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MULTIPLICITY OF SOLUTIONS FOR FRACTIONAL $\kappa(X)$-LAPLACIAN EQUATIONS

J. Vanterler da C. Sousa[1,†], Gabriela L. Araújo[1],
Maria V. S. Sousa[1] and Amália R. E. Pereira[1]

Abstract In this present paper, we first discuss some results of the energy functional on the Nehari manifold. Furthermore, we are interested in a compactness result and in estimates involving minimax levels over the ψ-fractional space $\mathcal{H}^{\alpha,\beta;\psi}_{\kappa(\xi)}(\Omega)$. In this sense, the condition of Palais-Smale is discussed. In other words, we are concerned with the multiplicity of solutions to a class of quasilinear fractional problems with super-linear growth involving variable exponents through the previously discussed results, in particular via the Lions concentration-compaction principle.

Keywords Fractional $p(x)$-Laplacian equations, existence, multiplicity, Ekeland variational principle, Nehari manifold.

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1. Introduction and motivation

Variational methods is one of the main tools used to tackle problems in the theory of nonlinear ordinary and partial differential equations. The central idea is the formulation of a variational problem equivalent, in a sense, to the differential equation problem. The variational problem consists of obtaining critical points for an associated functional I, such that the Euler-Lagrange equation is the proposed problem. It is interesting to observe that the problem of minimization of functionals is the central objective of the classical calculus of variations, and that in its study, differential equations naturally appear as sufficient conditions that the function that minimizes the functional must satisfy. Thus, in the classical calculus of variations, the issue of minimization of a functional is reduced to the study of a problem in the theory of differential equations. The direct method of calculus of variations emerged in the mid-nineteenth century, and consists of directly studying the functional and seeking to obtain its minimum (or a critical point) without resorting to its differential equation. Here are some interesting and important works that emerge from variational problems: [2, 7, 8, 11, 12, 14, 18, 27, 29, 32, 34] and the references therein.

Since the last decade of the last century, considerable attention has been given

[†]The corresponding author.
[1]Aerospace Engineering, PPGEA-UEMA, Department of Mathematics, DEMATI-UEMA, São Luís, MA 65054, Brazil
Email: vanterler@ime.unicamp.br(J. Vanterler da C. Sousa),
gabi.araujolm@gmail.com(Gabriela L. Araújo),
mariavictoriadesousa03@gmail.com(Maria V. S. Sousa),
reginaamalia679@gmail.com(Amália R. E. Pereira)

to problems involving the $p(x)$-Laplacian operator, i.e.,

$$\Delta_{p(x)}u = \operatorname{div}\left(|\nabla u|^{p(x)-2}\ \nabla u\right).$$

This differential operator is a natural generalization of the defined p-Laplacian operator $\Delta_p u = \operatorname{div}\left(|\nabla u|^{p-2}\ \nabla u\right)$, with $p > 1$ being a real constant. However, in some situations the $p(x)$-Laplacian operator is more complex than the p-Laplacian operator, due to the fact that $\Delta_{p(x)}$ is inhomogeneous.

In recent years, we have observed a growing interest in the study of equations and systems of equations with growth conditions involving variable exponents. Interest in studying such problems was stimulated by their applications in electrorheological fluids (see Acerbi & Mingione [1], Ruzicka [38]), flow in porous media (see Antontseva & Shmarevb [9]). These physical problems were facilitated by the development of Lebesgue and Sobolev spaces with variable exponents. Lebesgue spaces with variable exponents appeared for the first time in the literature, as early as 1931, in an article by Orlicz [37]. On image restoration problems, Y. Chen, S. Levine & R. Rao in [16], proposed a model based on the $p(x)$-Laplacian. There are numerous other works of great relevance in the area, see for example: [3–5, 13, 31, 39, 47] and the references therein. We can highlight the work carried out by Nyamoradi [36], on the existence and multiplicity of positive solutions for a singular elliptical problem using variational methods. See also the interesting work [30] and the references therein.

On the other hand, we highlight the fractional operators that over the last few years have gained a lot of attention in several areas, in particular, involving problems like p-Laplacian, $p(x)$-Laplacian and problems like Kirchhoff [10, 15, 26, 33, 48] and the references therein. The study of the existence and multiplicity of solutions to such problems via variational and topological methods, in fact, are of great relevance both in the analytical aspect and in the applicable aspect. Although there is an interesting range of work in this regard, it is still an area that is experiencing exponential growth. In addition, we also highlight a class of fractional operators so-called ψ-Hilfer, which plays a fundamental role in the study of Laplace problems. Since 2019 Sousa and researchers have been using the fractional operator ψ-Hilfer and variational, topological and nonlinear analysis tools to discuss properties of weak solutions of differential equations with p-Laplacian and $p(x)$-Laplacian [19, 20, 28, 40–42, 44–46] and the references therein. Although there are interesting results, the path is still unclear since it requires new results, new tools and care.

Before commenting on some work that motivated this paper, it is worth highlighting an interesting work on the existence of at least two non-trivial and non-negative solutions to the fractional boundary value problem via the Nehari method, i.e., the following problem

$$\begin{cases} -\dfrac{d}{dt}\left(\dfrac{1}{2}D_t^{-\beta}(u'(t)) + \dfrac{1}{2}D_T^{-\beta}(u'(t))\right) = \lambda f(t)(u(t))^{p-1} + g(t)(u(t))^{q-1}, \\ u(0) = u(T) = 0 \end{cases}$$

a.e. $t \in [0, T]$. For more details, see [35].

Yin And Yang [47] consider the following problem

$$\Delta_p u + \lambda |u|^{p-2}u = \theta_1(x)|u|^{s-2}u + h(x)|x|^{r-2}u \tag{1.1}$$

with $u > 0$, $x \in \mathbb{R}^N$, $u \in W^{1,p}(\mathbb{R}^N)$.

Cao and Noussair [13], discussed the existence and multiplicity of solutions for problem given by

$$\begin{cases} -\Delta u + u = \theta_1(\varepsilon x)|u|^{r-2}u, \text{ in } \mathbb{R}^N, \\ u \in H^{1,2}(\mathbb{R}^N). \end{cases} \tag{1.2}$$

In 2012, Hsu, Lin and Hu [31] investigated the multiple positive solutions of quasilinear elliptic equations

$$\begin{cases} -\Delta_p u + |u|^{p-2}u = \theta_1(\varepsilon x)|u|^{r-2}u + \lambda\theta_2(\varepsilon x), \text{ in } \mathbb{R}^N, \\ u \in W^{1,p}(\mathbb{R}^N). \end{cases} \tag{1.3}$$

In 2016, Alves and Barreiro [4], discussed the multiplicity of solutions for a problem involving variable exponents

$$\begin{cases} -\Delta_{p(x)} u + |u|^{p(x)-2}u = \lambda\theta_2(k^{-1}x)|u|^{q(x)-2}u + \theta_1(k^{-1}x)|u|^{r(x)-2}u, \\ u \in W^{1,p(x)}(\mathbb{R}^N). \end{cases} \tag{1.4}$$

Motivated by the works (1.1)-(1.4), in this paper, we concern in the new class of fractional differential equations with $\kappa(\xi)$-Laplacian given by

$$ {}^{\mathrm{H}}\mathfrak{D}_T^{\alpha;\beta;\psi}\left(\left|{}^{\mathrm{H}}\mathfrak{D}_{0+}^{\alpha;\beta;\psi}u\right|^{\kappa(\xi)-2}\ {}^{\mathrm{H}}\mathfrak{D}_{0+}^{\alpha;\beta;\psi}u\right) + |u|^{\kappa(\xi)-2}u = \mathbf{L}_{k,x}u, \tag{1.5}$$

where $\mathbf{L}_{k,\xi}u =: \lambda\theta_1(k^{-1}\xi)|u|^{q(\xi)-2}u + \theta_2(k^{-1}\xi)|u|^{r(\xi)-2}u$, ${}^{\mathrm{H}}\mathfrak{D}_T^{\alpha;\beta;\psi}(\cdot)$ and ${}^{\mathrm{H}}\mathfrak{D}_{0+}^{\alpha;\beta;\psi}(\cdot)$ are the right and left ψ-Hilfer fractional derivatives of order $0 < \alpha < 1$ and type β ($0 \leq \beta \leq 1$) with $1 < \alpha\kappa(\xi) < 3$, $u \in \mathcal{H}_{\kappa(\xi)}^{\alpha,\beta;\psi}(\Omega)$ (ψ-fractional space, see Section 2), where λ, ξ and k are non-negative parameters with $k \in \mathbb{N}$ and $\Omega = [0,T] \times [0,T] \times [0,T] \subset \mathbb{R}^3$.

Suppose that $\kappa, q, r : \Omega \to \mathbb{R}$ are Lipschitz functions, $\mathbb{Z}^3$-periodic and satisfying:

$$1 < \kappa_- \leq \kappa(\xi) \leq \kappa_+ < q_- \leq q(\xi) \leq r(\xi) \ll \kappa_\alpha^*(\xi), \text{ a.e. on } \Omega. \tag{1.6}$$

Furthermore, we assume that the functions κ and q satisfy the following condition:

(H_1): There exists $\iota > 0$ such that

$$\int_\Omega \left(\frac{1}{\kappa(\xi)}\left|{}^{\mathrm{H}}\mathfrak{D}_{0+}^{\alpha,\beta;\psi}u\right|^{\kappa(\xi)} - a(\xi)|u|^{\kappa(\xi)}\right) d\xi \geq \iota \int_\Omega \frac{1}{\kappa(\xi)}|u|^{\kappa(\xi)}d\xi.$$

The measurable function $h : \Omega \to \mathbb{R}$ is $\mathbb{Z}^3$-periodic if

$$h(\xi + z) = h(\xi), \ \forall \xi \in \Omega \text{ and } \forall z \in \mathbb{Z}^3.$$

To discuss what is the main result, we consider $\theta_1, \theta_2 : \Omega \to \mathbb{R}$ are functions: continuous, positive and satisfy the following conditions:

(g_3) $\lim_{|\xi|\to\infty} \theta_2(\xi) = 0$.

(f_1) $\lim_{|\xi|\to\infty} \theta_1(\xi) = \theta_{1,\infty}$.

(f_2) There are l points $a_1, a_2, ..., a_l$ in $\mathbb{Z}^3$ with $a_1 = 0$ such that

$$1 = \theta_1(a_i) = \max_{\mathbb{R}^N} \theta_1(\xi), \text{ for } 1 \leq i \leq l.$$

Furthermore, we assume that $0 < \theta_{1,\infty} < \theta_1(\xi)$ for all $\xi \in \Omega$.

The main result of this present paper is to investigate the multiplicity of solutions to the problem (1.5), in other words, we are interested in discussing the proof of the following result:

Theorem 1.1. *Suppose the conditions* (1.6)*,* $(g_3), (f_1)$ *and* (f_2) *are satisfied. Then, there exists* $\Lambda^* > 0$ *and* $k^* \in \mathbb{N}$ *such that the problem* (1.5) *admits at least* l *solutions for* $0 \leq \lambda < \Lambda^*$ *and* $k \geq k^*$*.*

The idea of proof of **Theorem 1.1** will be based on Ekeland's variational principle, some properties involving the Nehari manifold and Lions' principle of concentration-compactness.

Otherwise, the paper is organized as follows. Section 2, we present an approach to fractional operators and some variational setting properties. In this sense, we also investigated results of the energy functional related to the main problem of this paper about the Nehari manifold. In Section 3, we covered a result of compactness. In Section 4, we discuss estimates involving minimax levels. In Section 5, we investigate the Palais-Smale condition and the main result of this paper, i.e., the proof of **Theorem 1.1**.

2. Mathematical background and variational setting

Let $\Omega \subset \mathbb{R}^N$ be an open set. We denote by $|\Omega|$ the N-dimensional Lebesgue measure of Ω. For this aim, let us introduce the space

$$C_+(\overline{\Omega}) = \left\{ h \in C(\overline{\Omega}; \mathbb{R}) : \inf_{\xi \in \Omega} h(\xi) > 1 \right\}.$$

The variable exponent Lebesgue space $\mathscr{L}^{h(\cdot)}(\Omega)$ is defined by

$$\mathscr{L}^{h(\cdot)}(\Omega) = \left\{ u : \Omega \to \mathbb{R} \text{ measurable: } \int_\Omega |u(\xi)|^{h(\xi)} d\xi < \infty \right\}.$$

$\mathscr{L}^{h(\cdot)}$ is a Banach space when endowed with the Luxemburg norm defined by

$$||u||_{h(\cdot)} := \inf \left\{ \lambda > 0 : \int_\Omega \left| \frac{u(\xi)}{\lambda} \right|^{h(\xi)} d\xi \leq 1 \right\}.$$

The variable exponent Lebesgue space $\mathscr{L}^{h(\cdot)}(\Omega)$ is a special case of an Orlicz-Musielak space.

For each $h \in \mathscr{L}^\infty_+(\Omega)$, we define the numbers h_- and h_+ given by

$$h_- := ess \inf_{\xi \in \Omega} h(\xi) \text{ and } h_+ := ess \sup_{\xi \in \Omega} h(\xi).$$

It is well known that for each $h_1,\ h_2 \in C_+(\overline{\Omega})$ such that $h_1 \leq h_2$ in Ω, the embedding $\mathscr{L}^{h_2(\cdot)}(\Omega) \hookrightarrow \mathscr{L}^{h_1(\cdot)}(\Omega)$ is continuous. Furthermore, For $h(\xi) \in C_+(\overline{\Omega})$, if we take h' such that $\frac{1}{h(\xi)} + \frac{1}{h'(\xi)} = 1$, then the Hölder inequality is given as follows

$$\left|\int_\Omega uv d\xi\right| \leq \left(\frac{1}{h^-} + \frac{1}{h'^-}\right) ||u||_{h(\xi)}||v||_{h'(\xi)} \leq 2||u||_{h(\xi)}||v||_{h'(\xi)},$$

for any $u \in \mathscr{L}^{h(\xi)}(\Omega)$ and $v \in \mathscr{L}^{h'(\xi)}(\Omega)$.

On the space $\mathscr{L}^{h(\xi)}(\Omega)$, consider the modular function $\rho(u) := \int_\Omega |u(\xi)|^{h(\xi)} d\xi$.

Proposition 2.1. [21, 22, 24] *Let $u \in \mathscr{L}^{h(\xi)}(\Omega)$*

1. *If $u \neq 0$, $\|u\|_{h(\xi)} = \lambda$ if and only if $\rho\left(\frac{u}{\lambda}\right) = 1$;*
2. *$\|u\|_{h(\xi)} < 1$ $(= 1, > 1)$ if and only if $\rho(u) < 1$ $(= 1, > 1)$.*
3. $$||u||_{h(\xi)} \geq 1 \ \Rightarrow \ ||u||^{h^-}_{h(\xi)} \leq \int_\Omega |u(\xi)|^{h(\xi)} d\xi \leq ||u||^{h^+}_{h(\xi)}.$$
4. $$||u||_{h(\xi)} < 1 \ \Rightarrow \ ||u||^{h^+}_{h(\xi)} \leq \int_\Omega |u(\xi)|^{h(\xi)} d\xi \leq ||u||^{h^-}_{h(\xi)}.$$

Let $\theta = (\theta_1, \theta_2, \theta_3)$, $T = (T_1, T_2, T_3)$ and $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ where $0 < \alpha_1, \alpha_2, \alpha_3 < 1$ with $\theta_j < T_j$, for all $j \in \{1, 2, 3\}$. Also put $\Lambda = I_1 \times I_2 \times I_3 = [\theta_1, T_1] \times [\theta_2, T_2] \times [\theta_3, T_3]$, where T_1, T_2, T_3 and $\theta_1, \theta_2, \theta_3$ are positive constants. Let $u, \psi \in C^n(\Lambda)$ two functions such that ψ is increasing and $\psi'(\xi_j) \neq 0$ with $\xi_j \in [\theta_j, T_j]$, $j \in \{1, 2, 3\}$. The left and right-sided ψ-Hilfer fractional partial derivative of 3-variables of $u \in AC^n(\Lambda)$ of order $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ $(0 < \alpha_1, \alpha_2, \alpha_3 \leq 1)$ and type $\beta = (\beta_1, \beta_2, \beta_3)$ where $0 \leq \beta_1, \beta_2, \beta_3 \leq 1$, are defined by

$$\begin{aligned}&{}^{\mathbf{H}}\mathfrak{D}^{\alpha,\beta;\psi}_{\theta} u(\xi_1, \xi_2, \xi_3)\\ &= \mathbf{I}^{\beta(1-\alpha),\psi}_{\theta}\left(\frac{1}{\psi'(\xi_1)\psi'(\xi_2)\psi'(\xi_3)}\left(\frac{\partial^3}{\partial\xi_1\partial\xi_2\partial\xi_3}\right)\right)\mathbf{I}^{(1-\beta)(1-\alpha),\psi}_{\theta} u(\xi_1, \xi_2, \xi_3)\end{aligned}$$

and

$$\begin{aligned}&{}^{\mathbf{H}}\mathfrak{D}^{\alpha,\beta;\psi}_{T} u(\xi_1, \xi_2, \xi_3)\\ &= \mathbf{I}^{\beta(1-\alpha),\psi}_{T}\left(-\frac{1}{\psi'(\xi_1)\psi'(\xi_2)\psi'(\xi_3)}\left(\frac{\partial^3}{\partial\xi_1\partial\xi_2\partial\xi_3}\right)\right)\mathbf{I}^{(1-\beta)(1-\alpha),\psi}_{T} u(\xi_1, \xi_2, \xi_3),\end{aligned}$$

where $\mathbf{I}^{\alpha,\psi}_{\theta} u(\xi_1, \xi_2, \xi_3)$ and $\mathbf{I}^{\alpha,\psi}_{T} u(\xi_1, \xi_2, \xi_3)$ there are the ψ-Riemann-Liouville fractional integrals of $u \in \mathscr{L}^1(\Lambda)$ of order α $(0 < \alpha < 1)$ given

$$\begin{aligned}&\mathbf{I}^{\alpha,\psi}_{\theta} u(\xi_1, \xi_2, \xi_3)\\ &= \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\alpha_3)}\int_{\theta_1}^{\xi_1}\int_{\theta_2}^{\xi_2}\int_{\theta_3}^{\xi_3} \psi'(s_1)\psi'(s_2)\psi'(s_3)(\psi(\xi_1) - \psi(s_1))^{\alpha_1 - 1}\\ &\quad \times(\psi(\xi_2) - \psi(s_2))^{\alpha_2 - 1}(\psi(\xi_3) - \psi(s_3))^{\alpha_3 - 1} u(s_1, s_2, s_3) ds_3 ds_2 ds_1,\end{aligned}$$

to $\theta_1 < s_1 < \xi_1, \theta_2 < s_2 < \xi_2, \theta_3 < s_3 < \xi_3$ and

$$\begin{aligned}
&\mathbf{I}_T^{\alpha,\psi} u(\xi_1,\xi_2,\xi_3) \\
&= \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\alpha_3)} \int_{\xi_1}^{T_1}\int_{\xi_2}^{T_2}\int_{\xi_3}^{T_3} \psi'(s_1)\psi'(s_2)\psi'(s_3)(\psi(s_1)-\psi(\xi_1))^{\alpha_1-1} \\
&\times(\psi(s_2)-\psi(\xi_2))^{\alpha_2-1}(\psi(s_3)-\psi(\xi_3))^{\alpha_3-1}u(s_1,s_2,s_3)ds_3ds_2ds_1,
\end{aligned}$$

with $\xi_1 < s_1 < T_1, \xi_2 < s_2 < T_2, \xi_3 < s_3 < T_3$, $\xi_1 \in [\theta_1, T_1]$, $\xi_2 \in [\theta_2, T_2]$ and $\xi_3 \in [\theta_3, T_3]$. For a study of N-variables, see [40].

Let $\theta = (\theta_1, \theta_2, \theta_3)$, $T = (T_1, T_2, T_3)$ and $\alpha = (\alpha_1, \alpha_2, \alpha_3)$. The relation

$$\begin{aligned}
&\int_{\theta_1}^{T_1}\int_{\theta_2}^{T_2}\int_{\theta_3}^{T_3} \left(\mathbf{I}_\theta^{\alpha;\psi}\varphi(\xi_1,\xi_2,\xi_3)\right)\phi(\xi_1,\xi_2,\xi_3)\,d\xi_3d\xi_2d\xi_1 \\
&= \int_{\theta_1}^{T_1}\int_{\theta_2}^{T_2}\int_{\theta_3}^{T_3} \varphi(\xi_1,\xi_2,\xi_3)\,\psi'(\xi_1)\,\psi'(\xi_2)\psi'(\xi_3)\mathbf{I}_T^{\alpha;\psi} \\
&\times\left(\frac{\phi(\xi_1,\xi_2,\xi_3)}{\psi'(\xi_1)\,\psi'(\xi_2)\psi'(\xi_3)}\right)d\xi_3d\xi_2d\xi_1
\end{aligned} \tag{2.1}$$

is valid.

On the other hand, let $\psi(\cdot)$ be an increasing and positive monotone function on $[\theta_1, T_1] \times [\theta_2, T_2] \times [\theta_3, T_3]$, having a continuous derivative $\psi'(\cdot) \neq 0$ on $(\theta_1, T_1) \times (\theta_2, T_2) \times (\theta_3, T_3)$. If $0 < \alpha = (\alpha_1, \alpha_2, \alpha_3) < 1$ and $0 \leq \beta = (\beta_1, \beta_2) \leq 1$, then

$$\begin{aligned}
&\int_{\theta_1}^{T_1}\int_{\theta_2}^{T_2}\int_{\theta_3}^{T_3} \left({}^{\mathbf{H}}\mathfrak{D}_\theta^{\alpha,\beta;\psi}\varphi(\xi_1,\xi_2,\xi_3)\right)\phi(\xi_1,\xi_2,\xi_3)\,d\xi_2d\xi_1 \\
&= \int_{\theta_1}^{T_1}\int_{\theta_2}^{T_2}\int_{\theta_3}^{T_3} \varphi(\xi_1,\xi_2,\xi_3)\,\psi'(\xi_1)\,\psi'(\xi_2)\,\psi'(\xi_3) \\
&\times {}^{\mathbf{H}}\mathfrak{D}_T^{\alpha,\beta;\psi}\left(\frac{\phi(\xi_1,\xi_2,\xi_3)}{\psi'(\xi_1)\,\psi'(\xi_2)\,\psi'(\xi_3)}\right)d\xi_3d\xi_2d\xi_1
\end{aligned} \tag{2.2}$$

for any $\varphi \in AC^1$ and $\phi \in C^1$ satisfying the boundary conditions $\varphi(\theta_1, \theta_2, \theta_3) = 0 = \varphi(T_1, T_2, T_3)$.

The ψ-fractional space is given by [46]

$$\mathcal{H}_{h(\xi)}^{\alpha,\beta;\psi}(\Omega) = \left\{u \in \mathscr{L}^{h(\xi)}(\Omega) \;:\; \left|{}^{\mathrm{H}}\mathfrak{D}_{0+}^{\alpha,\beta;\,\psi}u\right| \in \mathscr{L}^{h(\xi)}(\Omega)\right\}$$

with the norm

$$||u|| = ||u||_{\mathcal{H}_{h(\xi)}^{\alpha,\beta;\psi}(\Omega)} = ||u||_{\mathscr{L}^{h(\xi)}(\Omega)} + \left\|{}^{\mathrm{H}}\mathfrak{D}_{0+}^{\alpha,\beta;\psi}u\right\|_{\mathscr{L}^{h(\xi)}(\Omega)}.$$

The space $\mathcal{H}_{h(\xi),0}^{\alpha,\beta;\psi}(\Omega)$ is defined as the closure of $C_0^\infty(\Omega)$ in $\mathcal{H}_{h(\xi)}^{\alpha,\beta;\psi}(\Omega)$ with respect to the above norm.

In $\mathcal{H}_{h(\xi)}^{\alpha,\beta;\psi}(\Omega)$ let consider the modular function $\rho_1 : \mathcal{H}_{h(\xi)}^{\alpha,\beta;\psi}(\Omega) \to \mathbb{R}$ given by

$$\rho_1(u) = \int_\Omega \left(\left|{}^{\mathrm{H}}\mathfrak{D}_{0+}^{\alpha,\beta;\psi}u\right|^{h(\xi)} + |u|^{h(\xi)}\right)d\xi.$$

Define

$$||u||_1 = \inf \left\{ t > 0 : \int_\Omega \frac{\left(\left| {}^{\mathrm{H}}\mathfrak{D}_{0+}^{\alpha,\beta;\psi} u \right|^{h(\xi)} + |u|^{h(\xi)} \right)}{t^{h(\xi)}} d\xi \leq 1 \right\},$$

then $||\cdot||_{\mathcal{H}_{h(\xi)}^{\alpha,\beta;\psi}(\Omega)}$ and $||\cdot||_1$ are equivalent in $\mathcal{H}_{h(\xi)}^{\alpha,\beta;\psi}(\Omega)$. In this paper, we will consider the norm $||u|| = ||u||_1$.

Next, we will present some results for the space ψ-fractional $\mathcal{H}_{h(\xi)}^{\alpha,\beta;\psi}(\Omega)$.

Proposition 2.2. [46] *The spaces $\mathscr{L}^{h(\xi)}(\Omega)$ and $\mathcal{H}_{h(\xi)}^{\alpha,\beta;\,\psi}(\Omega)$ are separable and reflexive Banach spaces.*

Theorem 2.1. [40] *Let Ω be a domain with the cone property, $h : \tilde{\Omega} \to \mathbb{R}$ be a Lipschitz function checking (H_1) and $q \in \mathscr{L}_+^\infty(\Omega)$ satisfying $h(\xi) \leq q(\xi) \leq h^*(\xi)$ a.e. in $\tilde{\Omega}$. So there is a continuous embedding*

$$\mathcal{H}_{h(\xi)}^{\alpha,\beta;\psi}(\Omega) \to \mathscr{L}^{q(\xi)}(\Omega).$$

In the space ψ-fractional $\mathcal{H}_{h(\xi),0}^{\alpha,\beta;\psi}(\Omega)$ let consider the modular function $\rho_0 : \mathcal{H}_{h(\xi),0}^{\alpha,\beta;\psi}(\Omega) \to \mathbb{R}$ given by

$$\rho_0(u) = \int_\Omega \left| {}^{\mathrm{H}}\mathfrak{D}_{0+}^{\alpha,\beta;\psi} u(\xi) \right|^{h(\xi)} d\xi.$$

Proposition 2.3. *Let $u \in \mathcal{H}_{h(\xi),0}^{\alpha,\beta;\psi}(\Omega)$ and $\{u_n\} \subset \mathcal{H}_{h(\xi),0}^{\alpha,\beta;\psi}(\Omega)$. Then, the same conclusion of* **Proposition 2.1** *occurs considering $||\cdot||$ and ρ_0.*

Proposition 2.4. *Let $v \in \mathcal{H}_{h(\xi)}^{\alpha,\beta;\psi}(\Omega)$. Then, the same conclusion as* **Proposition 2.1** *occurs considering $||\cdot||_1$ and ρ_1.*

Lemma 2.1. [21, 22, 24] *Let $h, r \in \mathscr{L}_+^\infty(\Omega)$ with $h(\xi) \leq r(\xi)$ a.e. in Ω and $u \in \mathscr{L}^{r(\xi)}(\Omega)$. Then $|u|^{h(\xi)}$, $\mathscr{L}^{\frac{r(\xi)}{h(\xi)}}(\Omega)$ and*

$$\left|\left| |u|^{h(\xi)} \right|\right|_{\mathscr{L}^{\frac{r(\xi)}{h(\xi)}}(\Omega)} \leq ||u||^{h+}_{\mathscr{L}^{\frac{r(\xi)}{h(\xi)}}(\Omega)} + ||u||^{h-}_{\mathscr{L}^{\frac{r(\xi)}{h(\xi)}}(\Omega)}$$

or yet

$$\left|\left| |u|^{h(\xi)} \right|\right|_{\mathscr{L}^{\frac{r(\xi)}{h(\xi)}}(\Omega)} \leq \max \left\{ ||u||^{h+}_{\mathscr{L}^{\frac{r(\xi)}{h(\xi)}}(\Omega)},\ ||u||^{h-}_{\mathscr{L}^{\frac{r(\xi)}{h(\xi)}}(\Omega)} \right\}.$$

Lemma 2.2. [6, 25] (Brezis-Lieb lemma) *Let $\{\mu_n\} \subset \mathscr{L}^{h(\xi)}(\Omega, \mathbb{R}^m)$ with $m \in \mathbb{N}$, verify*

1. *$\mu_n(\xi) \to \mu(\xi)$, a.e. in Ω;*
2. *$\sup\limits_{n\in\mathbb{N}} |\mu_n|_{\mathscr{L}^{h(\xi)}(\Omega,\mathbb{R}^m)} < \infty$.*

Then

$$\mu_n \rightharpoonup \mu \text{ in } \mathscr{L}^{h(\xi)}(\Omega, \mathbb{R}^m).$$

Lemma 2.3. [23] *Suppose that $h:\mathbb{R}^N\to\mathbb{R}$ is a uniformly continuous function with $1<\kappa_-\leq\kappa_+<N$. If $\{u_n\}$ is bounded by $\mathcal{H}^{\alpha,\beta;\psi}_{\kappa(\xi)}(\Omega)$ and*

$$\limsup_{n\to\infty}\sup_{y\in\mathbb{R}^N}\int_{Br(y)}|u_n|^{q(\xi)}\,d\xi=0$$

for some $r>0$ and some function $q\in\mathscr{L}^{\infty}_{+}(\mathbb{R}^N)$ satisfying $\kappa\leq q\ll\kappa^*_\alpha$, then $u_n\to 0$ in $\mathscr{L}^{s(\xi)}(\mathbb{R}^N)$ for every measurable function $s:\mathbb{R}^N\to\mathbb{R}$ with $\kappa\ll s\ll\kappa^*_\alpha$.

Lemma 2.4. [6,25] *Let $\{\mu_n\}\subset\mathscr{L}^{h(\xi)}(\Omega,\mathbb{R}^m)$ with $m\in\mathbb{N}$, such that*

1. *$\mu_n(\xi)\to\mu(\xi)$, a.e. in Ω.*
2. *$\sup\limits_{n\in\mathbb{N}}|\mu_n|_{\mathscr{L}^{h(\xi)}(\Omega,\mathbb{R}^m)}<\infty$.*

Then,

$$\int_\Omega\left||\mu_n|^{h(\xi)-2}\mu_n|-|\mu_n-\mu|^{h(\xi)-2}(\mu_n-\mu)-|\mu|^{h(\xi)-2}\right|^{h'(\xi)}d\xi=o_n(1).$$

Associated with the **Problem (1.5)**, we have the energy functional $\mathbf{\Theta}^{\alpha,\beta}_{\lambda,k}:\mathcal{H}^{\alpha,\beta;\psi}_{\kappa(\xi)}(\Omega)\to\mathbb{R}$ defined by

$$\mathbf{\Theta}^{\alpha,\beta}_{\lambda,k}(u)=\int_\Omega\left(\frac{1}{\kappa(\xi)}\left|{}^{\mathrm{H}}\mathfrak{D}^{\alpha,\beta;\psi}_{0+}u\right|^{\kappa(\xi)}+|u|^{\kappa(\xi)}\right)d\xi-\lambda\int_\Omega\frac{\theta_2(k^{-1}\xi)}{q(\xi)}|u|^{q(\xi)}d\xi$$
$$-\int_\Omega\frac{\theta_1(k^{-1}\xi)}{r(\xi)}|u|^{r(\xi)}d\xi.$$

No that $\mathbf{\Theta}^{\alpha,\beta}_{\lambda,k}\in C^1\left(\mathcal{H}^{\alpha,\beta;\psi}_{\kappa(\xi)}(\Omega),\mathbb{R}\right)$ with

$$\mathbf{\Theta}^{\alpha,\beta}_{\lambda,k}(u)v=\int_\Omega\left(\left|{}^{\mathrm{H}}\mathfrak{D}^{\alpha,\beta;\psi}_{0+}u\right|^{\kappa(\xi)-2}\ {}^{\mathrm{H}}\mathfrak{D}^{\alpha,\beta;\psi}_{0+}u\ {}^{\mathrm{H}}\mathfrak{D}^{\alpha,\beta;\psi}_{0+}v+|u|^{\kappa(\xi)-2}uv\right)d\xi$$
$$-\lambda\int_\Omega\theta_2(k^{-1}\xi)|u|^{q(\xi)-2}uv\,d\xi-\int_\Omega\theta_1(k^{-1}\xi)|u|^{r(\xi)-2}uv\,d\xi,$$

for all $u,v\in\mathcal{H}^{\alpha,\beta;\psi}_{\kappa(\xi)}(\Omega)$. Thus, the critical points of the functional $\mathbf{\Theta}^{\alpha,\beta}_{\lambda,k}$ are solutions to the **Problem (1.5)**. Since $\mathbf{\Theta}^{\alpha,\beta}_{\lambda,k}$ is not bounded lower about $\mathcal{H}^{\alpha,\beta;\psi}_{\kappa(\xi)}(\Omega)$, consider the functional $\mathbf{\Theta}^{\alpha,\beta}_{\lambda,k}$ restricted to the Nehari manifold $\mathcal{N}^{\alpha,\beta}_{\lambda,k}$, given by

$$\mathcal{N}^{\alpha,\beta}_{\lambda,k}=\left\{u\in\mathcal{H}^{\alpha,\beta;\psi}_{\kappa(\xi)}(\Omega)/\{0\}:\left(\mathbf{\Theta}^{\alpha,\beta}_{\lambda,k}\right)'(u)u=0\right\}$$

and the level

$$c_{\lambda,k}=\inf_{u\in\mathcal{N}^{\alpha,\beta}_{\lambda,k}}\mathbf{\Theta}^{\alpha,\beta}_{\lambda,k}(u).$$

Note that $c_{\lambda,k}$ is the mountain pass level of the functional $\mathbf{\Theta}^{\alpha,\beta}_{\lambda,k}$.

Choosing $\theta_1 = 1$ and $\lambda = 0$ in the **Problem (1.5)**, we have the fractional problem given by

$$ {}^{\mathrm{H}}\mathfrak{D}_T^{\alpha,\beta;\psi}\left(\left|{}^{\mathrm{H}}\mathfrak{D}_{0+}^{\alpha,\beta;\psi}u\right|^{\kappa(\xi)-2}\ {}^{\mathrm{H}}\mathfrak{D}_{0+}^{\alpha,\beta;\psi}u\right) + |u|^{\kappa(\xi)-2}u = |u|^{r(\xi)-2}u,\ u\in\mathcal{H}_{\kappa(\xi)}^{\alpha,\beta;\psi}(\Omega). \tag{2.3}$$

Consider energy functional $\mathbf{\Theta}_\infty^{\alpha,\beta} : \mathcal{H}_{\kappa(\xi)}^{\alpha,\beta;\psi}(\Omega)\to\mathbb{R}$ associated the **Problem (2.3)**, given by

$$\mathbf{\Theta}_\infty^{\alpha,\beta}(u) = \int_\Omega\left(\frac{1}{\kappa(\xi)}\left|{}^{\mathrm{H}}\mathfrak{D}_{0+}^{\alpha,\beta;\psi}u\right|^{\kappa(\xi)} + |u|^{\kappa(\xi)}\right)d\xi - \int_\Omega\frac{1}{r(\xi)}|u|^{r(\xi)}d\xi$$

and the level

$$c_\infty = inf_{u\in\mathcal{N}_\infty}\mathbf{\Theta}_\infty^{\alpha,\beta}(u).$$

Also, consider the Nehari manifold

$$\mathcal{N}_\infty^{\alpha,\beta} = \left\{u\in\mathcal{H}_{\kappa(\xi)}^{\alpha,\beta;\psi}(\Omega)/\left\{0\right\} : \left(\mathbf{\Theta}_\infty^{\alpha,\beta}\right)'(u)u = 0\right\}. \tag{2.4}$$

For $\theta_1\equiv\theta_{1,\infty}$ and $\lambda = 0$, we also consider the following problem

$$\begin{aligned} &{}^{\mathrm{H}}\mathfrak{D}_T^{\alpha,\beta;\psi}\left(\left|{}^{\mathrm{H}}\mathfrak{D}_{0+}^{\alpha,\beta;\psi}u\right|^{\kappa(\xi)-2}\ {}^{\mathrm{H}}\mathfrak{D}_{0+}^{\alpha,\beta;\psi}u\right) + |u|^{\kappa(\xi)-2}u \\ =&\theta_{1,\infty}|u|^{r(\xi)-2}u,\ u\in\mathcal{H}_{\kappa(\xi)}^{\alpha,\beta;\psi}(\Omega) \end{aligned} \tag{2.5}$$

and as above, denote by $\mathbf{\Theta}_{\theta_{1,\infty}}^{\alpha,\beta}, c_{\theta_{1,\infty}}$ and $\mathcal{N}_{\theta_{1,\infty}}^{\alpha,\beta}$ the energy functional, the level of the mountain pass and the Nehari manifold associated with (2.5), respectively.

Lemma 2.5. (Local Property) *Given $\Lambda > 0$, there exists positive constants $\overline{\gamma}$ and σ* (independent of k)*, such that $\mathbf{\Theta}_{\lambda,k}^{\alpha,\beta}(u) > \overline{\gamma} > 0$ for all $\lambda\in(0,\Lambda)$ with $||u|| = \sigma$.*

Proof. Using the definition of $\mathbf{\Theta}_{\lambda,k}^{\alpha,\beta}$ and the conditions (g_3) and (f_2), it results

$$\begin{aligned}\mathbf{\Theta}_{\lambda,k}^{\alpha,\beta}(u) \geq& \frac{1}{\kappa_+}\int_\Omega\left(\left|{}^{\mathrm{H}}\mathfrak{D}_{0+}^{\alpha,\beta;\psi}u\right|^{\kappa(\xi)} + |u|^{\kappa(\xi)}\right)d\xi - \frac{\lambda}{q_-}||\theta_2||_\infty\int_\Omega|u|^{q(\xi)}d\xi \\ &-\frac{1}{r_-}\int_\Omega|u|^{r(\xi)}d\xi.\end{aligned}$$

If $||u|| < 1$, using the **Proposition 2.4** and **Theorem 2.1**, yields

$$\mathbf{\Theta}_{\lambda,k}^{\alpha,\beta}(u) \geq \frac{1}{\kappa_+}||u||^{\kappa_+} - \frac{\lambda}{q_-}||\theta_2||_\infty c_1||u||^{q_-} - \frac{c_2}{r_-}||u||^{r_-}$$

where c_1 and c_2 are positive constants. Since $\kappa_+ < q_- \leq r_-$, setting $\sigma > 0$ small enough such that

$$\frac{1}{\kappa_+}\sigma^{\kappa_+} - \frac{\Lambda}{q-}||\theta_2||_\infty c_1\sigma^{q-} - \frac{c_2}{r_-}\sigma^{r_-} \geq \frac{1}{2\kappa_+}\sigma^{\kappa_+}.$$

If $0<\lambda<\Lambda$, $\mathbf{\Theta}_{\lambda,k}^{\alpha,\beta}(u) \geq \dfrac{1}{2\kappa_+}\sigma^{\kappa_+} = \overline{\gamma} > 0$ on $\partial B_\sigma(0)$ establishing the result. □

The following result refers to the behavior of $\mathbf{\Theta}_{\lambda,k}^{\alpha,\beta}$ over $\mathcal{N}_{\lambda,k}^{\alpha,\beta}$.

Lemma 2.6. *The functional $\mathbf{\Theta}_{\lambda,k}^{\alpha,\beta}$ is bounded from below and coercive on $\mathcal{N}_{\lambda,k}^{\alpha,\beta}$.*

Proof. For every $u \in \mathcal{N}_{\lambda,k}^{\alpha,\beta}$, we have $\left(\mathbf{\Theta}_{\lambda,k}^{\alpha,\beta}\right)'(u)u = 0$. So,

$$\lambda \int_\Omega \theta_2(k^{-1}\xi)|u|^{q(\xi)}d\xi = \left(\int_\Omega \left|{}^{\mathrm{H}}\mathfrak{D}_{0+}^{\alpha,\beta;\psi}u\right|^{\kappa(\xi)} + |u|^{\kappa(\xi)}\right)d\xi - \int_\Omega \theta_1(k^{-1}\xi)|u|^{r(\xi)}d\xi. \tag{2.6}$$

Using the definition of $\mathbf{\Theta}_{\lambda,k}^{\alpha,\beta}$ and Eq.(2.6), one has

$$\begin{aligned}\mathbf{\Theta}_{\lambda,k}^{\alpha,\beta}(u) &\geq \frac{1}{\kappa_+}\int_\Omega\left(\left|{}^{\mathrm{H}}\mathfrak{D}_{0+}^{\alpha,\beta;\psi}u\right|^{\kappa(\xi)} + |u|^{\kappa(\xi)}\right)d\xi - \frac{\lambda}{q_-}\int_\Omega \theta_2(k^{-1}\xi)|u|^{q(\xi)}d\xi \\ &\quad -\frac{1}{r_-}\int_\Omega \theta_1(k^{-1}\xi)|u|^{r(\xi)}d\xi \\ &= \left(\frac{1}{\kappa_+} - \frac{1}{q_-}\right)\int_\Omega\left(\left|{}^{\mathrm{H}}\mathfrak{D}_{0+}^{\alpha,\beta;\psi}u\right|^{\kappa(\xi)} + |u|^{\kappa(\xi)}\right)d\xi \\ &\quad + \left(\frac{1}{q_-} - \frac{1}{r_-}\right)\int_\Omega \theta_1(k^{-1}\xi)|u|^{r(\xi)}d\xi.\end{aligned}$$

By hypothesis $\kappa_+ < q_- \leq r_-$, so

$$\mathbf{\Theta}_{\lambda,k}^{\alpha,\beta}(u) \geq \left(\frac{1}{\kappa_+} - \frac{1}{q_-}\right)\int_\Omega\left(\left|{}^{\mathrm{H}}\mathfrak{D}_{0+}^{\alpha,\beta;\psi}u\right|^{\kappa(\xi)} + |u|^{\kappa(\xi)}\right)d\xi \tag{2.7}$$

for all $u \in \mathcal{N}_{\lambda,k}^{\alpha,\beta}$. In this sense, we have that $\mathbf{\Theta}_{\lambda,k}^{\alpha,\beta}$ is bounded from below by $\mathcal{N}_{\lambda,k}^{\alpha,\beta}$. □

Corollary 2.1. *If $\{u_n\}$ is a sequence in $\mathcal{N}_{\lambda,k}^{\alpha,\beta}$ with $\mathbf{\Theta}_{\lambda,k}^{\alpha,\beta}(u_n) \to c_{\lambda,k}$, then $\{u_n\}$ is bounded in $\mathcal{H}_{\kappa(\xi)}^{\alpha,\beta;\psi}(\Omega)$.*

Proof. From (2.7), one has

$$\int_\Omega\left(\left|{}^{\mathrm{H}}\mathfrak{D}_{0+}^{\alpha,\beta;\psi}u_n\right|^{\kappa(\xi)} + |u_n|^{\kappa(\xi)}\right)d\xi \leq \left(\frac{1}{\kappa_+} - \frac{1}{q_-}\right)^{-1}(c_{\lambda,k}+1) \tag{2.8}$$

for n sufficiently great. Applying the **Proposition 2.4** we conclude that $\{u_n\}$ is bounded in $\mathcal{H}_{\kappa(\xi)}^{\alpha,\beta;\psi}(\Omega)$. □

The next lemma states that the Nehari manifold $\mathcal{N}_{\lambda,k}^{\alpha,\beta}$ has a positive distance from the origin.

Lemma 2.7. *Given $\Lambda > 0$ there is $\delta > 0$ such that*

$$||u|| > \delta,\ \forall (u,\lambda,k) \in \mathcal{N}_{\lambda,k}^{\alpha,\beta} \times [0,\Lambda] \times \mathbb{N}. \tag{2.9}$$

So, using **Proposition 2.4**, *there exists $\mu > 0$ such that*

$$\rho_1(u) \geq \mu, \forall (u,\lambda,k) \in \mathcal{N}_{\lambda,k}^{\alpha,\beta} \times [0,\Lambda] \times \mathbb{N}. \tag{2.10}$$

Proof. Assume by contradiction that the inequality (2.9) not hold. So there is a sequence $\{u_n\} \subset \mathcal{N}_{\lambda,k}^{\alpha,\beta}$ such that $||u_n|| \to 0$ when $n \to \infty$. Since that $\{u_n\} \subset \mathcal{N}_{\lambda,k}^{\alpha,\beta}$ and $||\theta_1||_\infty \leq 1$, from (2.6), one has

$$\int_\Omega\left(\left|{}^{\mathrm{H}}\mathfrak{D}_{0+}^{\alpha,\beta;\psi}u_n\right|^{\kappa(\xi)} + |u_n|^{\kappa(\xi)}\right)d\xi \leq \lambda||\theta_2||_\infty\int_\Omega |u_n|^{q(\xi)}d\xi + \int_\Omega |u_n|^{r(\xi)}d\xi.$$

From **Proposition 2.3** and **Lemma 2.1**, yields

$$\begin{aligned}\min\{||u_n||^{\kappa_-},||u_n||^{\kappa_+}\} &\leq \int_\Omega \left(\left|{}^{\mathrm{H}}\mathfrak{D}_{0+}^{\alpha,\beta;\psi}u_n\right|^{\kappa(\xi)}+|u_n|^{\kappa(\xi)}\right)d\xi\\ &\leq \lambda||\theta_2||_\infty \max\left\{||u_n||_{q(\xi)}^{q_-},||u_n||_{q(\xi)}^{q_+}\right\}\\ &\quad+\max\left\{||u_n||_{r(\xi)}^{r_-},||u_n||_{r(\xi)}^{r_+}\right\}.\end{aligned}$$

Using Sobolev embedding, there exists positive constants c_1 and c_2 such that

$$\begin{aligned}\min\{||u_n||^{\kappa_-},||u_n||^{\kappa_+}\} \leq{}& \Lambda||\theta_2||_\infty c_1 \max\{||u_n||^{q_-},||u_n||^{q_+}\}\\ &+c_2\max\{||u_n||^{r_-},||u_n||^{r_+}\}.\end{aligned}$$

Therefore, for n large enough, yields

$$||u_n||^{\kappa_+}\leq \Lambda c_1||\theta_2||_\infty||u_n||^{q_-}+c_2||u_n||^{r_-}\leq(\Lambda c_1||\theta_2||_\infty+c_2)||u_n||^{q_-}$$

or equivalently,

$$(\Lambda c_1||\theta_2||_\infty+c_2)^{-1}\leq||u_n||^{q_--\kappa_+},$$

from which we get an contradiction, since $\kappa_+<q_-$. Therefore, we complete the proof. □

Corollary 2.2. *Let* $\mathcal{E}_{\lambda,k}^{\alpha,\beta}(u)u=\left(\mathbf{\Theta}_{\lambda,k}^{\alpha,\beta}\right)'(u)u$. *Then, there exists* $\mu_0>0$ *such that*

$$\left(\mathcal{E}_{\lambda,k}^{\alpha,\beta}\right)'(u)<-\mu_0,\ \forall(u,\lambda,k)\in\mathcal{N}_{\lambda,k}^{\alpha,\beta}\times[0,\Lambda]\times\mathbb{N}.$$

Proof. Note that

$$\begin{aligned}\left(\mathcal{E}_{\lambda,k}^{\alpha,\beta}\right)'(u)u={}&\int_\Omega\kappa(\xi)\left(\left|{}^{\mathrm{H}}\mathfrak{D}_{0+}^{\alpha,\beta;\psi}u\right|^{\kappa(\xi)}+|u|^{\kappa(\xi)}\right)d\xi-\lambda\int_\Omega q(\xi)\theta_2(k^{-1}\xi)|u|^{q(\xi)}d\xi\\ &-\int_\Omega r(\xi)\theta_1(k^{-1}\xi)|u|^{r(\xi)}d\xi\end{aligned}$$

for all $u\in\mathcal{H}_{\kappa(\xi)}^{\alpha,\beta;\psi}(\Omega)$. From the definition of $\mathcal{N}_{\lambda,k}^{\alpha,\beta}$, it follows that

$$\begin{aligned}\left(\mathcal{E}_{\lambda,k}^{\alpha,\beta}\right)'(u)u\leq{}&(\kappa_+-q_-)\int_\Omega\left(\left|{}^{\mathrm{H}}\mathfrak{D}_{0+}^{\alpha,\beta;\psi}u\right|^{\kappa(\xi)}+|u|^{\kappa(\xi)}\right)d\xi\notin\\ &+(q_--r_-)\int_\Omega\theta_1(k^{-1}\xi)|u|^{r(\xi)}d\xi.\end{aligned}$$

Since $\kappa_+<q_-\leq r_-$ and f is a non-negative function, we get

$$\left(\mathcal{E}_{\lambda,k}^{\alpha,\beta}\right)'(u)u\leq(\kappa_+-q_-)\int_\Omega\left(\left|{}^{\mathrm{H}}\mathfrak{D}_{0+}^{\alpha,\beta;\psi}u\right|^{\kappa(\xi)}+|u|^{\kappa(\xi)}\right)d\xi.$$

Applying the **Lemma 2.7**, we conclude that

$$\left(\mathcal{E}_{\lambda,k}^{\alpha,\beta}\right)'(u)<-(q_--\kappa_+)\mu.$$

Therefore, we conclude the proof. □

Lemma 2.8. *If $u \in \mathcal{N}^{\alpha,\beta}_{\lambda,k}$ is a critical point of $\mathbf{\Theta}^{\alpha,\beta}_{\lambda,k}$ restricted to $\mathcal{N}^{\alpha,\beta}_{\lambda,k}$, so u is critical point of $\mathbf{\Theta}^{\alpha,\beta}_{\lambda,k}$ in $\mathcal{H}^{\alpha,\beta;\psi}_{\kappa(\xi)}(\Omega)$.*

Proof. Let $u \in \mathcal{N}^{\alpha,\beta}_{\lambda,k}$ a critical point of $\mathbf{\Theta}^{\alpha,\beta}_{\lambda,k}$ restricted to the manifold $\mathcal{N}^{\alpha,\beta}_{\lambda,k}$. Then, there exists $\tau \in \mathbb{R}$ such that

$$\left(\mathbf{\Theta}^{\alpha,\beta}_{\lambda,k}\right)'(u) = \tau\left(\mathcal{E}^{\alpha,\beta}_{\lambda,k}\right)'(u).$$

Since $(\mathbf{\Theta}^{\alpha,\beta}_{\lambda,k})'(u)u = 0$, we have that $\tau\left(\mathcal{E}^{\alpha,\beta}_{\lambda,k}\right)'(u) = 0$. From the **Corollary 2.2**, we know that $(\mathbf{\Theta}^{\alpha,\beta}_{\lambda,k})'(u)u < 0$, so we should have $\tau = 0$. Therefore,

$$\left(\mathbf{\Theta}^{\alpha,\beta}_{\lambda,k}\right)'(u) = 0$$

implying that u is the critical point of $\mathbf{\Theta}^{\alpha,\beta}_{\lambda,k}$ in $\mathcal{H}^{\alpha,\beta;\psi}_{\kappa(\xi)}(\Omega)$. □

Theorem 2.2. *Let u_n a sequence in $\mathcal{H}^{\alpha,\beta;\psi}_{\kappa(\xi)}(\Omega)$ such that $u_n \rightharpoonup u$ in $\mathcal{H}^{\alpha,\beta;\psi}_{\kappa(\xi)}(\Omega)$ and $\left(\mathbf{\Theta}^{\alpha,\beta}_{\lambda,k}\right)'(u_n) \to 0$ with $n \to \infty$. So, for some subsequence, ${}^{\mathrm{H}}\mathfrak{D}^{\alpha,\beta;\psi}_{0+}u_n \to {}^{\mathrm{H}}\mathfrak{D}^{\alpha,\beta;\psi}_{0+}u$ a.e. in Ω. Also, $\left(\mathbf{\Theta}^{\alpha,\beta}_{\lambda,k}\right)'(u) = 0$.*

Proof. Let $R > 0$ and $\phi \in C^\infty_0(\Omega)$ such that

$$\begin{cases} \phi \equiv 0 \text{ if } |\xi| \geq 2R, \\ \phi \equiv 1 \text{ if } |\xi| \leq R \end{cases}$$

and $0 \leq \phi(\xi) \leq 1$, $\forall \xi \in \Omega$.

Using the same arguments as the proof of **Lemma 13** (see [46]), considering the sequence

$$\begin{aligned} &P_n(\xi) \\ &= \left(\left|{}^{\mathrm{H}}\mathfrak{D}^{\alpha,\beta;\psi}_{0+}u_n\right|^{\kappa(\xi)-2}\mathfrak{D}^{\alpha,\beta;\psi}_{0+}u_n - \left|\mathfrak{D}^{\alpha,\beta;\psi}_{0+}u\right|^{\kappa(\xi)-2}\mathfrak{D}^{\alpha,\beta;\psi}_{0+}u\right)\mathfrak{D}^{\alpha,\beta;\psi}_{0+}(u_n-u) \end{aligned} \tag{2.11}$$

it is shown that

$$\begin{aligned} &\int_{B_R} P_n d\xi \\ &\leq \int_\Omega \left|{}^{\mathrm{H}}\mathfrak{D}^{\alpha,\beta;\psi}_{0+}u_n\right|^{\kappa(\xi)}\phi d\xi - \int_\Omega \left|{}^{\mathrm{H}}\mathfrak{D}^{\alpha,\beta;\psi}_{0+}u\right|^{\kappa(\xi)-2}{}^{\mathrm{H}}\mathfrak{D}^{\alpha,\beta;\psi}_{0+}u_n\,{}^{\mathrm{H}}\mathfrak{D}^{\alpha,\beta;\psi}_{0+}u\phi d\xi + o_n(1). \end{aligned}$$

From $\left(\mathbf{\Theta}^{\alpha,\beta}_{\lambda,k}\right)'(u_n)(\phi u_n) = o_n(1)$, $\left(\mathbf{\Theta}^{\alpha,\beta}_{\lambda,k}\right)'(u)(\phi u_n) = o_n(1)$, and using the triangular and Cauchy-Schwarz inequality, ones has

$$\begin{aligned} &\int_{B_R} P_n d\xi \\ &\leq o_n(1) - \int_\Omega \left|{}^{\mathrm{H}}\mathfrak{D}^{\alpha,\beta;\psi}_{0+}u_n\right|^{\kappa(\xi)-2}(u_n-u)\,{}^{\mathrm{H}}\mathfrak{D}^{\alpha,\beta;\psi}_{0+}u_n\,{}^{\mathrm{H}}\mathfrak{D}^{\alpha,\beta;\psi}_{0+}\phi d\xi \end{aligned}$$

$$
\begin{aligned}
& -\int_\Omega |u_n|^{\kappa(\xi)-2}u_n(u_n-u)\phi d\xi + \lambda\int_\Omega \theta_2(k^{-1}\xi)|u_n|^{q(\xi)-2}u_n(u_n-u)\phi d\xi \\
& + \int_\Omega \theta_1(k^{-1}\xi)|u_n|^{r(\xi)-2}u_n(u_n-u)\phi d\xi \\
\leq\; & o_n(1) + c_1\int_{B_{2R}} \left|{}^{\mathrm{H}}\mathfrak{D}_{0+}^{\alpha,\beta;\psi}u_n\right|^{\kappa(\xi)-1}|u_n-u| \\
& + \int_{B_{2R}} |u_n|^{\kappa(\xi)-1}|u_n-u| + \lambda||\theta_2||_\infty \int_{B_{2R}} |u_n|^{q(\xi)-1}|u_n-u| \\
& + \int_{B_{2R}} |u_n|^{r(\xi)-1}|u_n-u| \\
\leq\; & o_n(1) + 2c_1\left\|\left|\mathfrak{D}_{0+}^{\alpha,\beta;\psi}u_n\right|^{\kappa(\xi)-1}\right\|_{\mathscr{L}^{p'(\xi)}(B_{2R})}\|u_n-u\|_{\mathscr{L}^{p'(\xi)}(B_{2R})} \\
& +2\left\||u_n|^{\kappa(\xi)-1}\right\|_{\mathscr{L}^{p'(\xi)}(B_{2R})}\|u_n-u\|_{\mathscr{L}^{p'(\xi)}(B_{2R})} \\
& +2\left\||u_n|^{q(\xi)-1}\right\|_{\mathscr{L}^{p'(\xi)}(B_{2R})}\|u_n-u\|_{\mathscr{L}^{p'(\xi)}(B_{2R})} \\
& +2\left\||u_n|^{r(\xi)-1}\right\|_{\mathscr{L}^{p'(\xi)}(B_{2R})}\|u_n-u\|_{\mathscr{L}^{p'(\xi)}(B_{2R})}.
\end{aligned}
$$

Since that $\mathcal{H}_{\kappa(\xi)}^{\alpha,\beta;\psi}(\Omega) \rightharpoonup \mathscr{L}_{loc}^{s(\xi)}(\Omega)$ is compact embedding for every measurable function s, satisfying $\kappa \leq s \ll \kappa_\alpha^*$ and $\{u_n\}$ is a bounded sequence, we conclude that

$$\int_{B_R} P_n d\xi \to 0 \text{ with } n\to\infty.$$

Consider the sets

$$B_R^+ = \{\xi \in B_R : \kappa(\xi) \geq 2\} \text{ and } B_R^- = \{\xi\in B_R : 1 < \kappa(\xi) < 2\}$$

and proceeding as in the proof of **Lemma 13** (see [46]), it is shown that $\mathfrak{D}_{0+}^{\alpha,\beta;\psi}u_n \to \mathfrak{D}_{0+}^{\alpha,\beta;\psi}u_n$ a.e. in B_R. Since R is arbitrary, it follows that for some subsequence

$$\mathfrak{D}_{0+}^{\alpha,\beta;\psi}u_n \to \mathfrak{D}_{0+}^{\alpha,\beta;\psi}u_n \text{ a.e. in } \Omega.$$

Since $\left\{\left|\mathfrak{D}_{0+}^{\alpha,\beta;\psi}u_n\right|^{\kappa(\xi)-2}\mathfrak{D}_{0+}^{\alpha,\beta;\psi}u_n\right\}$ is bounded in $(\mathscr{L}^{p'(\xi)}(\Omega))^3$ and $\left|{}^{\mathrm{H}}\mathfrak{D}_{0+}^{\alpha,\beta;\psi}u_n\right|^{\kappa(\xi)-2}{}^{\mathrm{H}}\mathfrak{D}_{0+}^{\alpha,\beta;\psi}u_n \to \left|{}^{\mathrm{H}}\mathfrak{D}_{0+}^{\alpha,\beta;\psi}u\right|^{\kappa(\xi)-2}{}^{\mathrm{H}}\mathfrak{D}_{0+}^{\alpha,\beta;\psi}u$ a.e. in Ω, from the **Lemma 2.2** implies

$$\left|\mathfrak{D}_{0+}^{\alpha,\beta;\psi}u_n\right|^{\kappa(\xi)-2}\mathfrak{D}_{0+}^{\alpha,\beta;\psi}u_n \rightharpoonup \left|\mathfrak{D}_{0+}^{\alpha,\beta;\psi}u\right|^{\kappa(\xi)-2}\mathfrak{D}_{0+}^{\alpha,\beta;\psi}u \text{ in } (\mathscr{L}^{p'(\xi)}(\Omega))^3.$$

Analogously, we get

$$|u_n|^{s(\xi)-2}u_n \rightharpoonup |u|^{s(\xi)-2}u \text{ in } \mathscr{L}^{s'(\xi)}(\Omega)$$

for every measurable function s checking $\kappa \leq s \leq \kappa_\alpha^*$.

Using the fact that $\left(\mathbf{\Theta}_{\lambda,k}^{\alpha,\beta}\right)'(u_n)v = o_n(1)$ for all $v \in \mathcal{H}_{\kappa(\xi)}^{\alpha,\beta;\psi}(\Omega)$ together with the last two limits, we get that $\left(\mathbf{\Theta}_{\lambda,k}^{\alpha,\beta}\right)'(u_n)v = 0$ for all $v\in\mathcal{H}_{\kappa(\xi)}^{\alpha,\beta,\psi}(\Omega)$. $\square$

3. A result of compactness

Theorem 3.1. *Suppose the condition* (κ_3) *holds and* $\{u_n\} \subset \mathcal{N}_\infty^{\alpha,\beta}$ *is a sequence with* $\mathbf{\Theta}_\infty^{\alpha,\beta}(u_n) \to c_\infty$. *Then,*

1. $u_n \to u$ *in* $\mathcal{H}_{\kappa(\xi)}^{\alpha,\beta;\psi}(\Omega)$;
2. *Exist* $\{y_n\} \subset \mathbb{Z}^N$ *with* $|y_n| \to +\infty$ *and* $w \in \mathcal{H}_{\kappa(\xi)}^{\alpha,\beta;\psi}(\Omega)$ *such that* $w_n(\xi) = u_n(\xi + y_n) \to w \in \mathcal{H}_{\kappa(\xi)}^{\alpha,\beta;\psi}(\Omega)$ *and* $\mathbf{\Theta}_\infty^{\alpha,\beta}(w) = c_\infty$.

Proof. Similar to **Corollary 2.1**, we have that $\{u_n\}$ is a bounded sequence and, from the reflexivity of $\mathcal{H}_{\kappa(\xi)}^{\alpha,\beta;\psi}(\Omega)$, there exists $u \in \mathcal{H}_{\kappa(\xi)}^{\alpha,\beta;\psi}(\Omega)$ and a subsequence of $\{u_n\}$, still denoted by $\{u_n\}$, such that $u_n \rightharpoonup u$ in $\mathcal{H}_{\kappa(\xi)}^{\alpha,\beta;\psi}(\Omega)$. From Ekeland variational principle, there is a sequence $\{w_n\}$ in $\mathcal{N}_\infty^{\alpha,\beta}$ satisfying

$$w_n = u_n + o_n(1),\ \ \mathbf{\Theta}_\infty^{\alpha,\beta}(w_n) \to c_\infty,$$

and

$$\left(\mathbf{\Theta}_\infty^{\alpha,\beta}\right)'(w_n) - \tau_n \left(\mathcal{E}_\infty^{\alpha,\beta}\right)'(w_n) = o_n(1), \tag{3.1}$$

where $(\tau_n) \subset \mathbb{R}$ and $\left(\mathcal{E}_\infty^{\alpha,\beta}\right)'(w) = \left(\mathbf{\Theta}_\infty^{\alpha,\beta}\right)'(w)w$, for all $w \in \mathcal{H}_{\kappa(\xi)}^{\alpha,\beta;\psi}(\Omega)$. Since that $\{u_n\} \subset \mathcal{N}_\infty^{\alpha,\beta}$, it follows from (3.1) that

$$\tau_n \left(\mathcal{E}_\infty^{\alpha,\beta}\right)'(w_n)w_n = o_n(1).$$

Using arguments from the **Lemma 2.2**, there exists a $\delta > 0$ such that

$$\left|\left(\mathcal{E}_\infty^{\alpha,\beta}\right)'(w_n)w_n\right| > \delta \qquad \forall n \in \mathbb{N}.$$

Using (3.1), it follows that $\tau \to 0$ when $n \to \infty$. Since $\{w_n\}$ is a bounded sequence, $\left\{\left(\mathcal{E}_\infty^{\alpha,\beta}\right)'(w_n)\right\}$ is also bounded, so we can say that

$$\left(\mathbf{\Theta}_\infty^{\alpha,\beta}\right)'(w_n) \to 0 \text{ in } \left(\mathcal{H}_{\kappa(\xi)}^{\alpha,\beta;\psi}(\Omega)\right)^*.$$

Without loss of generality assume that

$$\mathbf{\Theta}_\infty^{\alpha,\beta}(u_n) \to c_\infty \text{ and } \left(\mathbf{\Theta}_\infty^{\alpha,\beta}\right)'(w_n) \to 0 \text{ in } \left(\mathcal{H}_{\kappa(\xi)}^{\alpha,\beta;\psi}(\Omega)\right)^*.$$

Next, we discuss the possibilities: $u \neq 0$ or $u = 0$.

First case. $u \neq 0$.

Similar to **Theorem 2.2**, u is shown to be the critical point of $\mathbf{\Theta}_\infty^{\alpha,\beta}$. Applying the Fatou lemma, it follows that

$$\begin{aligned} c_\infty &\leq \mathbf{\Theta}_\infty^{\alpha,\beta}(u) - \frac{1}{q_-}\left(\mathbf{\Theta}_\infty^{\alpha,\beta}\right)'(u)u \\ &= \int_\Omega \left(\frac{1}{\kappa(\xi)} - \frac{1}{q_-}\right)\left(\left|{}^{\mathrm{H}}\mathfrak{D}_{0+}^{\alpha,\beta;\psi}u\right|^{\kappa(\xi)} + |u|^{\kappa(\xi)}\right)d\xi + \int_\Omega \left(\frac{1}{q_-} - \frac{1}{r(\xi)}\right)|u|^{r(\xi)}d\xi \end{aligned}$$

$$\leq \liminf_{n\to\infty} \int_\Omega \left(\frac{1}{\kappa(\xi)} - \frac{1}{q_-}\right)\left(\left|{}^{\mathrm{H}}\mathfrak{D}_{0+}^{\alpha,\beta;\psi} u\right|^{\kappa(\xi)} + |u|^{\kappa(\xi)}\right) d\xi$$
$$+ \liminf_{n\to\infty} \int_\Omega \left(\frac{1}{q_-} - \frac{1}{r(\xi)}\right) |u|^{r(\xi)} d\xi$$
$$= \liminf_{n\to\infty} \left\{\mathbf{\Theta}_\infty^{\alpha,\beta}(u_n) - \frac{1}{q_-}\left(\mathbf{\Theta}_\infty^{\alpha,\beta}\right)'(u_n)u_n\right\}$$
$$= c_\infty.$$

Consequently,

$$\lim_{n\to\infty} \int_\Omega \left(\left|{}^{\mathrm{H}}\mathfrak{D}_{0+}^{\alpha,\beta;\psi} u_n\right|^{\kappa(\xi)} + |u_n|^{\kappa(\xi)}\right) d\xi = \int_\Omega \left(\left|{}^{\mathrm{H}}\mathfrak{D}_{0+}^{\alpha,\beta;\psi} u\right|^{\kappa(\xi)} + |u|^{\kappa(\xi)}\right) d\xi.$$

Let

$$\theta_{1,n} = \left|{}^{\mathrm{H}}\mathfrak{D}_{0+}^{\alpha,\beta;\psi} u_n - {}^{\mathrm{H}}\mathfrak{D}_{0+}^{\alpha,\beta;\psi} u\right|^{\kappa(\xi)} + |u_n - u|^{\kappa(\xi)}$$

and

$$\theta_{2,n} = 2^{\kappa^+}\left(\left|{}^{\mathrm{H}}\mathfrak{D}_{0+}^{\alpha,\beta;\psi} u_n\right|^{\kappa(\xi)} + \left|{}^{\mathrm{H}}\mathfrak{D}_{0+}^{\alpha,\beta;\psi} u\right|^{\kappa(\xi)} + |u_n|^{\kappa(\xi)} + |u|^{\kappa(\xi)}\right).$$

So, it is immediate that $\theta_{1,n} \leq \theta_{2,n}$, $\theta_{1,n} \to 0$ and

$$\theta_{2,n} \to \theta_2 = 2^{\kappa_+ + 1}\left|\left|{}^{\mathrm{H}}\mathfrak{D}_{0+}^{\alpha,\beta;\psi} u\right|^{\kappa(\xi)} + |u|^{\kappa(\xi)}\right|$$

a.e. in $\mathbb{R}$. Applying Lebesgue dominated convergence theorem, we conclude that $\theta_{1,n} \to 0$ in $\mathcal{H}_p^{\alpha,\beta;\psi}(\Omega)$ from which follows the result.

Second case. $u = 0$.

Suppose that exists $R, \tau > 0$ and a sequence $\{y_n\} \subset \Omega$ satisfying

$$\limsup_{n\to\infty} \int_{B_R(y_n)} |y_n|^{\kappa(\xi)} d\xi \geq \tau. \tag{3.2}$$

If the statement is false, yields

$$\limsup_{n\to\infty} \sup_{y\in\Omega} \int_{B_R(y_n)} |y_n|^{\kappa(\xi)} d\xi = 0.$$

So, using the **Lemma 2.3**, yields

$$u_n \to 0 \text{ in } \mathscr{L}^{s(\xi)}(\Omega)$$

for every measurable function $s : \Omega \to \mathbb{R}$ with $\kappa \ll s \ll \kappa_\alpha^*$.

Since $\left(\mathbf{\Theta}_\infty^{\alpha,\beta}\right)'(u_n)u_n = o_n(1)$, the last bound implies that

$$\int_\Omega \left(\left|{}^{\mathrm{H}}\mathfrak{D}_{0+}^{\alpha,\beta;\psi} u_n\right|^{\kappa(\xi)} + |u_n|^{\kappa(\xi)}\right) d\xi = o_n(1),$$

or equivalently

$$u_n \to 0 \text{ in } \mathcal{H}^{\alpha,\beta;\psi}_{\kappa(\xi)}(\Omega)$$

resulting in $c_\infty = 0$, which is absurd. Therefore, the inequality (3.2) is true.

Note that $|y_n| \to \infty$ when $n \to \infty$ otherwise there would be a bounded subsequence of $\{y_n\}$, which we will denote further by $\{y_n\}$. Let say that $|y_n| \leq M$. Hence

$$\int_{B_{R+M}} |u|^{\kappa(\xi)} d\xi = \limsup_{n\to\infty} \int B_{R+M} |u_n|^{\kappa(\xi)} d\xi \geq \limsup_{n\to\infty} \int_{B_R(y_n)} |y_n|^{\kappa(\xi)} d\xi \geq \tau > 0$$

which contradicts the hypothesis $u = 0$.

Let $\overline{y_n} \in \mathbb{Z}^3$ such that

$$\|y_n - \overline{y_n}\| < \sqrt{3}$$

and

$$w_n(\xi) = u_n(\xi + \overline{y}_n), \ \forall \xi \in \mathbb{R}^3.$$

Then, from the translation invariance of $\mathbb{R}^3$ and κ and r being $\mathbb{Z}^3$-periodic, we deduce that $\mathbf{\Theta}^{\alpha,\beta}_\infty(w_n) = \mathbf{\Theta}^{\alpha,\beta}_\infty(u_n)$. Now let $\psi \in \mathcal{H}^{\alpha,\beta;\psi}_{\kappa(\xi)}(\Omega)$ with $\|\psi\| \leq 1$, so

$$\left| \left(\mathbf{\Theta}^{\alpha,\beta}_\infty\right)' (w_n)\psi \right| = \left| \left(\mathbf{\Theta}^{\alpha,\beta}_\infty\right)' (u_n)\psi(\cdot - \overline{y}_n) \right|$$

implying

$$\left\| \left(\mathbf{\Theta}^{\alpha,\beta}_\infty\right)' (w_n) \right\|_{\left(\mathcal{H}^{\alpha,\beta;\psi}_{\kappa(\xi)}(\Omega)\right)^*} \leq \left\| \left(\mathbf{\Theta}^{\alpha,\beta}_\infty\right)' (u_n) \right\|_{\left(\mathcal{H}^{\alpha,\beta;\psi}_{\kappa(\xi)}(\Omega)\right)^*}.$$

Analogously, yields

$$\left| \left(\mathbf{\Theta}^{\alpha,\beta}_\infty\right)' \psi \right| = \left| \left(\mathbf{\Theta}^{\alpha,\beta}_\infty\right)' (w_n)\psi(\cdot - +\overline{y}_n) \right|.$$

Therefore

$$\left\| \left(\mathbf{\Theta}^{\alpha,\beta}_\infty\right)' (w_n) \right\|_{\left(\mathcal{H}^{\alpha,\beta;\psi}_{\kappa(\xi)}(\Omega)\right)^*} = \left\| \left(\mathbf{\Theta}^{\alpha,\beta}_\infty\right)' (w_n) \right\|_{\left(\mathcal{H}^{\alpha,\beta;\psi}_{\kappa(\xi)}(\Omega)\right)^*}$$

showing that $\{w_n\}$ is a sequence $(PS)_{c_\infty}$ for $\mathbf{\Theta}^{\alpha,\beta}_\infty$. If $w \in \mathcal{H}^{\alpha,\beta;\psi}_{\kappa(\xi)}(\Omega)$, denotes the weak limit of $\{w_n\}$, considering $\overline{R} = R + \sqrt{3}$, we get

$$\int_{B_{\overline{R}}} |w|^{\kappa(\xi)} d\xi \geq \limsup_{n\to\infty} \int_{B_R} |u_n|^{\kappa(\xi)} d\xi \geq \tau > 0$$

which implies $w \neq 0$. Following the same steps as in the first case for the sequence $\{w_n\}$, it follows that $w_n \to w$ in $\mathcal{H}^{\alpha,\beta;\psi}_{\kappa(\xi)}(\Omega)$, $w \in \mathcal{N}^{\alpha,\beta}_\infty$ and $\mathbf{\Theta}^{\alpha,\beta}_\infty(w) = c_\infty$. $\square$

Lemma 3.1. *If the function g satisfies (g_3), then the functionals $\Psi_1, \Psi_2 : \mathcal{H}^{\alpha,\beta;\psi}_{\kappa(\xi)}(\Omega) \to \mathbb{R}$ given by*

$$\Psi_1(u) = \int_\Omega \theta_2(\xi)|u|^{q(\xi)} d\xi \ \text{ and } \ \Psi_2(u) = \int_\Omega \frac{\theta_2(\xi)}{q(\xi)} |u|^{q(\xi)} d\xi$$

are weakly continuous.

Proof. The proof will be done only for Ψ_1, because the same arguments apply to Ψ_2. Let $\{u_n\}$ be a sequence in $\mathcal{H}^{\alpha,\beta;\psi}_{\kappa(\xi)}(\Omega)$ such that $u_n \rightharpoonup u$ in $\mathcal{H}^{\alpha,\beta;\psi}_{\kappa(\xi)}(\Omega)$. From the assumption (g_3), for every $\epsilon > 0$, $\exists R > 0$ such that $|\theta_2(\xi)| < \epsilon$ for $|\xi| > R$. In that sense, we have

$$\int_{|\xi|>R} \theta_2(\xi)|u|^{q(\xi)}d\xi \leq \epsilon \int_{|\xi|>R} |u|^{q(\xi)}d\xi.$$

Since $u_n \rightharpoonup u$ in $\mathcal{H}^{\alpha,\beta;\psi}_{\kappa(\xi)}(\Omega)$, we have that $\{u_n\}$ is bounded by $\mathcal{H}^{\alpha,\beta;\psi}_{\kappa(\xi)}(\Omega)$. It follows from Sobolev embedding that the sequence $\{u_n\}$ is also bounded in $\mathscr{L}^{q(\xi)}(\Omega)$, and therefore

$$\int_{|\xi|>R} \theta_2(\xi)|u_n|^{q(\xi)}d\xi \leq \epsilon \int_{|\xi|>R} |u|^{q(\xi)}d\xi \leq \epsilon M, \ \forall n \in \mathbb{N}, \tag{3.3}$$

for M a positive constant. Again, Sobolev embedding imply

$$u_n \to u \text{ in } \mathscr{L}^{q(\xi)}(B_R). \tag{3.4}$$

Using the inequalities (3.3) and (3.4), one has

$$\int_{\Omega} \theta_2(\xi)|u_n|^{q(\xi)}d\xi \to \int_{\Omega} \theta_2(\xi)|u|^{q(\xi)}d\xi$$

which completes the proof. □

4. Estimates involving minimax levels

In this present section, we investigate some estimates involving minimax levels $c_{\lambda,k}, c_{0,k}$ and c_∞.

Firstly, note that

$$\mathbf{\Theta}^{\alpha,\beta}_{\lambda,k}(u) \leq \mathbf{\Theta}^{\alpha,\beta}_{0,k}(u) \text{ and } \mathbf{\Theta}^{\alpha,\beta}_{\infty}(u) \leq \mathbf{\Theta}^{\alpha,\beta}_{0,k}(u), \quad \forall u \in \mathcal{H}^{\alpha,\beta;\psi}_{\kappa(\xi)}(\Omega)$$

which implies

$$c_{\lambda,k} \leq c_{0,k} \text{ and } c_\infty \leq c_{0,k}. \tag{4.1}$$

Lemma 4.1. *The minimax levels $c_{0,k}$ and $c_{\theta_1\infty}$ satisfy the inequality*

$$c_{0,k} < c_{\theta_{1,\infty}}.$$

Consequently, $c_\infty < c_{\theta_{1,\infty}}$.

Proof. In a similar way to **Theorem 3.1**, there exists $V \in \mathcal{H}^{\alpha,\beta;\psi}_{\kappa(\xi)}(\Omega)/\{0\}$ checking

$$\mathbf{\Theta}^{\alpha,\beta}_{\theta_{1,\infty}}(V) = c_{\theta_{1,\infty}} \text{ and } \left(\mathbf{\Theta}^{\alpha,\beta}_{\theta_{1,\infty}}\right)'(V) = 0.$$

It follows from **Lemma 6.1** that there is $t > 0$ such that $tV \in \mathcal{N}^{\alpha,\beta}_{0,k}$. Thus, we have

$$c_{0,k} \leq \int_{\Omega} \frac{t^{\kappa}(\xi)}{\kappa(\xi)} \left(\left| {}^{\mathrm{H}}\mathfrak{D}^{\alpha,\beta,\psi}_{0+} V \right|^{\kappa(\xi)} + |V|^{\kappa(\xi)} \right) d\xi - \int_{\Omega} \theta_1\left(k^{-1}\xi\right) \frac{t^{r(\xi)}}{r(\xi)} |V|^{r(\xi)} d\xi.$$

Since that $0 < \theta_{1,\infty} < \theta_1(\xi)$ for all $\xi \in \mathbb{R}^3$, we get

$$c_{0,k} < \mathbf{\Theta}^{\alpha,\beta}_{\theta_{1,\infty}}(tV) \leq \max_{s\leq 0} \mathbf{\Theta}^{\alpha,\beta}_{\theta_{1,\infty}}(sV) = \mathbf{\Theta}^{\alpha,\beta}_{\theta_{1,\infty}}(V) = c_{\theta_{1,\infty}}.$$

Combining the last inequality with (4.1), it follows that $c_\infty < c_{\theta_{1,\infty}}$. In this sense, we conclude the proof. □

Proposition 4.1. *The level* $c_{0,k}$ *is a critical value of* $\mathbf{\Theta}^{\alpha,\beta}_{0,k}$, *i.e., there exists* $v \in \mathcal{H}^{\alpha,\beta;\psi}_{\kappa(\xi)}(\Omega)$ *such that*

$$\mathbf{\Theta}^{\alpha,\beta}_{0,k}(v) = c_{0,k} \text{ and } \left(\mathbf{\Theta}^{\alpha,\beta}_{0,k}\right)'(v) = 0.$$

Proof. In a similar way to **Theorem 3.1**, there is a sequence u_n in $\mathcal{N}^{\alpha,\beta}_{0,k}$ with

$$\mathbf{\Theta}^{\alpha}_{0,k}\beta(u_n) \to c_{0,k} \text{ and } \left(\mathbf{\Theta}^{\alpha}_{0,k}\beta\right)'(u_n) \to 0.$$

As in **Corollary 2.1**, $\{u_n\}$ is a bounded sequence in $\mathcal{H}^{\alpha,\beta;\psi}_{p(\xi)}(\Omega)$, and because the ψ-fractional space $\mathcal{H}^{\alpha,\beta;\psi}_{\kappa(\xi)}(\Omega)$ is reflexive, it follows that up to subsequence $u_n \rightharpoonup u$ in $\mathcal{H}^{\alpha,\beta;\psi}_{\kappa(\xi)}(\Omega)$.

Affirmation: $u \neq 0$.

Assume by contradiction that $u = 0$. So $u_n \rightharpoonup 0$ in $\mathcal{H}^{\alpha,\beta;\psi}_{\kappa(\xi)}(\Omega)$. We claim that there are positive numbers R and τ and a sequence $\{y_n\}$ in Ω^3 such that the inequality (3.2) holds. Otherwise, by **Lemma 2.3** it follows that $u \to 0$ in $\mathscr{L}^{r(\xi)}(\Omega)$. From the definition of $\mathbf{\Theta}^{\alpha,\beta}_{0,k}$, it follows that

$$\int_\Omega \left(\left| {}^{\mathrm{H}}\mathfrak{D}^{\alpha,\beta;\psi}_{0+} u_n \right|^{p(\xi)} + |u_n|^{\kappa(\xi)} \right) d\xi = o_n(1).$$

Therefore, $u_n \to 0$ in $\mathcal{H}^{\alpha,\beta;\psi}_{\kappa(\xi)}(\Omega)$ implying in $c_{0,k} = 0$, which is contradiction. Therefore, the inequality (3.2) holds.

It follows from the same arguments as the **Theorem 3.1** that the sequence $\{y_n\}$ is unbounded. Now, define the function $u_n(\xi) = u_n(\xi + y_n)$ for all $\xi \in \mathbb{R}^3$. Remembering that $\left(\mathbf{\Theta}^{\alpha,\beta}_{0,k}\right)'(u_n)\phi(\cdot + y_n) = o_n(1)$ for all $\phi \in \mathcal{H}^{\alpha,\beta;\psi}_{\kappa(\xi)}(\Omega)$, we get

$$\begin{aligned}&\int_\Omega \left(\left| {}^{\mathrm{H}}\mathfrak{D}^{\alpha,\beta,\psi}_{0+} v_n \right|^{\kappa(\xi)-2} {}^{\mathrm{H}}\mathfrak{D}^{\alpha,\beta;\psi}_{0+} v_n \; {}^{\mathrm{H}}\mathfrak{D}^{\alpha,\beta;\psi}_{0+} \phi + |v_n|^{\kappa(\xi)-2} v_n \phi \right) d\xi \\ &- \int_\Omega \theta_1(k^{-1}(\xi + y_n)) |v_n|^{\tau(\xi)-2} v_n \phi d\xi = o_n(1).\end{aligned}$$

Adapting the arguments used in the proof of **Theorem 2.2**, it is shown that for some subsequence

$${}^{\mathrm{H}}\mathfrak{D}^{\alpha,\beta;\psi}_{0+} v_n \to {}^{\mathrm{H}}\mathfrak{D}^{\alpha,\beta;\psi}_{0+} v \text{ and } v_n \to v \text{ a.e. in } \mathbb{R}^3.$$

Taking the limit $n \to \infty$, yields

$$\int_\Omega \left(\left| {}^{\mathrm{H}}\mathfrak{D}^{\alpha,\beta;\psi}_{0+} v \right|^{\kappa(\xi)-2} \mathfrak{D}^{\alpha,\beta;\psi}_{0+} v \; {}^{\mathrm{H}}\mathfrak{D}^{\alpha,\beta;\psi}_{0+} \phi \right) d\xi$$

$$-\int_\Omega \left(\theta_{1,\infty}|v|^{r(\xi)-2}+|v|^{\kappa_\alpha^*(\xi)-2}\right)v\phi=0,$$

proving that v is a weak solution to the **Problem (2.5)**. Applying Fatou lemma, one has

$$\begin{aligned}
c_{\theta_{1,\infty}} &\leq \boldsymbol{\Theta}^{\alpha,\beta}_{\theta_{1,\infty}}(v)-\frac{1}{q-}\boldsymbol{\Theta}'_{\theta_{1,\infty}}(v)\\
&=\int_\Omega \lim_{n\to\infty}\inf\left(\frac{1}{\kappa(\xi)}-\frac{1}{q-}\right)\left(\left|{}^{\mathrm{H}}\mathfrak{D}^{\alpha,\beta;\psi}_{0+}v_n\right|^{\kappa(\xi)}+|v_n|^{\kappa(\xi)}\right)d\xi\\
&\quad+\int_\Omega \lim_{n\to\infty}\inf\left(\frac{1}{q-}-\frac{1}{r(\xi)}\right)\xi\theta_1(k^{-1}(\xi+y_n))|v_n|^{r(\xi)}d\xi\\
&\leq \lim_{n\to\infty}\inf\int_\Omega\left(\frac{1}{\kappa(\xi)}-\frac{1}{q-}\right)\left(\left|{}^{\mathrm{H}}\mathfrak{D}^{\alpha,\beta;\psi}_{0+}v_n\right|^{\kappa(\xi)}+|u_n|^{\kappa(\xi)}\right)d\xi\\
&\quad+\lim_{n\to\infty}\inf\left(\frac{1}{q-}-\frac{1}{r(\xi)}\right)\xi\theta_1(k^{-1}\xi)|u_n|^{r(\xi)}d\xi\\
&=\lim_{n\to\infty}\inf\left(\boldsymbol{\Theta}^{\alpha,\beta}_{0,k}(u_n)-\frac{1}{q-}\left(\boldsymbol{\Theta}^{\alpha,\beta}_{0,k}\right)(u_n)u_n\right)\\
&=c_{0,k}.
\end{aligned}$$

Therefore, $c_{\theta_{1,\infty}}\leq c_{0,k}$, which contradicts the **Lemma 4.1**. Therefore, $u\neq 0$. Since $\left(\boldsymbol{\Theta}^{\alpha,\beta}_{0,k}\right)'(u_n)=o_n(1)$ it's follows that $\left(\boldsymbol{\Theta}^{\alpha,\beta}_{0,k}\right)'(u_n)=0$, i.e., $u\in\mathcal{N}^{\alpha,\beta}_{0,k}$. Applying Fatou Lemma again, we conclude that

$$\begin{aligned}
c_{0,k} &\leq \boldsymbol{\Theta}^{\alpha,\beta}_{0,k}(u)-\frac{1}{q-}\left(\boldsymbol{\Theta}^{\alpha,\beta}_{0,k}\right)'(u)u\\
&\leq \lim_{n\to\infty}\inf \boldsymbol{\Theta}^{\alpha,\beta}_{0,k}(u_n)-\frac{1}{q-}\left(\boldsymbol{\Theta}^{\alpha,\beta}_{0,k}\right)'(u_n)u_n\\
&=c_{0,k}
\end{aligned}$$

from which it follows that $\boldsymbol{\Theta}^{\alpha,\beta}_{0,k}(u)=c_{0,k}$. □

Throughout this section, we denote by $U\in\mathcal{H}^{\alpha,\beta;\psi}_{\kappa(\xi)}(\Omega)/\{0\}$ a minimum energy solution of the problem (2.3), i.e.,

$$\boldsymbol{\Theta}^{\alpha,\beta}_{\infty}(U)=c_\infty \text{ and } \boldsymbol{\Theta}^{\alpha,\beta}_{\infty}(U)=0.$$

For $1\leq i\leq$ and $k\in\mathbb{N}$, consider the function $U^i_k:\Omega\to\mathbb{R}$ by

$$U^i_k(\xi)=U(\xi-ka_i).$$

The next result establishes an important relationship involving the energy of functions U^i_k with c_∞.

Lemma 4.2. *For every $i\in$ 1,...,l, yields*

$$\lim_{k\to+\infty}\sup\left(\sup_{t\geq 0}\boldsymbol{\Theta}^{\alpha,\beta}_{\lambda,k}\left(tU^i_k\right)\right)\leq c_\infty.$$

Proof. Since the functions κ, q, and r are $\mathbb{Z}^3$-periodic and $a_i\in\mathbb{Z}^3$, we get

$$\boldsymbol{\Theta}^{\alpha,\beta}_{\lambda,k}(tU^i_k)=\int_\Omega\frac{t^{\kappa(\xi)}}{\kappa(\xi)}\left(\left|{}^{\mathrm{H}}\mathfrak{D}^{\alpha,\beta;\psi}_{0+}U\right|^{\kappa(\xi)}+|U(\xi-ka_i)|^{\kappa(\xi)}\right)d\xi$$

$$
\begin{aligned}
&-\lambda\int_\Omega \frac{\theta_2(k^{-1}\xi)}{\kappa(\xi)} t^{q(\xi)}|U(\xi-ka_i)|^{q(\xi)}d\xi\\
&-\int_\Omega \theta_1\left(k^{-1}\xi\right)\frac{t^{r(\xi)}}{r(\xi)}\left|U(\xi-ka_i)\right|^{r(\xi)}d\xi\\
=&\int_\Omega \frac{t^{\kappa(\xi)}}{\kappa(\xi)}\left(\left|{}^{\mathrm{H}}\mathfrak{D}_{0+}^{\alpha,\beta;\psi}U\right|^{\kappa(\xi)}+|U|^{\kappa(\xi)}\right)d\xi\\
&-\lambda\int_\Omega \theta_2(k^{-1}\xi+a_i)\frac{t^{q(\xi)}}{q(\xi)}|U|^{q(\xi)}d\xi\\
&-\int_\Omega \theta_1(k^{-1}\xi+a_i)\frac{t^{r(\xi)}}{r(\xi)}|U|^{r(\xi)}d\xi.
\end{aligned}
$$

Furthermore, by **Lemma 6.1** there is $t_k>0$ such that

$$
\max_{t\geq 0}\mathbf{\Theta}_{\lambda,k}^{\alpha,\beta}\left(tU_k^i\right)=\mathbf{\Theta}_{\lambda,k}^{\alpha,\beta}\left(t_kU_k^i\right)\geq\beta, \tag{4.2}
$$

where β was given by **Lemma 2.5**. Note that if $t_k\to 0$ when $k\to\infty$ then $\mathbf{\Theta}_{\lambda,k}^{\alpha,\beta}(t_kU_k^i)\to 0$ when $k\to\infty$, which contradicts (4.2). On the other hand if $t_k\to\infty$ when $k\to\infty$, it shows that $\mathbf{\Theta}_{\lambda,k}^{\alpha,\beta}(t_kU_k^i)\to-\infty$ and again we have a contradiction with (4.2). So, we can assume $t_k\to t_0>0$ (without loss of generality) with $k\to\infty$. So, one has

$$
\begin{aligned}
\lim_{k\to\infty}\left(\max_{t\geq 0}\mathbf{\Theta}_{\lambda,k}^{\alpha,\beta}\left(tU_K^i\right)\right)=&\int_\Omega \frac{t_0^{\kappa(\xi)}}{\kappa(\xi)}\left(\left|{}^{\mathrm{H}}\mathfrak{D}_{0+}^{\alpha,\beta;\psi}U\right|^{\kappa(\xi)}+|U|^{\kappa(\xi)}\right)d\xi\\
&-\lambda\int_\Omega \theta_2(a_i)\frac{t_0^{q(\xi)}}{q(\xi)}|U|^{q(\xi)}d\xi\\
&-\int_\Omega \theta_1(a_i)\frac{t_0^{r(\xi)}}{r(\xi)}|U|^{r(\xi)}d\xi\\
\leq&\,\mathbf{\Theta}_\infty^{\alpha,\beta}(t_0U)\leq\max\mathbf{\Theta}_\infty^{\alpha,\beta}(sU)\\
=&\,\mathbf{\Theta}_\infty^{\alpha,\beta}(U)=c_\infty.
\end{aligned}
$$

Consequently,

$$
\lim_{k\to+\infty}\sup\left(\sup_{t\geq 0}\mathbf{\Theta}_{\lambda,k}^{\alpha,\beta}\left(tU_k^i\right)\right)\leq c_\infty \text{ for } i\in\{1,...,l\},
$$

completing the proof of the result. □

Now, consider the positive numbers R_0 and r_0 satisfying:

- $\overline{B_{R_0}(a_i)}\cap\overline{B_{R_0}(a_j)}=\emptyset$ for $i\neq j$ and $i,j\in\{1,...,l\}$;
- $U_{i=1}^l B_{R_0}(a_i)\subset B_{R_0}(0)$;
- $K_{\frac{R_0}{2}}=U_{i=1}^l\overline{B_{\frac{R_0}{2}}(a_i)}$.

Consider the following barycenter function $Q_k:\mathcal{H}_{\kappa(\xi)}^{\alpha,\beta,\psi}(\Omega)\backslash\{0\}\to\mathbb{R}$ given by

$$
Q_k(u)=\frac{\displaystyle\int_\Omega \chi(k^{-1}\xi)|u|^{\kappa_+}d\xi}{\displaystyle\int_\Omega |u|^{\kappa_+}d\xi}
$$

where $\chi : \mathbb{R}^3 \to \mathbb{R}^3$ is given by

$$\chi(\xi) = \begin{cases} \xi, \text{ if } \ |\xi| \leq r_0, \\ \dfrac{r_0}{|\xi|}\xi, \text{ if } |\xi| > r_0. \end{cases} \tag{4.3}$$

Lemma 4.3. *There exist* $\delta_0 > 0$ *and* $k_1 \in \mathbb{N}$ *such that if* $u \in \mathcal{N}_{0,k}^{\alpha,\beta}$ *and* $\mathbf{\Theta}_{0,k}^{\alpha,\beta}(u) \leq c_\infty + r_0$*, then*

$$Q_k(u) \in K_{\frac{R_0}{2}}, \ k \geq k_1.$$

Proof. Suppose the lemma is not true, then there are $\delta_n \to 0, k_n \to +\infty$ and $u_n \in \mathcal{N}_{0,k_n}^{\alpha,\beta}$ satisfying

$$\mathbf{\Theta}_{0,k_n}^{\alpha,\beta}(u_n) \leq c_\infty + \delta_n$$

and

$$Q_{k_n}(u_n) \notin K_{\frac{R_0}{2}}.$$

Take $\zeta_n > 0$ in such a way that $\zeta_n u_n \in \mathcal{N}_\infty^{\alpha,\beta}$, yields

$$c_\infty \leq \mathbf{\Theta}_\infty^{\alpha,\beta}(\zeta_n u_n) \leq \ \mathbf{\Theta}_{0,k_n}^{\alpha,\beta}(\zeta_n u_n) \leq \ \max_{t\geq 0} \mathbf{\Theta}_{0,k_n}^{\alpha,\beta}(t u_n) = \mathbf{\Theta}_{0,k_n}^{\alpha,\beta}(u_n) \leq c_\infty + \delta_n.$$

Therefore

$$\{\zeta_n u_n\} \subset \mathcal{N}_\infty^{\alpha,\beta} \text{ and } \mathbf{\Theta}_\infty^{\alpha,\beta}(\zeta_n u_n) \to c_\infty.$$

Applying Ekeland variational principle, we can assume that $\{\zeta_n u_n\} \subset \mathcal{N}_\infty^{\alpha,\beta}$ (without loss of generality) is a sequence $(PS)_{c_\infty}$ to $\mathbf{\Theta}_\infty^{\alpha,\beta}$, i.e.,

$$\mathbf{\Theta}_\infty^{\alpha,\beta}(\zeta_n u_n) \to c_\infty \text{ and } \left(\mathbf{\Theta}_\infty^{\alpha,\beta}\right)'(\zeta_n u_n) \to 0.$$

Applying the **Theorem 3.1**, we have some cases to be considered, namely:

1. $\zeta_n u_n \to U \neq 0$ in $\mathcal{H}_{\kappa(\xi)}^{\alpha,\beta;\psi}(\Omega)$;
2. There are $\{y_n\} \subset \mathbb{Z}^3$ with $|y_n| \to +\infty$ such that $v_n(\xi) = \zeta_n u(\xi + y_n)$ converges on $\mathcal{H}_{\kappa(\xi)}^{\alpha,\beta;\psi}(\Omega)$ for some $V \in \mathcal{H}_{\kappa(\xi)}^{\alpha,\beta;\psi}(\Omega)\backslash\{0\}$.

Proceeding as in the **Lemma 4.2**, it is shown that $\zeta_n \to \zeta_0$ for some $\zeta_0 > 0$. Therefore, we can assume that

$$u_n \to U \text{ or } v_n = u_n(\cdot + y_n) \to V \text{ in } \mathcal{H}_{\kappa(\xi)}^{\alpha,\beta;\psi}(\Omega).$$

Next, we will analyze (1) and (2).

1. **Analysis of (1).** By Lebesgue dominated convergence theorem it follows that

$$Q_{k_n}(u_n) = \frac{\displaystyle\int_\Omega \chi(k_n^{-1}\xi)|u_n|^{\kappa_+} d\xi}{\displaystyle\int_\Omega |u_n|^{\kappa_+} d\xi} \to \frac{\displaystyle\int_\Omega \chi(0)|U|^{\kappa_+} d\xi}{\displaystyle\int_\Omega |U|^{\kappa_+} d\xi} = 0,$$

implying $Q_{k_n}(u_n) \in K_{\frac{R_0}{2}}$ for n large, because $0 \in K_{\frac{R_0}{2}}$.

2. **Analysis of (2).** Using Ekeland variational principle again, we assume that $\left(\mathbf{\Theta}^{\alpha,\beta}_{0,k_n}\right)'(u_n) = o_n(1)$. Hence $\left(\mathbf{\Theta}^{\alpha,\beta}_{0,k_n}\right)\phi(\cdot - y_n) = o_n(1)$ for all $\phi \in \mathcal{H}^{\alpha,\beta;\psi}_{\kappa(\xi)}(\omega)$, and so

$$o_n(1) = \int_\Omega \left(\left| {}^{\mathrm{H}}\mathfrak{D}^{\alpha,\beta;\psi}_{0+} v_n \right|^{\kappa(\xi)-2} + {}^{\mathrm{H}}\mathfrak{D}^{\alpha,\beta;\psi}_{0+} v_n \; {}^{\mathrm{H}}\mathfrak{D}^{\alpha,\beta;\psi}_{0+} \phi + |v_n|^{\kappa(\xi)-2} v_n \phi \right) d\xi$$
$$- \int_\Omega \theta_1 \left(k_n^{-1}(\xi + y_n) \right) |v_n|^{r(\xi)-2} v_n \phi d\xi. \tag{4.4}$$

It follows from the last limit that up to subsequence,

$${}^{\mathrm{H}}\mathfrak{D}^{\alpha,\beta;\psi}_{0+} v_n(\xi) \to {}^{\mathrm{H}}\mathfrak{D}^{\alpha,\beta;\psi}_{0+} V(\xi) \text{ and } v_n(\xi) \to V(\xi) \text{ a.e. in } \Omega.$$

Consider the following cases:

(a) $\left| k_n^{-1} y_n \right| \to +\infty$;

(b) $k_n^{-1} y_n \to y$, for some $y \in \Omega$.

Assuming that **(a)** is valid, it follows that

$$\int \left(\left| {}^{\mathrm{H}}\mathfrak{D}^{\alpha,\beta;\psi}_{0+} V \right|^{\kappa(\xi)-2} {}^{\mathrm{H}}\mathfrak{D}^{\alpha,\beta;\psi}_{0+} V \; {}^{\mathrm{H}}\mathfrak{D}^{\alpha,\beta;\psi}_{0+} \phi + |V|^{\kappa(\xi)-2} V \phi \right) d\xi$$
$$= \int \theta_{1,\infty} |V|^{r(\xi)-2} V \phi d\xi.$$

In this sense, V is a non-trivial weak solution to the **Problem (2.5)**. Combining the condition $\theta_{1,\infty} < 1$ with Fatou lemma, yields

$$\begin{aligned} c_{\theta_{1,\infty}} \leq & \mathbf{\Theta}^{\alpha,\beta}_{\theta_{1,\infty}}(V) - \frac{1}{q-} \left(\mathbf{\Theta}^{\alpha,\beta}_{\theta_{1,\infty}} \right)'(V)V \\ \leq & \lim_{n\to\infty} \inf \left\{ \mathbf{\Theta}^{\alpha,\beta}_{\infty}(u_n) - \frac{1}{q-} \left(\mathbf{\Theta}^{\alpha,\beta}_{\infty} \right)'(u_n) u_n \right\} \\ = & c_\infty \end{aligned}$$

or equivalent, $c_{\theta_{1,\infty}} \leq c_\infty$, which contradicts the **Lemma 4.1**.

If $k_n^{-1} y_n \to y$ for some $y \in \mathbb{R}^N$, then V is a weak solution of

$${}^{\mathrm{H}}\mathfrak{D}^{\alpha,\beta;\psi}_{0} u + |u|^{\kappa(\xi)-2} u = \theta_1(y) |u|^{r(\xi)-2} u, \; u \in \mathcal{H}^{\alpha,\beta;\psi}_{\kappa(\xi)}(\Omega).$$

Repeating the previous argument, we deduce that

$$c_{\theta_1(y)} \leq c_\infty \tag{4.5}$$

where $c_{\theta_1(\xi)}$ is the mountain pass level of the functional $\mathbf{\Theta}^{\alpha,\beta}_{\theta_1(y)}$: $\mathcal{H}^{\alpha,\beta;\psi}_{\kappa(\xi)}(\Omega) \to \mathbb{R}$ given by

$$\mathbf{\Theta}^{\alpha,\beta}_{\theta_1(y)}(u) = \int_\Omega \frac{1}{\kappa(\xi)} \left(\left| {}^{\mathrm{H}}\mathfrak{D}^{\alpha,\beta;\psi}_{\kappa(\xi)} u \right|^{\kappa(\xi)} + |u|^{\kappa(\xi)} \right) d\xi - \int_\Omega \frac{\theta_1(y)}{r(\xi)} |u|^{r(\xi)} d\xi.$$

Note that

$$c_{\theta_1(y)} = \inf_{u \in \mathcal{N}_{\theta_1(y)}} \mathbf{\Theta}^{\alpha,\beta}_{\theta_1(y)}(u)$$

where

$$\mathcal{N}^{\alpha,\beta}_{\theta_1(\xi)} = \left\{ u \in \mathcal{H}^{\alpha,\beta;\psi}_{\kappa(\xi)}(\Omega)\backslash\{0\} : \left(\mathbf{\Theta}^{\alpha,\beta}_{\theta_1(y)}\right)'(u)u = 0 \right\}.$$

If $\theta_1(y) < 1$, an argument similar to the one explored in the proof of **Lemma 4.1** shows that $c_{\theta_1(y)} > c_\infty$, contradicting the inequality (4.5). Therefore, $\theta_1(y) = 1$ and $y = a_i$ for some $i = 1, ..., l$. Then,

$$Q_{k_n}(u_n) = \frac{\displaystyle\int_\Omega \chi\left(k_n^{-1}\xi\right)|u_n|^{\kappa_+} d\xi}{\displaystyle\int_\Omega |u_n|^{\kappa_+} d\xi} = \frac{\displaystyle\int_\Omega \chi\left(k_n^{-1}\xi + k_n^{-1}y_n\right)|v_n|^{\kappa_+} d\xi}{\displaystyle\int_\Omega |v_n|^{\kappa_+} d\xi}.$$

In the previous equality passing to the limit when $n \to \infty$, one has

$$\lim_{n\to\infty} Q_{k_n}(u_n) = \frac{\displaystyle\int_\Omega \chi(y)|V|^{\kappa_+} d\xi}{\displaystyle\int_\Omega |V|^{\kappa_+} d\xi} = a_i$$

which implies $Q_{k_n}(u_n) \in K_{\frac{R_0}{2}}$ for n large enough, resulting in a contradiction, because by hypothesis $Q_{k_n}(u_n) \notin K_{\frac{R_0}{2}}$. □

Lemma 4.4. *There exists a constant $R > 0$ such that*

$$\mathcal{A}_{\lambda,k} = \left\{ u \in \mathcal{N}^{\alpha,\beta}_{\lambda,k} : \mathbf{\Theta}^{\alpha,\beta}_{\lambda,k}(u) < c_\infty + \frac{\delta_0}{2} \right\} \subset B_R$$

for $k \geq k_1$, i.e. $\mathcal{A}_{\lambda,k}$ is a bounded set, where k_1 was given in the **Lemma 4.3**. *Furthermore, R is independent of λ and k.*

Proof. Let $u \in \mathcal{N}^{\alpha,\beta}_{\lambda,k}$ such that $\mathbf{\Theta}^{\alpha,\beta}_{\lambda,k}(u) < c_\infty + \dfrac{\delta_0}{2}$ for $k \geq k_1$. Then,

$$\begin{aligned} &\int_\Omega \left(\left| {}^{\mathrm{H}}\mathfrak{D}^{\alpha,\beta,\psi}_{0+} u \right|^{\kappa(\xi)} + |u|^{\kappa(\xi)} \right) d\xi - \lambda \int_\Omega \theta_2(k^{-1}\xi)|u|^{q(\xi)} d\xi \\ &- \int_\Omega \theta_1(k^{-1}\xi)|u|^{r(\xi)} d\xi = 0 \end{aligned} \tag{4.6}$$

and

$$\begin{aligned} &\int_\Omega \frac{1}{\kappa(\xi)} \left(\left| {}^{\mathrm{H}}\mathfrak{D}^{\alpha,\beta,\psi}_{0+} u \right|^{\kappa(\xi)} + |u|^{\kappa(\xi)} \right) d\xi \\ &-\lambda \int_\Omega \frac{\theta_2(k^{-1}\xi)}{q(\xi)}|u|^{q(\xi)} d\xi - \int_\Omega \frac{\theta_1(k^{-1}\xi)}{r(\xi)}|u|^{r(\xi)} d\xi \\ &< c_\infty + \frac{\delta_0}{2}. \end{aligned} \tag{4.7}$$

From Eq.(4.6)-Eq(4.7), yields

$$\begin{aligned}&\left(\frac{1}{\kappa_+}-\frac{1}{\kappa_-}\right)\int_\Omega\left(\left|{}^{\mathrm{H}}\mathfrak{D}_{0+}^{\alpha,\beta,\psi}u\right|^{\kappa(\xi)}+|u|^{\kappa(\xi)}\right)d\xi\\&+\left(\frac{1}{q+}-\frac{1}{r-}\right)\int_\Omega\theta_1(k^{-1}\xi)|u|^{r(\xi)}d\xi\\&<c_\infty+\frac{\delta_0}{2}.\end{aligned}$$

Therefore,

$$\int_\Omega\left(\left|{}^{\mathrm{H}}\mathfrak{D}_{0+}^{\alpha,\beta,\psi}u\right|^{\kappa(\xi)}+|u|^{\kappa(\xi)}\right)d\xi<\left(c_\infty+\frac{\delta_0}{2}\right)\left(\frac{1}{\kappa_+}-\frac{1}{q-}\right)^{-1}.$$

In this sense, we concluded the proof. □

Lemma 4.5. *Let $u\in\mathcal{A}_{\lambda,k}$ and $t_u>0$ such that $t_u\,u\in\mathcal{N}_{0,k}$. Then, given $\Lambda>0$, there exists constants $C>0$ and $k_2\in\mathbb{N}$ such that $0\le t_u\le C$ for all $(u,\lambda,k)\in\mathcal{A}_{\lambda,k}\times[0,\Lambda]\times([k_2,+\infty)\cap\mathbb{N})$.*

Proof. Assume that the lemma is not true. Then there must be $\{u_n\}\subset\mathcal{A}_{\lambda_n,k_n}$ with $\lambda_n\to0$ and $k_n\to+\infty$ such that $t_{u_n}u_n\in\mathcal{N}_{0,k_n}^{\alpha,\beta}$ and $t_{u_n}\to\infty$ with $n\to\infty$. We can assume that $t_{u_n}\ge1$ (without loss of generality). Since $t_{u_n}u_n\in\mathcal{N}_{0,k_n}^{\alpha,\beta}$, it follows that

$$(t_{u_n})^{\kappa_+}\int_\Omega\left(\left|{}^{\mathrm{H}}\mathfrak{D}_{0+}^{\alpha,\beta,\psi}u_n\right|^{\kappa(\xi)}+|u_n|^{\kappa(\xi)}\right)d\xi\ge\theta_{1,\infty}(t_{u_n})^{r-}\int_\Omega|u_n|^{r(\xi)}d\xi$$

or equivalent,

$$\int_\Omega\left(\left|{}^{\mathrm{H}}\mathfrak{D}_{0+}^{\alpha,\beta,\psi}u_n\right|^{\kappa(\xi)}+|u_n|^{\kappa(\xi)}\right)d\xi\ge\theta_{1,\infty}t_{u_n}^{r--\kappa_+}\int_\Omega|u_n|^{r(\xi)}d\xi.\tag{4.8}$$

Affirmation: There is $\mu_1>0$ such that $\displaystyle\int_\Omega|u_n|^{r(\xi)}d\xi>\mu_1,\ \forall n\in\mathbb{N}$.

Indeed, arguing by contradiction, if $\displaystyle\int_\Omega|u_n|^{r(\xi)}d\xi\to0$, by interpolation it follows that $\displaystyle\int_\Omega|u_n|^{q(\xi)}d\xi\to0$. Since $u_n\in\mathcal{N}_{\lambda_n,k_n}$,

$$\int_\Omega\left(\left|{}^{\mathrm{H}}\mathfrak{D}_{0+}^{\alpha,\beta,\psi}u_n\right|^{\kappa(\xi)}+|u_n|^{\kappa(\xi)}\right)d\xi\le\lambda||\theta_2||_\infty\int_\Omega|u_n|^{q(\xi)}d\xi+\int_\Omega|u_n|^{r(\xi)}d\xi=o_n(1),$$

or yet, $u_n\to0$ in $\mathcal{H}_{\kappa(\xi)}^{\alpha,\beta,\psi}(\Omega)$ which contradicts the **Lemma 4.3**, proving the statement. Using the inequality (4.8), it follows that

$$\rho_1(u_n)=\int_\Omega\left(\left|{}^{\mathrm{H}}\mathfrak{D}_{0+}^{\alpha,\beta,\psi}u_n\right|^{\kappa(\xi)}+|u_n|^{\kappa(\xi)}\right)d\xi\to+\infty.$$

Therefore, we have that $\{u_n\}$ is an unbounded sequence. However, this is impossible, because by **Lemma 4.4** $\{u_n\}$ is bounded. □

Lemma 4.6. *Let* $\delta_0 > 0$ *be given by* **Lemma 4.3** *and* $k_3 = max\{k_1, k_2\}$. *Then, there exists* $\Lambda^* > 0$ *such that*

$$Q_k(u) \in K_{\frac{R_0}{2}}, \forall (u, \lambda, k) \in \mathcal{A}_{\lambda,k} \times [0, \Lambda^*) \times ([k_3, +\infty) \cap \mathbb{N}).$$

Proof. Note that

$$\mathbf{\Theta}_{\lambda,k}^{\alpha,\beta}(u) = \mathbf{\Theta}_{0,k}^{\alpha,\beta}(u) - \lambda \int_\Omega \frac{\theta_2\left(k^{-1}\xi\right)}{q(\xi)} |u|^{q(\xi)} d\xi, \ \forall u \in \mathcal{H}_{\kappa(\xi)}^{\alpha,\beta;\psi}(\Omega).$$

Consider $t_u > 0$ with $t_u u \in \mathcal{N}_{0,k}^{\alpha,\beta}$. So, get

$$\begin{aligned}\mathbf{\Theta}_{0,k}^{\alpha,\beta}(t_u u) &= \mathbf{\Theta}_{\lambda,k}^{\alpha,\beta}(t_u u) + \lambda \int_\Omega \frac{\theta_2\left(k^{-1}\xi\right)}{q(\xi)} (t_u)^{q(\xi)} |u|^{q(\xi)} d\xi \\ &\leq \max_{t \geq 0} \mathbf{\Theta}_{\lambda,k}^{\alpha,\beta}(tu) + \lambda \int_\Omega \frac{\theta_2\left(k^{-1}\xi\right)}{q(\xi)} (t_u)^{q(\xi)} |u|^{q(\xi)} d\xi.\end{aligned}$$

Using the **Lemma 2.8**, yields

$$\mathbf{\Theta}_{0,k}^{\alpha,\beta}(t_u u) \leq \mathbf{\Theta}_{\lambda,k}^{\alpha,\beta}(u) + \frac{\lambda}{q-} ||\theta_2||_\infty C^{q+} \int_\Omega |u|^{q(\xi)} d\xi.$$

Since $u \in \mathcal{A}_{\lambda,k}$, we get

$$\mathbf{\Theta}_{0,k}^{\alpha,\beta}(t_u u) < c_\infty + \frac{\delta_0}{2} + \lambda c_2 \int_\Omega |u|^{q(\xi)} d\xi.$$

Using Sobolev embedding and the **Lemma 4.4**, we obtain

$$\mathbf{\Theta}_{0,k}^{\alpha,\beta}(t_u u) < c_\infty + \frac{\delta_0}{2} + c_3 \lambda, \ \forall u \in \mathcal{A}_{\lambda,k}$$

where c_3 is a positive constant. Taking $\Lambda^* := \delta_0 / 2c_3$ and $\lambda \in [0, \Lambda^*)$, we conclude that $t_u u \in \mathcal{M}_{0,k}^{\alpha,\beta}$ and $\mathbf{\Theta}_{0,k}^{\alpha,\beta}(t_u u) < c_\infty + \delta_0$.

So, by the **Lemma 2.7**, we have $Q_k(t_u u) \in K_{\frac{R_0}{2}}$. Note note that $Q_k(u) = Q_k(t_u u)$. Thus, we complete the proof. □

5. The Palais-Smale condition

In this section, we prove the existence of a sequence $(PS)_{\beta^i_{\lambda,k}}$ in $\theta^i_{\lambda,k}$ for the functional $\mathbf{\Theta}_{\lambda,k}^{\alpha,\beta}$.

Before starting the discussion of some technical lemmas, essential for the investigation of the main result of this section. Consider the following notations:

- $\theta^i_{\lambda,k} = \{u \in \mathcal{N}_{\lambda,k}; |Q_k(u) - a_i| < R_0\}$;
- $\partial\theta^i_{\lambda,k} = \{u \in \mathcal{N}_{\lambda,k} | Q_k(u) - a_i| = R_0\}$;
- $\beta^i_{\lambda,k} = \inf\limits_{u \in \theta^i_{\lambda,k}} \mathbf{\Theta}_{\lambda,k}^{\alpha,\beta}(u)$;
- $\tilde{\gamma}^i_{\lambda,k} = \inf\limits_{u \in \partial\theta^i_{\lambda,k}} \mathbf{\Theta}_{\lambda,k}^{\alpha,\beta}(u)$.

Lemma 5.1. *Setting $\varrho = \frac{1}{2}(c_{\theta_{1,\infty}} - c_\infty)$ exists $k^* \in \mathbb{N}$ such that $\beta^i_{\lambda,k} < c_\infty + \varrho$ and $\beta^i_{\lambda,k}$, $\tilde{\gamma}^i_{\lambda,k}$, for all $\lambda \in [0, \Lambda^*), i \in \{1, ...l\}$ and $k \leq k^*$.*

The next result establishes an important relationship between functionals $\mathbf{\Theta}^{\alpha,\beta}_{\lambda,k}$ and $\mathbf{\Theta}^{\alpha,\beta}_{\infty}$.

Lemma 5.2. *Let $\{v_n\}$ be a sequence $(PS)_d$ for the functional $\mathbf{\Theta}^{\alpha,\beta}_{\lambda,k}$ with $v_n \rightharpoonup v$ in $\mathcal{H}^{\alpha,\beta,\psi}_{\kappa(\xi)}(\Omega)$. Then*

$$\mathbf{\Theta}^{\alpha,\beta}_{\lambda,k}(v_n) - \mathbf{\Theta}^{\alpha,\beta}_{0,k}(w_n) - \mathbf{\Theta}^{\alpha,\beta}_{\lambda,k}(v) = o_n(1) \tag{5.1}$$

and

$$\left\| \left(\mathbf{\Theta}^{\alpha,\beta}_{\lambda,k}\right)'(v_n) - \left(\mathbf{\Theta}^{\alpha,\beta}_{0,k}\right)'(w_n) - \left(\mathbf{\Theta}^{\alpha,\beta}_{\lambda,k}\right)'(v) \right\| = o_n(1) \tag{5.2}$$

where $w_n = v_n - v$.

Proof. Proceeding as in the **Theorem 2.2** , we have the following convergences:
(1) ${}^{\mathrm{H}}\mathfrak{D}^{\alpha,\beta,\psi}_{0+} v_n \to {}^{\mathrm{H}}\mathfrak{D}^{\alpha,\beta,\psi}_{0+} v$ a.e. in Ω;
(2) $v_n \to v$ a.e. in Ω;
(3)

$$\begin{aligned} &\int_\Omega \left| {}^{\mathrm{H}}\mathfrak{D}^{\alpha,\beta,\psi}_{0+} v_n \right|^{\kappa(\xi)-2} {}^{\mathrm{H}}\mathfrak{D}^{\alpha,\beta,\psi}_{0+} v_n \; {}^{\mathrm{H}}\mathfrak{D}^{\alpha,\beta,\psi}_{0+} \phi d\xi \\ \to &\int_\Omega \left| {}^{\mathrm{H}}\mathfrak{D}^{\alpha,\beta,\psi}_{0+} v \right|^{\kappa(\xi)-2} {}^{\mathrm{H}}\mathfrak{D}^{\alpha,\beta,\psi}_{0+} v \; {}^{\mathrm{H}}\mathfrak{D}^{\alpha,\beta,\psi}_{0+} \phi d\xi; \end{aligned}$$

(4)

$$\int_\Omega \theta_1\left(k^{-1}\xi\right) |v_n|^{r(\xi)-2} v_n \phi d\xi \to \int_\Omega \theta_1\left(k^{-1}\xi\right) |v|^{r(\xi)-2} v \phi d\xi,$$

for all $\phi \in \mathcal{H}^{\alpha,\beta,\psi}_{\kappa(\xi)}(\Omega)$. Applying the Brezis-Lieb lemma to variable exponents, one has

$$\begin{aligned} \mathbf{\Theta}^{\alpha,\beta}_{\lambda,k}(v_n) = &\, o_n(1) + \int_\Omega \frac{1}{\kappa(\xi)} \left(\left| {}^{\mathrm{H}}\mathfrak{D}^{\alpha,\beta,\psi}_{0+} w_n \right|^{\kappa(\xi)} + |w_n|^{\kappa(\xi)} \right) d\xi \\ &+ \int_\Omega \frac{1}{\kappa(\xi)} \left(\left| {}^{\mathrm{H}}\mathfrak{D}^{\alpha,\beta,\psi}_{0+} v \right|^{\kappa(\xi)} + |v|^{\kappa(\xi)} \right) d\xi \\ &- \lambda \int_\Omega \frac{\theta_2\left(k^{-1}\xi\right)}{q(\xi)} |w_n|^{\kappa(\xi)} d\xi - \lambda \int_\Omega \frac{\theta_2\left(k^{-1}\xi\right)}{q(\xi)} |v|^{q(\xi)} d\xi \\ &- \int_\Omega \frac{\theta_1\left(k^{-1}\xi\right)}{r(\xi)} |w_n|^{r(\xi)} d\xi - \int_\Omega \frac{\theta_1\left(k^{-1}\xi\right)}{r(\xi)} |v|^{r(\xi)} d\xi. \end{aligned}$$

Soon

$$\mathbf{\Theta}^{\alpha,\beta}_{\lambda,k}(v_n) = \mathbf{\Theta}^{\alpha,\beta}_{0,k}(w_n) + \mathbf{\Theta}^{\alpha,\beta}_{\lambda,k}(v) + o_n(1)$$

proving (5.1).

Now, let's prove (5.2). Consider $\varphi \in \mathcal{H}^{\alpha,\beta;\psi}_{\kappa(\xi)}(\Omega)$ with $||\varphi|| = 1$. Carrying out some calculations, one has

$$\left| \left[\left(\mathbf{\Theta}^{\alpha,\beta}_{\lambda,k}\right)'(v_n) - \left(\mathbf{\Theta}^{\alpha,\beta}_{\lambda,k}\right)'(w_n) - \left(\mathbf{\Theta}^{\alpha,\beta}_{\lambda,k}\right)'(v) \right] \varphi \right|$$

$$
\begin{aligned}
\leq & \left| \int_\Omega \left(\left| {}^{\mathrm{H}}\mathfrak{D}_{0+}^{\alpha,\beta;\psi} v_n \right|^{\kappa(\xi)-2} {}^{\mathrm{H}}\mathfrak{D}_{0+}^{\alpha,\beta;\psi} v_n - \left| {}^{\mathrm{H}}\mathfrak{D}_{0+}^{\alpha,\beta;\psi} w_n \right|^{\kappa(\xi)-2} {}^{\mathrm{H}}\mathfrak{D}_{0+}^{\alpha,\beta;\psi} w_n \right.\right. \\
& \left.\left. - \left| {}^{\mathrm{H}}\mathfrak{D}_{0+}^{\alpha,\beta;\psi} v \right|^{\kappa(\xi)-2} {}^{\mathrm{H}}\mathfrak{D}_{0+}^{\alpha,\beta;\psi} v \right) {}^{\mathrm{H}}\mathfrak{D}_{0+}^{\alpha,\beta;\psi} \varphi \, d\xi \right| \\
& + \left| \int_\Omega \left(|v_n|^{\kappa(\xi)-2} v_n - |w_n|^{\kappa(\xi)-2} w_n - |v|^{\kappa(\xi)-2} v \right) \varphi \, d\xi \right| \\
& + \lambda \left| \int_\Omega \theta_2(k^{-1}\xi) \left(|v_n|^{q(\xi)-2} v_n - |v|^{q(\xi)-2} v \right) \varphi \, d\xi \right| \\
& + \left| \int_\Omega \theta_1(k^{-1}\xi) \left(|v_n|^{r(\xi)-2} v_n - |v_n|^{r(\xi)-2} v_n - |v|^{r(\xi)-2} v \right) \varphi d\xi \right|.
\end{aligned}
$$

Applying Holder's inequality, yields

$$
\begin{aligned}
& \left| \left[\mathbf{\Theta}_{\lambda,k}^{\alpha,\beta}(v_n) - \mathbf{\Theta}_{\lambda,k}^{\alpha,\beta}(w_n) - \mathbf{\Theta}_{\lambda,k}^{\alpha,\beta}(v) \right] \varphi \right| \\
\leq & \; 2 \left\| {}^{\mathrm{H}}\mathfrak{D}_{0+}^{\alpha,\beta;\psi} \varphi \right\|_{\kappa(\xi)} \left\| \left| {}^{\mathrm{H}}\mathfrak{D}_{0+}^{\alpha,\beta;\psi} v_n \right|^{\kappa(\xi)-2} {}^{\mathrm{H}}\mathfrak{D}_{0+}^{\alpha,\beta;\psi} v_n \right. \\
& - \left| {}^{\mathrm{H}}\mathfrak{D}_{0+}^{\alpha,\beta;\psi} w_n \right|^{\kappa(\xi)-2} {}^{\mathrm{H}}\mathfrak{D}_{0+}^{\alpha,\beta;\psi} w_n \\
& \left. - \left| {}^{\mathrm{H}}\mathfrak{D}_{0+}^{\alpha,\beta;\psi} v \right|^{\kappa(\xi)-2} {}^{\mathrm{H}}\mathfrak{D}_{0+}^{\alpha,\beta;\psi} v \right\|_{p'(\xi)} \\
& + 2 \|\varphi\|_{\kappa(\xi)} \left\| |v_n|^{\kappa(\xi)-2} v_n - |w_n|^{\kappa(\xi)-2} w_n - |v|^{\kappa(\xi)-2} v \right\|_{p'(\xi)} \\
& + 2\xi ||\varphi||_{r(\xi)} \left\| |v_n|^{r(\xi)-2} v_n - |w_n|^{r(\xi)-2} w_n - |v|^{r(\xi)-2} v \right\|_{r(\xi)} \\
& + \lambda \left| \int_\Omega \theta_2\left(k^{-1}\xi\right) \left(|v_n|^{q(\xi)-2} v_n - |v|^{q(\xi)-2} v \right) \varphi d\xi \right|.
\end{aligned}
$$

Using the **Proposition 2.4** and **Proposition 2.1** it follows that the first three terms on the right side of the previous inequality converge to 0 when $n \to \infty$. To conclude the proof, it remains to show that the last term of the above inequality is $o_n(1)$. From Holder inequality, we have

$$
\begin{aligned}
& \int_\Omega \theta_2\left(k^{-1}\xi\right) \left(|v_n|^{q(\xi)-2} v_n - |v|^{q(\xi)-2} v \right) \varphi d\xi \\
= & \int_\Omega \theta_2\left(k^{-1}\xi\right)^{\frac{1}{q'(\xi)}} \left[|v_n|^{q(\xi)-2} v_n - |v|^{q(\xi)-2} v \right] \theta_2\left(k^{-1}\xi\right)^{\frac{1}{q(\xi)}} \varphi d\xi \\
\leq & \; C \left\| \theta_2\left(k^{-1}\xi\right)^{\frac{1}{q'(\xi)}} \left(|v_n|^{q(\xi)-2} v_n - |v|^{q(\xi)-2} v \right) \right\|_{q'(\xi)}.
\end{aligned}
$$

Note that $|v_n|^{q(\xi)} \to |v|^{q(\xi)}$ a.e. in Ω with $n \to \infty$. As

$$
\left| |v_n|^{q(\xi)-2} v_n - |v|^{q(\xi)-2} v||^{q'(\xi)} \right| \leq 2^{q'_+} + \left(|v_n|^{q'(\xi)} |v|^{q'(\xi)} \right).
$$

Applying Fatou lemma, and conclude that

$$
\int_\Omega 2^{1+q'_+} \theta_2(k^{-1}\xi) |v|^{q(\xi)} d\xi
$$

$$= \int_\Omega \liminf_{n\to\infty} \theta_2(k^{-1}\xi)\left(2^{q'_+}|v_n|^{q(\xi)} + 2^{q'_+}|v|^{q(\xi)} - \left||v_n|^{q(\xi)-2}v_n - |v|^{q(\xi)-2}v\right|^{q'(\xi)}\right) d\xi$$
$$\leq \liminf_{n\to\infty} \int_\Omega \left[2^{q'_+}\theta_2(k^{-1}\xi)|v_n|^{q(\xi)} + \theta_2(k^{-1}\xi)|v|^{q(\xi)}\right] d\xi$$
$$-\liminf_{n\to\infty}\int_\Omega \theta_2(k^{-1}\xi)\left||v_n|^{q(\xi)-2}v_n - |v|^{q(\xi)-2}v\right|^{q'(\xi)}.$$

Using **Lemma 5.1**, we obtain

$$\int_\Omega 2^{1+q'_+}\theta_2\left(k^{-1}\xi\right)|v|^{q(\xi)}d\xi$$
$$\leq \int_\Omega 2^{1+q'}\theta_2\left(k^{-1}\xi\right)|v|^{q(\xi)}d\xi$$
$$-\lim\sup_{n\to\infty}\int_\Omega \theta_2\left(k^{-1}\xi\right)|v_n|^{q(\xi)-2}v_n d\xi - |v|^{q(\xi)-2}v|^{q'(\xi)}d\xi$$

which implies

$$\lim\sup_{n\to\infty}\int_\Omega \theta_2\left(k^{-1}\xi\right)\left||v_n|^{q(\xi)-2}v_n - |v|^{q(\xi)-2}v\right|^{q'(\xi)} d\xi = 0,$$

since the function θ_2 is not negative and the theorem is proved. □

Lemma 5.3. *The functional* $\mathbf{\Theta}^{\alpha,\beta}_{\lambda,k}$ *satisfies the condition* $(\mathrm{PS})_d$ *for* $d \leq c_\infty + \varrho$, *where* ϱ *is given in the* **Lemma 5.1**.

Proof. Let $\{v_n\} \subset \mathcal{H}^{\alpha,\beta;\psi}_{\kappa(\xi)}(\Omega)$ be a sequence $(PS)_d$ for the functional $\mathbf{\Theta}^{\alpha,\beta}_{\lambda,k}$ with $d \leq c_\infty + \varrho$. Similar to **Corollary 2.1**, $\{v_n\}$ is a sequence bounded in $\{v_n\} \subset \mathcal{H}^{\alpha,\beta;\psi}_{\kappa(\xi)}(\Omega)$ and therefore, for some subsequence $\{v_n\}$, we have that $v_n \to v$ in $\mathcal{H}^{\alpha,\beta;\psi}_{\kappa(\xi)}(\Omega)$, for some $v \in \mathcal{H}^{\alpha,\beta;\psi}_{\kappa(\xi)}(\Omega)$. As $\left(\mathbf{\Theta}^{\alpha,\beta}_{\lambda,k}\right)' = 0$ and $\left(\mathbf{\Theta}^{\alpha,\beta}_{\lambda,k}\right) \geq 0$, it follows from (5.1) and (5.2) that $w_n = v_n - v$ is a sequence $(PS)_{d^*}$ for the functional $\mathbf{\Theta}^{\alpha,\beta}_{0,k}$ with $d^* = d - \mathbf{\Theta}^{\alpha,\beta}_{\lambda,k} \leq c_\infty + \varrho$.

Statement 1. There is $R > 0$ such that $\limsup_{n\to\infty}\sup_{y\in\mathbb{R}^N}\int_{B_R(y)}|w_n|^{\kappa(\xi)}d\xi = 0.$

If the statement is true, then $\int_\Omega |w_n|^{r(\xi)}d\xi \to 0$. On the other hand, by (5.2), we know that $\left(\mathbf{\Theta}^{\alpha,\beta}_{0,k}\right)' = o_n(1)$, so

$$\int_\Omega \left(\left|{}^{\mathrm{H}}\mathfrak{D}^{\alpha,\beta;\psi}_{0+}w_n\right|^{\kappa(\xi)} + |w_n|^{\kappa(\xi)}\right) d\xi = o_n(1)$$

showing that $w_n \to 0$ in $\mathcal{H}^{\alpha,\beta;\psi}_{\kappa(\xi)}(\Omega)$, and therefore $v_n \to v$ in $\mathcal{H}^{\alpha,\beta;\psi}_{\kappa(\xi)}(\Omega)$.

Let's now prove **Statement 1**. If the assertion does not hold, given $R > 0$, we can find $\mu > 0$ and $\{y_n\} \subset \mathbb{Z}^N$ by checking

$$\limsup_{n\to\infty}\int_{B_R(y_n)}|w_n|^{\kappa(\xi)}d\xi \geq \mu > 0.$$

Since $w_n \rightharpoonup 0$ in $\mathcal{H}^{\alpha,\beta;\psi}_{\kappa(\xi)}(\Omega)$, it follows that $\{y_n\}$ is an unbounded sequence. It is

$$\tilde{w}_n(\xi) = w_n(\xi + y_n) \text{ for all } \xi \in \Omega.$$

Then, $\{\tilde{w}_n\}$ is also a sequence $(PS)_{d*}$ for $\mathbf{\Theta}_{0,k}^{\alpha,\beta}$, and therefore bounded. So there are $\tilde{w} \in \mathcal{H}_{\kappa(\xi)}^{\alpha,\beta;\psi}(\Omega)$ and a subsequence $\{\tilde{w}_n\}$, still denoted by $\{\tilde{w}_n\}$, such that

$$\tilde{w}_n(\xi) \rightarrow \tilde{w} \in \mathcal{H}_{\kappa(\xi)}^{\alpha,\beta;\psi}(\Omega).$$

Arguing as in the proof of **Theorem 3.1** we have that $w \neq 0$. Furthermore, since $\left(\mathbf{\Theta}_{0,k}^{\alpha,\beta}\right)'(w_n)\phi(\cdot - y_n) = o_n(1)$ for all $\phi \in \mathcal{H}_{\kappa(\xi)}^{\alpha,\beta;\psi}(\Omega)$ it is shown that ${}^{\mathrm{H}}\mathfrak{D}_{0+}^{\alpha,\beta;\psi}\tilde{w}_n \rightarrow {}^{\mathrm{H}}\mathfrak{D}_{0+}^{\alpha,\beta;\psi}\tilde{w}$ a.e. in $\mathbb{R}^N$, and therefore

$$\begin{aligned} &\int_\Omega \left(\left| {}^{\mathrm{H}}\mathfrak{D}_{0+}^{\alpha,\beta;\psi}\tilde{w} \right|^{\kappa(\xi)-2} \ {}^{\mathrm{H}}\mathfrak{D}_{0+}^{\alpha,\beta;\psi}\tilde{w} \ {}^{\mathrm{H}}\mathfrak{D}_{0+}^{\alpha,\beta;\psi}\phi + |\tilde{w}|^{\kappa(\xi)-2}\tilde{w}\phi \right) d\xi \\ = &\int_\Omega \theta_{1,\infty}|\tilde{w}|^{r(\xi)-2}\tilde{w}\phi d\xi \end{aligned}$$

from which it follows that $\tilde{w}$ is a weak solution to the **Problem (2.5)**. Consequently, we have

$$\begin{aligned} c_{f_\infty} &\leq \mathbf{\Theta}_{\theta_{1,\infty}}^{\alpha,\beta}(\tilde{w}) - \frac{1}{q_-}\left(\mathbf{\Theta}_{\theta_{1,\infty}}^{\alpha,\beta}\right)'(\tilde{w})\tilde{w} \\ &\leq \liminf_{n\to\infty} \left\{ \mathbf{\Theta}_{0,k}^{\alpha,\beta}(w_n) - \frac{1}{q_-}\left(\mathbf{\Theta}_{0,k}^{\alpha,\beta}\right)'(w_n)w_n \right\} = d^* \end{aligned}$$

implying $c_{\theta_{1,\infty}} \leq c_\infty + \varrho$, which is contradiction, because $\varrho < c_{\theta_{1,\infty}} - c_\infty$. Therefore, **Statement 1** is true. □

Lemma 5.4. *For every $u \in \theta_{\lambda,k}^i$, there is a constant $\mu > 0$ and a differentiable function $\zeta : B_\mu \subset \mathcal{H}_{\kappa(\xi)}^{\alpha,\beta;\psi}(\Omega) \rightarrow \mathbb{R}^+$ such that*

$$\zeta(0) = 1, \quad \zeta(v)(u - v) \in \theta_{\lambda,k}^i, \quad \forall v \in B_\mu$$

and

$$\zeta'(0)\phi = \frac{\left(\mathcal{E}_{\lambda,k,\xi}^{\alpha,\beta}\right)'(u)\phi}{\left(\mathcal{E}_{\lambda,k}^{\alpha,\beta}\right)'(u)u}, \ \ \forall \phi \in \mathcal{H}_{\kappa(\xi)}^{\alpha,\beta;\psi}(\Omega),$$

where $\mathcal{E}_{\lambda,k}^{\alpha,\beta}(u) = \left(\mathbf{\Theta}_{\lambda,k}^{\alpha,\beta}\right)'(u)u$.

Proof. Let $\varphi : \mathbb{R} \times \mathcal{H}_{\kappa(\xi)}^{\alpha,\beta;\psi}(\Omega) \rightarrow \mathbb{R}$ given by $\varphi(t,w) = \mathcal{E}_{\lambda,k}^{\alpha,\beta}(t(u-w))$. So it's easy to see that

$$D_1\varphi(t,w) = \left(\mathcal{E}_{\lambda,k}^{\alpha,\beta}\right)'(t(u-w))(u-w)$$

and

$$D_2\varphi(t,w)\phi = -\left(\left(\mathcal{E}_{\lambda,k}^{\alpha,\beta}\right)' t(u-w)\right)\phi, \ \ \forall \phi \in \mathcal{H}_{\kappa(\xi)}^{\alpha,\beta;\psi}(\Omega).$$

It follows from **Corollary 2.2**, that there exists $\mu > 0$ such that $D_1\varphi(1,0) = \left(\mathcal{E}_{\lambda,k}^{\alpha,\beta}\right)'(u)u < \mu_0$. Since $u \in \mathcal{N}_{\lambda,k}^{\alpha,\beta}$, then $\varphi(1,0) = \mathcal{E}_{\lambda,k}^{\alpha,\beta}(u) = \left(\mathbf{\Theta}_{\lambda,k}^{\alpha,\beta}\right)'(u)u =$

0. Applying the implicit function theorem, it follows that there exists an open neighborhood $B_\mu \subset \mathcal{H}^{\alpha,\beta;\psi}_{\kappa(\xi)}(\Omega)$ and a differentiable function $\zeta : B_\mu \to \mathbb{R}^+$ such that $\zeta(0) = 1$ and $\varphi(\zeta(w), w) = 0$ for all $w \in B_\mu$.

Deriving the above equation, we have

$$D_1\varphi(\zeta(w), w)\zeta'(w)\phi + D_2\varphi(\zeta(w), w)\zeta'(w)\phi.$$

Therefore,

$$\zeta'(0)\phi = \frac{-D_2\varphi(1,0)\phi}{D_1\varphi(1,0)} = \frac{\left(\mathcal{E}^{\alpha,\beta}_{\lambda,k}\right)'(u)\phi}{\left(\mathcal{E}^{\alpha,\beta}_{\lambda,k,\xi}\right)'(u)u}.$$

Like $u \neq 0$, we can choose μ small enough so that $u \notin B_\mu$. Using the definition of the function φ we conclude that $\zeta(w)(u - w) \in \mathcal{N}^{\alpha,\beta}_{\lambda,k}$ for all $w \in B_\mu$. From the continuity of the function Q_K it follows that $\zeta(w)(u - w) \in \theta^i_{\lambda,k}$. Therefore, we concluded the proof. □

Lemma 5.5. *For every $1 \leq i \leq l$, there is a sequence $(PS)_{\beta^i_{\alpha,k}}, \{u^i_{\alpha,k}\} \subset \theta^i_{\lambda,k}$ for the functional $\mathbf{\Theta}^{\alpha,\beta}_{\lambda,k}$.*

Proof. For every $1 \leq i \leq l$, the **Lemma 5.1** implies

$$\beta^i_{\lambda,k} < \tilde{\gamma}^i_{\lambda,k}, \text{ for all } k \geq k_0. \tag{5.3}$$

So,

$$\beta^i_{\lambda,k} = \inf\left\{\mathbf{\Theta}^{\alpha,\beta}_{\lambda,k}(u) : u \in \theta^i_{\lambda,k} \cup \partial\theta^i_{\lambda,k}\right\}, \text{ for all } k \geq k_0.$$

Let $\{u^i_n\} \subset \theta^i_{\lambda,k} \cup \partial\theta^i_{\lambda,k}$ a sequence minimally to $\beta^i_{\lambda,k}$. Applying Ekeland variational principle, there exists a subsequence of $\{u^i_n\}$ still denoted by $\{u^i_n\}$ such that

$$\mathbf{\Theta}^{\alpha,\beta}_{\lambda,k}(u^i_n) = \beta^i_{\lambda,k} + \frac{1}{n}$$

and

$$\mathbf{\Theta}^{\alpha,\beta}_{\lambda,k}(u^i_n) \leq \mathbf{\Theta}^{\alpha,\beta}_{\lambda,k}(w) + \frac{1}{n}\left\|w - u^i_n\right\| \text{ for all } w \in \theta^i_{\lambda,k} \cup \partial\theta^i_{\lambda,k}. \tag{5.4}$$

Using (5.3), we can assume that $u^i_n \in \theta^i_{\lambda,k}$ for n is large enough. Indeed, if $u^i_n \subset \partial\theta^i_{\lambda,k}$ for an infinite number of terms, then $\mathbf{\Theta}^{\alpha,\beta}_{\lambda,k}(u^i_n) \geq \tilde{\gamma}^i_{\lambda,k} > \beta^i_{\lambda,k}$. In this sense, we have a contradiction, because $\mathbf{\Theta}^{\alpha,\beta}_{\lambda,k}(u^i_n) \to \beta^i_{\lambda,k}$. By the **Lemma 5.4**, there are $\mu^i_n > 0$ and a differentiable function $\zeta^i_n : B_{n^i_n} \to \mathbb{R}^+$ with $B_{n^i_n} \subset \mathcal{H}^{\alpha,\beta;\psi}_{\kappa(\xi)}(\Omega)$ such that $\zeta^i_n(0) = 1$ and $\zeta^i_n(v)(u^i_n - v) \in \theta^i_{\lambda,k}$ for all $v \in B_{\mu^i_n}$. Let $v_\sigma = \sigma v$ with $||v|| = 1$ and $0 < \sigma < \mu^i_n$. Then, $v_\sigma \in B_{n^i_n}$ and $w^i_{\sigma,n} := \zeta^i_{\lambda,k}(v_\sigma)(u^i_n - v_\sigma) \in \theta^i_{\lambda,k}$.

Since $\mathbf{\Theta}^{\alpha,\beta}_{\lambda,k}$ is C^1, it follows from inequality (5.4) that

$$\frac{1}{n}\left\|w^i_{\sigma,n} - u^i_n\right\|$$

$$
\begin{aligned}
&\geq \boldsymbol{\Theta}_{\lambda,k}^{\alpha,\beta}(u_n^i) - \boldsymbol{\Theta}_{\lambda,k}^{\alpha,\beta}\left(w_{\sigma,n}^i\right) \\
&= \left(\boldsymbol{\Theta}_{\lambda,k}^{\alpha,\beta}\right)'\left(u_n^i\right)\left(u_n^i - w_{\sigma,n}^i\right) + o\left(\left\|u_n^i - w_{\sigma,n}^i\right\|\right) \\
&= \sigma\zeta_n^i(v_\sigma)\boldsymbol{\Theta}_{\lambda,k}^{\alpha,\beta}\left(u_n^i\right)v + (1-\zeta_n^i(v_\sigma))\boldsymbol{\Theta}_{\lambda,k}^{\alpha,\beta}(u_n^i)(u_n^i) + o\left(\left\|u_n^i - w_{\sigma,n}^i\right\|\right) \\
&= \sigma\zeta_n^i(v_\sigma)\boldsymbol{\Theta}_{\lambda,k}^{\alpha,\beta}(u_n^i)v + o\left(\left\|u_n^i - w_{\sigma,n}^i\right|\right).
\end{aligned}
$$

So,

$$
\begin{aligned}
\left(\boldsymbol{\Theta}_{\lambda,k}^{\alpha,\beta}\right)'\left(u_n^i\right) &\leq \frac{\left\|w_{\sigma,n}^i - u_n^i\right\|}{\sigma\zeta_n^i(v_\sigma)}\left(\frac{1}{n} - \frac{o\left(\left\|u_n^i - w_{\sigma,n}^i\right\|\right)}{\left\|u_n^i - w_{\sigma,n}^i\right\|}\right) \\
&= \frac{\left\|u_n^i\left(\zeta_n^i(v_\sigma) - \zeta_n^i(0)\right) - \sigma v\zeta_n^i(v_\sigma)\right\|}{\sigma\zeta_n^i(v_\sigma)}\left(\frac{1}{n} + o_\sigma(1)\right) \\
&\leq \frac{\left\|u_n^i\right|\left|\zeta_n^i(v_\sigma) - \zeta_n^i(0) + \sigma\right|\left|v\right|\left|\zeta_n^i(v_\sigma)\right\|}{\sigma\zeta_n^i(v_\sigma)}\left(\frac{1}{n} + o_\sigma(1)\right) \\
&= \frac{\left\|u_n^i\right\|\left|(\zeta_n^i)'(0)v_\sigma + o(v_\sigma)\right| + \sigma||v||\zeta_n^i(v_\sigma)}{\sigma\zeta_n^i(v_\sigma)}\left(\frac{1}{n} + o_\sigma(1)\right).
\end{aligned}
$$

Taking the limit $\sigma \to 0$, we get

$$
\left(\boldsymbol{\Theta}_{\lambda,k}^{\alpha,\beta}\right)'(u_n^i)v \leq \frac{1}{n}.
$$

Consequently,

$$
\left\|\left(\boldsymbol{\Theta}_{\lambda,k}^{\alpha,\beta}\right)'(u_n^i)\right\| = \sup_{\substack{v\in\mathcal{H}_{\kappa(\xi)}^{\alpha,\beta;\psi}(\Omega)\\ ||v||=1}} \left(\boldsymbol{\Theta}_{\lambda,k}^{\alpha,\beta}\right)'(u_n^i)v \leq \frac{1}{n}.
$$

Therefore, $\left(\boldsymbol{\Theta}_{\lambda,k}^{\alpha,\beta}\right)'(u_n^i) \to 0$ in $\left(\mathcal{H}_{\kappa(\xi)}^{\alpha,\beta;\psi}(\Omega)\right)^*$ when $n \to \infty$ proving the result. □

Finally, we will prove the main result of this paper.

Proof. (Proof of Theorem 1.1). Let $\{u_n^i\} \subset \theta_{\lambda,k}^i$ a sequence $(PS)_{\beta_{\lambda,k}^i}$ for $\boldsymbol{\Theta}_{\lambda,k}^{\alpha,\beta}$ (energy functional) given by **Lemma 5.5**. Since $\beta_{\lambda,k}^i < c_\infty + \varrho$, at **Lemma 5.3** there exists u^i such that $u_n^i \to u^i$ in $\mathcal{H}_{\kappa(\xi)}^{\alpha,\beta;\psi}(\Omega)$. So,

$$
u^i \in \theta_{\lambda,k}^i, \left(\boldsymbol{\Theta}_{\lambda,k}^{\alpha,\beta}\right)'(u^i) = \beta_{\lambda,k}^i \text{ and } \left(\boldsymbol{\Theta}_{\lambda,k}^{\alpha,\beta}\right)'(u^i) = 0.
$$

Now, we can infer that $u^i \neq u^j$ for $i \neq j$ with $1 \leq i, j \leq l$. To see why, it remains to observe that

$$
Q_k(u^i) \in \overline{B_{R_0}(a_i)} \text{ and } Q_k(u^j) \in \overline{B_{R_0}(a_j)}.
$$

Since

$$
\overline{B_{R_0}(a_i)} \cap \overline{B_{R_0}(a_j)} = \emptyset \text{ to } i \neq j
$$

it follows that $u^i \neq u^j$ for $i \neq j$. Therefore, $\boldsymbol{\Theta}_{\lambda,k}^{\alpha,\beta}$ has at least ℓ non-trivial critical points for $\lambda \in [0, \Lambda^*)$ and $k \geq k^*$. □

6. Appendix

Consider the following conditions:

(P) $h : \mathbb{R}^N \to \mathbb{R}$ is a continuous Lipschitz function with $1 < h_- \leq h_+ < N$.

(P_1) $\theta : \mathbb{R}^N \to \mathbb{R}$ is a continuous function satisfying

$$|\theta_1(\xi, t)| \leq C \left(|t|^{\kappa(\xi)-1} + |t|^{q(\xi)-1} \right),$$

where C is a positive constant and $q \in C(\mathbb{R}^N, \mathbb{R})$ with $\kappa \leq q \ll \kappa_\alpha^*$.

(P_2) $\theta_1(\xi, t) = o(|t|^{\kappa_+ - 1})$ with $t \to 0$ uniformly in ξ;

(P_3) There is a positive constant $\beta > \kappa_+$ such that

$$0 < \beta F(\xi, t) \leq t\theta_1(\xi, t), \ \forall \xi \in \mathbb{R}^N \text{ and } t \neq 0;$$

where $F(\xi, t) = \displaystyle\int_0^t \theta_1(\xi, s) ds$.

(P_4) For each $\xi \in \mathbb{R}^N$, the function $\dfrac{\theta_1(\xi, t)}{|t|^{\kappa_+ - 1}}$ is increasing by t in $\mathbb{R}^N / \{0\}$.

Let $\mathcal{I} : \mathcal{H}^{\alpha,\beta;\psi}_{\kappa(\xi)}(\Omega) \to \mathbb{R}$ the class functional C^1 defined by

$$\mathcal{I}(u) = \int_\Omega \left(\left| {}^{\mathrm{H}}\mathfrak{D}^{\alpha,\beta;\psi}_{0+} u \right|^{\kappa(\xi)} + |u|^{\kappa(\xi)} \right) d\xi - \int_\Omega F(\xi, u) d\xi$$

for all $u \in \mathcal{H}^{\alpha,\beta;\psi}_{\kappa(\xi)}(\Omega)$.

Consider the Nehari manifold given by

$$\mathcal{S} = \left\{ u \in \mathcal{H}^{\alpha,\beta;\psi}_{\kappa(\xi)}(\Omega) / \{0\} : \mathcal{I}'(u)u = 0 \right\}.$$

Lemma 6.1. *Under the conditions (P)-(P_3), for every $u \in \mathcal{H}^{\alpha,\beta;\psi}_{\kappa(\xi)}(\Omega)/\{0\}$ there exists a unique $t_u > 0$ such that $t_u u \in \mathcal{S}$. Furthermore, the maximum of $\mathcal{I}(tu)$ for $t > 0$ is reached at $t = t_u$.*

Proof. Fixed $u \in \mathcal{H}^{\alpha,\beta;\psi}_{\kappa(\xi)}(\Omega)/\{0\}$ arbitrary, we consider the function $\varphi : [0, \infty) \to \mathbb{R}$ given by $\varphi(t) = \mathcal{I}(tu)$. Note that $\varphi(0) = 0$ and that φ verifies the geometry of the mountain pass, i.e. $\varphi(t) > 0$ for $t > 0$ small enough and $\varphi(t) < 0$ for large $t > 0$. Thus, the maximum of $\varphi(t)$ in $[0, \infty)$ is reached at some point $t_u = t(u) > 0$. Hence, yields

$$\varphi(t_u) = \mathcal{I}'(t_u u)u = 0.$$

Making $v = t_u u$, we have $\mathcal{I}'(v)v = 0$, therefore $v \in \mathcal{S}$. Now we will prove the uniqueness of t_u. Define the function $\Phi : [0, \infty) \to \mathbb{R}$ given by $\Phi(t) = \mathcal{I}(tv)$. Note that

$$\Phi(1) = \mathcal{I} = \varphi(t_u) = \max_{t \in [0,+\infty)} \varphi(t) = \max_{t \in [0,+\infty)} \mathcal{I}(st_u u)$$

$$= \max_{t\in[0,+\infty)} \mathcal{I}(su) = \max_{t\in[0,+\infty)} \Phi(t).$$

Hence, $0 = \Phi'(1) = I'(v)v$ or equivalent

$$\int_\Omega \left(\left| {}^{\mathrm{H}}\mathfrak{D}_{0+}^{\alpha,\beta;\psi} v \right|^{\kappa(\xi)} + |v|^{\kappa(\xi)} \right) d\xi = \int_\Omega \theta_1(\xi, v) v d\xi. \tag{6.1}$$

Assuming $t \geq 1$, one has

$$\begin{aligned} \Phi'(t) = \mathcal{I}'(tv)v &= \int_\Omega t^{\kappa(\xi)-1} \left(\left| {}^{\mathrm{H}}\mathfrak{D}_{0+}^{\alpha,\beta;\psi} v \right|^{\kappa(\xi)} + |v|^{\kappa(\xi)} \right) d\xi - \int_\Omega \theta_1(\xi, tv) v d\xi \\ &\leq t^{\kappa_+-1} \left(\int_\Omega \theta_1(\xi, v) v d\xi - \int_\Omega \frac{1}{t^{\kappa_+-1}} \theta_1(\xi, tv) v d\xi \right). \end{aligned}$$

Statement: $\theta_1(\xi, v)v < \frac{1}{t^{\kappa_+-1}} \theta_1(\xi, tv)v$.
Indeed, if $v > 0$ then $tv > v$ and by (P_4), we have

$$\frac{\theta_1(\xi, tv)}{|tv|^{\kappa_+-1}} > \frac{\theta_1(\xi, v)}{|v|^{\kappa_+-1}} > \theta_1(\xi, v)v.$$

On the other hand if $v < 0$ then $tv < v$ and by (P_4) the statement follows.

Consequently, $\Phi'(t) > 0$ for $t > 1$. Analogously we conclude that $\Phi'(t) < 0$ if $t \in (0, 1)$. This shows that the positive number t_u satisfying $\varphi'(t_u) = \mathcal{I}'(t_u) = \mathcal{I}'(t_u u)u = 0$ is unique. In this sense, we complete the proof. □

Declarations

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Authors' contributions.
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