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ON EPIDEMIOLOGICAL TRANSITION MODEL OF THE EBOLA VIRUS IN FRACTIONAL SENSE

I. Masti[1], K. Sayevand[1] and H. Jafari[2,3,4,†]

Abstract Recently, many researchers have focused on modeling and analyzing various problems in biological phenomena and life sciences such as viruses and nervous system. One of these cases can be seen in the modeling of the Ebola virus. In this paper, we present an efficient method based on properties of Bernstein's operational matrices as well as dual Bernstein for the system of nonlinear equations of Ebola virus in the Caputo fractional sense. The operational matrix of the fractional derivative of order v is obtained based on the dual Bernstein. The proposed dual Bernstein method reduces the solution of the Ebola virus in fractional sense to the solution of a system of nonlinear algebraic equations. The unknown coefficients are obtained by solving the final system of nonlinear equations using the Newton-Raphson method. Another feature of this method is that a reasonable approximate solution can be found with a small number of bases. Moreover, some numerical treatments of fractional models of Ebola Virus are examined. The existence, uniqueness and stability of the suggested methodologies are discussed and proven. Numerical simulations are reported for various fractional orders and by using comparisons between the simulated and measured data, we find the best value of the fractional order. Finally, we will use the data provided by the World Health Organization (WHO) and we compare the fractional Mellin transform, real data, Caputo's derivative, and the classical model. According to the obtained results, the ordinary derivative is less accurate than the fractional order model. In other words, the results showed that fractional order derivatives are superior to classical orders, more reliable and effective in describing biological processes.

Keywords Ebola virus nonlinear equation, Caputo's fractional-order derivative, dual Bernstein polynomials, operational matrix.

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[†]The corresponding author.
[1]Faculty of Mathematical Sciences and Statistics, Malayer University, Malayer, Iran
[2]Department of Applied Mathematics, University of Mazandaran, Babolsar, Iran
[3]Department of Mathematical Sciences, University of South Africa, UNISA 0003, South Africa
[4]Department of Medical Research, China Medical University Hospital, China Medical University, Taichung 110122, Taiwan
Email: iman.masty@gmail.com(I. Masti), ksayehvand@malayeru.ac.ir(K. Sayevand), jafari.usern@gmail.com(H. Jafari)

1. Introduction

Mathematical modeling is the description of a system using mathematical rules and its theorems and concepts. Modeling helps scientists to analyze a system and predict its properties. Mathematical modeling is widely used in physical sciences, geology, meteorology, artificial intelligence, psychology, economics, sociology and biology. For more details regarding the recent history of fractional calculus and applied sciences, the reader is advised to consult the research works presented in [23, 24] and [38].

In recent years, many researchers have focused on modeling and analyzing various problems in life sciences and biological phenomena such as viruses, nervous system, etc. With the help of these modeling, scientists can investigate and predict behavior phenomena separately in an equipped laboratory.

In [9], a new mathematical model in a generalized fractional framework is proposed to investigate the transmission dynamics of HIV/AIDS. In [8] a new and efficient fractional model is explored for the investigation of COVID-19 dynamics.

In [3], the authors provide the historical information about Ebola to primary care nurses to inform future treatment options and algorithm development. The results of a randomized clinical trial of investigational therapeutics for Ebola demonstrated survival benefits from two monoclonal antibody products targeting the Ebola membrane glycoprotein [17]. For instance, the colorized scanning electron micrograph of Ebola virus particles (green) found both as extracellular particles and budding particles from a chronically infected African green monkey kidney cell (blue). For further information see special reference [45].

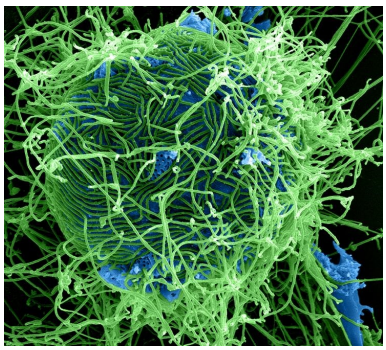

Figure 1. Ebola virus [45].

The Ebola virus (Figure 1) was first identified in 1976. This year, Ebola infected about 300 people and 9 out of 10 people died due to its spread. With the re-emergence of this virus in 1995, more than 80% of the infected died. In 2014, the release of this virus started from Guinea and spread to neighboring countries, including Liberia, and about 400 people lost their lives. Between 2013 and 2016, and during the Ebola virus epidemic in West Africa, about 22,000 suspected cases were identified. In this epidemic, at least 11,000 people were also killed by this disease [26]. The Ebola virus causes severe fevers in humans and other mammals, which itself leads to dangerous and usually fatal bleeding [21]. Researchers in this field believed that the natural and main source of Ebola viruses are bats and especially fruit bats. Bats infect other animals, especially monkeys. Monkeys also transmit

this virus to humans through direct contact with the blood, body fluids and tissues of infected animals. Then, the Ebola virus spreads between humans through direct contact with the body fluids of the infected person, and through the mouth, nose, and eyes [32].

After the Ebola epidemic in Liberia in 2014, a mathematical model of the spread of this virus was presented with the help of data simulation provided by the WHO [33]. After that, by presenting a fractional model for the spread of this virus, a study and comparison was made between the classical and fractional Ebola outbreak model [4]. In [6], considering classical and fractional models, and with the help of the Jacobian method, they numerically solved these models by an iterative method. For a better understanding of this model, the existence and uniqueness of the solution in the fractional model was presented in [19]. In the following, the effect of various factors on the efficiency of the proposed models was investigated, and numerical methods were also presented to solve this fractional system [12, 14, 16, 18].

Ebola is a rare but deadly human virus that causes symptoms inside and outside the body. As the virus spreads through the body, it damages the immune system and organs. There is no cure for Ebola, although researchers are working on one. Therefore, investigating this virus can be a very important issue for researchers. However, considering the actual cases reported in the world, it seems that more research is needed in this area. It was shown in [9] that numerical methods are effective for understanding epidemiological patterns as well as for optimal control of epidemic models, and scientific computing is useful in medicine and has led to disease control.

Inspired by the above discussion, in this article we intend to propose a numerical method to solve the fractional order model of Ebola virus. The suggested numerical scheme is based on properties of Bernstein's operational matrices as well as dual Bernstein for the system of nonlinear equations of Ebola virus in the Caputo fractional sense. Numerical analysis is performed using a real case registered by WHO [4].

For this purpose, consider the fractional model as follows:

$$\begin{aligned}
&D^v S_1(t) = -\alpha S_1(t)S_2(t) + \beta S_3(t) - \gamma N, && S_1(0) = \widehat{s_0},\\
&D^v S_2(t) = \alpha S_1(t)S_2(t) - \zeta S_2(t) - \delta S_2(t), && S_2(0) = \widehat{i_0},\\
&D^v S_3(t) = \delta S_2(t) - \beta S_3(t), && S_3(0) = \widehat{r_0},\\
&D^v S_4(t) = \zeta S_2(t) + \gamma N, && S_4(0) = \widehat{d_0},
\end{aligned} \tag{1.1}$$

where D^v is the Caputo fractional operator of order $v \in (0, 1]$. The general fractional derivative is considered in the Caputo sense in order to be coincidence with the initial conditions. On the other hand, it helps us to find out the main characteristics of the dynamics of the studied disease. $S_1(t)$ represents people susceptible to this disease, $S_2(t)$ represents infected people, $S_3(t)$ represents recovered people, and finally $S_4(t)$ represents people who died due to this disease. This equation has been investigated and solved in different ways. In [44], the authors presented the spectral collocation method using Chebyshev polynomial to solve this equation and also investigated the existence and uniqueness of the solution. In [11], the authors solved this model numerically by using the Sinc Legendre collocation method. In [43], the authors present three different and new models of the fractional order Ebola virus propagation process and solve them numerically. However, in many numerical works, no comparison has been made between the presented numerical method and

real data. Therefore, the accuracy of the methods is not comparable. In this article, we will present a numerical scheme for Eq. (1.1) using dual Bernstein operational matrices. The main contributions of this study are highlighted as below:

• Obtaining the derivative of a fractional value in the form of a matrix, which makes calculations easy.

• The studied problem becomes a system of nonlinear algebraic equations, which is solved by Newton's method.

• A reasonable approximate solution can be found with a small number of bases.

• The existence and uniqueness of the answer have been investigated.

• The stability of the method is also presented.

• The simulation results are compared with a real case of Ebola virus epidemic in Liberia in 2014.

• Numerical simulations are reported for various fractional orders. By using these comparisons between the simulated and measured data, we find the best value of the fractional order.

• Using real-world data for the Ebola virus, we compared the fractional Mellin transform, real data, Caputo's derivative, and the classical model.

2. Definitions and properties

This section is dedicated to stating the basic definitions and some of their features. First, the essential definitions of fractional calculus is briefly reviewed and we introduce the fractional Riemann-Liouville integral, the Riemann-Liouville fractional derivative and the Caputo fractional derivative. Then Bernstein polynomials and dual Bernstein polynomials are defined. For more information about fractional calculus and its application see [2, 13, 15, 25, 27, 28, 30, 34, 37, 39, 42].

Definition 2.1. A real function $S(t)$, $t > 0$, is said to be in the space $C_v, v \in \mathbb{R}$, if there exists a real number $p\,(> v)$, such that $S(t) = t^p S^\star(t)$, where $S^\star \in C[0, \infty)$, and it is said to be in the space $C_v^m, m \in \mathbb{N} \bigcup \{0\}$, if and only if $S^{(m)}(t) \in C_v$.

Definition 2.2. [31] Let $t, v \in (0, \infty)$, the fractional Riemann-Liouville integral of order v is defined as:

$$ {}_R I_t^v S(t) = (I_t^v S)(t) = \frac{1}{\Gamma(v)} \int_0^t (t-u)^{v-1} S(u) du. \tag{2.1} $$

Eq. (2.1) can be written in the following form:

$$ {}_R I_t^v S(t) = \int_0^t S(u) dg_t(u), \quad \text{where} \quad g_t(u) = \frac{1}{\Gamma(v+1)} (t^v - (t-u)^v). \tag{2.2} $$

As mentioned in [31] for describing the geometric interpretation of fractional Riemann-Liouville integration, in the plane (u, g) the function $g_t(u)$, $0 \le u \le t$ is plotted. Also, a fence of the varying height $f(u)$ is built, where the top edge of the fence is a three-dimensional line $(u, g_t(u), f(u))$, $0 \le u \le t$. A Matlab program for calculating and analyzing the snapshots of the changing shadow of changing fence, the fence and its shadows and the process of change of the fence basis shape for $S(t) = \frac{1}{2} sin(t) + t$ and $v = 0.75$ are given in Figures 2-4 respectively.

```
clc
clear
close all
v=0.75;
S = @(t)(t+0.5*sin(t));
g = @(u,t) 1/gamma(v+1)*(t.^v-(t-u).^v);
t = 10;
u = 0:0.5:t;
G = g(u,t);
F=S(u);
stem3(u,G,F,'.r'); hold on
stem3(u,G*0,F,'.b');
stem3(u*0,G,F,'.g');
plot3(u,G,F,'r')
plot3(u,G*0,F,'b');
plot3(u*0,G,F,'g');
xlabel('t,u');
ylabel('g_t(u)');
zlabel('S(u)');
```

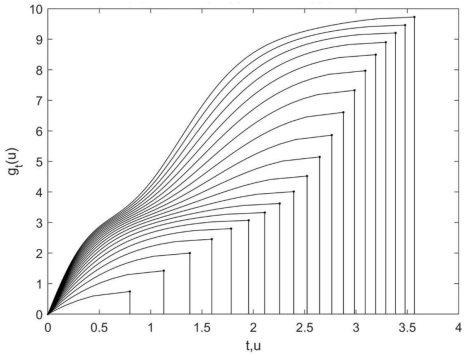

Figure 2. Snapshots of the changing shadow of changing fence for I_t^v, $v = 0.75$, $S(t) = \frac{1}{2}\sin(t) + t$, $\Delta t = 0.5$.

```
clc
clear
close all
v=0.75;
g = @(u,t) 1/gamma(v+1)*(t^v-(t-u).^v);
t=0:0.5:10;
for i=1:length(t)
    u= 0:0.1:t(i);
    G = g(u,t(i));
    plot3(u,G,u*0,'k');
    hold on
end
xlabel('t,u');
ylabel('g_t(u)');
```

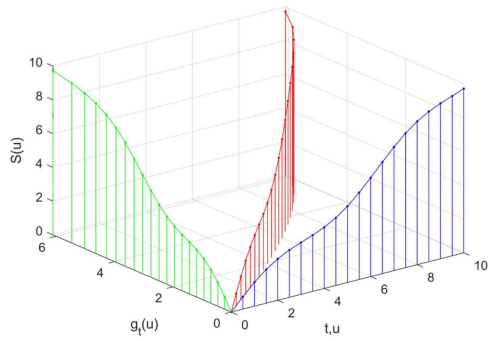

Figure 3. The fence and its shadows I_t^1 and I_t^v, $v = 0.75$, $S(t) = \frac{1}{2}\sin(t) + t$.

```
clc
clear
close all
warning off
v=0.5;
f = @(t)(t+0.5*sin(t));
g = @(u,t) 1/gamma(v+1)*(t^v-(t-u).^v);
for t=0:0.5:10
    u = 0:0.1:t;
    G = g(u,t);
    F = f(u);
    stem(G(end),F(end),'.k');hold on;
    plot(G,F,'k');
end
xlabel('t,u');
ylabel('g_t(u)');
zlabel('S(t)');
```

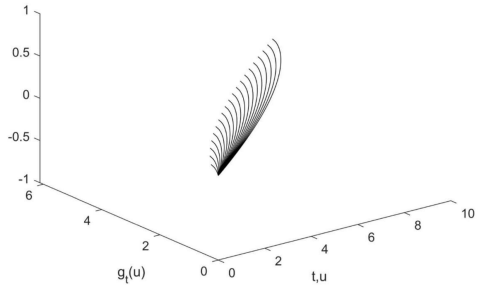

Figure 4. The process of change of the fence basis shape for I_t^v, $v = 0.75$.

Definition 2.3. [31] The Riemann-Liouville fractional derivative of order $n-1 \leqslant v < n$ is defined as:

$$ {}_R D_t^v S(t) = \frac{d^n}{dt^n} I_t^{n-v} S(t) = \frac{1}{\Gamma(n-v)} \frac{d^n}{dt^n} \int_0^t \frac{S(u)}{(t-u)^{v-n+1}} du. \tag{2.3} $$

Definition 2.4. [31] The Caputo fractional derivative of order $v > 0$ is given by:

$$ {}_C D_t^v S(t) = D^v S(t) = \begin{cases} \dfrac{1}{\Gamma(n-v)} \displaystyle\int_0^t \frac{S^{(n)}(u)}{(t-u)^{v-1+n}} du, & n-1 < v < n, \quad n \in \mathbb{N}, \\ \dfrac{d^n}{dt^n} S(t), & v = n \in \mathbb{N}. \end{cases} \tag{2.4} $$

2.1. Bernstein polynomials (BPs)

Definition 2.5. [10] The Bernstein basis polynomials of degree r in the interval $[0,1]$ are defined as:

$$ \varphi_i^r(u) = \binom{r}{i} u^i (1-u)^{r-i} = \sum_{h=i}^{r} (-1)^{h-i} \binom{r}{i} \binom{r-i}{h-i} u^h, \quad i = 0, 1, 2, \ldots, r, \tag{2.5} $$

where $\binom{r}{i} = \dfrac{r!}{i!(r-i)!}$.

Definition 2.6. [40] The dual to the Bernstein basis of degree r on $[0,1]$ is defined as:

$$ \psi_k^r(u) = \sum_{i=0}^{r} \lambda_{hi} \varphi_i^r(u), \quad k = 0, 1, 2, \ldots, r, \tag{2.6} $$

where λ_{hi} defined as:

$$ \lambda_{hi} = \frac{(-1)^{h+i}}{\binom{r}{h}\binom{r}{i}} \sum_{k=0}^{\min(h,i)} (2k+1) \binom{r+k+1}{r-h} \binom{r-h}{r-k} \binom{r+k+1}{r-i} \binom{r-k}{r-i}, $$
$$ h, i = 0, 1, 2, \ldots, r. \tag{2.7} $$

Proposition 2.1. *Since* (2.6) *is the dual of polynomial* (2.5)*, we will have the following property:*

$$ \int_0^1 \varphi_h^r(u) \psi_k^r(u) du = \begin{cases} 1, & \text{if } h = k, \\ 0, & \text{if } h \neq k. \end{cases} \tag{2.8} $$

For more details about properties of dual BP see [41].

3. Approximation of functions

Let us consider the set of BP of n-th degree:

$$ \phi(t) = [\varphi_0^n(t), \varphi_1^n(t), ..., \varphi_n^n(t)]^T \subset L^2[0,1], \tag{3.1} $$

and assume that

$$W = span\{\varphi_0^n(t), \varphi_1^n(t), ..., \varphi_n^n(t)\}.$$

Considering g as an arbitrary function in the space $L^2[0,1]$ and W as a vector space with finite dimension, g has the best approximation out of W such as $g_n \in W$ [20]. In other hands,

$$\forall\ w \in W,\ \ \| g - g_n \|_2 \ \leq\ \| g - w \|_2 \ . \tag{3.2}$$

Since $g_n \in W$, there exist the unique vector $C = [c_0, c_1, \cdots, c_n]^T$ so that

$$g(v) \simeq g_n(t) = \sum_{p=0}^{n} c_p \varphi_p^n(t) = C^T \phi(t). \tag{3.3}$$

The coefficients c_p in Eq. (3.3) are determined as:

$$c_p = \langle g, \psi_p^n \rangle = \int_0^1 g(t)\psi_p^n(t)dt. \tag{3.4}$$

Lemma 3.1. [41] *Suppose that $S(t)$ is a continuous and bounded function such that $| S(t) | \leqslant \xi$. If function $S(t)$ is approximated with Bernstein functions, coefficients s_j can be bounded as:*

$$| s_j | \leq \xi \sum_{i=0}^{m} \lambda_{j,i} \binom{m}{i} 2^{m-i}. \tag{3.5}$$

Theorem 3.1. *Let $\phi(t)$ be BPs vector. So:*

$$D^v \phi(t) \simeq D^{(v)} \phi(t), \tag{3.6}$$

where $D^{(v)}_{(n+1)\times(n+1)}$ is the operational matrix of fractional derivative of order v in the following form

$$D^{(v)} = \begin{pmatrix} \sum\limits_{j=\lceil v \rceil}^{n} w_{0,j,0} & \sum\limits_{j=\lceil v \rceil}^{n} w_{0,j,1} & \cdots & \sum\limits_{j=\lceil v \rceil}^{n} w_{0.j,n} \\ \vdots & \vdots & \cdots & \vdots \\ \sum\limits_{j=\lceil v \rceil}^{n} w_{i,j,0} & \sum\limits_{j=\lceil v \rceil}^{n} w_{i,j,1} & \cdots & \sum\limits_{j=\lceil v \rceil}^{n} w_{i.j,n} \\ \vdots & \vdots & \cdots & \vdots \\ \sum\limits_{j=\lceil v \rceil}^{n} w_{n,j,0} & \sum\limits_{j=\lceil v \rceil}^{n} w_{n,j,1} & \cdots & \sum\limits_{j=\lceil v \rceil}^{n} w_{n,j,n} \end{pmatrix}, \tag{3.7}$$

where $w_{h,j,l}$ is given by:

$$w_{h,j,l} = (-1)^{j-h} \binom{n}{h} \binom{n-h}{j-h} \frac{\Gamma(j+1)}{\Gamma(j-v+1)} \sum_{k=0}^{n} \lambda_{lk}\mu_{kj}, \tag{3.8}$$

and

$$\mu_{rj} = \sum_{s=r}^{n} (-1)^{s-r} \binom{n}{r} \binom{s-r}{n-r} \frac{1}{j-\beta+s+1}.$$

Proof. Let $\phi(t)$ be the Bernstein vector defined in Eq. (3.1) and suppose that $v > 0$. Then, by using Eqs. (2.5) and the Caputo's fractional differentiation we have:

$$\begin{aligned} D^v\varphi_{p,m}(t) &= \sum_{j=p}^{m}(-1)^{j-p}\binom{m}{p}\binom{m-p}{j-p}D^v(t^j) \\ &= \sum_{j=\lceil v\rceil}^{m}(-1)^{j-p}\binom{m}{p}\binom{m-p}{j-p}\frac{\Gamma(j+1)}{\Gamma(j-v+1)}t^{j-v},\ p=0,1,...,m. \end{aligned} \tag{3.9}$$

Approximating t^{j-v} by means of the BP, leads to:

$$t^{j-v} \simeq \sum_{l=0}^{m} u_{lj}\varphi_{l,m}(t). \tag{3.10}$$

By using Eq. (3.4) we have:

$$\begin{aligned} u_{lj} &= \int_0^1 t^{j-v}\psi_{l,m}(t)dv \\ &= \sum_{k=0}^{m}\lambda_{lk}\int_0^1 t^{j-v}\varphi_{m,k}(t)dt \\ &= \sum_{k=0}^{m}\lambda_{lk}\sum_{s=k}^{m}(-1)^{s-k}\binom{m}{k}\binom{m-k}{s-k}\int_0^1 t^{j-v+s}dt \\ &= \sum_{k=0}^{m}\lambda_{lk}\sum_{s=k}^{m}(-1)^{s-k}\binom{m}{k}\binom{m-k}{s-k}\frac{1}{j-v+s+1} \\ &:= \sum_{k=0}^{m}\lambda_{lk}\mu_{kj}, \end{aligned}$$

where λ_{lk} is given in Eq. (2.7) and

$$\mu_{kj} = \sum_{s=k}^{m}(-1)^{s-k}\binom{m}{k}\binom{s-k}{m-k}\frac{1}{j-v+s+1}.$$

Therefore:

$$\begin{aligned} D^v\varphi_{p,m}(t) &\simeq \sum_{j=\lceil v\rceil}^{m}\sum_{l=0}^{m}(-1)^{j-p}\binom{m}{p}\binom{m-p}{j-p}\frac{\Gamma(j+1)}{\Gamma(j-v+1)}u_{lj}\varphi_{l,m}(t) \\ &= \sum_{l=0}^{m}\left(\sum_{j=\lceil v\rceil}^{m} w_{p,j,l}\right)\varphi_{l,m}(t), \end{aligned} \tag{3.11}$$

where $w_{p,j,l}$ is given by:

$$w_{p.j,l} = (-1)^{j-p}\binom{m}{p}\binom{m-p}{j-p}\frac{\Gamma(j+1)}{\Gamma(j-v+1)}\sum_{k=0}^{m}\lambda_{lk}\mu_{kj}. \tag{3.12}$$

Let us rewrite Eq. (3.11) in the vector form:

$$D^v\varphi_{p,m}(t) \simeq \big[\sum_{j=\lceil v\rceil}^{m} w_{p,j,0}, \sum_{j=\lceil v\rceil}^{m} w_{p,j,1}, ..., \sum_{j=\lceil v\rceil}^{m} w_{p,j,m}\big]\phi(t), p = 0, 1, ..., m. \quad (3.13)$$

Therefore:

$$D^v\phi(t) \simeq D^{(v)}\phi(t). \quad (3.14)$$

□

In the follow up using the operational matrix method based on BPs, Eq. (1.1) is analyzed. Therefore by using Eq. (3.2) and Eq. (3.6) we have:

$$\begin{aligned}
S_1(t) &= \sum_{j=0}^{n} s_j\varphi_j^n(t) = S_1^T\phi(t), \qquad D^vS_1(t) = S_1^TD^{(v)}\phi(t),\\
S_2(t) &= \sum_{j=0}^{n} i_j\varphi_j^n(t) = S_2^T\phi(t), \qquad D^vS_2(t) = S_2^TD^{(v)}\phi(t),\\
S_3(t) &= \sum_{j=0}^{n} r_j\varphi_j^n(t) = S_3^T\phi(t), \qquad D^vS_3(t) = S_3^TD^{(v)}\phi(t),\\
S_4(t) &= \sum_{j=0}^{n} d_j\varphi_j^n(t) = S_4^T\phi(t), \qquad D^vS_4(t) = S_4^TD^{(v)}\phi(t).
\end{aligned} \quad (3.15)$$

By inserting Eq. (3.15) in Eq. (1.1), one will set:

$$\begin{aligned}
S_1^TD^{(v)}\phi(t) &= -\alpha\left(S_1^T\phi(t)\right)\left(S_2^T\phi(t)\right) + \beta\left(S_3^T\phi(t)\right) - \gamma N,\\
S_2^TD^{(v)}\phi(t) &= \alpha\left(S_1^T\phi(t)\right)\left(S_2^T\phi(t)\right) - \zeta\left(S_2^T\phi(t)\right) - \delta\left(S_2^T\phi(t)\right),\\
S_3^TD^{(v)}\phi(t) &= \delta\left(S_2^T\phi(t)\right) - \beta\left(S_3^T\phi(t)\right),\\
S_4^TD^{(v)}\phi(t) &= \zeta\left(S_2^T\phi(t)\right) + \delta N.
\end{aligned} \quad (3.16)$$

On the other hand, considering that $\phi(0) = [1, 0, 0, \ldots, 0]$, the initial values will be as follows:

$$\begin{aligned}
S_1(0) = \widehat{s_0} &\longrightarrow \sum_{j=0}^{n} s_j\varphi_j^n(0) = \widehat{s_0} \longrightarrow s_0 = \widehat{s_0},\\
S_2(0) = \widehat{i_0} &\longrightarrow \sum_{j=0}^{n} i_j\varphi_j^n(0) = \widehat{i_0} \longrightarrow i_0 = \widehat{i_0},\\
S_3(0) = \widehat{r_0} &\longrightarrow \sum_{j=0}^{n} r_j\varphi_j^n(0) = \widehat{r_0} \longrightarrow r_0 = \widehat{r_0},\\
S_4(0) = \widehat{d_0} &\longrightarrow \sum_{j=0}^{n} d_j\varphi_j^n(0) = \widehat{d_0} \longrightarrow d_0 = \widehat{d_0}.
\end{aligned} \quad (3.17)$$

To calculate unknown coefficients in nonlinear system (3.16) with boundary conditions (3.17), we use the suitable collocation points. Therefore, Eq. (1.1) becomes a nonlinear system with $4(n + 1)$ equations. Hence, this system is solved by using the Newton-Raphson method.

4. Existence and uniqueness of solution

In the follow up, we show that the solution to Eq. (1.1) exists and is unique. For this purpose, first consider the following nomenclature

$$Z_i(t, S_i) := D^v S_i(t),\ i = 1, 2, 3, 4, \tag{4.1}$$

where

$$\begin{aligned} Z_1(t, S_1) &= -\alpha S_1(t)S_2(t) + \beta S_3(t) - \gamma N, \\ Z_2(t, S_2) &= \alpha S_1(t)S_2(t) - \zeta S_2(t) - \delta S_2(t), \\ Z_3(t, S_3) &= \delta S_2(t) - \beta S_3(t), \\ Z_4(t, S_4) &= \zeta S_2(t) + \gamma N. \end{aligned} \tag{4.2}$$

Theorem 4.1. *Assuming that functions $S_i(t)$ for $i = 1, 2, 3, 4$ are bounded ($\exists r_i : \|S_i(t)\| \leq r_i$), then the functions $Z_i(t, S_i)$ for $i = 1, 2, 3, 4$ applies to the Lipshitz condition.*

Proof. According to Eq. (4.2) for $Z_1(t, S_1)$, we have

$$\begin{aligned} & Z_1(t, S_1) - Z_1(t, S_1^*) \\ =& (-\alpha S_1(t)S_2(t) + \beta S_3(t) - \gamma N) - (-\alpha S_1^*(t)S_2(t) + \beta S_3(t) - \gamma N) \\ =& -\alpha S_1(t)S_2(t) + \alpha S_1^*(t)S_2(t) \\ =& -\alpha S_2(t)\left(S_1(t) - S_1^*(t)\right). \end{aligned} \tag{4.3}$$

Therefore:

$$\begin{aligned} \|Z_1(t, S_1) - Z_1(t, S_1^*)\| &\leq \|\alpha S_2(t)\|\|S_1(t) - S_1^*(t)\| \\ &\leq \alpha r_2 \|S_1(t) - S_1^*(t)\| \\ &:= R_1 \|S_1(t) - S_1^*(t)\|. \end{aligned} \tag{4.4}$$

Definition 4.1. A function S, defined on $[a, b]$, is said to satisfy a Lipschitz condition on $[a, b]$ ([20]) if there exists a constant $L > 0$ such that

$$|S(t_1) - S(t_2)| \leq L\,|t_1 - t_2|\,, \tag{4.5}$$

for all $t_1, t_2 \in [a, b]$ and L is called the Lipschitz constant.

With a similar process for other $Z_i(t, S_i),\ i = 1, 2, 3, 4$ we will have

$$\|Z_i(t, S_i) - Z_i(t, S_i^*)\| \leq R_i \|S_i(t) - S_i^*(t)\|,\ \text{for } i = 1, 2, 3, 4, \tag{4.6}$$

where $R_1 = \alpha r_2$, $R_2 = \alpha r_1 + \zeta + \delta$, $R_3 = \beta$ and $R_4 = 0$, which indicates that the Lipschitz condition for $Z_i(t, S_i)$ are satisfied. □

Lemma 4.1. [7] *If $D^v g(t) = f(t)$ and $g(0) = g_0$, then*

$$g(t) = g_0 + \frac{1}{\Gamma(v)} \int_0^t (t - \eta)^{v-1} f(\eta) d\eta. \tag{4.7}$$

Theorem 4.2. *Assume that $Z_i(t, S_i)$ for $i = 1, 2, 3, 4$ apply in the Lipschitz conditions. In this case, there is a unique solution for Eq. (1.1).*

Proof. By using Lemma (4.1), the model (1.1) can be written as:

$$\begin{aligned}
S_1(t) - \widehat{s_0} &= \frac{1}{\Gamma(v)} \int_0^t (t-\eta)^{v-1} Z_1(\eta, S_1) d\eta, \\
S_2(t) - \widehat{i_0} &= \frac{1}{\Gamma(v)} \int_0^t (t-\eta)^{v-1} Z_2(\eta, S_2) d\eta, \\
S_3(t) - \widehat{r_0} &= \frac{1}{\Gamma(v)} \int_0^t (t-\eta)^{v-1} Z_3(\eta, S_3) d\eta, \\
S_4(t) - \widehat{d_0} &= \frac{1}{\Gamma(v)} \int_0^t (t-\eta)^{v-1} Z_4(\eta, S_4) d\eta.
\end{aligned} \tag{4.8}$$

The recurrence form and the initial conditions of the model (4.8) are given as:

$$\begin{aligned}
S_1^n(t) &= \frac{1}{\Gamma(v)} \int_0^t (t-\eta)^{v-1} Z_1(\eta, S_1^{n-1}) d\eta, \quad S_1(0) = \widehat{s_0}, \\
S_2^n(t) &= \frac{1}{\Gamma(v)} \int_0^t (t-\eta)^{v-1} Z_2(\eta, S_2^{n-1}) d\eta, \quad S_2(t) = \widehat{i_0}, \\
S_3^n(t) &= \frac{1}{\Gamma(v)} \int_0^t (t-\eta)^{v-1} Z_3(\eta, S_3^{n-1}) d\eta, \quad S_3(t) = \widehat{r_0}, \\
S_4^n(t) &= \frac{1}{\Gamma(v)} \int_0^t (t-\eta)^{v-1} Z_4(\eta, S_4^{n-1}) d\eta, \quad S_4(t) = \widehat{d_0}.
\end{aligned} \tag{4.9}$$

In this case, the difference between two consecutive sentences will be as follows

$$\begin{aligned}
\psi_i^n(t) :=& S_i^n(t) - S_i^{n-1}(t) = \frac{1}{\Gamma(v)} \int_0^t (t-\eta)^{v-1} (Z_i(\eta, S_i^{n-1}) - Z_i(\eta, S_i^{n-2})) d\eta, \\
& i = 1, 2, 3, 4.
\end{aligned} \tag{4.10}$$

In this case, it is clear that:

$$S_i^n(t) = \sum_{j=1}^{n} \psi_i^j(t), \ i = 1, 2, 3, 4. \tag{4.11}$$

From Eq. (4.6) for $\psi_1^n(t)$

$$\begin{aligned}
\|\psi_1^n(t)\| &= \|S_1^n(t) - S_1^{n-1}(t)\| \\
&\leq \frac{1}{\Gamma(v)} \int_0^t (t-\eta)^{v-1} \|Z_1(\eta, S_1^{n-1}) - Z_1(\eta, S_1^{n-2})\| d\eta \\
&\leq \frac{R_1}{\Gamma(v)} \int_0^t (t-\eta)^{v-1} \|S_1^{n-1} - S_1^{n-2}\| d\eta.
\end{aligned} \tag{4.12}$$

With a similar process for other $\psi_i^n(t)$ for $i = 1, 2, 3, 4$ we will have

$$\|\psi_i^n(t)\| \leq \frac{R_i}{\Gamma(v)} \int_0^t (t-\eta)^{v-1} \|\psi_i^{n-1}(t)\| d\eta, \ i = 1, 2, 3, 4. \tag{4.13}$$

By continuing this process for $\psi_i^{n-1}(t)$, we get

$$\|\psi_i^n(t)\| \leq \left(\frac{R_i t^v}{\Gamma(v+1)} \right)^n, \ i = 1, 2, 3, 4. \tag{4.14}$$

In order to show that Eqs. (4.14) are solutions for Eq. (1.1), we will have the following assumption

$$S_i(t) - \widehat{s_0} = S_i^n(t) - \Upsilon_i^n(t), \ i = 1, 2, 3, 4. \tag{4.15}$$

Consider the following conditions

$$\|\Upsilon_i^n(t)\| \leq \|\frac{1}{\Gamma(v)} \int_0^t (t-\eta)^{v-1} \left(Z_i(\eta, S_i) - Z_i(\eta, S_i^{n-1}) \right) d\eta\|, \ i = 1, 2, 3, 4. \tag{4.16}$$

Therefore

$$\|\Upsilon_i^n(t)\| \leq \left(\frac{R_i t^v}{\Gamma(v+1)} \right)^n, \ i = 1, 2, 3, 4. \tag{4.17}$$

This shows that when $n \longrightarrow \infty$, then $\|\Upsilon_i^n(t)\| \longrightarrow 0$ for $i = 1, 2, 3, 4$. Now, to show the uniqueness of the solution, we assume that there are S_1^*, S_2^*, S_3^* and S_4^* other solutions of the Eq. (1.1). Therefore

$$\begin{aligned} &S_1(t) = S_1^n(t) + \widehat{s_0}, \\ &S_1^*(t) = S_1^m(t) + \widehat{s_0}, \\ &\|S_1(t) - S_1^*(t)\| = \|S_1^n(t) - S_1^m(t)\| \leq \left(\frac{R_1 t^v}{\Gamma(v+1)} \right)^l, \end{aligned} \tag{4.18}$$

where $l = \max\{m, n\}$. This shows that when $l \longrightarrow \infty$, then $\|S_1(t) - S_1^*(t)\| \longrightarrow 0$. With a similar process we have

$$S_i(t) - S_i^*(t) = 0, \ i = 1, 2, 3, 4. \tag{4.19}$$

□

5. Stability

In this section, the stability for the model (1.1) is investigated. For this purpose, we use the Ulam-Hyers stability. So first consider the following definition:

Definition 5.1. [1, 35, 36] Suppose that $\Lambda = E \times E \times E \times E$ is a Banach space where $E = C[0, T]$. In this case, we call model (1.1) Ulam-Hyers stable whenever if there exists $\lambda > 0$ and $\varepsilon > 0$ for each $\widehat{S}_i(t)$, $i = 1, 2, 3, 4$ with the following inequalities

$$|D^v \widehat{S}_i(t) - Z_i(t, \widehat{S}_i)| \leq \varepsilon_i, \tag{5.1}$$

then there exists $S_i(t)$ satisfying the Eqs. (1.1) with the following initial conditions

$$\begin{aligned} S_1(0) &= \widehat{s_0}, \\ S_2(0) &= \widehat{i_0}, \\ S_3(0) &= \widehat{r_0}, \\ S_4(0) &= \widehat{d_0}, \end{aligned} \tag{5.2}$$

such that

$$\|(\widehat{S_1}, \widehat{S_2}, \widehat{S_3}, \widehat{S_4}) - (S_1, S_2, S_3, S_4)\|_\Lambda \leq \lambda\varepsilon. \tag{5.3}$$

Lemma 5.1. *The solution of the perturbed problem*

$$D^v \widehat{S}(t) = w_1(t, \widehat{S}) + f_1(t), \ \widehat{S}(0) = \widehat{s_0}, \tag{5.4}$$

implies $|\widehat{S_{f_1}}(t) - \widehat{S}(t)| \leq k\varepsilon_1$, *where* $\widehat{S_{f_1}}(t)$ *is a solution of (5.4) and* $k = \dfrac{T^v}{\Gamma(v+1)}$.

Proof. Using Lemma (4.1), the proof is straightforward. □

Theorem 5.1. *Under the presumptions of (4.8) and conditions (4.6), the Eqs. (1.1) are Ulam-Hyers stable in* Λ.

Proof. Let $S \in E$ be a unique solution of Eq. (1.1) and $\widehat{S} \in E$ be the solution of the inequality (5.1) with the condition

$$S(0) = \widehat{S}(0), \tag{5.5}$$

that is

$$S(t) = S_0 + \frac{1}{\Gamma(v)} \int_0^t (t-\eta)^{v-1} Z_1(\eta, S) d\eta. \tag{5.6}$$

From Eq. (5.5) one will set

$$S(t) = \widehat{S}_0 + \frac{1}{\Gamma(v)} \int_0^t (t-\eta)^{v-1} Z_1(\eta, S) d\eta. \tag{5.7}$$

Therefore

$$\begin{aligned} |\widehat{S}(t) - S(t)| &\leq |\widehat{S}(t) - \widehat{S_{f_1}}(t)| + |\widehat{S_{f_1}}(t) - S(t)| \\ &\leq k\varepsilon_1 + \frac{1}{\Gamma(v)} \int_0^t (t-\eta)^{v-1} |Z_1(\eta, \widehat{S}) - Z_1(\eta, S)| d\eta \\ &\leq 2k\varepsilon_1 + \frac{T^v}{\Gamma(v+1)} R_1 \|\widehat{S} - S\|. \end{aligned} \tag{5.8}$$

In other words

$$\|\widehat{S} - S\|_E \leq \frac{2k\varepsilon_1}{1-\chi_1}, \text{ where } \chi_1 = \frac{T^v}{\Gamma(v+1)} R_1. \tag{5.9}$$

Now, for $\lambda_1 = \dfrac{2k}{1-\chi_1}$ we have

$$\|\widehat{S} - S\|_E \leq \lambda_1 \varepsilon_1. \tag{5.10}$$

By continuing this process, we will have

$$\|\widehat{S}_i - S_i\|_E \leq \lambda_i \varepsilon_i, \ i = 1, 2, 3, 4, \tag{5.11}$$

where $\lambda_j = \dfrac{2k}{1-\chi_j}$, and $\chi_j = \dfrac{T^v}{\Gamma(v+1)} R_j$ for $j = 1, 2, 3, 4$. Thus, Eq. (1.1) is Ulam-Hyers stable. □

6. Numerical results

Hereunder, the numerical treatment of the proposed method is investigated. For this purpose, we have considered the data of the WHO, following the spread and epidemic of the Ebola virus in West Africa and especially in Liberia in 2014. Some of these data are given in Table 1 (see [4]). In Figure 5, a comparison between the

Table 1. Confirmed cases of Ebola virus

Date	Total	Date	Total
2014/03/27	15	2014/07/12	706
2014/03/31	24	2014/07/20	786
2014/04/05	54	2014/07/30	953
2014/04/09	66	2014/08/11	1176
2014/04/14	71	2014/08/22	1516
2014/04/20	112	2014/09/05	2364
2014/04/26	121	2014/09/16	2997
2014/05/01	127	2014/09/26	3606
2014/05/07	129	2014/10/08	4440
2014/05/23	152	2014/10/17	5159
2014/06/01	217	2014/10/29	7606
2014/06/05	249	2014/11/07	8142
2014/06/16	364	2014/11/19	9397
2014/06/20	441	2014/11/16	10018
2014/07/02	557	2014/12/01	10553

actual data in 200 days and the proposed method in [4] is presented. In the left figure, the parameter values are $m = 90$, $q = 0.058$ and $v = 0.9$, and in the right figure, $m = 85$, $q = 0.09$ and $v = 0.9$. In Figure 6, a comparison is made between the proposed method for number of infected people ($S_2(t)$) and real data in a period of 250 days. In the left figure, the parameter values are $m = 4$ and $v = 0.9$, and in the right figure, $m = 5$ and $v = 0.9$. Other parameters are as follows: $\alpha = 0.001$, $\beta = 0.02$, $\gamma = 0.01$, $\zeta = 0.06$ and $\delta = 0$.

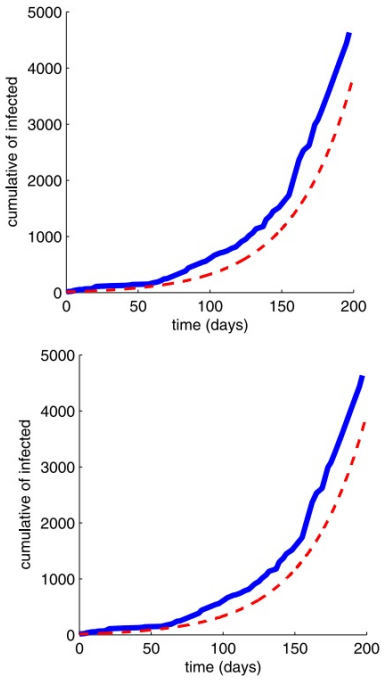

Figure 5. Comparison of real data and approximate method for number of infected people ($S_2(t)$) in [4].

The operational matrix $D^{(v)}$ of order $v = 0.9$ and $m = 4$ and $m = 5$ are as follows:

$$D_4^{(v)} = \begin{pmatrix} -2.7537 & -2.2577 & 0.4754 & -0.5195 & -0.0719 \\ 2.7968 & -0.0562 & -2.7096 & 0.2678 & -0.1333 \\ -0.0482 & 2.3639 & 0.6374 & -2.5723 & -0.1445 \\ 0.0057 & -0.0539 & 1.6181 & 2.0332 & -3.1734 \\ -0.0005 & 0.0039 & -0.0212 & 0.7908 & 3.5232 \end{pmatrix}, \tag{6.1}$$

$$D_5^{(v)} = \begin{pmatrix} -3.3140 & -2.9023 & 1.2528 & -1.2649 & 0.2365 & -0.1914 \\ 3.3618 & -0.0746 & -3.7504 & 1.1083 & -0.5311 & 0.0369 \\ -0.0542 & 3.0480 & 0.1544 & -2.7875 & -0.0816 & -0.0845 \\ 0.0074 & -0.0803 & 2.3952 & 1.3737 & -3.2282 & -0.1900 \\ -0.0011 & 0.0100 & -0.0558 & 1.5881 & 2.8315 & -3.8672 \\ 0.0001 & -0.0008 & 0.0038 & -0.0178 & 0.7728 & 4.2963 \end{pmatrix}. \tag{6.2}$$

As it can be seen from the comparison of Figures 5 and 6, our proposed method has more appropriate accuracy for approximating data and can approximate data with less error, which shows the efficiency of the proposed method. Figure 6 shows

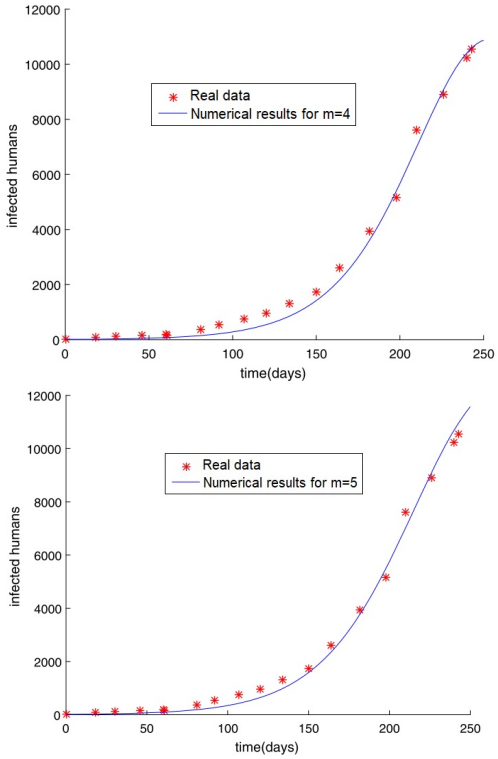

Figure 6. Comparison of real data and suggested method for number of infected people ($S_2(t)$).

that it is possible to have a reasonable approximate solution with a small number of bases, which will reduce calculations and increase the speed of program execution. Also, from the examination of Figure 6, we can see that by increasing the value of m, the accuracy of the method and on the other hand its approximation will be better. In Figure 7, a comparison has been made between the solution of the classic

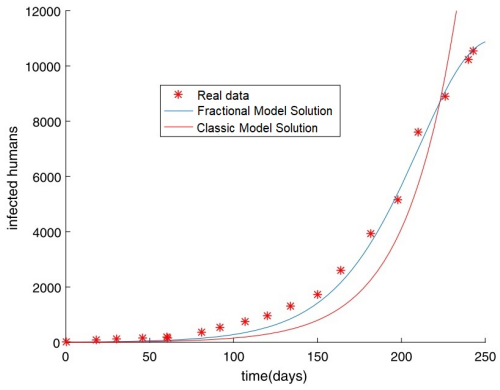

Figure 7. Comparison of real data and solution of the classic model and the solution of the fractional model.

model and the solution of the fractional model for the real data in Table 1. As it is

clear from the figure, the fractional model shows a much better and more suitable approximation for the data, which shows the importance of examining and modeling problems in fractional form. In Figure 8, numerical results are shown for different

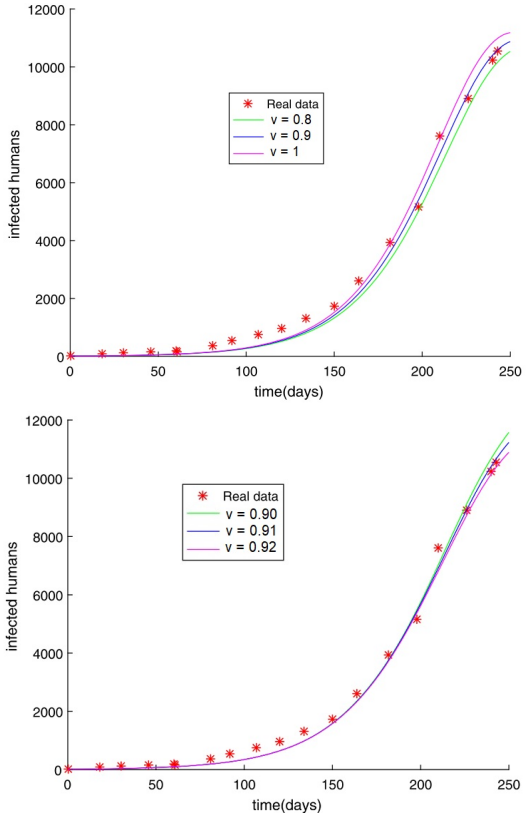

Figure 8. Comparison of real data and suggested method for number of infected people ($S_2(t)$) for $m = 4$ and different values of v.

values of fractional derivative v with $m = 4$. From examining the left figure, it is clear that the optimal result for $v = 0.9$ has been obtained, so approximations of $v = 0.9$ are presented in the right figure.

6.1. A comparison between Caputo fractional derivatives and fractional Mellin transform

Definition 6.1. Let $f(x)$ be locally Lebesgue integrable over $[0, \infty)$. The Mellin transform of $f(x)$ is defined by

$$M[f(x); s] = f^*(s) = \int_0^\infty x^{s-1} f(x) dx, \ s = \eta_1 + i\mu_1, \ \eta_1, \mu_1 \in \mathbb{R}, \ i^2 = -1. \quad (6.3)$$

The largest open strip (a, b) in which the integral converges is called the fundamental strip, and its ainverse transform is

$$f(x) = M^{-1}[f^*(s); x] = \frac{1}{2\pi i}\int_{\eta_1 - i\infty}^{\eta_1 + i\infty} f^*(s)x^{-s}ds,\ x > 0,\ \eta_1 = Re(s). \tag{6.4}$$

For more details, the reader is advised to consult the research works presented in [5, 22, 29].

Theorem 6.1. *Let $f(x)$ be Mellin transformable function on $\mathbb{R}^+$, and f be a fractional derivative function for all $n-1 < v < n,\ n \in \mathbb{N}$, then*

$$M[f^v(x); s] = \frac{\Gamma(s)}{\Gamma(s-v)}M[f(x); s-v]. \tag{6.5}$$

Corollary 6.1. *Let $s = \eta_1 + i\mu_1$, $z = \eta_2 + i\mu_2$ and assume that $Re(s) + Re(z) < 0$ where*

$$f(t) = H(t - t_0)t^z, \tag{6.6}$$

and H is the Heaviside step function. Then we have

$$\int_0^\infty t^{s-1}f(t)dt = \frac{-t_0^{z+s}}{z+s}. \tag{6.7}$$

Proof. By applying the properties of Mellin transform and Heaviside step function, one will set

$$\int_0^\infty t^{s-1}f(t)dt = \int_0^\infty t^{s-1}t^z dt = \int_{t_0}^\infty t^{z+s-1}dt = \frac{t^{z+s}}{z+s}\Big|_{t_0}^\infty. \tag{6.8}$$

Assume know that

$$\frac{t^{z+s}}{z+s} = \frac{t^{(\eta_1+\eta_2)+i(\mu_1+\mu_2)}}{z+s} = \frac{t^{(\eta_1+\eta_2)}t^{i(\mu_1+\mu_2)}}{z+s}, \tag{6.9}$$

as we know, $\eta_1 + \eta_2 < 0$, then, when t tends to infinity, $t^{\eta_1+\eta_2} \to 0$ which implies that:

Eq. (6.9) be equal to zero, which completes the proof. □

In Fig. 9 and using a real-world data for the Ebola virus, we compared the fractional Mellin transform, real data, Caputo derivative and classic model. According to our research, the ordinary derivative has less accuracy than the fractional order model. In other words, results demonstrated that fractional order derivations are superior than classical orders, more dependable, and more effective in describing biological processes. The numerical simulation of the findings shows that the Caputo derivative in comparing with some other applicable methods (for instance; fractional Mellin transform) gives a more precise numerical results. Finally, we point out that this idea should be generalized and verified for more complicated linear and nonlinear problems. In other words, the present note is only an introduction to the topic, and there remains a lot of issues to do.

7. Concluding remarks

Recently, the modeling of problems and their analysis have attracted the attention of many scientists in different fields. In biological sciences and biological phenomena such as viruses, the need for modeling is felt more to recognize and predict the

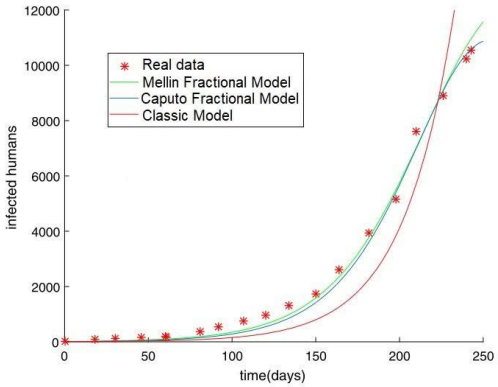

Figure 9. Comparison of fractional Mellin transform, real data, Caputo derivative and classic model for presented example in Section 6.

function of viruses. One of these cases can be seen in the modeling of the Ebola virus. The model presented for the Ebola virus and even many models presented for different problems do not have an analytical solution. Therefore, it is necessary to evaluate them with different numerical methods in order to make a correct prediction of the performance of the virus. In this paper, we presented an effective method based on properties of Bernstein operational matrices as well as dual Bernstein for the system of fractional order Ebola virus nonlinear equations. Also, for the simplicity of work and calculations, the operational matrix of fractional derivative of order v was obtained based on dual Bernstein. The proposed dual Bernstein method leads to the transformation of the Ebola virus into a system of nonlinear algebraic equations. The unknown coefficients of the final system are obtained using the Newton-Raphson method. One of the features of this method is that a reasonable approximate solution can be found with a small number of bases. Then, the data provided by the WHO were used to obtain numerical results. Numerical simulations are reported for various fractional orders. By using these comparisons between the simulated and measured data, we find the best value of the fractional order. These comparisons, additionally, indicated that the fractional order model follows the reality more precisely than the classical framework, a fact which justifies the use of fractional calculus modeling in our case under study. Finally, we compared the fractional Mellin transform, real data, Caputo's derivative, and the classical model. According to our research, the ordinary derivative is less accurate than the fractional order model. In other words, the results showed that fractional order derivatives are superior to classical orders, more reliable and effective in describing biological processes. Finally, it should be mentioned that, the results can be extended to other scientific areas involving the treatment of COVID-19, Zombie models and SARS and etc. Furthermore, the future works will focus on the similarities, differences and numerical treatments of fractional models of Ebola virus, Zombie models, SARS and coronavirus. Also, the application of this method for solving the stochastic fractional equations as well as solving the multi dimensional differential-integral stochastic equations can be investigated. It is also possible to evaluate the derivative of the fractional Mellin transform or other fractional derivatives in these problems.

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Declarations conflict of interest

The authors declare that there is no conflict of interest regarding the publication of this article.

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