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EXISTENCE AND UNIQUENESS OF CONSTRAINED MINIMIZERS FOR FRACTIONAL KIRCHHOFF TYPE PROBLEMS IN HIGH DIMENSIONS*

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Abstract In this paper, we investigate the existence and uniqueness of solutions with prescribed L^2-norm for a class of fractional Kirchhoff type problems. Firstly, we prove the existence of global constraint minimizers for the exponent $2 < p < 2+\frac{4\theta s}{N}$. Secondly, we obtain the existence of solutions with prescribed L^2-norm for the exponent $2+\frac{4\theta s}{N} \le p < 2_s^*$ by mountain pass theorem. Furthermore, all these solutions are unique up to translations and our methods only rely on scaling transformations and simply energy estimates. We point out that these obtained results extend the previous results for $0 < s < 1$ and $\theta = 2$ or $s = 1$ and $\theta = 2$ in low dimensions. To the best of our knowledge, with respect to the L^2-subcritical or L^2-critical constrained variational problem for fractional Kirchhoff type problems, the critical exponent $p = 2+\frac{4\theta s}{N}$ is properly established for the first time.

Keywords Fractional Kirchhoff type problems, L^2-normalized critical point, existence and uniqueness, mountain pass theorem.

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1. Introduction

In this paper, we discuss the existence and uniqueness of solutions with prescribed L^2-norm for the following fractional Kirchhoff type problems

$$\Big[a+b\Big(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}dxdy\Big)^{\theta-1}\Big](-\Delta)^s u-|u|^{p-2}u=\lambda u,\ x\in\mathbb{R}^N, \tag{1.1}$$

where $a,b>0,\theta>1,1\le N<\frac{2\theta}{\theta-1},s\in(0,1),2<p<2_s^*=\frac{2N}{N-2s}$ with

$$2_s^*=\begin{cases}\frac{2N}{N-2s}, & \text{if } 0<s<\frac{N}{2},\\ +\infty, & \text{if } s\ge\frac{N}{2},\end{cases}$$

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and $(-\Delta)^s$ is the fractional Laplacian operator which is defined as

$$(-\Delta)^s u(x) = C_{N,s} P.V. \int_{\mathbb{R}^N} \frac{u(x)-u(y)}{|x-y|^{N+2s}} dy, \ x \in \mathbb{R}^N,$$

where $C_{N,s}$ is a constant depending only on N, s and $P.V.$ is the Cauchy principal value.

Notice that if $\theta = 2$ and $s = 1$, equation (1.1) is related to the stationary solutions of equation

$$u_{tt} - (a + b\int_{\mathbb{R}^N} |\nabla u|^2 dx)\Delta u = f(x,u), \tag{1.2}$$

where $f(x,u)$ is a general nonlinearity. Equation (1.2) models free vibrations of elastic strings by taking into account the changes in length of the string produced by transverse vibrations. After the pioneering works of [20,27], equation (1.1) has attracted considerable attention. For instance, if we take $\theta = 2$ and $s = 1$, then (1.1) turns into

$$-(a + b\int_{\mathbb{R}^N} |\nabla u|^2 dx)\Delta u - |u|^{p-2}u = \lambda u, \ x \in \mathbb{R}^N. \tag{1.3}$$

In [33,34], Ye firstly obtained the existence of normalized solutions for equation (1.3) with L^2-subcritical or L^2-critical growth. Subsequently, Zeng and Zhang [37] proved the existence and uniqueness of normalized solutions for equation (1.3) by simple energy estimates and avoiding the concentration compactness principles. For the Kirchhoff equations involving critical growth, please see [11,19,38] and the references therein. Moreover, the supercritical growth problems were studied in [7,8].

Additionally, if we take $\theta = 2$ and $0 < s < 1$, then (1.1) is reduced to the following fractional Kirchhoff equation

$$(a + b\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}} dxdy)(-\Delta)^s u - |u|^{p-2}u = \lambda u, \ x \in \mathbb{R}^N. \tag{1.4}$$

In [14], Huang and Zhang obtained the existence and uniqueness of normalized solutions for equation (1.4) by some simple energy estimates. In what following, Liu, Chen and Yang [21] considered the following fractional Kirchhoff equations with a perturbation

$$\begin{aligned}&(a + b\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}} dxdy)(-\Delta)^s u \\ =&\lambda u + \mu|u|^{q-2}u + |u|^{p-2}u, \ x \in \mathbb{R}^3.\end{aligned} \tag{1.5}$$

They obtained the existence and properties of normalized ground states for equation (1.5) with prescribed L^2-norm by decomposing Pohozaev set and constructing fiber map. Other critical results for fractional Kirchhoff problems, please see [9,18] and the references therein.

In recent years, finding the existence, uniqueness, multiplicity of normalized solutions and properties of normalized solutions is one of the hot topics, which has attracted much attention from physics and mathematics. For example, Schrödinger equations [1,2,4,6,15,17,29], quasilinear Schrödinger equations [25,31,35], fractional

Schrödinger equations [10,22,24], Schrödinger-Poisson equations [5,16,26], Kirchhoff equations [13, 28, 36, 39], the related to Choquard equations [3, 12, 23, 32].

Motivated by the aforementioned works, in particular by [14, 28, 33, 34, 37], we study the existence and uniqueness of normalized solutions for equation (1.1). As far as we know, there is no work concerning fractional Kirchhoff type problems with a L^2-subcritical or L^2-critical growth in high dimensions.

In order to obtain the existence and uniqueness of normalized solutions for equation (1.1), we consider the following minimization problem

$$\phi(c) := \inf_{u\in S_c} \Phi(u), \tag{1.6}$$

where

$$\begin{aligned}\Phi(u) = &\frac{a}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}dxdy\\ &+\frac{b}{2\theta}\Big(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}dxdy\Big)^{\theta}-\frac{1}{p}\int_{\mathbb{R}^N}|u|^p dx,\end{aligned} \tag{1.7}$$

and

$$S_c = \{u \in H^s(\mathbb{R}^N) : \int_{\mathbb{R}^N}|u|^2dx = c^2\}.$$

The fractional Sobolev space $H^s(\mathbb{R}^N)$ is defined by

$$H^s(\mathbb{R}^N) = \Big\{u\in L^2(\mathbb{R}^N) : \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}dxdy < +\infty\Big\}.$$

Clearly, it is easy to see that for any $c > 0$, $(u_c, \lambda_c) \in H^s(\mathbb{R}^N)\times\mathbb{R}$ is a solution of equation (1.1) if and only if u_c is a critical point of $\Phi(u)|_{S_c}$ and λ_c is a Lagrange multiplier.

Inspired by [30], we state the well-known Gagliardo-Nirenberg inequality of fractional Laplacian type, which is given as follows: set $p\in(2, 2_s^*)$, then

$$\begin{aligned}&\int_{\mathbb{R}^N}|u|^p dx\\ \le&\frac{p\alpha_p\beta_p}{|Q|_2^{p-2}}\Big(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}dxdy\Big)^{\frac{N(p-2)}{4s}}\Big(\int_{\mathbb{R}^N}|u|^2dx\Big)^{\frac{2ps-Np+2N}{4s}},\end{aligned} \tag{1.8}$$

where $\alpha_p = \frac{2s}{2ps-Np+2N}$, $\beta_p = (\frac{2ps-Np+2N}{N(p-2)})^{\frac{N(p-2)}{4s}}$. Similar to [30], we can prove that all optimizers of (1.8) are the scalings and translations of $Q(x)$, i.e., belong to the following set

$$\{kQ(lx + m) : k, l, m \in \mathbb{R}^+, x \in \mathbb{R}^N\}. \tag{1.9}$$

Moreover, the function $Q(x)$ is the unique ground state solution of the following equation

$$(-\Delta)^s u + u - |u|^{p-2}u = 0, \quad x \in \mathbb{R}^N. \tag{1.10}$$

Equation (1.10) combining with the Pohozaev and Nehari identity, we have

$$\begin{aligned}&\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}dxdy = \frac{N(p-2)}{2N+2ps-Np}\int_{\mathbb{R}^N}|u|^2dx,\\ &\int_{\mathbb{R}^N}|u|^p dx = \frac{2ps}{2N+2ps-Np}\int_{\mathbb{R}^N}|u|^2dx.\end{aligned} \tag{1.11}$$

From (1.7) and (1.8), we can obtain the L^2-critical exponent for (1.6) as

$$p = 2 + \frac{4\theta s}{N},$$

i.e. for any $c > 0$,

$$\begin{aligned} &\phi(c) > -\infty, \text{ if } p \in (2, 2 + \frac{4\theta s}{N}), \\ &\phi(c) = -\infty, \text{ if } p \in (2 + \frac{4\theta s}{N}, 2_s^*). \end{aligned}$$

However, in the case of $p = 2 + \frac{4\theta s}{N}$, we can not get that $\phi(c) > -\infty$ or $\phi(c) = -\infty$.

Now, we first describe a complete classification with respect to the exponent p with the L^2-normalized solutions of problem (1.6).

Set

$$\begin{aligned} c_* :=& \Big[\frac{|Q|_2^{p-2}}{\alpha_p \beta_p} \Big(\frac{2as(\theta - 1)}{4\theta s - Np + 2N} \Big)^{\frac{4\theta s - Np + 2N}{4s(\theta-1)}} \\ &\times \Big(\frac{2bs(\theta - 1)}{\theta(Np - 2N - 4s)} \Big)^{\frac{Np - 2N - 4s}{4s(\theta - 1)}} \Big]^{\frac{2s}{2ps - Np + 2N}}. \end{aligned} \tag{1.12}$$

Theorem 1.1. *(i) When $2 < p < 2 + \frac{2s}{N}$, problem (1.6) has a unique minimizer u_c (up to translations). Moreover, the function u_c has the following form*

$$u_c = \frac{c^{1-\frac{N}{2s}}}{|Q|_2} \Big(\frac{(2N + 2ps - Np)t_p}{N(p-2)} \Big)^{\frac{N}{4s}} Q\Big(\Big[\frac{(2N + 2ps - Np)t_p}{N(p-2)c^2} \Big]^{\frac{1}{2s}} x \Big),$$

where t_p is the unique minimum point of the following function

$$\varphi(t) = \frac{a}{2} t + \frac{b}{2\theta} t^\theta - \frac{\alpha_p \beta_p c^{\frac{2ps - Np + 2N}{2s}}}{|Q|_2^{p-2}} t^{\frac{N(p-2)}{4s}}, \ t \in (0, +\infty). \tag{1.13}$$

(ii) When $p = 2 + \frac{4s}{N}$, if $c > a^{\frac{N}{4s}} |Q|_2$, problem (1.6) has a unique minimizer (up to translations)

$$u_c = \frac{c}{|Q|_2^2} \Big(\frac{2s(c^{\frac{4s}{N}} - a|Q|_2^{\frac{4s}{N}})}{Nbc^2} \Big)^{\frac{N}{4s}} Q\Big(\Big[\frac{2s(c^{\frac{4s}{N}} - a|Q|_2^{\frac{4s}{N}})}{Nbc^2 |Q|_2^{\frac{4s}{N}}} \Big]^{\frac{1}{2s}} x \Big).$$

In addition, problem (1.6) has no minimizer if $c \leq a^{\frac{N}{4s}} |Q|_2$.
(iii) When $2 + \frac{4s}{N} < p < 2 + \frac{4\theta s}{N}$ and $s > \frac{N(\theta-1)}{2\theta}$, problem (1.6) has no minimizer if

$$c < c_*.$$

On the contrary, if $c \geq c_*$, problem (1.6) has a unique minimizer (up to translations)

$$u_c = \frac{c^{1-\frac{N}{2s}}}{|Q|_2} \Big(\frac{(2N + 2ps - Np)t_p}{N(p-2)} \Big)^{\frac{N}{4s}} Q\Big(\Big[\frac{(2N + 2ps - Np)t_p}{N(p-2)c^2} \Big]^{\frac{1}{2s}} x \Big)$$

with $t_p = \left[\frac{\theta a(Np-2N-4s)}{b(4\theta s-Np+2N)}\right]^{\frac{1}{\theta-1}}$ *and*

$$\phi(c) = \frac{\alpha_p \beta_p}{|Q|_2^{p-2}}\Big(c_*^{\frac{2ps-Np+2N}{2s}} - c^{\frac{2ps-Np+2N}{2s}}\Big)\Big[\frac{\theta a(Np-2N-4s)}{b(4\theta s - Np + 2N)}\Big]^{\frac{N(p-2)}{4s(\theta-1)}}.$$

(iv) When $p \geq 2 + \frac{4\theta s}{N}$*, problem (1.6) has no minimizer for all* $c > 0$*.*

Note that if $2 < p < 2 + \frac{4\theta s}{N}$, equation (1.1) has a unique normalized solution (up to translations), i.e., problem (1.6) has a unique minimizer. But if $p \geq 2 + \frac{4\theta s}{N}$, equation (1.1) has no normalized solution for all $c > 0$, i.e., problem (1.6) has no minimizer. Inspired by [5,15], we search the mountain pass solution for $\Phi(\cdot)$ on S_c for $p \geq 2 + \frac{4\theta s}{N}$. For this reason, we give the following definition [15,37].

Definition 1.1.3 Given $c > 0$, there exists $K(c) > 0$ such that

$$\rho(c) := \inf_{h \in \Upsilon(c)} \max \Phi(h(t))_{t\in[0,1]} > \max\{\Phi(h(0)), \Phi(h(1))\}, \tag{1.14}$$

where $\Upsilon(c) = \{h \in C([0,1]; S_c) | h(0) \in A_{K(c)}, \Phi(h(1)) < 0\}$ and

$$A_{K(c)} = \Big\{u \in S_c : \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}} dxdy \leq K(c)\Big\},$$

then the functional $\Phi(\cdot)$ satisfies the mountain pass geometry on the constraint set S_c.

Theorem 1.2. *Assume* $p > 2 + \frac{4\theta s}{N}$ *or* $p = 2 + \frac{4\theta s}{N}$ *and* $c > c^* := \left(\frac{b|Q|_2^{\frac{4\theta s}{N}}}{2\theta\alpha_p\beta_p}\right)^{\frac{N}{4\theta s+2N-2N\theta}}$*, then equation (1.1) has a unique solution (up to translations)*

$$u_c = \frac{c^{1-\frac{N}{2s}}}{|Q|_2}\Big(\frac{(2N+2ps-Np)\bar{t}_p}{N(p-2)c^2}\Big)^{\frac{N}{4s}} Q\Big(\Big[\frac{(2N+2ps-Np)\bar{t}_p}{N(p-2)c^2}\Big]^{\frac{1}{2s}} x\Big),$$

where $\bar{t}_p$ *is the maximum value of the function* $\varphi(t)$ *and* $\rho(c) = \varphi(\bar{t}_p)$*.*

Remark 1.1. In theorem 1.2, $u \in S_c$ is the unique solution of (1.14) in the following sense: if $u \in S_c$ is a critical point of $\Phi(u)$ on the constraint set S_c and its energy equals to $\rho(c)$, that is,

$$\Phi'(u)|_{S_c} = 0 \text{ and } \Phi(u) = \rho(c). \tag{1.15}$$

Remark 1.2. On the one hand, in [14,33,37], the dimension demands $1 \leq N < 4$, but here it can be greater than 4 dimension. Since $1 \leq N < \frac{2\theta}{\theta-1}(\theta > 1)$, when $\theta = 2$, we get that $1 \leq N < 4$. But when $\theta = 1.2$, we have that $1 \leq N < 12$. So our results extend the existing results [14,33,37], where $0 < s < 1$ and $\theta = 2$ or $s = 1$ and $\theta = 2$ in low dimensions. On the other hand, the main difficulty lies in that a minimizing sequence of $\phi(c)$ may lack of the compactness. The usual approach [33] is to apply concentration compactness principle to obtain the compactness of a minimizing sequence by excluding the cases of vanishing and dichotomy. But there are some difficulties in ruling out the cases of vanishing and dichotomy, it is necessary to set up some functional inequalities, which needs to be more complex analysis. To avoid the complexity of concentration compactness principle, we adopt the method used in [14,37] to obtain the existence and uniqueness of normalized solutions for equation (1.1)

Throughout this paper, $L^p(\mathbb{R}^N)(1 \leq p < +\infty)$ is usual Lebesgue space with the standard norm $|\cdot|_p$.

2. Some Lemmas

Lemma 2.1. *Assume $u \in S_c$ and $2 < p < 2_s^*$, then the following facts hold:*
(i) If $2 < p < 2 + \frac{4\theta s}{N}$, then $\phi(c)$ is well defined, $\phi(c) > -\infty$ and $\phi(c) \le 0$ for any $c > 0$;
(ii) If $2 < p < 2 + \frac{4s}{N}$, then we have $\phi(c) < 0$ for any $c > 0$;
(iii) If $2 + \frac{4s}{N} \le p < 2 + \frac{4\theta s}{N}$ and $s > \frac{N(\theta-1)}{2\theta}$, then there is $\phi(c) < 0$ for c large enough;
(iv) In the case of $p = 2 + \frac{4\theta s}{N}$, if $c \le c^*$, we have $\phi(c) > -\infty$ and if $c > c^*$, we get $\phi(c) = -\infty$;
(v) In the case of $p > 2 + \frac{4\theta s}{N}$, we have $\phi(c) = -\infty$ for any $c > 0$.

Proof. From (1.7) and (1.8) , we deduce that for any $u \in S_c$,

$$\begin{aligned}\Phi(u) \ge& \frac{a}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}dxdy + \frac{b}{2\theta}\Big(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}dxdy\Big)^\theta \\ &- \frac{\alpha_p\beta_p c^{\frac{2ps-Np+2N}{2s}}}{|Q|_2^{p-2}}\Big(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}dxdy\Big)^{\frac{N(p-2)}{4s}} \\ =& \varphi(t),\end{aligned} \tag{2.1}$$

where $\varphi(\cdot)$ is given by (1.13) and let $t = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}dxdy$.

In addition, set $u_t = t^{\frac{N}{2}}u(tx), t > 0$, by direct computations, we have that

$$\int_{\mathbb{R}^N}|u_t|^2dx = c^2, \quad \int_{\mathbb{R}^N}|u_t|^pdx = t^{\frac{N(p-2)}{2}}\int_{\mathbb{R}^N}|u|^pdx,$$

and

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u_t(x)-u_t(y)|^2}{|x-y|^{N+2s}}dxdy = t^{2s}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}dxdy.$$

Therefore, $u_t \in S_c$ for all c and

$$\begin{aligned}\Phi(u_t) =& \frac{at^{2s}}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}dxdy \\ &+ \frac{bt^{2\theta s}}{2\theta}\Big(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}dxdy\Big)^\theta - \frac{t^{\frac{N(p-2)}{2}}}{p}\int_{\mathbb{R}^N}|u|^pdx.\end{aligned} \tag{2.2}$$

(i) If $2 < p < 2 + \frac{4\theta s}{N}$, then $0 < \frac{N(p-2)}{4s} < \theta$. By (2.1), we have

$$\begin{aligned}\Phi(u) \ge& \frac{b}{2\theta}\Big(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}dxdy\Big)^\theta \\ &- \frac{\alpha_p\beta_p c^{\frac{2ps-Np+2N}{2s}}}{|Q|_2^{p-2}}\Big(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}dxdy\Big)^{\frac{N(p-2)}{4s}}.\end{aligned}$$

Clearly, it is easy to see that $\Phi(u)$ is bounded below on S_c. Moreover, we deduce that $\phi(c) > -\infty$ for any $c > 0$. Using again (2.2), we get that $\Phi(u_t) \to 0$ as $t \to 0$, which implies that $\phi(c) < 0$ for all $c > 0$.

(ii) If $2 < p < 2 + \frac{4s}{N}$, then $0 < \frac{N(p-2)}{4s} < 1$. By (2.2), it is not hard to find that $\frac{t^{\frac{N(p-2)}{2}}}{p}\int_{\mathbb{R}^N}|u|^p dx$ is the dominant term in (2.2) as $t \to 0^+$. Thus we deduce that $\phi(c) < 0$ for any $c > 0$.

(iii) If $2 + \frac{4s}{N} \le p < 2 + \frac{4\theta s}{N}$, then $1 \le \frac{N(p-2)}{4s} < \theta$. In view of the fact that $\Phi(u_t) \to 0$ as $t \to 0^+$, by (2.2), we get that $\phi(c) \le 0$ for any $c > 0$. Moreover, for any $u \in S_1$, set $u_\eta(x) = u(\eta^{-\frac{1}{N}}x)$, we have

$$\int_{\mathbb{R}^N}|u_\eta|^2 dx = \eta, \quad \int_{\mathbb{R}^N}|u_\eta|^p dx = \eta\int_{\mathbb{R}^N}|u|^p dx,$$

and

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u_\eta(x)-u_\eta(y)|^2}{|x-y|^{N+2s}}dxdy = \eta^{1-\frac{2s}{N}}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}dxdy.$$

Then

$$\begin{aligned}\Phi(u_\eta) = &\frac{a\eta^{1-\frac{2s}{N}}}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}dxdy\\ &+\frac{b\eta^{\theta-\frac{2\theta s}{N}}}{2\theta}\Big(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}dxdy\Big)^\theta - \frac{\eta}{p}\int_{\mathbb{R}^N}|u|^p dx.\end{aligned}$$

Thus, $\Phi(u_\eta) \to -\infty$ as $\eta \to +\infty$ because $s > \frac{N(\theta-1)}{2\theta}$. This means that $\phi(c) < 0$ for some c large enough.

(iv) In the case of $p = 2 + \frac{4\theta s}{N}$, $\alpha_p = \frac{N}{2N+4\theta s-2N\theta}$, $\beta_p = (\frac{N+2\theta s-N\theta}{N\theta})^\theta$, by (2.1), we have

$$\begin{aligned}\Phi(u) \ge &\frac{a}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}dxdy\\ &+\Big(\frac{b}{2\theta} - \frac{\alpha_p\beta_p c^{\frac{4\theta s+2N-2N\theta}{N}}}{|Q|_2^{\frac{4\theta s}{N}}}\Big)\Big(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}dxdy\Big)^\theta,\end{aligned}$$

which implies that $\phi(c) > -\infty$ if $c \le c^*$ and $\phi(c) = -\infty$ if $c > c^*$, where c^* is defined in Theorem 1.2.

(v) If $p > 2 + \frac{4\theta s}{N}$, then $\frac{N(p-2)}{4s} > \theta$. We know that $\frac{t^{\frac{N(p-2)}{2}}}{p}\int_{\mathbb{R}^N}|u|^p dx$ is the dominant term in (2.2) as $t \to +\infty$ and $\Phi(u_t) \to -\infty$ as $t \to +\infty$. This means that $\phi(c) = -\infty$ for $c > 0$. $\square$

Lemma 2.2. *Assume $p > 2 + \frac{4\theta s}{N}$ or $p = 2 + \frac{4\theta s}{N}$ and $c > c^*$, then there exists $K(c) \in (0,1)$ such that (1.14) holds.*

Proof. For any $u \in S_c$ and $\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}dxdy$ small enough, by (2.1), we have

$$\begin{aligned}\Phi(u) \ge &\frac{a}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}dxdy + \frac{b}{2\theta}\Big(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}dxdy\Big)^\theta\\ &-\frac{\alpha_p\beta_p c^{\frac{2ps-Np+2N}{2s}}}{|Q|_2^{p-2}}\Big(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}dxdy\Big)^{\frac{N(p-2)}{4s}}\\ \ge &\frac{a}{4}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}dxdy.\end{aligned} \tag{2.3}$$

Additionally, if $\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}dxdy \le (\frac{\theta a}{b})^{\frac{1}{\theta-1}}$, we obtain

$$\begin{aligned}\Phi(u) &\le \frac{a}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}dxdy + \frac{b}{2\theta}\Big(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}dxdy\Big)^\theta \\ &\le a\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}dxdy.\end{aligned} \tag{2.4}$$

Combining (2.3) and (2.4), it is easy to know that

$$\Phi(u)\to 0 \quad \text{as} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}dxdy \to 0,$$

and for $K(c)$ small enough. Furthermore, $K(c) \le (\frac{\theta a}{b})^{\frac{1}{\theta-1}}$, we deduce that

$$\begin{aligned}\sup_{u\in A_{K(c)}}\Phi(u) &\le a\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}dxdy \le aK(c) \\ &= \frac{a}{4}\cdot 4K(c) \le \inf_{u\in\partial A_{4K(c)}}\Phi(u),\end{aligned}$$

where $\partial A_{4K(c)} = \{u\in S_c : \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}dxdy = 4K(c)\}$. Moreover, for all $u\in A_{4K(c)}$, by (2.3), we have

$$\Phi(u)\ge 0. \tag{2.5}$$

Next we prove that $\gamma(c)\ne\emptyset$. Set

$$u_\varepsilon(x) = \frac{c\varepsilon^{\frac{N}{2}}}{|Q|_2}Q(\varepsilon x), \tag{2.6}$$

where $\varepsilon>0$ will be determined later. Then $u_\varepsilon\in S_c$ and by simple computations, we have

$$\begin{aligned}&\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u_\varepsilon(x)-u_\varepsilon(y)|^2}{|x-y|^{N+2s}}dxdy = \frac{N(p-2)c^2\varepsilon^{2s}}{2N+2ps-Np}, \\ &\int_{\mathbb{R}^N}|u_\varepsilon|^p dx = \frac{2psc^p\varepsilon^{\frac{N(p-2)}{2}}}{(2N+2ps-Np)|Q|_2^{p-2}},\end{aligned} \tag{2.7}$$

and

$$\begin{aligned}\Phi(u_\varepsilon) = {}& \frac{a}{2}\frac{N(p-2)c^2\varepsilon^{2s}}{2N+2ps-Np} + \frac{b}{2\theta}(\frac{N(p-2)c^2\varepsilon^{2s}}{2N+2ps-Np})^\theta \\ &- \frac{\alpha_p\beta_p c^{\frac{2ps-Np+2N}{2s}}}{|Q|_2^{p-2}}(\frac{N(p-2)c^2\varepsilon^{2s}}{2N+2ps-Np})^{\frac{N(p-2)}{4s}}.\end{aligned} \tag{2.8}$$

By (2.7), we get that

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u_{\varepsilon_1}(x)-u_{\varepsilon_1}(y)|^2}{|x-y|^{N+2s}}dxdy \le K(c), \text{ if } \varepsilon_1 \le \Big(\frac{K(c)(2N+2ps-Np)}{Nc^2(p-2)}\Big)^{\frac{1}{2s}}.$$

From (2.8), if $\frac{N(p-2)}{4s}>\theta$, that is, $p>2+\frac{4\theta s}{N}$, then we deduce that $\Phi(u_\varepsilon)\to-\infty$ as $\varepsilon\to+\infty$. If $\frac{N(p-2)}{4s}=\theta$, that is, $p=2+\frac{4\theta s}{N}$, then by (2.8), we get that

$$\Phi(u_\varepsilon) = \frac{a}{2}\frac{N(p-2)c^2\varepsilon^{2s}}{2N+2ps-Np} + \Big(\frac{b}{2\theta} - \frac{\alpha_p\beta_p c^{\frac{2ps-Np+2N}{2s}}}{|Q|_2^{\frac{4\theta s}{N}}}\Big)\Big(\frac{N(p-2)c^2\varepsilon^{2s}}{2N+2ps-Np}\Big)^\theta,$$

which implies that $\Phi(u_\varepsilon) \to -\infty$ as $\varepsilon \to +\infty$ and $c > c^*$. Thus, there is a $\varepsilon_2 > 0$ large enough such that $\Phi(u_{\varepsilon_2}) < 0$.

Taking $h(t) = u_{(1-t)\varepsilon_1 + t\varepsilon_2}$, then we get $h(0) = u_{\varepsilon_1} \in A_{K(c)}, h(1) = u_{\varepsilon_2}$ and $\Phi(u_{\varepsilon_2}) < 0$, which implies that $h(t) \in \Upsilon(c) \neq \emptyset$.

For any $h(t) \in \Upsilon(c)$, we show that $h(0) \in A_{K(c)}$ and $\Phi(h(1)) < 0$. Thus, there is a $t_0 \in (0,1)$ such that $h(t_0) \in \partial A_{4K(c)}$ since $h(t)$ is continuous and (2.5) holds. Moreover,

$$\max_{t\in[0,1]} \Phi(h(t)) \geq \Phi(h(t_0)) > \max\{\Phi(h(0)), \Phi(h(1))\}.$$

We complete the proof. □

3. Proof of main results

Proof of Theorem 1.1. (i) Since $2 < p < 2 + \frac{4s}{N}$, it is easy to check that $\varphi(t)$ has a unique minimum point, denoted by t_p. Thus, by (2.1), we have

$$\phi(c) = \inf_{u\in S_c} \Phi(u) \geq \varphi(t_p). \tag{3.1}$$

In addition, choosing $t_p = \frac{N(p-2)c^2\varepsilon^{2s}}{2N+2ps-Np}$, i.e., $\varepsilon = (\frac{(2N+2ps-Np)t_p}{N(p-2)c^2})^{\frac{1}{2s}}$, by (2.8), we conclude that

$$\phi(c) \leq \Phi(u_\varepsilon) = \varphi(t_p). \tag{3.2}$$

Combining with (3.1) and (3.2), we get that

$$\phi(c) = \varphi(t_p) = \inf_{t\in\mathbb{R}^+} \varphi(t). \tag{3.3}$$

Therefore, problem (1.6) has a minimizer when $2 < p < 2 + \frac{4s}{N}$. Moreover, the minimizer of (1.6) has the following form

$$u_\varepsilon = u_c = \frac{c^{1-\frac{N}{2s}}}{|Q|_2}\Big(\frac{(2N+2ps-Np)t_p}{N(p-2)}\Big)^{\frac{N}{4s}} Q\Big(\Big[\frac{(2N+2ps-Np)t_p}{N(p-2)c^2}\Big]^{\frac{1}{2s}} x\Big).$$

Next, we prove that u_ε is the unique minimizer of problem (1.6) in the sense of translation. Indeed, if $u_0 \in S_c$ is a minimizer of problem (1.6), by (2.1), we have

$$\phi(c) = \Phi(u_0) \geq \varphi(t_0),\ t_0 = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u_0(x) - u_0(y)|^2}{|x-y|^{N+2s}} dxdy,$$

where the second " = " if and only if u_0 is an optimizer of (1.8). Together with (3.3), we have

$$\varphi(t_p) = \phi(c) = \Phi(u_0) \geq \varphi(t_0).$$

Thus, we conclude that $t_0 = t_p, \varphi(t_0) = \Phi(u_0)$ and u_0 is an optimizer of (1.8). Moreover, by (1.9), we see that u_0 must be the form of $u_0(x) = \alpha Q(\beta x)$. Utilizing

$$\int_{\mathbb{R}^N} |u_0|^2 dx = c^2, \quad \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u_0(x) - u_0(y)|^2}{|x-y|^{N+2s}} dxdy = t_p,$$

and combining with (1.11), we get

$$\alpha = \frac{c^{1-\frac{N}{2s}}}{|Q|_2}(\frac{(2N+2ps-Np)t_p}{N(p-2)})^{\frac{N}{4s}}, \ \beta = (\frac{(2N+2ps-Np)t_p}{N(p-2)c^2})^{\frac{1}{2s}},$$

which implies that $u_0 = u_c$.

(*ii*) In the case of $p = 2 + \frac{4s}{N}$, $\alpha_p = \frac{N}{4s}$, $\beta_p = \frac{2s}{N}$ by (2.1), we have

$$\varphi(t) = (\frac{a}{2} - \frac{c^{\frac{4s}{N}}}{2|Q|_2^{\frac{4s}{N}}})t + \frac{b}{2\theta}t^\theta. \tag{3.4}$$

If $c \le a^{\frac{N}{4s}}|Q|_2$, by (2.1), we have

$$\Phi(u) \ge \varphi(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}dxdy) > 0 \ \text{ for any } \ u \in S_c,$$

which implies that $\phi(c) = \inf_{u\in S_c}\Phi(u) > 0$ for $c \le a^{\frac{N}{4s}}|Q|_2$. By Lemma 2.1-(*i*), we know that $\phi(c) \le 0$ for any $c > 0$. Thus problem (1.6) has no minimizer on S_c for $c \le a^{\frac{N}{4s}}|Q|_2$.

On the other hand, if $c > a^{\frac{N}{4s}}|Q|_2$, it is also easy to know from (3.4) that $\varphi(t)$ attains its minimum at the unique point $t_p = \frac{c^{\frac{4s}{N}} - a|Q|_2^{\frac{4s}{N}}}{b|Q|_2^{\frac{4s}{N}}}$. Similar to the case (*i*), set $u_\varepsilon(x) = \frac{c\varepsilon^{\frac{N}{2}}}{|Q|_2}Q(\varepsilon x)$, $\varepsilon > 0$ is given later, then $\Phi(u_\varepsilon) = \varphi(\frac{Nc^2\varepsilon^{2s}}{2s})$. Taking $t_p = \frac{Nc^2\varepsilon^{2s}}{2s}$, that is, $\varepsilon = [\frac{2s(c^{\frac{4s}{N}} - a|Q|_2^{\frac{4s}{N}})}{Nbc^2|Q|_2^{\frac{4s}{N}}}]^{\frac{1}{2s}}$. Thus, we deduce that

$$u_\varepsilon = u_c = \frac{c}{|Q|_2^2}\Big(\frac{2s(c^{\frac{4s}{N}} - a|Q|_2^{\frac{4s}{N}})}{Nbc^2}\Big)^{\frac{N}{4s}}Q\Big(\Big[\frac{2s(c^{\frac{4s}{N}} - a|Q|_2^{\frac{4s}{N}})}{Nbc^2|Q|_2^{\frac{4s}{N}}}\Big]^{\frac{1}{2s}}x\Big).$$

The uniqueness of the minimizer of problem (1.6) is similar to (*i*).

(*iii*) If $2 + \frac{4s}{N} < p < 2 + \frac{4\theta s}{N}$, set $\tau = \frac{4\theta s - Np + 2N}{4s(\theta-1)}$, $\sigma = 1 - \tau = \frac{Np-2N-4s}{4s(\theta-1)}$, then using Yong's inequality, for any $t > 0$,

$$\begin{aligned}\frac{a}{2}t + \frac{b}{2\theta}t^\theta &= \tau(\frac{a}{2\tau}t) + \sigma(\frac{b}{2\theta\sigma}t^\theta)\\ &\ge (\frac{a}{2\tau})^\tau(\frac{b}{2\theta\sigma})^\sigma t^{\tau+\theta\sigma}\\ &= (\frac{2as(\theta-1)}{4\theta s - Np + 2N})^{\frac{4\theta s-Np+2N}{4s(\theta-1)}}(\frac{2bs(\theta-1)}{\theta(Np-2N-4s)})^{\frac{Np-2N-4s}{4s(\theta-1)}}t^{\frac{N(p-2)}{4s}},\end{aligned} \tag{3.5}$$

where the "=" in the second inequality holds if and only if

$$\frac{a}{2\tau}t = \frac{b}{2\theta\sigma}t^\theta, \ \text{ i.e., } \ t_p = [\frac{\theta a(Np-2N-4s)}{b(4\theta s-Np+2N)}]^{\frac{1}{\theta-1}}.$$

From (2.1) and c_* is defined by (1.12), we have

$$\Phi(u) \ge \frac{\alpha_p\beta_p}{|Q|_2^{p-2}}\Big(c_*^{\frac{2ps-Np+2N}{2s}} - c^{\frac{2ps-Np+2N}{2s}}\Big)t_p^{\frac{N(p-2)}{4s}} = \varphi(t_p), \text{ for all } u \in S_c. \tag{3.6}$$

If $c < c_*$, by (3.5) and (3.6), we know that $\Phi(u) > 0$ for all $u \in S_c$, which means that $\phi(c) > 0$. This contradicts with Lemma 2.1-(iii) because $\phi(c) \leq 0$. Thus, problem (1.6) has no minimizer for $c < c_*$. If $c \geq c_*$, by (3.6), we have $\phi(c) \geq \varphi(t_p)$. Besides, taking $u_\varepsilon(x) = \frac{c\varepsilon^{\frac{N}{2}}}{|Q|_2} Q(\varepsilon x)$, then $\Phi(u_\varepsilon) = \varphi(\frac{N(p-2)c^2\varepsilon^{2s}}{2N+2ps-Np})$. Set $t_p = \frac{N(p-2)c^2\varepsilon^{2s}}{2N+2ps-Np}$, that is, $\varepsilon = \left[\frac{(2N+2ps-Np)t_p}{N(p-2)c^2}\right]^{\frac{1}{2s}}$. Furthermore, we deduce that $\phi(c) \leq \Phi(u_\varepsilon) = \varphi(t_p)$. This means that u_ε is a minimizer of problem (1.6) and

$$\phi(c) = \varphi(t_p) = \frac{\alpha_p \beta_p}{|Q|_2^{p-2}} \Big(c_*^{\frac{2ps-Np+2N}{2s}} - c^{\frac{2ps-Np+2N}{2s}} \Big) \Big[\frac{\theta a(Np-2N-4s)}{b(4\theta s - Np + 2N)} \Big]^{\frac{N(p-2)}{4s(\theta-1)}}$$

for any $c \geq c_*$. The uniqueness of minimizers can be proved by the same argument of the case (i).

(iv) If $p > 2 + \frac{4\theta s}{N}$ or $p = 2 + \frac{4\theta s}{N}$ and $c > c^*$, by Lemma 2.1-(iv), (v), we have $\phi(c) = -\infty$. Thus, problem (1.6) has no minimizer. If $p = 2 + \frac{4\theta s}{N}$ and $c \leq c^*$, from (2.1), we deduce that $\Phi(u) > 0$ for any $u \in S_c$. It is easy to know that problem (1.6) also has no minimizer. $\square$

Proof of Theorem 1.2. In the case of $p > 2 + \frac{4\theta s}{N}$ or $p = 2 + \frac{4\theta s}{N}$ and $c > c^*$, by lemma 2.2, there exists $K(c) > 0$ such that $\Phi(u)$ satisfying mountain pass geometry on S_c. Moreover, it is easy to find that $\varphi(t)$ gets its maximun in $(0, +\infty)$, denoted by $\bar{t}_p$. In what follows, we always assume that $K(c) < \bar{t}_p$.

For any $h(t) \in \Upsilon(c)$, by (2.1), we get that

$$\Phi(h(t)) \geq \varphi(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|(h(t))(x) - (h(t))(y)|^2}{|x-y|^{N+2s}} dxdy), \tag{3.7}$$

where "=" holds if and only if $h(t) \in S_c$ is an optimizer of (1.8), i.e., up to translations,

$$(h(t))(x) = \frac{c\alpha^{\frac{N}{2}}}{|Q|_2} Q(\alpha x) \quad \text{for some } \alpha > 0. \tag{3.8}$$

Since $h(0) \in A_{K(c)}$, $K(c) < \bar{t}_p$, and note that $\varphi(t) > 0, \forall t \in [0, \bar{t}_p]$, we have

$$\begin{aligned} &\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|(h(0))(x) - (h(0))(y)|^2}{|x-y|^{N+2s}} dxdy \\ <&\bar{t}_p \\ <&\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|(h(1))(x) - (h(1))(y)|^2}{|x-y|^{N+2s}} dxdy. \end{aligned} \tag{3.9}$$

In view of (3.7) and (3.9), we have that

$$\max_{t\in[0,1]} \Phi(h(t)) \geq \varphi(\bar{t}_p) = \max_{t\in\mathbb{R}^+} \varphi(t). \tag{3.10}$$

Thus,

$$\rho(c) \geq \varphi(\bar{t}_p). \tag{3.11}$$

On the other hand, let $u_\varepsilon(x)$ be the test function given by (2.6) with

$$\varepsilon = (\frac{(2N+2ps-Np)\bar{t}_p}{N(p-2)c^2})^{\frac{1}{2s}}.$$

Set $\overline{h}(r) = r^{\frac{N}{4s}} u_\varepsilon(r^{\frac{1}{2s}}x)$, it is not hard to know that $\Phi(\overline{h}(r)) = \varphi(\bar{t}_p r)$. We can choose $0 < \widehat{t} < \bar{t}_p$ small enough such that $\overline{h}(\widehat{t}/\bar{t}_p) \in A_{K(c)}$ and $\widetilde{t} > \bar{t}_p$ such that $\varphi(\widetilde{t}) < 0$. Set $h(r) = \overline{h}((1-r)\widehat{t}/\bar{t}_p + \widetilde{t}r/\bar{t}_p)$, then $h(0) = \overline{h}(\widehat{t}/\bar{t}_p) \in A_{K(c)}$ and $\Phi(h(1)) = \Phi(\overline{h}(\widetilde{t}/\bar{t}_p)) = \varphi(\bar{t}) < 0$. This means that $h \in \Upsilon(c)$ and

$$\rho(c) \le \max_{r\in[0,1]} \Phi(h(r)) = \Phi(u_\varepsilon) = \varphi(\bar{t}_p).$$

Moreover, we conclude from (3.11) that $\rho(c) = \varphi(\bar{t}_p)$ and

$$u_\varepsilon = u_c = \frac{c^{1-\frac{N}{2s}}}{|Q|_2}\Big(\frac{(2N+2ps-Np)\bar{t}_p}{N(p-2)c^2}\Big)^{\frac{N}{4s}} Q\Big(\Big[\frac{(2N+2ps-Np)\bar{t}_p}{N(p-2)c^2}\Big]^{\frac{1}{2s}} x\Big),$$

is a solution of problem (1.14).

Next, we prove that u_ε is a solution of equation (1.1) for some $\lambda \in \mathbb{R}^-$. Indeed, in view of $\varphi'(\bar{t}_p) = 0$, we obtain

$$\begin{aligned}
&a + b\Big(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u_\varepsilon(x)-u_\varepsilon(y)|^2}{|x-y|^{N+2s}}dxdy\Big)^{\theta-1}\\
=&a + b\bar{t}_p^{\theta-1}\\
=&\frac{N(p-2)\alpha_p\beta_p c^{\frac{2ps-Np+2N}{2s}}}{2s|Q|_2^{p-2}} \bar{t}_p^{\frac{Np-2N-4s}{4s}}\\
=&\frac{c^{\frac{2ps-Np+2N}{2s}}}{|Q|_2^{p-2}}\Big(\frac{2ps-Np+2N}{N(p-2)}\Big)^{\frac{Np-2N-4s}{4s}} \bar{t}_p^{\frac{Np-2N-4s}{4s}}.
\end{aligned} \tag{3.12}$$

Since $u_\varepsilon(x) = \frac{c\varepsilon^{\frac{N}{2}}}{|Q|_2}Q(\varepsilon x)$, we have $Q(x) = \frac{|Q|_2}{c\varepsilon^{\frac{N}{2}}}u(\frac{x}{\varepsilon})$. According to (1.10), the function u_ε satisfies the following equation

$$\frac{c^{p-2}\varepsilon^{\frac{Np-2N-4s}{2}}}{|Q|_2^{p-2}}(-\Delta)^s u_\varepsilon - |u_\varepsilon|^{p-2}u_\varepsilon = -\frac{c^{p-2}\varepsilon^{\frac{N(p-2)}{2}}}{|Q|_2^{p-2}}u_\varepsilon. \tag{3.13}$$

Due to $\varepsilon = (\frac{(2N+2ps-Np)\bar{t}_p}{N(p-2)c^2})^{\frac{1}{2s}}$, we can get that

$$\frac{c^{p-2}\varepsilon^{\frac{Np-2N-4s}{2}}}{|Q|_2^{p-2}} = \frac{c^{\frac{2ps-Np+2N}{2s}}}{|Q|_2^{p-2}}(\frac{2ps-Np+2N}{N(p-2)})^{\frac{Np-2N-4s}{4s}}\bar{t}_p^{\frac{Np-2N-4s}{4s}}. \tag{3.14}$$

From (3.12)-(3.14), we know that u_ε is a solution of equation (1.1) with

$$\lambda = -\frac{c^{p-2}\varepsilon^{\frac{N(p-2)}{2}}}{|Q|_2^{p-2}}.$$

Finally, we prove that u_ε is the uniqueness of solution for problem (1.14). Assume $\overline{u}$ is a solution of (1.14) and satisfies (1.15), then there exists a $\lambda \in \mathbb{R}$ such that $\Phi'(u) = \lambda u$. We adopt some ideas used in [22] to obtain the following Nehari-Pohozaev identities

$$\begin{aligned}
&as\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|\bar{u}(x)-\bar{u}(y)|^2}{|x-y|^{N+2s}}dxdy + bs\Big(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|\bar{u}(x)-\bar{u}(y)|^2}{|x-y|^{N+2s}}dxdy\Big)^{\theta}\\
&-\frac{N(p-2)}{2p}\int_{\mathbb{R}^N}|\bar{u}|^p dx = 0.
\end{aligned} \tag{3.15}$$

Set $\widehat{h}(r)=r^{\frac{N}{4s}}\overline{u}(r^{\frac{1}{2s}}x)$, then we get that

$$\Phi(\widehat{h}(r))=\frac{ar}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|\bar{u}(x)-\bar{u}(y)|^2}{|x-y|^{N+2s}}dxdy+\frac{br^\theta}{2\theta}\Big(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|\bar{u}(x)-\bar{u}(y)|^2}{|x-y|^{N+2s}}dxdy\Big)^\theta$$
$$-\frac{r^{\frac{N(p-2)}{4s}}}{p}\int_{\mathbb{R}^N}|\overline{u}|^p dx.$$

From (3.15), we know that the function $\Phi(\widehat{h}(r))$ has its maximum at the unique point $r=1$ and $\Phi(\widehat{h}(r))\to-\infty$ as $r\to+\infty$. Choosing $0<\widetilde{r}<1<\widehat{r}$ such that $\widehat{h}(\widetilde{r})\in A_{K(c)}$ and $\Phi(\widehat{h}(\widehat{r}))<0$, we deduce that $h_0(r):=\widehat{h}((1-r)\widetilde{r}+r\widehat{r})\in\Upsilon(c)$ and $\max_{r\in[0,1]}\Phi(h_0(r))=\Phi(\overline{u})$. Similar to the arguments of (3.7) and (3.10), we have that

$$\varphi(\bar{t}_p)=\gamma(c)=\Phi(\overline{u})=\max_{r\in[0,1]}\Phi(h_0(r))\geq\max_{t\in\mathbb{R}^+}\varphi(t)=\varphi(\bar{t}_p).$$

By (3.8), this implies that $\overline{u}$ must be the form of $\frac{c\alpha^{\frac{N}{2}}}{|Q|_2}Q(\alpha x)$ for some $\alpha>0$. Since $\varphi(\bar{t}_p)=\Phi(\overline{u})$, we deduce that $\alpha=\varepsilon$ and $\overline{u}=u_\varepsilon$. □

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