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SOLVABILITY OF HADAMARD FRACTIONAL DIFFERENTIAL EQUATIONS ON A HALF-LINE WITH LOGARITHMIC TYPE INITIAL DATA

Nemat Nyamoradi[1,†] and Bashir Ahmad[2]

Abstract In this paper, by using the Leray-Schauder nonlinear alternative and contraction mapping principle, we study the existence and uniqueness of solutions to a new class of Hadamard fractional differential equations on a half-line supplemented with logarithmic type initial conditions.

Keywords Hadamard fractional derivative, logarithmic type initial data, existence of solutions, fixed point.

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1. Introduction

In the past few decades, there has been shown a considerable interest in studying differential equations on a half-line, for instance, see the papers [8, 9, 11, 13, 16] and the references cited therein. In [10], Liu applied the Schauder fixed point theorem to prove the existence of at least one positive solution for the following boundary value problem of fractional differential equations:

$$\begin{cases} D_{0^+}^{\zeta_1}\mathcal{J}_1(\varsigma) = f_1(\varsigma, \mathcal{J}_2(\varsigma), D_{0^+}^{p}\mathcal{J}_2(\varsigma)), & \varsigma \in (0,\infty), \\ D_{0^+}^{\zeta_2}\mathcal{J}_2(\varsigma) = f_2(\varsigma, \mathcal{J}_1(\varsigma), D_{0^+}^{q}\mathcal{J}_1(\varsigma)), & \varsigma \in (0,\infty), \\ \lim\limits_{t\to 0}\varsigma^{2-\zeta_1}\mathcal{J}_1(\varsigma) = a_0, \quad \lim\limits_{\varsigma\to 0}\varsigma^{2-\zeta_2}\mathcal{J}_2(\varsigma) = b_0, \\ \lim\limits_{\varsigma\to 0} D_{0^+}^{\zeta_1-1}\mathcal{J}_1(\varsigma) = a_1, \quad \lim\limits_{\varsigma\to 0} D_{0^+}^{\zeta_2-1}\mathcal{J}_2(\varsigma) = b_1, \end{cases}$$

where $\zeta_1, \zeta_2 \in (1,2)$, $p \in (\zeta_2 - 1, \zeta_2)$, $q \in (\zeta_1 - 1, \zeta_1)$, $a_0, b_0, a_1, b_1 \in \mathbb{R}$, $D_{0^+}^{(.)}$ is the Riemann-Liouville (R-L) fractional derivative operator of order (.), and $f_1, f_2 \in C((0,\infty)\times\mathbb{R}^2, \mathbb{R})$.

In [15], the authors obtained the sufficient conditions for the existence of solutions to a system of R-L type fractional differential equations on a infinite interval by using the Banach contraction mapping theorem and Schauder's fixed point theorem.

†The corresponding author.
[1]Department of Mathematics, Faculty of Sciences, Razi University, 67149 Kermanshah, Iran
[2]Nonlinear Analysis and Applied Mathematics (NAAM)-Research Group, Department of Mathematics, Faculty of Science, King Abdulaziz University, P.O. Box 80203, Jeddah 21589, Saudi Arabia
Email: neamat80@yahoo.com(N. Nyamoradi),
bashirahmad_qau@yahoo.com(B. Ahmad)

In 1892, Hadamard [6] introduced a fractional derivative with its kernel depending on a logarithmic function with an arbitrary exponent, which is known as Hadamard fractional derivative in the literature. One can find a useful information on Hadamard-type fractional differential equations and inclusions supplemented with different kinds of initial and boundary conditions in the text [3]. In [14], the authors studied a class of Hadamard fractional differential equations equipped with Hadamard integral and multipoint discrete boundary conditions on a half-line by using monotone iterative method. By using fixed point theorems, the existence results for fractional differential equations on bounded as well as unbounded domains were obtained in [2, 4, 12, 15].

Inspired by aforementioned works, in this article, we consider the following Hadamard fractional differential equation on the half-line with with logarithmic type initial data:

$$\begin{cases} {}_HD^{\zeta_1}\mathcal{Q}(\omega) = g(\omega, \mathcal{Q}(\omega), (S\mathcal{Q})(\omega), (H\mathcal{Q})(\omega)), & \omega \in (1, \infty), \\ \lim\limits_{\omega\to 1}(\log\omega)^{2-\zeta_1}\mathcal{Q}(\omega) = \eta_1, \quad \lim\limits_{\omega\to 1} {}_HD^{\zeta_1-1}\mathcal{Q}(\omega) = \eta_2, \end{cases} \tag{1.1}$$

where $1 < \zeta_1 \leq 2$, $\eta_1, \eta_2 \in \mathbb{R}$, ${}_HD^{\zeta_1}$ is the Hadamard fractional derivative of order ζ_1, $g \in C((1,\infty)\times\mathbb{R}^3;\mathbb{R})$ and

$$(S\mathcal{Q})(\omega) = \int_1^\omega K(\omega, s)\mathcal{Q}(s)\frac{ds}{s}, \quad (H\mathcal{Q})(\omega) = \int_1^\infty U(\omega, s)\mathcal{Q}(s)\frac{ds}{s},$$

with $K, U : (1,\infty)\times(1,\infty) \to [0,\infty)$.

The objective of this article is to establish the existence and uniqueness of solutions to the problem (1.1). We make use of the Leray-Schauder nonlinear alternative and contraction mapping principle to derive the desired results, which are new in the given configuration and enhance the related literature on the topic.

The rest of the manuscript is organized as follows. Section 2 contains some preliminary concepts related to the problem investigated in this work. The main results together with an illustrative example are presented in Section 3. The paper concludes with some interesting observations.

2. Preliminaries

In this section, we set our notation and present basic definitions and lemmas.

Definition 2.1. ([7]) For a function $h : [1,\infty) \longrightarrow \mathbb{R}$, the Hadamard derivative of fractional order ζ_1 is defined as

$${}_HD^{\zeta_1}h(\omega) = \frac{1}{\Gamma(n-\zeta_1)}\Big(\omega\frac{d}{d\omega}\Big)^n \int_1^\omega \Big(\log\frac{\omega}{\vartheta}\Big)^{n-\zeta_1-1}\frac{h(\vartheta)}{\vartheta}d\vartheta, \ n-1 \leq \zeta_1 < n,$$

where $\log(\cdot) = \log_e(\cdot)$.

Definition 2.2. ([7]) For a function h, the Hadamard fractional integral of order ζ_1 is defined as

$${}_HI^{\zeta_1}h(\omega) = \frac{1}{\Gamma(\zeta_1)}\int_1^\omega \Big(\log\frac{\omega}{\vartheta}\Big)^{\zeta_1-1}\frac{h(\vartheta)}{\vartheta}d\vartheta, \ \zeta_1 > 0.$$

Lemma 2.1. ([7]). *If* $0 < z_0 < \infty$ *and* $\zeta_1, \nu > 0$, *then*

$$\left({}_H I_{z_0^+}^{\zeta_1}\left(\log\frac{\omega}{z_0}\right)^{\nu-1}\right)(\mathcal{Q}) = \frac{\Gamma(\nu)}{\Gamma(\nu+\zeta_1)}\left(\log\frac{\mathcal{Q}}{z_0}\right)^{\nu+\zeta_1-1},$$

$$\left({}_H D_{z_0^+}^{\zeta_1}\left(\log\frac{\omega}{z_0}\right)^{\nu-1}\right)(\mathcal{Q}) = \frac{\Gamma(\nu)}{\Gamma(\nu-\zeta_1)}\left(\log\frac{\mathcal{Q}}{z_0}\right)^{\nu-\zeta_1-1}.$$

Choosing $\sigma > -1$, we define a real Banach space

$$X := \left\{\mathcal{Q} \in C(1,\infty) : \frac{(\log\omega)^{2-\zeta_1}}{1+(\log\omega)^{\sigma+2}}\mathcal{Q}(\omega) \text{ is bounded on } (1,\infty)\right\},$$

endowed with the norm

$$\|\mathcal{Q}\|_X := \sup_{\omega\in(1,\infty)}\left(\frac{(\log\omega)^{2-\zeta_1}}{1+(\log\omega)^{\sigma+2}}|\mathcal{Q}(\omega)|\right).$$

Lemma 2.2. *Suppose that* $\xi : (1,\infty) \to \mathbb{R}$ *is a given function such that* $|\xi(\omega)| \le M$ *when there exists a numbers* $M > 0$ *and* $1 < \zeta_1 < 2$, $\eta_1, \eta_2 \in \mathbb{R}$. *Then, the solution of the following problem:*

$$\begin{cases} {}_H D^{\zeta_1}\mathcal{Q}(\omega) = \xi(\omega), \qquad \omega \in (1,\infty), \\ \lim\limits_{\omega\to 1}(\log\omega)^{2-\zeta_1}\mathcal{Q}(\omega) = \eta_1, \quad \lim\limits_{\omega\to 1}{}_H D^{\zeta_1-1}\mathcal{Q}(\omega) = \eta_2, \end{cases} \tag{2.1}$$

is given by

$$\mathcal{Q}(\omega) = \frac{1}{\Gamma(\zeta_1)}\int_1^\omega\left(\log\frac{\omega}{\vartheta}\right)^{\zeta_1-1}\frac{\xi(\vartheta)}{\vartheta}d\vartheta + \frac{\eta_2}{\Gamma(\zeta_1)}(\log\omega)^{\zeta_1-1} + \eta_1(\log\omega)^{\zeta_1-2}. \tag{2.2}$$

Proof. By employing an argument similar to the one used in [3], we can write the solution of (2.1)as

$$\mathcal{Q}(\omega) = {}_H I^{\zeta_1}\xi(\omega) + c_1(\log\omega)^{\zeta_1-1} + c_2(\log\omega)^{\zeta_1-2}, \tag{2.3}$$

for constants $c_1, c_2 \in \mathbb{R}$. Since

$$\begin{aligned} &\left|(\log\omega)^{2-\zeta_1}\int_1^\omega\left(\log\frac{\omega}{\vartheta}\right)^{\zeta_1-1}\frac{\xi(\vartheta)}{\vartheta}d\vartheta\right| \\ &\le M(\log\omega)^{2-\zeta_1}\int_1^\omega\left(\log\frac{\omega}{\vartheta}\right)^{\zeta_1-1}\frac{1}{\vartheta}d\vartheta \\ &= \frac{M}{\zeta_1}(\log\omega)^{2-\zeta_1}(\log\omega)^{\zeta_1} \\ &= \frac{M}{\zeta_1}(\log\omega)^2 \to 0 \text{ as } \omega \to 1, \end{aligned}$$

therefore, it follows by the condition $\lim_{\omega\to 1}(\log\omega)^{2-\zeta_1}\mathcal{Q}(\omega) = \eta_1$ that $c_2 = \eta_1$.

Also, by Lemma 2.1, we have

$${}_H D^{\zeta_1-1}\mathcal{Q}(\omega) = \int_1^\omega\frac{\xi(\vartheta)}{\vartheta}d\vartheta + c_1\Gamma(\zeta_1). \tag{2.4}$$

Observe that

$$\left|\int_1^\omega\frac{\xi(\vartheta)}{\vartheta}d\vartheta\right| \le M\int_1^\omega\frac{d\vartheta}{\vartheta} = M\log\omega \to 0 \text{ as } \omega \to 1. \tag{2.5}$$

Using (2.4), (2.5) and $\lim_{\omega\to 1}{}_H D^{\zeta_1-1}\mathcal{Q}(\omega) = \eta_2$, we get $c_1 = \dfrac{\eta_2}{\Gamma(\zeta_1)}$. This finishes the proof. □

3. Main results

This section is concerned with the existence and uniqueness results for the problem (1.1). In the subsequent analysis, we need the following assumptions:

(G1) $l_0 := \sup_{\omega\in(1,\infty)} \frac{(\log\omega)^{2-\zeta_1}}{1+(\log\omega)^{\sigma+2}} \int_1^\omega \frac{1+(\log\vartheta)^{\sigma+2}}{(\log\vartheta)^{2-\zeta_1}} K(\omega,\vartheta)\frac{d\vartheta}{\vartheta} < \infty$ and
$$k_0 := \sup_{\omega\in(1,\infty)} \frac{(\log\omega)^{2-\zeta_1}}{1+(\log\omega)^{\sigma+2}} \int_1^\infty U(\omega,\vartheta)\frac{1+(\log\vartheta)^{\sigma+2}}{(\log\vartheta)^{2-\zeta_1}} \frac{d\vartheta}{\vartheta} < \infty;$$

(G2) There exist three positive functions $\varpi_i(\omega)$, $i=1,2,3$ such that
$$\Bigg|g\left(\omega, \frac{1+(\log\omega)^{\sigma+2}}{(\log\omega)^{2-\zeta_1}}\mathcal{Q}(\omega), (S\frac{1+(\log\omega)^{\sigma+2}}{(\log\omega)^{2-\zeta_1}}\mathcal{Q})(\omega), (H\frac{1+(\log\omega)^{\sigma+2}}{(\log\omega)^{2-\zeta_1}}\mathcal{Q})(\omega)\right)$$
$$-g\left(\omega, \frac{1+(\log\omega)^{\sigma+2}}{(\log\omega)^{2-\zeta_1}}\mathcal{G}(\omega), (S\frac{1+(\log\omega)^{\sigma+2}}{(\log\omega)^{2-\zeta_1}}\mathcal{G})(\omega), (H\frac{1+(\log\omega)^{\sigma+2}}{(\log\omega)^{2-\zeta_1}}\mathcal{G})(\omega)\right)\Bigg|$$
$$\le \varpi_1(\omega)|\mathcal{Q}-\mathcal{G}| + \varpi_2(\omega)|S\mathcal{Q}-S\mathcal{G}| + \varpi_3(\omega)|H\mathcal{Q}-H\mathcal{G}|,$$

for all $\mathcal{Q},\mathcal{G}\in\mathbb{R}$, $\omega\in(1,\infty)$, where
$$\left(S\frac{1+(\log\omega)^{\sigma+2}}{(\log\omega)^{2-\zeta_1}}\mathcal{Q}\right)(\omega) = \int_1^\omega K(\omega,\vartheta)\frac{1+(\log\vartheta)^{\sigma+2}}{(\log\vartheta)^{2-\zeta_1}}\mathcal{Q}(\vartheta)\frac{d\vartheta}{\vartheta},$$
$$\left(H\frac{1+(\log\omega)^{\sigma+2}}{(\log\omega)^{2-\zeta_1}}\mathcal{Q}\right)(\omega) = \int_1^\infty U(\omega,\vartheta)\frac{1+(\log\vartheta)^{\sigma+2}}{(\log\vartheta)^{2-\zeta_1}}\mathcal{Q}(\vartheta)\frac{d\vartheta}{\vartheta};$$

(G3) There exists a number Υ such that $\varrho_0 \le \Upsilon < 1$, $\omega\in(1,\infty)$, where
$$\varrho_0 = (1+l_0+k_0)\,{}_H I_{\varpi}^{\zeta_1},$$

and
$${}_H I_{\varpi}^{\zeta_1} = \max\left\{\sup_{\omega\in(1,\infty)} \frac{(\log\omega)^{2-\zeta_1}}{1+(\log\omega)^{\sigma+2}}\,{}_H I^{\zeta_1}\varpi_i(\omega),\ i=1,2,3\right\}.$$

Theorem 3.1. *Suppose that $g\in C((1,\infty)\times\mathbb{R}^3,\mathbb{R})$ satisfies the conditions (G1) and (G2) and that there exists a number $M>0$ such that $|g(\omega,\mathcal{Q}(\omega),(S\mathcal{Q})(\omega),(H\mathcal{Q})(\omega))| \le M$. Then, the problem (1.1) has a unique solution when $\varrho_0<1$ (ϱ_0 is defined in (G3)).*

Proof. Define an operator $\Phi: X\to X$ as
$$\Phi\mathcal{Q}(\omega) = \frac{1}{\Gamma(\zeta_1)}\int_1^\omega \left(\log\frac{\omega}{\vartheta}\right)^{\zeta_1-1} g(\vartheta,\mathcal{Q}(\vartheta),(S\mathcal{Q})(\vartheta),(H\mathcal{Q})(\vartheta))\frac{d\vartheta}{\vartheta}$$
$$+\frac{\eta_2}{\Gamma(\zeta_1)}(\log\omega)^{\zeta_1-1} + \eta_1(\log\omega)^{\zeta_1-2}. \tag{3.1}$$

Let us set $\sup_{\omega\in(1,\infty)}\|g(\omega,0,0,0)\| = \Lambda$, $\omega_1 = \frac{1}{\Gamma(\zeta_1+1)}\sup_{\omega\in(1,\infty)}\frac{(\log\omega)^2}{1+(\log\omega)^{\sigma+2}}$, $\vartheta_1 = \sup_{\omega\in(1,\infty)}\frac{\log\omega}{1+(\log\omega)^{\sigma+2}}$, $\vartheta_2 = \sup_{\omega\in(1,\infty)}\frac{1}{1+(\log\omega)^{\sigma+2}}$ and choose
$$\rho \ge \frac{1}{1-\Upsilon}|\omega_1\Lambda + \frac{\eta_2}{\Gamma(\zeta_1)}\vartheta_1 + \eta_1\vartheta_2|,$$

where $\varrho_0 \leq \Upsilon < 1$. Introduce $B_\rho = \{u \in X : \|u\|_X \leq \rho\}$. For any $\mathcal{Q} \in B_\rho$ and $\vartheta \in (1, \infty)$, by the triangle inequality and (G2), we obtain

$$\begin{aligned}
&|g(\vartheta, \mathcal{Q}(\vartheta), (S\mathcal{Q})(\vartheta), (H\mathcal{Q})(\vartheta))| \\
=& \Bigg| g\Bigg(\vartheta, \frac{1+(\log\vartheta)^{\sigma+2}}{(\log\vartheta)^{2-\zeta_1}} \cdot \frac{(\log\vartheta)^{2-\zeta_1}}{1+(\log\vartheta)^{\sigma+2}}\mathcal{Q}(\vartheta), (S\frac{1+(\log\vartheta)^{\sigma+2}}{(\log\vartheta)^{2-\zeta_1}} \\
&\times \frac{(\log\vartheta)^{2-\zeta_1}}{1+(\log\vartheta)^{\sigma+2}}\mathcal{Q})(\vartheta), (H\frac{1+(\log\vartheta)^{\sigma+2}}{(\log\vartheta)^{2-\zeta_1}} \cdot \frac{(\log\vartheta)^{2-\zeta_1}}{1+(\log\vartheta)^{\sigma+2}}\mathcal{Q})(\vartheta)\Bigg)\Bigg| \\
\leq& \Bigg| g\Bigg(\vartheta, \frac{1+(\log\vartheta)^{\sigma+2}}{(\log\vartheta)^{2-\zeta_1}} \cdot \frac{(\log\vartheta)^{2-\zeta_1}}{1+(\log\vartheta)^{\sigma+2}}\mathcal{Q}(\vartheta), (S\frac{1+(\log\vartheta)^{\sigma+2}}{(\log\vartheta)^{2-\zeta_1}} \\
&\times \frac{(\log\vartheta)^{2-\zeta_1}}{1+(\log\vartheta)^{\sigma+2}}\mathcal{Q})(\vartheta), (H\frac{1+(\log\vartheta)^{\sigma+2}}{(\log\vartheta)^{2-\zeta_1}} \cdot \frac{(\log\vartheta)^{2-\zeta_1}}{1+(\log\vartheta)^{\sigma+2}}\mathcal{Q})(\vartheta)\Bigg) \\
&-g(\vartheta, 0, 0, 0)\Bigg| + |g(\vartheta, 0, 0, 0)| \\
\leq& \varpi_1(\vartheta)\frac{(\log\vartheta)^{2-\zeta_1}}{1+(\log\vartheta)^{\sigma+2}}|\mathcal{Q}| + \varpi_2(\vartheta)\frac{(\log\vartheta)^{2-\zeta_1}}{1+(\log\vartheta)^{\sigma+2}}|S\mathcal{Q}| \\
&+\varpi_3(\vartheta)\frac{(\log\vartheta)^{2-\zeta_1}}{1+(\log\vartheta)^{\sigma+2}}|H\mathcal{Q}| + \Lambda \\
\leq& \varpi_1(\vartheta)|\|\mathcal{Q}\|_X + \varpi_2(\vartheta)\|S\mathcal{Q}\|_X + \varpi_3(\vartheta)\|H\mathcal{Q}\|_X + \Lambda \\
\leq& [\varpi_1(\vartheta) + l_0\varpi_2(\vartheta) + k_0\varpi_3(\vartheta)]\|\mathcal{Q}\| + \Lambda \\
\leq& [\varpi_1(\vartheta) + l_0\varpi_2(\vartheta) + k_0\varpi_3(\vartheta)]\rho + \Lambda. \qquad (3.2)
\end{aligned}$$

Now, we will show that $\Phi B_\rho \subset B_\rho$. For all $\mathcal{Q} \in B_\rho$, by (G1), (G2), (G3) and (3.2), we have

$$\begin{aligned}
\|(\Phi\mathcal{Q})\| \leq& \sup_{\omega\in(1,\infty)} \frac{(\log\omega)^{2-\zeta_1}}{1+(\log\omega)^{\sigma+2}}\Big[\int_1^\omega \frac{\left(\log\frac{\omega}{\vartheta}\right)^{\zeta_1-1}}{\Gamma(\zeta_1)}\{[\varpi_1(\vartheta) + l_0\varpi_2(\vartheta) \\
&+k_0\varpi_3(\vartheta)]\rho + \Lambda\}\frac{d\vartheta}{\vartheta} + \frac{\eta_2}{\Gamma(\zeta_1)}(\log\omega)^{\zeta_1-1} + \eta_1(\log\omega)^{\zeta_1-2}\Big] \\
\leq& \Big\{\sup_{\omega\in(1,\infty)} \frac{(\log\omega)^{2-\zeta_1}}{1+(\log\omega)^{\sigma+2}}\,{}_HI^{\zeta_1}\varpi_1(\omega) \\
&+l_0\sup_{t\in(1,\infty)} \frac{(\log\omega)^{2-\zeta_1}}{1+(\log\omega)^{\sigma+2}}\,{}_HI^{\zeta_1}\varpi_2(\omega) \\
&+k_0\sup_{\omega\in(1,\infty)} \frac{(\log\omega)^{2-\zeta_1}}{1+(\log\omega)^{\sigma+2}}\,{}_HI^{\zeta_1}\varpi_3(\omega)\Big\}\rho \\
&+\frac{1}{\Gamma(\zeta_1+1)}\sup_{\omega\in(1,\infty)} \frac{(\log\omega)^{\zeta_1}(\log\omega)^{2-\zeta_1}}{1+(\log\omega)^{\sigma+2}}\Lambda \\
&+\frac{\eta_2}{\Gamma(\zeta_1)}\sup_{\omega\in(1,\infty)} \frac{(\log\omega)^{\zeta_1-1}(\log\omega)^{2-\zeta_1}}{1+(\log\omega)^{\sigma+2}} \\
&+\eta_1\sup_{\omega\in(1,\infty)} \frac{(\log\omega)^{\zeta_1-2}(\log\omega)^{2-\zeta_1}}{1+(\log\omega)^{\sigma+2}}
\end{aligned}$$

$$\begin{aligned}&\le (1+l_0+k_0)\,_HI^{\zeta_1}_{\varpi}\rho+\omega_1\Lambda+\frac{\eta_2}{\Gamma(\zeta_1)}\vartheta_1+\eta_1\vartheta_2\\&\le \varrho_0\rho+(1-\Upsilon)\rho\le\rho.\end{aligned}\tag{3.3}$$

Therefore, $\|(\Phi\mathcal{Q})\|\le\rho$.

Now, we show that Φ is a contraction. For $\mathcal{Q},\mathcal{G}\in X$ and $\omega\in(1,\infty)$, it follows by (G1) and (G2) that

$$\begin{aligned}&\|(\Phi\mathcal{Q})-(\Phi\mathcal{G})\|\\
\le&\sup_{\omega\in(1,\infty)}\frac{(\log\omega)^{2-\zeta_1}}{1+(\log\omega)^{\sigma+2}}\Big[\int_1^\omega\frac{\left(\log\frac{\omega}{\vartheta}\right)^{\zeta_1-1}}{\Gamma(\zeta_1)}\\
&\times[g(\vartheta,\mathcal{Q}(\vartheta),(S\mathcal{Q})(\vartheta),(H\mathcal{Q})(\vartheta))-g(\omega,\mathcal{G}(\vartheta),(S\mathcal{G})(\vartheta),(H\mathcal{G})(\vartheta))]\frac{d\vartheta}{\vartheta}\Big]\\
\le&\sup_{\omega\in(1,\infty)}\frac{(\log\omega)^{2-\zeta_1}}{1+(\log\omega)^{\sigma+2}}\int_1^\omega\frac{\left(\log\frac{\omega}{\vartheta}\right)^{\zeta_1-1}}{\Gamma(\zeta_1)}\{\varpi_1(\vartheta)\|\mathcal{Q}-\mathcal{G}\|_X\\
&+\varpi_2(\omega)\|S\mathcal{Q}-S\mathcal{G}\|_X+\varpi_3(\omega)\|H\mathcal{Q}-H\mathcal{G}\|_X\}\frac{d\vartheta}{\vartheta}\\
\le&\Big\{\sup_{\omega\in(1,\infty)}\frac{(\log\omega)^{2-\zeta_1}}{1+(\log\omega)^{\sigma+2}}\,_HI^{\zeta_1}\varpi_1(\omega)\\
&+l_0\sup_{\omega\in(1,\infty)}\frac{(\log\omega)^{2-\zeta_1}}{1+(\log\omega)^{\sigma+2}}\,_HI^{\zeta_1}\varpi_2(\omega)\\
&+k_0\sup_{\omega\in(1,\infty)}\frac{(\log\omega)^{2-\zeta_1}}{1+(\log\omega)^{\sigma+2}}\,_HI^{\zeta_1}\varpi_3(\omega)\Big\}\|\mathcal{Q}-\mathcal{G}\|_X\\
\le&(1+l_0+k_0)\,_HI^{\zeta_1}_{\varpi}\|\mathcal{Q}-\mathcal{G}\|_X=\varrho_0\|\mathcal{Q}-\mathcal{G}\|_X,\end{aligned}$$

where ϱ_0 is given in (G3). Since $\varrho_0<1$, therefore Φ is a contraction. Hence, the assumptions of the contraction mapping principle are satisfied. This leads to the conclusion. □

Now, we prove the next existence result by applying Leray-Schauder nonlinear alternative [1,5].

Lemma 3.1. (See [1,5]). *Let E be a Banach space, C a closed, convex subset of E, U an open subset of C and $0\in U$. Suppose that $F:\bar{U}\to C$ is a completely continuous operator. Then, either:*

- *(i) F has a fixed point in $\bar{U}$, or*
- *(ii) there is an element $u\in\partial U$ (the boundary of U in C) and $\lambda\in(0,1)$ with $u=\lambda F(u)$.*

Theorem 3.2. *Suppose that g is a continuous function with $g\in X$. In addition, let the following conditions hold:*

(H1) There exist three functions $\mathcal{L}_i\in X$ and nondecreasing functions $\mathcal{Y}_i:\mathbb{R}^+\to\mathbb{R}^+$ $(i=1,2,3)$, such that

$$\left|g\left(\omega,\frac{1+(\log\omega)^{\sigma+2}}{(\log\omega)^{2-\zeta_1}}\mathcal{Q}(\omega),(S\frac{1+(\log\omega)^{\sigma+2}}{(\log\omega)^{2-\zeta_1}}\mathcal{Q})(\omega),(H\frac{1+(\log\omega)^{\sigma+2}}{(\log\omega)^{2-\zeta_1}}\mathcal{Q})(\omega)\right)\right|$$

$$\leq \sum_{i=1}^{3} \mathcal{L}_i(\omega)\mathcal{Y}_i(\mathcal{Q}),$$

for all $\omega \in [1, \infty)$ *and* $\mathcal{Q} \in \mathbb{R}$*;*

(H2) There exists a constant $M > 0$ *satisfying the the inequality:*

$$\frac{M}{\sum_{i=1}^{3} \mathcal{Y}_i(M) \sup_{\omega\in(1,\infty)} \frac{(\log\omega)^{2-\zeta_1}}{1+(\log\omega)^{\sigma+2}} {}_H I^{\zeta_1}\mathcal{L}_i(\omega) + \frac{\eta_2}{\Gamma(\zeta_1)}\vartheta_1 + \eta_1\vartheta_2} > 1.$$

Then, the problem (1.1) has at least one solution.

Proof. We firstly establish that Φ is uniformly bounded in X. Set $B_r = \{\mathcal{Q} \in X : \|\mathcal{Q}\|_X \leq r\}$. We firstly get

$$\begin{aligned}
& g\left(\omega, \mathcal{Q}(\omega), (S\mathcal{Q})(\omega), (H\mathcal{Q})(\omega)\right) \\
&= \Bigg| g\Bigg(\omega, \frac{1+(\log\vartheta)^{\sigma+2}}{(\log\vartheta)^{2-\zeta_1}} \cdot \frac{(\log\vartheta)^{2-\zeta_1}}{1+(\log\vartheta)^{\sigma+2}}\mathcal{Q}(\omega), \\
& (S\frac{1+(\log\vartheta)^{\sigma+2}}{(\log\vartheta)^{2-\zeta_1}} \cdot \frac{(\log\vartheta)^{2-\zeta_1}}{1+(\log\vartheta)^{\sigma+2}}\mathcal{Q})(\omega), \\
& (H\frac{1+(\log\vartheta)^{\sigma+2}}{(\log\vartheta)^{2-\zeta_1}} \cdot \frac{(\log\vartheta)^{2-\zeta_1}}{1+(\log\vartheta)^{\sigma+2}}\mathcal{Q})(\omega)\Bigg)\Bigg| \\
&\leq \sum_{i=1}^{3} \mathcal{L}_i(\omega)\mathcal{Y}_i(\|\mathcal{Q}\|_X).
\end{aligned}$$

Consequently, we have

$$\begin{aligned}
& \|\Phi\mathcal{Q}\|_X \\
&\leq \sup_{\omega\in(1,\infty)} \frac{(\log\omega)^{2-\zeta_1}}{1+(\log\omega)^{\sigma+2}} \Bigg[\int_1^{\omega} \frac{1}{\Gamma(\zeta_1)}\Big(\log\frac{\omega}{\vartheta}\Big)^{\zeta_1-1} \sum_{i=1}^{3} \mathcal{L}_i(\vartheta)\mathcal{Y}_i(\|\mathcal{Q}\|_X)\frac{d\vartheta}{\vartheta} \\
&\quad + \frac{\eta_2}{\Gamma(\zeta_1)}(\log\omega)^{\zeta_1-1} + \eta_1(\log\omega)^{\zeta_1-2}\Bigg] \\
&\leq \sum_{i=1}^{3} \mathcal{Y}_i(r) \sup_{\omega\in(1,\infty)} \frac{(\log\omega)^{2-\zeta_1}}{1+(\log\omega)^{\sigma+2}} {}_H I^{\zeta_1}\mathcal{L}_i(\omega) \\
&\quad + \frac{\eta_2}{\Gamma(\zeta_1)} \sup_{\omega\in(1,\infty)} \frac{(\log\omega)^{\zeta_1-1}(\log\omega)^{2-\zeta_1}}{1+(\log\omega)^{\sigma+2}} + \eta_1 \sup_{\omega\in(1,\infty)} \frac{(\log\omega)^{\zeta_1-2}(\log\omega)^{2-\zeta_1}}{1+(\log\omega)^{\sigma+2}} \\
&\leq \sum_{i=1}^{3} \mathcal{Y}_i(r) \sup_{\omega\in(1,\infty)} \frac{(\log\omega)^{2-\zeta_1}}{1+(\log\omega)^{\sigma+2}} {}_H I^{\zeta_1}\mathcal{L}_i(\omega) + \frac{\eta_2}{\Gamma(\zeta_1)}\vartheta_1 + \eta_1\vartheta_2.
\end{aligned}$$

Setting $K := \sum_{i=1}^{3} \mathcal{Y}_i(r) \sup_{\omega\in(1,\infty)} \frac{(\log\omega)^{2-\zeta_1}}{1+(\log\omega)^{\sigma+2}} {}_H I^{\zeta_1}\mathcal{L}_i(\omega) + \frac{\eta_2}{\Gamma(\zeta_1)}\vartheta_1 + \eta_1\vartheta_2$, we get $\|\Phi\mathcal{Q}\|_X \leq K$. Thus, Φ is uniformly bounded in X.

Next, we show that Φ is equi-continuous. Let $\omega_1, \omega_2 \in [1, \infty)$ with $\omega_1 < \omega_2$ and $\mathcal{Q} \in B_r$. Then, we obtain

$$\left| \frac{(\log\omega_2)^{2-\zeta_1}}{1+(\log\omega_2)^{\sigma+2}}(\Phi\mathcal{Q}(\omega_2) - \frac{(\log\omega_1)^{2-\zeta_1}}{1+(\log\omega_1)^{\sigma+2}}(\Phi\mathcal{Q}(\omega_1) \right|$$

$$\begin{aligned}
&\leq \frac{1}{\Gamma(\zeta_1)} \int_1^{\omega_1} \left[\frac{(\log \omega_2)^{2-\zeta_1}}{1+(\log \omega_2)^{\sigma+2}} \Big(\log \frac{\omega_2}{\vartheta} \Big)^{\zeta_1 - 1} \right. \\
&\quad \left. - \frac{(\log \omega_1)^{2-\zeta_1}}{1+(\log \omega_1)^{\sigma+2}} \Big(\log \frac{\omega_1}{\vartheta} \Big)^{\zeta_1 - 1} \right] |g(\vartheta, \mathcal{Q}(\vartheta), (S\mathcal{Q})(\vartheta), (H\mathcal{Q})(\vartheta))| \frac{d\vartheta}{\vartheta} \\
&\quad + \frac{1}{\Gamma(\zeta_1)} \int_{\omega_1}^{\omega_2} \frac{(\log \omega_2)^{2-\zeta_1}}{1+(\log \omega_2)^{\sigma+2}} \Big(\log \frac{\omega_2}{\vartheta} \Big)^{\zeta_1 - 1} |g(\vartheta, \mathcal{Q}(\vartheta), (S\mathcal{Q})(\vartheta), (H\mathcal{Q})(\vartheta))| \frac{d\vartheta}{\vartheta} \\
&\quad + \frac{\eta_2}{\Gamma(\zeta_1)} \left[\frac{\log \omega_2}{1+(\log \omega_2)^{\sigma+2}} - \frac{\log \omega_1}{1+(\log \omega_1)^{\sigma+2}} \right] \\
&\quad + \eta_1 \left[\frac{1}{1+(\log \omega_2)^{\sigma+2}} - \frac{1}{1+(\log \omega_1)^{\sigma+2}} \right] \\
&\leq \sum_{i=1}^{3} \frac{\mathcal{Y}_i(r)}{\Gamma(\zeta_1)} \int_1^{\omega_1} \left[\frac{(\log \omega_2)^{2-\zeta_1}}{1+(\log \omega_2)^{\sigma+2}} \Big(\log \frac{\omega_2}{\vartheta} \Big)^{\zeta_1 - 1} \right. \\
&\quad \left. - \frac{(\log \omega_1)^{2-\zeta_1}}{1+(\log \omega_1)^{\sigma+2}} \Big(\log \frac{\omega_1}{\vartheta} \Big)^{\zeta_1 - 1} \right] \mathcal{L}_i(\vartheta) \frac{d\vartheta}{\vartheta} \\
&\quad + \sum_{i=1}^{3} \frac{\mathcal{Y}_i(r)}{\Gamma(\zeta_1)} \int_{\omega_1}^{\omega_2} \frac{(\log \omega_2)^{2-\zeta_1}}{1+(\log \omega_2)^{\sigma+2}} \Big(\log \frac{\omega_2}{\vartheta} \Big)^{\zeta_1 - 1} \mathcal{L}_i(\vartheta) \frac{d\vartheta}{\vartheta} \\
&\quad + \frac{\eta_2}{\Gamma(\zeta_1)} \left[\frac{\log \omega_2}{1+(\log \omega_2)^{\sigma+2}} - \frac{\log \omega_1}{1+(\log \omega_1)^{\sigma+2}} \right] \\
&\quad + \eta_1 \left[\frac{1}{1+(\log \omega_2)^{\sigma+2}} - \frac{1}{1+(\log \omega_1)^{\sigma+2}} \right] \\
&:= \digamma.
\end{aligned}$$

Clearly, $\digamma \to 0$ independently of $\mathcal{Q} \in B_r$ as $\omega_2 \to \omega_1$. So, by the Arzelá-Ascoli theorem, we deduce that Φ is completely continuous.

Let $\mathcal{Q}$ be a solution of the equation $\mathcal{G} = \lambda \Phi \mathcal{G}$ for $\lambda \in (0,1)$. Then, by straightforward computation, we have

$$\begin{aligned}
\|\mathcal{G}\|_X &= \|\lambda(\Phi\mathcal{G})\|_X \\
&\leq \sum_{i=1}^{3} \mathcal{Y}_i(r) \sup_{\omega \in (1,\infty)} \frac{(\log \omega)^{2-\zeta_1}}{1+(\log \omega)^{\sigma+2}} {}_H I^{\zeta_1} \mathcal{L}_i(\omega) + \frac{\eta_2}{\Gamma(\zeta_1)} \vartheta_1 + \eta_1 \vartheta_2,
\end{aligned}$$

which can alternatively be written as

$$\frac{\|\mathcal{G}\|_X}{\sum_{i=1}^{3} \mathcal{Y}_i(r) \sup_{\omega \in (1,\infty)} \frac{(\log \omega)^{2-\zeta_1}}{1+(\log \omega)^{\sigma+2}} {}_H I^{\zeta_1} \mathcal{L}_i(\omega) + \frac{\eta_2}{\Gamma(\zeta_1)} \vartheta_1 + \eta_1 \vartheta_2} \leq 1.$$

From (H2), there exists χ with $\|\mathcal{G}\| \neq \chi$. Set

$$W = \{\mathcal{G} \in X : \|\mathcal{G}\|_X < \chi\}.$$

Clearly, continuity of g implies that Φ is continuous. Also, Φ is completely continuous. By the definition of W, for some $\lambda \in (0,1)$ there is no $\mathcal{Q} \in \partial W$ such that $\mathcal{G} = \lambda \Phi(\mathcal{G})$. So, by Lemma 3.1, we get the conclusion. □

3.1. Special case

In case the nonlinearity in the problem (1.1) is of the form: $\widehat{g}(\omega, \mathcal{Q}(\omega))$, then it takes takes the form:

$$\begin{cases} {}_H D^{\zeta_1} \mathcal{Q}(\omega) = \widehat{g}(\omega, \mathcal{Q}(\omega)), & \omega \in (1, \infty), \\ \lim\limits_{\omega \to 1} (\log \omega)^{2-\zeta_1} \mathcal{Q}(\omega) = \eta_1, & \lim\limits_{\omega \to 1} {}_H D^{\zeta_1 - 1} \mathcal{Q}(\omega) = \eta_2, \end{cases} \tag{3.4}$$

where $\widehat{g} \in C((1, \infty) \times \mathbb{R}, \mathbb{R})$. In the following result, we prove the existence of solutions for the he problem (3.4).

Theorem 3.3. *Suppose that $\widehat{g} : [1, \infty) \times \mathbb{R} \to \mathbb{R}$ is a continuous function with $\widehat{g}(\omega, \mathcal{Q}) \in X$. In addition, let the following conditions hold:*

(H3) There exist a function $\mathcal{L} \in X$ and a nondecreasing function $\mathcal{Y} : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$\left| \widehat{g} \left(\omega, \frac{1 + (\log \omega)^{\sigma+2}}{(\log \omega)^{2-\zeta_1}} \mathcal{Q} \right) \right| \leq \mathcal{L}(\omega) \mathcal{Y}(\mathcal{Q}),$$

for all $(\omega, \mathcal{Q}) \in [1, \infty) \times \mathbb{R}$;

(H4) There exists a constant $M > 0$ such that

$$\frac{M}{\mathcal{Y}(M) \sup_{\omega \in (1,\infty)} \frac{(\log \omega)^{2-\zeta_1}}{1+(\log \omega)^{\sigma+2}} {}_H I^{\zeta_1} \mathcal{L}(\omega) + \frac{\eta_2}{\Gamma(\zeta_1)} \vartheta_1 + \eta_1 \vartheta_2} > 1.$$

Then, there exists at least one solution for the the problem (3.4) on $[1, \infty)$.

Proof. We only provide the outline of the proof as it is similar to that of Theorem 3.2. Let $\Phi_1 : X \to X$ be defined by :

$$\Phi_1 \mathcal{Q}(\omega) = \frac{1}{\Gamma(\zeta_1)} \int_1^\omega \left(\log \frac{\omega}{\vartheta} \right)^{\zeta_1 - 1} \widehat{g}(\vartheta, \mathcal{Q}(\vartheta)) \frac{d\vartheta}{\vartheta} + \frac{\eta_2}{\Gamma(\zeta_1)} (\log \omega)^{\zeta_1 - 1} + \eta_1 (\log \omega)^{\zeta_1 - 2}.$$

In order to show that Φ_1 is uniformly bounded in X, we consider the set $\overline{B}_\mu = \{\mathcal{Q} \in X : \|\mathcal{Q}\|_X \leq \mu\}$. Notice that

$$\widehat{g}(\omega, \mathcal{Q}) = \left| \widehat{g} \left(\omega, \frac{1 + (\log \omega)^{\sigma+2}}{(\log \omega)^{2-\zeta_1}} \frac{(\log \omega)^{2-\zeta_1}}{1 + (\log \omega)^{\sigma+2}} \mathcal{Q} \right) \right| \leq \mathcal{L}(\omega) \mathcal{Y}(\|\mathcal{Q}\|_X).$$

Then, as in the previous result, we get

$$\|\Phi_1 \mathcal{Q}\|_X \leq \mathcal{Y}(\mu) \sup_{\omega \in (1,\infty)} \frac{(\log \omega)^{2-\zeta_1}}{1 + (\log \omega)^{\sigma+2}} {}_H I^{\zeta_1} \mathcal{L}(\omega) + \frac{\eta_2}{\Gamma(\zeta_1)} \vartheta_1 + \eta_1 \vartheta_2 = K_s \text{ (say)}.$$

Thus, Φ_1 is uniformly bounded in X.

Now we verify that Φ_1 is equi-continuous. Let $\omega_1, \omega_2 \in [1, \infty)$ with $\omega_1 < \omega_2$ and $\mathcal{Q} \in \overline{B}_\mu$. Then, we obtain

$$\begin{aligned} &\left| \frac{(\log \omega_2)^{2-\zeta_1}}{1 + (\log \omega_2)^{\sigma+2}} (\Phi_1 \mathcal{Q}(\omega_2) - \frac{(\log \omega_1)^{2-\zeta_1}}{1 + (\log \omega_1)^{\sigma+2}} (\Phi_1 \mathcal{Q}(\omega_1) \right| \\ &\leq \frac{\mathcal{Y}(\mu)}{\Gamma(\zeta_1)} \int_1^{\omega_1} \left[\frac{(\log \omega_2)^{2-\zeta_1}}{1 + (\log \omega_2)^{\sigma+2}} \left(\log \frac{\omega_2}{\vartheta} \right)^{\zeta_1 - 1} \right. \end{aligned}$$

$$
\begin{aligned}
&-\frac{(\log\omega_1)^{2-\zeta_1}}{1+(\log\omega_1)^{\sigma+2}}\Big(\log\frac{\omega_1}{\vartheta}\Big)^{\zeta_1-1}\Bigg]\mathcal{L}(\vartheta)\frac{d\vartheta}{\vartheta}\\
&+\frac{\mathcal{Y}(\mu)}{\Gamma(\zeta_1)}\int_{\omega_1}^{\omega_2}\frac{(\log\omega_2)^{2-\zeta_1}}{1+(\log\omega_2)^{\sigma+2}}\Big(\log\frac{\omega_2}{\vartheta}\Big)^{\zeta_1-1}\mathcal{L}(\vartheta)\frac{d\vartheta}{\vartheta}\\
&+\frac{\eta_2}{\Gamma(\zeta_1)}\left[\frac{\log\omega_2}{1+(\log\omega_2)^{\sigma+2}}-\frac{\log\omega_1}{1+(\log\omega_1)^{\sigma+2}}\right]\\
&+\eta_1\left[\frac{1}{1+(\log\omega_2)^{\sigma+2}}-\frac{1}{1+(\log\omega_1)^{\sigma+2}}\right],
\end{aligned}
$$

which tends to 0, independently of $\mathcal{Q}\in\overline{B}_\mu$ as $\omega_2\to\omega_1$. So, by the Arzelá-Ascoli theorem, it follows that Φ_1 is completely continuous.

Let $\mathcal{Q}$ be a solution of the equation $\mathcal{G}_1=\lambda\Phi_1\mathcal{G}_1$ for $\lambda\in(0,1)$. Then, we have

$$
\frac{\|\mathcal{G}_1\|_X}{\mathcal{Y}(\mu)\sup_{\omega\in(1,\infty)}\frac{(\log\omega)^{2-\zeta_1}}{1+(\log\omega)^{\sigma+2}}\,{}_HI^{\zeta_1}\mathcal{L}(\omega)+\frac{\eta_2}{\Gamma(\zeta_1)}\vartheta_1+\eta_1\vartheta_2}\le 1.
$$

By using (H4), as argued in the last part of of the proof of the previous result, it can be shown that the problem (3.4) has at least one solution on $[1,\infty)$. □

Example 3.1. Consider the following Hadamard fractional differential equation with initial data:

$$
\begin{cases}
{}_HD^{\frac{3}{2}}\mathcal{Q}(\omega)=\frac{(\log\omega)^{\frac12})}{8\omega(1+(\log\omega)^2)}\mathcal{Q}(\omega)+\frac{2}{\omega(\log\omega)^2}\displaystyle\int_1^\omega(\log\vartheta)\frac{(\log\vartheta)^{\frac12}\mathcal{Q}(\vartheta)}{(1+(\log\vartheta)^2}\frac{d\vartheta}{\vartheta}\\
\qquad\qquad+\frac{1}{\omega(\omega+1)}\displaystyle\int_1^\infty e^{1-\log\vartheta}\cdot\frac{(\log\vartheta)^{\frac12}}{1+(\log\vartheta)^2}\mathcal{Q}(\vartheta)\frac{d\vartheta}{\vartheta},\qquad \omega\in(1,\infty),\\
\lim\limits_{\omega\to0}(\log\omega)^{\frac12}\mathcal{Q}(\omega)=\eta_1,\qquad \lim\limits_{\omega\to1}{}_HD^{\frac12}\mathcal{Q}(\omega)=\eta_2,
\end{cases}
\tag{3.5}
$$

Here $\zeta_1=\frac32$, $K(\omega,\vartheta)=\frac{2}{10}(\log\omega)^{-\frac12}\log\vartheta\cdot\frac{(\log\vartheta)^{\frac12}}{1+(\log\vartheta)^2}$ and $U(\omega,\vartheta)=\frac{1}{20}(\log\omega)^{-\frac12}e^{1-\log\vartheta}\cdot\frac{(\log\vartheta)^{\frac12}}{1+(\log\vartheta)^2}$. Choose $\sigma=0$. By direct calculations, we find that $l_0=0.1$, $k_0=0.05$ and ${}_HI_L^{\frac32}\approx0.7522756338$. Moreover, we find that

$$
\varrho_0=(1+l_0+k_0)\,{}_HI_{\varpi}^{\frac32}\approx0.8651169789<1.
$$

Clearly all the conditions of Theorem 3.1 are satisfied. So, Theorem 3.1 yields that the problem (3.5) has a unique solution on $(1,\infty)$.

4. Conclusion

We have discussed the solvability of a Hadamard type fractional differential equation involving nonlinearities with and without integral terms on a half-line complemented with logarithmic type initial data. The uniqueness of solutions for the given problem is established by applying a fixed point theorem due to Banach, while the existence of at least one solution is shown via Leray-Schauder nonlinear alternative. The results presented in this paper are new and opens a new avenue of research for

Hadamard type fractional integro-initial value problems on infinite domains. In our future work, we plan to investigate a system of coupled Hadamard type fractional differential equations of different orders on a half-line supplemented with logarithmic type initial conditions.

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Competing interests

The authors have no competing interests.

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