem 3.3. *Suppose that (1.5) holds and $x(t)$ is a solution of equation (1.3) with $x(0) = x_0 > 0$, then $r^+ > 0$ is necessary for almost sure persistence of the population.*

Proof. From Theorem 3.1, it remains to prove that the population can not be persistent under the assumption: $r^+ = 0$. We prove it by the indirect method. Suppose that the population is persistent a.s., that is, there exists a constant $c > 0$ such that $\lim_{t\to\infty} \inf x\left(t\right) \geq c$ a.s., then by (2.1),

$$\begin{aligned} x\left(t\right) =& x_0 e^{\int_0^t \left(r(u)-a(u)x^\theta(u)-b(u)x^\gamma([u])-\frac{\sigma^2(u)}{2}\right)du+\int_0^t \sigma(u)dB(u)} \\ \leq& x_0 e^{\int_0^t \left(r(u)-\frac{\sigma^2(u)}{2}\right)du-a_*\int_0^t x^\theta(u)du-b_*\int_0^t x^\gamma(u)du+\int_0^t \sigma(u)dB(u)}. \end{aligned}$$

Since (3.2) and $r^+ = 0$ are valid, for any $\varepsilon > 0$, we have

$$\int_0^t \left(r\left(u\right) - \frac{\sigma^2\left(u\right)}{2}\right) du + \int_0^t \sigma\left(u\right) dB\left(u\right) \leq 2\varepsilon t$$

for t sufficiently large. Recall (1.5), it follows

$$x\left(t\right) \leq x_0 e^{-\left(a_* c^\theta + b_* c^\gamma - 2\varepsilon\right)t} \to 0$$

as $t \to \infty$. This is a contradiction with $\lim_{t\to\infty} \inf x\left(t\right) \geq c > 0$ a.s. □

From Theorem 3.3, we immediately get,

Theorem 3.4. *Suppose that (1.5) holds and $x(t)$ is a solution of equation (1.3) with $x(0) = x_0 > 0$, then $r^+ > 0$ is a necessary condition for the population to be p-persistent a.s. by time average.*

Theorem 3.5. *Suppose that (1.5) holds and $x(t)$ is a solution of equation (1.3) with $x(0) = x_0 > 0$. If $r_- > 0$, the population $x(t)$ will be p-persistent a.s. by time average.*

Proof. Taking integration on formula (2.1) gives that for any $k \in \mathbf{N}$,

$$\begin{aligned}
&\int_k^{k+1} x^p(s)\,ds \\
=&\int_k^{k+1} x^p(k)\, e^{p\int_k^s \left(r(u)-a(u)x^\theta(u)-b(u)x^\gamma(k)-\frac{\sigma^2(u)}{2}\right)du+p\int_k^s \sigma(u)dB(u)} ds \qquad (3.6)\\
\le&\left(\frac{1}{b_*\gamma}\right)^{p/\gamma} e^{pr^*-\frac{p}{\gamma}+p^2(\sigma^*)^2}\cdot \int_k^{k+1} e^{-p^2\int_k^s \frac{\sigma^2(u)}{2}du+p\int_k^s \sigma(u)dB(u)} ds.
\end{aligned}$$

Denote $K_p=\left(\frac{1}{b_*\gamma}\right)^{p/\gamma} e^{pr^*-\frac{p}{\gamma}}$ and $M_p(k)=\int_k^{k+1} e^{-p^2\int_k^s \frac{\sigma^2(u)}{2}du+p\int_k^s \sigma(u)dB(u)} ds$, then $M_p(k)$, $k=0,1,2\cdots$ are independent of each other, with

$$E\left(M_p(k)\right)=\int_k^{k+1} E\left(e^{-p^2\int_k^s \frac{\sigma^2(u)}{2}du+p\int_k^s \sigma(u)dB(u)}\right) ds=1.$$

Due to Cauchy inequality: $\left(\int_a^b |f\cdot g|\,ds\right)^2 \le \int_a^b f^2 ds\cdot\int_a^b g^2 ds$, we have

$$\begin{aligned}
E\left(M_p^4(k)\right)=&E\left(\int_k^{k+1} e^{-p^2\int_k^s \frac{\sigma^2(u)}{2}du+p\int_k^s \sigma(u)dB(u)} ds\right)^4\\
\le&E\left(\int_k^{k+1} e^{-2p^2\int_k^s \frac{\sigma^2(u)}{2}du+2p\int_k^s \sigma(u)dB(u)} ds\right)^2\\
\le&E\left(\int_k^{k+1} e^{-4p^2\int_k^s \frac{\sigma^2(u)}{2}du+4p\int_k^s \sigma(u)dB(u)} ds\right) \qquad (3.7)\\
\le&\int_k^{k+1} e^{12p^2\int_k^s \frac{\sigma^2(u)}{2}du} ds\\
\le&e^{6p^2(\sigma^*)^2}.
\end{aligned}$$

The strong law of large numbers indicates

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n} M_p(k)=1\ a.s., \tag{3.8}$$

which implies part of the desired result:

$$\limsup_{t\to\infty}\frac{1}{t}\int_0^t x^p(s)\,ds \le K_p e^{p^2(\sigma^*)^2}\limsup_{t\to\infty}\frac{1}{t}\sum_{k=0}^{[t]} M_p(k)=K_p e^{p^2(\sigma^*)^2}\ a.s. \tag{3.9}$$

On the other hand, for any $t\in[k,k+1]$, the solution (2.1) tells

$$\begin{aligned}
x^p(t)=&x^p(k)\, e^{p\int_k^t \left(r(u)-a(u)x^\theta(u)-b(u)x^\gamma(k)-\frac{\sigma^2(u)}{2}\right)du+p\int_k^t \sigma(u)dB(u)}\\
\le&K_p e^{-p\int_k^t \frac{\sigma^2(u)}{2}du+p\int_k^t \sigma(u)dB(u)}.
\end{aligned} \qquad (3.10)$$

Or,

$$\frac{1}{x^p(t)} \geq \frac{1}{K_p} e^{p\int_k^t \frac{\sigma^2(u)}{2}du - p\int_k^t \sigma(u)dB(u)}. \tag{3.11}$$

Hence we have

$$x^p(t) = x^p(0)\, e^{p\int_0^t \left(r(u)-a(u)x^\theta(u)-b(u)x^\gamma([u])-\frac{\sigma^2(u)}{2}\right)du + p\int_0^t \sigma(u)dB(u)}. \tag{3.12}$$

Substituting (3.11) into (3.12),

$$\begin{aligned} e^{p\int_0^t \left(a(u)x^\theta(u)+b(u)x^\gamma([u])\right)du} =& \frac{x^p(0)}{x^p(t)} e^{p\int_0^t \left(r(u)-\frac{\sigma^2(u)}{2}\right)du + p\int_0^t \sigma(u)dB(u)} \\ \geq& \frac{x^p(0)}{K_p} e^{p\int_0^k \left(r(u)-\frac{\sigma^2(u)}{2}\right)du + p\int_0^k \sigma(u)dB(u)}. \end{aligned}$$

Compute that

$$\begin{aligned} &\int_0^t \left(a(u)x^\theta(u)+b(u)x^\gamma([u])\right)du \\ \geq& \frac{1}{p}\ln\frac{x^p(0)}{K_p} + \int_0^k \left(r(u)-\frac{\sigma^2(u)}{2}\right)du + \int_0^k \sigma(u)\,dB(u). \end{aligned} \tag{3.13}$$

With (3.2) and take limits on both sides of (3.13), we can get

$$\begin{aligned} \liminf_{t\to\infty}\frac{1}{t}\int_0^t \left(a(u)x^\theta(u)+b(u)x^\gamma([u])\right)du \geq& \liminf_{t\to\infty}\frac{1}{t}\int_0^{[t]}\left(r(u)-\frac{\sigma^2(u)}{2}\right)du \\ =& r_- \ a.s. \end{aligned} \tag{3.14}$$

Next we will prove that there exist constants $\hbar > 0$ and $0 < \xi < \frac{r_-}{2}$ such that

$$\hbar \liminf_{t\to\infty}\frac{1}{t}\int_0^t x(s)\,ds + \xi \geq \liminf_{t\to\infty}\frac{1}{t}\int_0^t \left(a(u)x^\theta(u)+b(u)x^\gamma([u])\right)du \ a.s., \tag{3.15}$$

holds, which suffices to prove the left of the desired result.

Let $q > 0$ be a constant satisfying $q > \max\left\{1, p, \frac{p}{\theta}, \frac{p}{2\gamma}\right\}$. Based on Young inequality, we have that

$$\begin{aligned} \int_0^t a(u)x^\theta(u)du =& \int_0^t a(u)\left(\frac{1}{\alpha}x^{\frac{p}{q}}(u)\right)\left(\alpha x^{\theta-\frac{p}{q}}(u)\right)du \\ \leq& \int_0^t a(u)\left\{\frac{1}{q\alpha^q}x^p(u) + \frac{q-1}{q}\alpha^{\frac{q}{q-1}}x^{\frac{\theta q-p}{q-1}}(u)\right\}du \\ \leq& \frac{a^*}{q\alpha^q}\int_0^t x^p(u)du + \frac{a^*(q-1)}{q}\alpha^{\frac{q}{q-1}}\int_0^t x^{\frac{\theta q-p}{q-1}}(u)du, \end{aligned} \tag{3.16}$$

where $\alpha > 0$ is a constant to be chosen later, and it is the same for positive constants

β and δ in the sequel. Now

$$
\begin{aligned}
&\int_0^t b(u)x^\gamma([u])\,du \\
&\le \sum_{k=0}^{[t]} x^\gamma(k)\int_k^{k+1} b(u)du \le b^*\sum_{k=0}^{[t]} x^\gamma(k) = b^*x_0 + b^*\sum_{k=1}^{[t]} x^\gamma(k) \\
&\le b^*x_0 + b^*\sum_{k=1}^{[t]}\int_{k-1}^k x^\gamma(k)\,ds \\
&= b^*x_0 + b^* \\
&\quad\times \sum_{k=1}^{[t]}\int_{k-1}^k x^\gamma(s)\,e^{\gamma\int_s^k\left(r(u)-\frac{\sigma^2(u)}{2}\right)du-\gamma\int_s^k\left(a(u)x^\theta(u)+b(u)x^\gamma([u])\right)du+\gamma\int_s^k\sigma(u)dB(u)}ds \\
&\le b^*x_0 + \frac{b^*}{2}\sum_{k=1}^{[t]}\left\{\int_{k-1}^k \frac{x^{2\gamma}(s)}{\delta^2}ds + \delta^2\int_{k-1}^k e^{2\gamma\int_s^k\left(r(u)-\frac{\sigma^2(u)}{2}\right)du+2\gamma\int_s^k\sigma(u)dB(u)}ds\right\} \\
&\le b^*x_0 + \frac{b^*}{2\delta^2}\sum_{k=1}^{[t]}\int_{k-1}^k x^{2\gamma}(s)\,ds + \frac{b^*\delta^2}{2}e^{2\gamma\left\{r^*+(\sigma^*)^2\right\}}\sum_{k=1}^{[t]}\hat{M}_{2\gamma}(k),
\end{aligned} \tag{3.17}
$$

where $\hat{M}_{2\gamma}(k) = \int_{k-1}^k e^{-4\gamma^2\int_s^k\frac{\sigma^2(u)}{2}du+2\gamma\int_s^k\sigma(u)dB(u)}ds$. Since

$$
\begin{aligned}
\sum_{k=1}^{[t]}\int_{k-1}^k x^{2\gamma}(s)\,ds &= \int_0^{[t]} x^{2\gamma}(s)\,ds \\
&\le \int_0^{[t]}\left\{\frac{1}{q\beta^q}x^p(s) + \frac{\beta^{\frac{q}{q-1}}(q-1)}{q}x^{\frac{2q\gamma-p}{q-1}}(s)\right\}ds \\
&\le \frac{1}{q\beta^q}\int_0^t x^p(s)\,ds + \frac{\beta^{\frac{q}{q-1}}(q-1)}{q}\int_0^{[t]} x^{\frac{2q\gamma-p}{q-1}}(s)\,ds
\end{aligned} \tag{3.18}
$$

holds, combining (3.16)-(3.18) we obtain

$$
\begin{aligned}
&\int_0^t\left(a(u)x^\theta(u)+b(u)x^\gamma([u])\right)du \\
&\le\left(\frac{a^*}{q\alpha^q}+\frac{b^*}{2q\beta^q\delta^2}\right)\int_0^t x^p(u)du + \frac{a^*(q-1)}{q}\alpha^{\frac{q}{q-1}}\int_0^t x^{\frac{\theta q-p}{q-1}}(u)du + b^*x_0 \\
&\quad+\frac{b^*}{2\delta^2}\frac{\beta^{\frac{q}{q-1}}(q-1)}{q}\int_0^{[t]} x^{\frac{2q\gamma-p}{q-1}}(s)\,ds + \frac{b^*\delta^2}{2}e^{2\gamma\left\{r^*+(\sigma^*)^2\right\}}\sum_{k=1}^{[t]}\hat{M}_{2\gamma}(k).
\end{aligned} \tag{3.19}
$$

Similarly to the proof of (3.8) and (3.9), one may get the following results:

$$\lim_{t\to\infty} \frac{1}{t} \sum_{k=1}^{[t]} \int_{k-1}^{k} e^{-4\gamma^2 \int_s^k \frac{\sigma^2(u)}{2} du + 2\gamma \int_s^k \sigma(u) dB(u)} ds = 1 \ a.s., \tag{3.20}$$

$$\limsup_{t\to\infty} \frac{1}{t} \int_0^t x^{\frac{\theta q - p}{q-1}}(s)\, ds \le K_{\frac{\theta q - p}{q-1}} e^{\left(\frac{\theta q - p}{q-1}\right)^2 (\sigma^*)^2} \ a.s., \tag{3.21}$$

$$\limsup_{t\to\infty} \frac{1}{t} \int_0^{[t]} x^{\frac{2q\gamma - p}{q-1}}(s)\, ds \le K_{\frac{2q\gamma - p}{q-1}} e^{\left(\frac{2q\gamma - p}{q-1}\right)^2 (\sigma^*)^2} \ a.s. \tag{3.22}$$

Put (3.20)-(3.22) into (3.19) and take limit, we have

$$\begin{aligned}
&\liminf_{x\to\infty} \frac{1}{t} \int_0^t \left(a(u)x^\theta(u) + b(u)x^\gamma([u])\right) du \\
\le& \hbar \liminf_{x\to\infty} \frac{1}{t} \int_0^t x^p(u) du + \frac{a^*(q-1)}{q} \alpha^{\frac{q}{q-1}} K_{\frac{\theta q - p}{q-1}} e^{\left(\frac{\theta q - p}{q-1}\right)^2 (\sigma^*)^2} \\
&+ \frac{b^*}{2\delta^2} \frac{\beta^{\frac{q}{q-1}}(q-1)}{q} K_{\frac{2q\gamma - p}{q-1}} e^{\left(\frac{2q\gamma - p}{q-1}\right)^2 (\sigma^*)^2} + \frac{b^*\delta^2}{2} e^{2\gamma\left\{r^* + (\sigma^*)^2\right\}} \ a.s.,
\end{aligned} \tag{3.23}$$

where $\hbar = \frac{a^*}{q\alpha^q} + \frac{b^*}{2q\beta^q\delta^2}$. Choose α, δ, β small enough such that

$$\begin{aligned}
\frac{r_-}{2} >& \frac{a^*(q-1)}{q} \alpha^{\frac{q}{q-1}} K_{\frac{\theta q - p}{q-1}} e^{\left(\frac{\theta q - p}{q-1}\right)^2 (\sigma^*)^2} \\
&+ \frac{b^*}{2\delta^2} \frac{\beta^{\frac{q}{q-1}}(q-1)}{q} K_{\frac{2q\gamma - p}{q-1}} e^{\left(\frac{2q\gamma - p}{q-1}\right)^2 (\sigma^*)^2} + \frac{b^*\delta^2}{2} e^{2\gamma\left\{r^* + (\sigma^*)^2\right\}},
\end{aligned} \tag{3.24}$$

then (3.15) holds, and hence

$$\liminf_{x\to\infty} \frac{1}{t} \int_0^t x^p(u) du \ge \frac{r_-}{2\hbar} \ a.s.$$

□

Remark 3.2. Let $r(t) \equiv r$ and $\sigma(t) \equiv \sigma$ be constants. Theorem 3.1 shows that if $r < \frac{\sigma^2}{2}$, the population modeled by (4.1) becomes a.s. extinct, and Theorem 3.5 shows that if $r > \frac{\sigma^2}{2}$, the population tends to be persistent by average. So $\frac{\sigma^2}{2}$ is the critical number between the extinction and persistence by average.

Remark 3.3. Let $r(t) \equiv r$ and $\sigma(t) \equiv \sigma$ be constants, and (1.5) holds. Theorem 3.4 and 3.5 indicate that the population becomes p-persistent a.s. by time average if and only if $r > \frac{\sigma^2}{2}$.

4. Oscillation

In this section, we study the oscillation property about solutions of the stochastic logistic equation with piecewise constant argument. For convenience, we suppose that all of the coefficients are constants. The equation is of the form:

$$dx(t) = x(t) \left\{ \left(r - ax^\theta(t) - bx^\gamma([t])\right) dt + \sigma dB(t) \right\}, \tag{4.1}$$

where r, a, b, θ, γ and σ are all positive constants. One definition stemming from the deterministic equation is given in the following.

Definition 4.1. Let $v > 0$ be a constant. Then the solution $x(t)$ of (4.1) is said to be oscillating around w a.s. if there exists $\Omega_1 \subset \Omega$ such that $P(\Omega_1) = 1$, and for any $\omega \in \Omega_1$ and $T > 0$, there are constants $T_1 = T_1(\omega) > T$ and $T_2 = T_2(\omega) > T$ such that $x(T_1)(\omega) < v$ and $x(T_2)(\omega) > v$.

Theorem 4.1. *Let $x(t)$ be a solution of (4.1) with $x(0) = x_0 > 0$. If $r > \frac{\sigma^2}{2}$ holds, then $x(t)$ oscillates around $\widetilde{x}$, which is a unique solution of equation:*

$$r - a\widetilde{x}^{\theta} - b\widetilde{x}^{\gamma} = \frac{\sigma^2}{2}. \tag{4.2}$$

Proof. By Theorem 2.1, $x(t)$ is the unique positive solution of (4.1). Define $f(x) = r - \frac{\sigma^2}{2} - ax^{\theta} - bx^{\gamma}$, there is

$$f(0) = r - \frac{\sigma^2}{2} > 0, \ \lim_{x\to\infty} f(x) = -\infty.$$

It means the existence of a solution of (4.2). Furthermore,

$$f'(x) = -a\theta x^{\theta-1} - b\gamma x^{\gamma-1} < 0$$

for all $x > 0$, and hence the solution of (4.2), denoted by $\widetilde{x}$, is unique.

To prove that the solution of (4.1) oscillates around the positive point $\widetilde{x}$. We first assume that $x(t) \geq \widetilde{x}$ *a.s.* for some constant $T = T(\omega) \geq 0$. Denote $\varphi(t) = x(t)e^{\sigma B(t)}$, from (4.1) we get

$$\begin{aligned}
\varphi(t) =& \varphi(T+1)\, e^{\int_{T+1}^{t}\left(r - ax^{\theta}(u) - bx^{\gamma}([u]) - \frac{\sigma^2}{2}\right)du} \\
\leq& \varphi(T+1) \\
=& x_0 e^{\int_0^{T+1}\left(r - ax^{\theta}(u) - bx^{\gamma}([u]) - \frac{\sigma^2}{2}\right)du + 2\sigma B(T+1)} \\
\leq& x_0 e^{r(T+1) + 2\sigma B(T+1)} \\
=& x_0 e^{\left(r+2\sigma^2\right)(T+1)} Q_2(T+1)
\end{aligned} \tag{4.3}$$

for all $t \geq T+1$, where function $Q_\lambda(t) = e^{-\frac{\lambda^2\sigma^2}{2}t + \lambda\sigma B(t)}$ for $t > 0$ and $\lambda > 0$. So there must be an a.s. finite random variable $\Gamma_\lambda = \Gamma_\lambda(\omega) > 0$ satisfying $Q_\lambda(t)(\omega) < \Gamma_\lambda(\omega)$ for all $t > 0$. It follows from (4.3) that

$$x(t) \leq x_0 e^{\left(r+2\sigma^2\right)(T+1)} \Gamma_2 e^{-\sigma B(t)}.$$

Therefore

$$\liminf_{t\to\infty} x(t) \leq \liminf_{t\to\infty} x_0 e^{\left(r+2\sigma^2\right)(T+1)} \Gamma_2 e^{-\sigma B(t)} = 0 \ a.s., \tag{4.4}$$

by the law of the iterated logarithm: $\limsup\limits_{t\to\infty} \frac{B(t)}{\sqrt{t \ln\ln t}} = 1$ *a.s.*. Now we get a contradiction between (4.4) and the assumption $x(t) \geq \widetilde{x} > 0$ *a.s.* for $T = T(\omega)$.

Secondly, we assume that $x(t) \leq \widetilde{x}$ $a.s.$ for some constant $T_2 = T_2(\omega) \geq 0$. Define

$$\widehat{T}(\omega) = \inf\left\{ t \mid x(s,\omega) \leq \widetilde{x} \text{ for all } s > t \text{ and } t \in R^{+} \right\}.$$

It's clear that $0 \leq \widehat{T}(\omega) < \infty$ $a.s.$ and $x\left(\widehat{T}(\omega)\right) = \begin{cases} \widetilde{x} \text{ for } \widehat{T}(\omega) > 0, \\ x_0 \text{ for } \widehat{T}(\omega) = 0. \end{cases}$ There are two cases. If $\widehat{T}(\omega) = 0$,

$$\varphi(t) = \varphi(0)\, e^{\int_0^t \left(r - ax^{\theta}(u) - bx^{\gamma}([u]) - \frac{\sigma^2}{2}\right) du} \geq x_0.$$

It follows that

$$x(t) \geq x_0 e^{-\sigma B(t)}. \tag{4.5}$$

If $\widehat{T}(\omega) > 0$, by (4.1)

$$\begin{aligned} x^{\gamma}\left(\left[\hat{T}\right]\right) =& x^{\gamma}\left(\left[\hat{T} - 1\right]\right) e^{\gamma \int_{[\hat{T}-1]}^{[\hat{T}]} \left(r - ax^{\theta}(u) - bx^{\gamma}\left(\left[\hat{T}-1\right]\right) - \frac{\sigma^2}{2}\right) du + \sigma\gamma\left(B\left(\left[\hat{T}\right]\right) - B\left(\left[\hat{T}-1\right]\right)\right)} \\ \leq& \frac{1}{b\gamma} e^{\gamma r - 1} \cdot e^{\sigma^2\gamma^2[\hat{T}]} \cdot e^{\left\{-\frac{\sigma^2\gamma^2}{2}\left[\hat{T}\right] + \sigma\gamma B\left(\left[\hat{T}\right]\right)\right\} + \left\{-\frac{\sigma^2\gamma^2}{2}\left[\hat{T}-1\right] - \sigma\gamma B\left(\left[\hat{T}-1\right]\right)\right\}} \\ \leq& \frac{1}{b\gamma} e^{\gamma r - 1 + \sigma^2\gamma^2\hat{T}} \Gamma_{\gamma}^2 \\ :=& \widetilde{\Gamma}, \end{aligned}$$

then $\widetilde{\Gamma} < \infty$ $a.s.$ and

$$\begin{aligned} \varphi(t) =& \varphi\left(\hat{T}\right) e^{\int_{\hat{T}}^{t} \left(r - ax^{\theta}(u) - bx^{\gamma}([u]) - \frac{\sigma^2}{2}\right) du} \\ =& \varphi\left(\hat{T}\right) e^{\int_{\hat{T}}^{[\hat{T}]+1} \left(r - ax^{\theta}(u) - bx^{\gamma}\left(\left[\hat{T}\right]\right) - \frac{\sigma^2}{2}\right) du} e^{\int_{[\hat{T}]+1}^{t} \left(r - ax^{\theta}(u) - bx^{\gamma}([u]) - \frac{\sigma^2}{2}\right) du} \\ \geq& \varphi\left(\hat{T}\right) e^{b \int_{\hat{T}}^{[\hat{T}]+1} \left(x^{\gamma} - x^{\gamma}\left(\left[\hat{T}\right]\right)\right) du} e^{\int_{\hat{T}}^{[\hat{T}]+1} \left(r - ax^{\theta}(u) - bx^{\gamma} - \frac{\sigma^2}{2}\right) du} \\ \geq& x\left(\hat{T}\right) e^{-\frac{\sigma^2}{2}\hat{T} + \frac{\sigma^2}{2}\hat{T} + \sigma B\left(\hat{T}\right)} \cdot e^{-b\widetilde{\Gamma}} \\ \geq& \tilde{x} \frac{1}{\Gamma_1} e^{-\frac{\sigma^2}{2}\hat{T} - b\widetilde{\Gamma}}. \end{aligned}$$

The last inequality obtained by using the formula: $e^{-\frac{\sigma^2}{2}\hat{T} - \sigma B\left(\hat{T}\right)} \leq \Gamma_1$. Thus we have

$$x(t) \geq \tilde{x} \frac{1}{\Gamma_1} e^{-\frac{\sigma^2}{2}\hat{T} - b\widetilde{\Gamma}} e^{-\sigma B(t)}. \tag{4.6}$$

In general, from (4.5) and (4.6), there is

$$\limsup_{t\to\infty} x(t) \geq \left(x_0 \wedge \tilde{x} \frac{1}{\Gamma_1} e^{-\frac{\sigma^2}{2}\hat{T} - b\widetilde{\Gamma}}\right) \limsup_{t\to\infty} e^{-\sigma B(t)} = \infty \ a.s.,$$

which is false because of the assumption $x(t) \leq \widetilde{x}$ $a.s.$ for all $t > T_2$. The proof is complete. $\square$

Remark 4.1. Let $\theta = 1$ with $\gamma = 1$, one can see that $\widehat{x} := \frac{r}{a+b}$ is the positive equilibrium of the deterministic equation of (4.1) with $\sigma = 0$. From Theorem 4.1, $\widehat{x}$ is not the equilibrium of the stochastic version (4.1), and the solution oscillates around $\widehat{\widehat{x}} := \frac{r-\frac{\sigma^2}{2}}{a+b}$ different from $\widehat{x}$.

5. Simulation and discussion

Three examples are given in this section for numerical experiments to illustrate our results by employing the Milstein method (see [10]). To begin, equation (1.3) can be discretized as follows: for $k = 0, 1, 2, ..., L$ and $j\Delta t \in (k, k+1]$,

$$\begin{aligned} x_{j+1} =& x_j + x_j \Big(r\left(j\Delta t\right) - a\left(j\Delta t\right)\left(x_j\right)^{\theta} - b\left(j\Delta t\right)\left(x_k\right)^{\gamma} \Big) \Delta t \\ &+ \sigma\left(j\Delta t\right) x_j \Delta B_j + \frac{1}{2}\sigma^2\left(j\Delta t\right) x_j \Big\{ \left(\Delta B_j\right)^2 - \Delta t \Big\}, \end{aligned} \tag{5.1}$$

where $x_j = x\left(j\Delta t\right)$, Δt is the step size and $\Delta B_j = B\left(\left(j+1\right)\Delta t\right) - B\left(j\Delta t\right)$ is an increments generated by using discretized Brownian paths like that in [10].

Example 5.1. Consider the stochastic logistic equation with piecewise constant argument

$$\begin{aligned} dx(t) =& x(t) \Bigg\{ \Bigg(0.4 + 0.2\sin\left(\frac{\pi}{2}t\right) - \left(0.6 + \frac{1}{t}\right)\sqrt{x(t)} \\ &- \left(0.4 + 0.1\cos\left(\frac{\pi}{2}t\right)\right) x([t]) \Bigg) dt + 0.96 dB(t) \Bigg\} \end{aligned} \tag{5.2}$$

with $x(0) = 0.3$ and it's determined version

$$\begin{aligned} &\frac{dx\left(t\right)}{dt} \\ =& x\left(t\right) \left\{ \left(0.4 + 0.2\sin\left(\frac{\pi}{2}t\right) - \left(0.6 + \frac{1}{t}\right)\sqrt{x(t)} - \left(0.4 + 0.1\cos\left(\frac{\pi}{2}t\right)\right) x\left([t]\right) \right) \right\} \end{aligned} \tag{5.3}$$

with $y(0) = 0.4$. The paths by Milstein method are as follows.

In Figure 1, from picture (a), the solution (denoted by green line) of (5.3) is always greater than zero and oscillates in an interval induced by the coefficients. Comparing the lines in picture (a) and (b), a result arises that noise induces the population to die out. Note that $\limsup\limits_{t\to\infty} \frac{1}{t}\int_0^t \left\{ 0.4 + 0.2\sin\left(\frac{\pi}{2}s\right) - \frac{(0.96)^2}{2} \right\} ds = -0.0608 < 0$, Theorem 3.1 confirms this.

Example 5.2. Consider the stochastic logistic equation with piecewise constant argument

$$\begin{aligned} dx(t) =& x(t) \Bigg\{ \Bigg(0.4 + 0.2\sin\left(\frac{\pi}{2}t\right) - \left(0.6 + \frac{1}{t}\right)\sqrt{x(t)} \\ &- \left(0.4 + 0.1\cos\left(\frac{\pi}{2}t\right)\right) x([t]) \Bigg) dt + 0.26 dB\left(t\right) \Bigg\} \end{aligned} \tag{5.4}$$

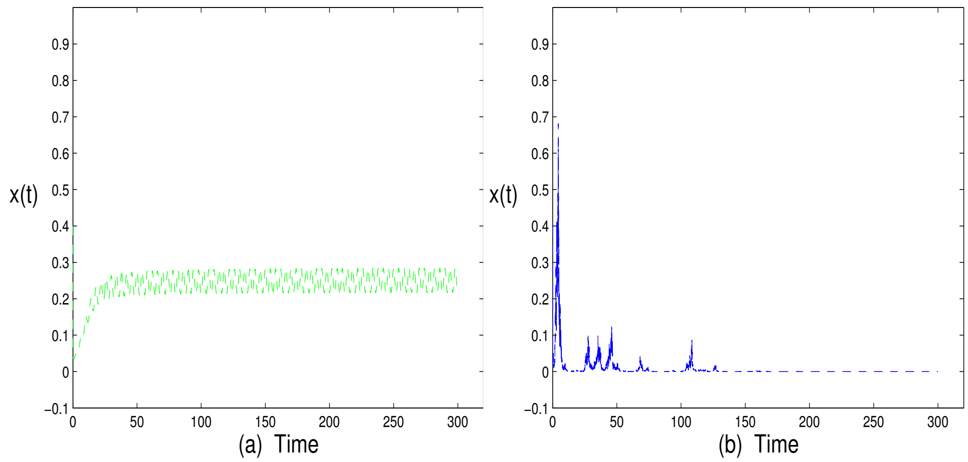

Figure 1. (a) : the trajectory of the determined equation (5.3). (b) : the trajectory of the stochastic equation (5.2) with $\Delta t = 0.001$.

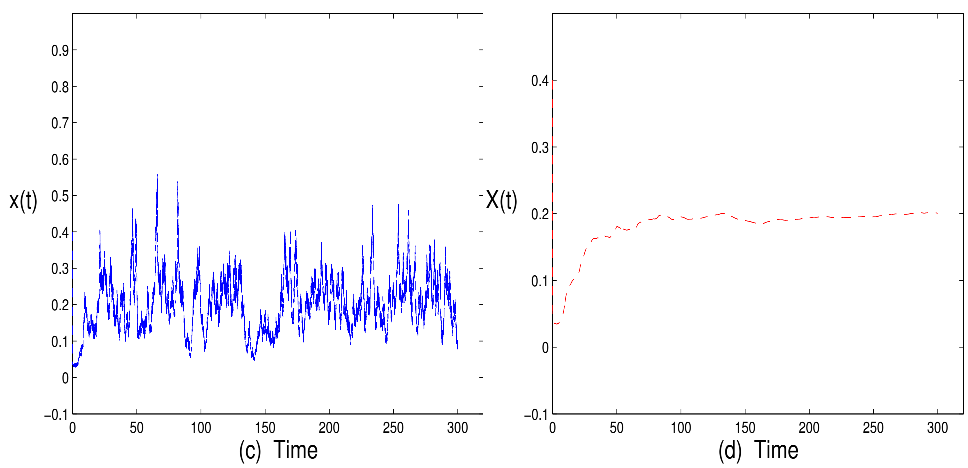

Figure 2. (c): the trajectory of equation (5.4) with $\Delta t = 0.001$. (d): the trajectory of $X(t)$.

with $x(0) = 0.3$. Define $X(t) = \frac{1}{t}\int_0^t x(s)ds$. In Figure 2, the blue line in picture (c) represents the solution of (5.4), and the red line in picture (d) is the path by time average of (5.4). The parameters in (5.2) and (5.4) are the same except with the only difference on value $\sigma(t)$. In (5.4), $\sigma(t) = 0.26$ means that $\liminf\limits_{t\to\infty} \frac{1}{t}\int_0^t \left\{0.4 + 0.2\sin\left(\frac{\pi}{2}t\right) - \frac{(0.26)^2}{2}\right\} = 0.3662 > 0$, and hence the population will be persistent by time average because of Theorem 3.5. Picture (d) in Figure 2 confirms this.

Example 5.3. Consider the stochastic logistic equation with constant coefficients

$$dx\left(t\right)=x\left(t\right)\left\{\left(0.6-0.2\sqrt{x(t)}-0.6x\left([t]\right)\right)dt+0.4dB\left(t\right)\right\} \tag{5.5}$$

with $x(0)=0.4$. $\widetilde{x}\approx 0.655$ is the solution of $0.6\widetilde{x}+0.2\sqrt{\widetilde{x}}=0.6-\frac{(0.4)^2}{2}$.

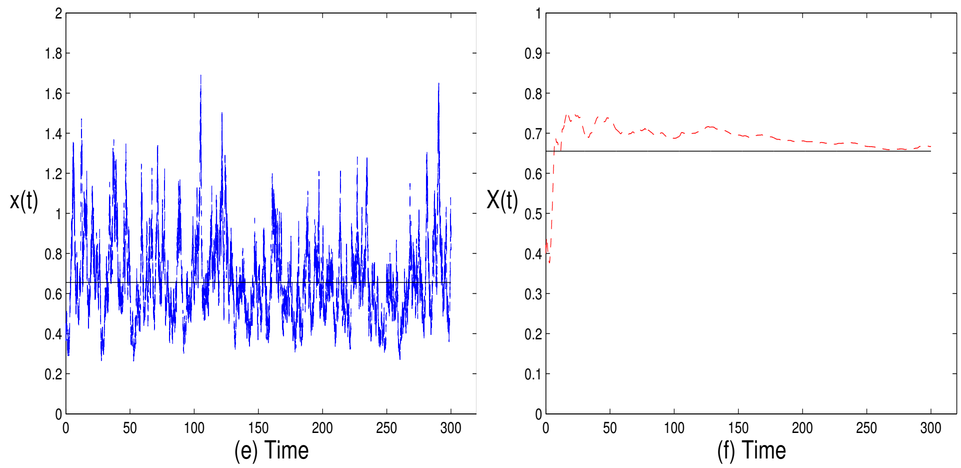

Figure 3. The blue line in (e) is the trajectory of equation (5.5) with $\Delta t=0.001$. The red line in (f) is the trajectory of $X(t)$ defined as before. The black line represents the formula: $\widetilde{x}=0.655$.

In Figure 3, the black line denotes the equilibrium level: $\widetilde{x}\approx 0.655$. The blue line in picture (e) represents the solution of (5.5), and the red line in picture (f) is the path by time average of (5.5). From picture (e), the solution of (5.5) oscillates around the point $\widetilde{x}$, which is consistent with the result presented by Theorem 4.1 for $r-\frac{\sigma^2}{2}=0.52>0$. From Remark 3.3, the solution of (5.5) is persistent a.s. by time average. Figure 3(f) supports this.

Remark 5.1. The method in this paper can be extended to the following general stochastic logistic equations with piecewise constant arguments:

$$\begin{aligned}&dx(t)\\=&x(t)\left\{\left(r(t)-a(t)x^{\theta}(t)-b(t)x^{\gamma}([t])-\sum_{j=1}^{K}b_j(t)x^{\gamma_j}([t-j])\right)dt+\sigma(t)dB(t)\right\}.\end{aligned}$$

In this paper, we give the conditions for the existence of the global positive solution and some criteria established for the almost sure extinction and persistence of the positive solutions about a stochastic population model with piecewise constant argument. In sense of persistence, necessary conditions are also given. By supposing that the coefficients are constants, we discuss the oscillation of the solution. The above results are supported by the numerical simulations. However, there are still some interesting problems needed to be studied, we list two of them for example.

Problem 5.1. *Applying Theorem 3.3 on equation (4.1), it's clear that* $r-\frac{\sigma^2}{2}>0$ *is necessary for the solution of (4.1) to be a.s. persistent. The question is whether* $r-\frac{\sigma^2}{2}>0$ *can be a sufficient condition for almost sure persistence.*

Problem 5.2. *A simulation for (5.5):*

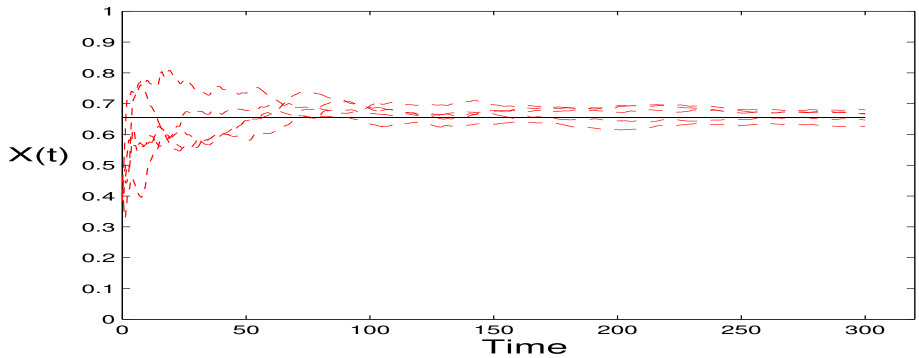

Figure 4. Five paths of the trajectory for equation (5.5) by time average. The black line represents the formula: $\widetilde{x} = 0.655$.

From the picture, it seems that the path by time average of (5.5) may converge to a point $\widetilde{x}$ *defined as before. We wonder if the statement is true, and whether* $r - \frac{\sigma^2}{2} > 0$ *is sufficient for the solution of (4.1) to be p-persistent a.s. by time average.*

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