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SHARP DECAY ESTIMATES FOR SMALL DATA SOLUTIONS TO THE MAGNETIZED VLASOV-POISSON SYSTEM AND MAGNETIZED VLASOV-YUKAWA SYSTEM*

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Abstract In this article, we present sharp decay estimates for small data solutions to the magnetized Vlasov-Poisson system and the magnetized Vlasov-Yukawa system in dimension three. Our arguments are based on the modification of the vector field method developed by Smulevici [42] for the Vlasov-Poisson system and improved by Duan [17] for the Vlasov-Poisson system and the Vlasov-Yukawa system. We extend the results in [17] to the magnetized case and slightly improve the decay estimates, our method improve the result in [48], in which a similar result was obtained but the norms considered have additional v-weighted L^p-norms. In our work, we do not need the extra L^p-norms.

Keywords Small data solution, magnetized Vlasov-Poisson system, magnetized Vlasov-Yukawa system, vector field method, decay estimate.

MSC(2010) 35B40, 35Q49, 35Q83, 82D10.

1. Introduction and main results

1.1. Introduction

The nonlinear interaction between particles and force field is one of the important subjects in particle physics and kinetic theory of many-body interacting particle systems. In a collisionless plasma, charged particles are affected by an electric field, the motion of particles can be governed by the following classical Vlasov-Poisson (VP) system (see [24])

$$\begin{cases} \partial_t f + v \cdot \nabla_x f + \mu \nabla_x \phi \cdot \nabla_v f = 0, \\ \triangle_x \phi = \rho(f), \quad \lim\limits_{|x|\to\infty} \phi(x) = 0, \\ f(0,x,v) = f_0(x,v). \end{cases} \tag{1.1}$$

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Here $f = f(t, x, v)$ is the density of particles at the time $t \in \mathbb{R}$ and at the phase space point $(x, v) \in \mathbb{R}^3 \times \mathbb{R}^3$, $\mu = \pm 1$ corresponds to an attractive or repulsive force and $\rho(f) = \rho(t, x) = \int_{\mathbb{R}^3} f(t, x, v) dv$ is the spatial density of charges. $\phi(t, x)$ is electric potential. $f_0(x, v) \geq 0$ is the initial phase-space density of particles which is supposed to be given.

When the Poisson equation $\triangle_x \phi = \rho(f)$ in (1.1) is replaced by the screened Poisson equation $\triangle_x \phi - m^2 \phi = \rho(f)$, we get the classical Vlasov-Yukawa (VY) system (see [7])

$$\begin{cases} \partial_t f + v \cdot \nabla_x f + \mu \nabla_x \phi \cdot \nabla_v f = 0, \\ \triangle_x \phi - m^2 \phi = \rho(f), \quad \lim\limits_{|x| \to \infty} \phi(x) = 0, \\ f(0, x, v) = f_0(x, v). \end{cases} \tag{1.2}$$

Here $m > 0$ is the mass of particles which is assumed to be a positive constant.

If we further assume that an external magnetic field $B(t, x)$ is applied to these particles. Then the magnetized Vlasov-Poisson (magnetized VP) system can be written as (see [25])

$$\begin{cases} \partial_t f + v \cdot \nabla_x f + \mu (\nabla_x \phi + v \times B) \cdot \nabla_v f = 0, \\ \triangle_x \phi = \rho(f), \quad \lim\limits_{|x| \to \infty} \phi(x) = 0, \\ f(0, x, v) = f_0(x, v). \end{cases} \tag{1.3}$$

Similarly, the magnetized Vlasov-Yukawa (magnetized VY) system can be written as(see [11])

$$\begin{cases} \partial_t f + v \cdot \nabla_x f + \mu (\nabla_x \phi + v \times B) \cdot \nabla_v f = 0, \\ \triangle_x \phi - m^2 \phi = \rho(f), \quad \lim\limits_{|x| \to \infty} \phi(x) = 0, \\ f(0, x, v) = f_0(x, v). \end{cases} \tag{1.4}$$

Without loss of generality, we will assume $m = 1$ throughout this paper. The electric potential $\phi(t, x)$ which solves either $\triangle_x \phi = \rho(f)$ or $\triangle_x \phi - \phi = \rho(f)$ can be expressed explicitly in the form

$$\phi(t, x) = G_m * \rho(f) = \int_{\mathbb{R}^3} G_m(x - y) \rho(f)(t, y) dy, \tag{1.5}$$

where G_m is the Green function with respect to the operator $\triangle$ ($m = 0$) or $\triangle - I$ ($m = 1$). More precisely, for the magnetized Vlasov-Poisson system (1.3), G_0 has the form

$$G_0(x) = \frac{-1}{4\pi |x|}. \tag{1.6}$$

For the magnetized Vlasov-Yukawa system (1.4), G_1 has the form

$$G_1(x) = -\left(\frac{1}{2\pi}\right)^{\frac{3}{2}} \frac{1}{|x|^{\frac{1}{2}}} K_{\frac{1}{2}}(|x|). \tag{1.7}$$

$K_{\frac{1}{2}}(r)$ is the modified Bessel function of the second kind, which has an integral expression of the form (see [45, p181]),

$$K_{\frac{1}{2}}(r) = \int_0^\infty e^{-r\cosh\lambda}\cosh(\frac{1}{2}\lambda)d\lambda, \quad r > 0. \tag{1.8}$$

Based on the arguments in [43], we can easily obtain the local classical solutions to the systems (1.3) and (1.4) when $B(t, x)$ verifies assumptions of this paper (see, e.g. [48]). The purpose of this paper is to derive the global-in-time solutions and their sharp decay estimates under a small data assumption.

The Vlasov-Poisson system is a classical system from plasma physics and we refer to [24] for an introduction. Global existence and uniqueness in three dimensions are known for large data thanks to the work of Pfaffelmoser [36] and Lions etc [35]. For small data, the first result was obtained by Bardos etc in [1], which was then strengthened first in [31] and then [42] where sharp decay estimates for derivatives were obtained. We also mention that there are many works on the Vlasov-Poisson system when initial data are spherical symmetric, cylindrical symmetric or almost spherical symmetric, see e.g.: [2, 29, 30, 37, 46, 47]. The Vlasov-Yukawa system was first studied by Caprino etc [13] as an approximation of the Vlasov-Poisson system for two charged particles. For small data, the global existence of classical solutions in three dimensions can be established by employing Bardos etc's framework [1] which uses the dispersive character of the free transport equation. And [28] presents an existence theory and uniform L^1 stability estimate for classical solutions to the relativistic Vlasov-Yukawa system, [14] presents dispersion estimates for the two dimensional Vlasov-Yukawa system.

As for the magnetized Vlasov-Poisson system, an important mathematical model in tokamak plasmas, there exists an abundant literature for strong magnetic fields, where the aim is to derive asymptotic models [10, 16, 22, 23, 25, 26] and devise numerical methods that capture this asymptotic behavior [15, 21]. The magnetized Vlasov-Poisson system has also been studied in a half-space, in a torus and in an infinite cylinder. In particular, for sufficiently small initial datum with compact support, Skubachevskii [39, 40] studied the existence and uniqueness of the classical solutions with different boundary conditions in a half space for an external magnetic field of large strength. If the initial distribution has compact support lying at some distance from the boundary with respect to x and has compact support with respect to v, then [41] concerned with the solvability with Dirichlet boundary condition in an infinite cylinder, and [12] gave the existence and uniqueness of solutions in a torus with an unbounded external magnetic field. Recently, by the vector field method [48] obtains the global existence and sharp decay estimates of the magnetized Vlasov-Poisson system when the initial datum is sufficient small and the magnetic field satisfies certain regularity conditions. For the magnetized Vlasov-Yukawa system, [11] considers the infinite mass problem in an infinite cylinder and proves the global existence and uniqueness of the classical solutions. The purpose of this paper is to improve the results in [48] and obtain the global existence and sharp decay estimates of small data solution to the Cauchy problem of magnetized Vlasov-Yukawa system by vector field method. As for recent development of vector field method, we refer the readers to [3–9, 17–20, 32, 34, 42, 44].

1.2. Main results

For any natural number N, we define the energy function $E_N(f)$ as

$$E_N(f) := \sum_{|\alpha|\le N, Y^\alpha \in \gamma_m^{|\alpha|}} \| Y^\alpha f \|_{L^1(\mathbb{R}^3_x\times\mathbb{R}^3_v)},$$

where γ_m is the corresponding set of modified vector fields as defined in Definition 2.3. Our main result is the follwing.

Theorem 1.1. *Let $N \ge 9$. Assume that the initial datum $f_0(x,v) \in C^N(\mathbb{R}^3 \times \mathbb{R}^3)$ has compact support in v and the magnetic field $B(t,x) \in C_b^N([0,\infty) \times \mathbb{R}^3)$. And assume there exists a sufficiently small constant $\varepsilon > 0$ such that*

$$E_N[f_0] := \sum_{|\alpha|\le N, Y^\alpha \in \gamma_m^{|\alpha|}} \|Y^\alpha f_0\|_{L^1(\mathbb{R}^3_x\times\mathbb{R}^3_v)} \le \varepsilon \tag{1.9}$$

and

$$|Z^\alpha(B(t,x))| \lesssim \frac{\varepsilon}{(1+t)^3}, \quad |\alpha| \le N, \ \forall\, t \ge 0, \tag{1.10}$$

where $Z \in \gamma$ and γ is the corresponding set of vector fields as defined in Definition 2.1. Then the solutions $f(t,x,v)$ to the magnetized VP system (1.3) *and the magnetized VY system* (1.4) *exist globally in time and satisfy the following estimates:*

1. *Global bounds*

$$E_N[f(t)] \le 2\varepsilon. \tag{1.11}$$

2. *For any multi-index α with $|\alpha| \le N-3$, we have*

$$|\rho(Y^\alpha f)(t,x)| \lesssim \frac{\varepsilon}{(1+t+|x|)^3}, \tag{1.12}$$

$$|\rho(\partial_x^\alpha f)(t,x)| \lesssim \frac{\varepsilon}{(1+t+|x|)^{3+|\alpha|}}. \tag{1.13}$$

3. *For any multi-index α with $|\alpha| \le N-4$, we have*

$$|\nabla Z^\alpha \phi(t,x)| \lesssim \frac{\varepsilon}{(1+t)(1+t+|x|)}, \tag{1.14}$$

$$|\partial_x^\alpha \nabla \phi(t,x)| \lesssim \frac{\varepsilon}{(1+t)(1+t+|x|)^{1+|\alpha|}} \tag{1.15}$$

for magnetized VP system and

$$|\nabla_x Z^\alpha \phi(t,x)| \lesssim \frac{\varepsilon}{(1+t+|x|)^3}, \tag{1.16}$$

$$|\partial_x^\alpha \nabla \phi(t,x)| \lesssim \frac{\varepsilon}{(1+t+|x|)^{3+|\alpha|}} \tag{1.17}$$

for magnetized VY system.

Remark 1.1. When the initial data $f_0(x,v) \in C_c^1(\mathbb{R}^3 \times \mathbb{R}^3)$ and the magnetic field $B(t,x) \in C([0,\infty); C_b^1(\mathbb{R}^3;\mathbb{R}^3))$, global existence of the magnetized VP system is already guaranteed from the works [33, 38], global existence of the magnetized VY system can also be obtained by similar methods, so the main points of the theorem, apart from providing an illustration of our method, are the propagation of the global bounds and the decay estimates for the solutions.

Remark 1.2. If we take $B(t,x) = 0$ in Theorem 1.1, then our results improve the decay estimates in [17], see (1.14)-(1.17), where we proved that $|\nabla Z^\alpha \phi(t,x)|$ as well as $|\partial_x^\alpha \nabla \phi(t,x)|$ decays not only with respect to t, but also with respect to x.

Remark 1.3. Theorem 1.1 improve the result in [48], where a similar result was obtained for the magnetized VP system, but the norms considered had additional v-weighted L^p-norms. More precisely, in [48], the energy function is defined for any $\delta > 0$ by

$$E_N[f] := \sum_{|\alpha|\le N, Y^\alpha \in \gamma_m^{|\alpha|}} \|Y^\alpha f\|_{L^1(\mathbb{R}_x^3\times\mathbb{R}_v^3)} + \sum_{|\alpha|\le N, Y^\alpha \in \gamma_m^{|\alpha|}} \| \, (1+v^2)^{\frac{\delta(\delta+3)}{2(1+\delta)}} Y^\alpha f \, \|_{L^{1+\delta}(\mathbb{R}_x^3\times\mathbb{R}_v^3)} \, .$$

It not only needs the boundedness of the L^1 norms of the modified vector fields $Y^\alpha f$, but also requires an additional integrability in some weighed L^p norms of the modified vector fields $Y^\alpha f$. In our work, we don't need the extra v-weighted L^p-norms.

Remark 1.4. Our method can also apply to magnetized VY system in dimension two.

2. Preliminary

2.1. Notation

In this paper, we denote by T the free transport operator i.e.

$$T(f) := \partial_t f + v \cdot \nabla_x f.$$

For any sufficiently regular potential ϕ, we denote by T_ϕ the perturbed transport operator i.e.

$$T_\phi(f) = \partial_t f + v \cdot \nabla_x f + \mu \nabla_x \phi \cdot \nabla_v f.$$

Similarly, for any sufficiently regular potential ϕ and magnetic field B, we denote by $T_{\phi,B}$ the perturbed transport operator

$$T_{\phi,B}(f) = \partial_t f + v \cdot \nabla_x f + \mu(\nabla_x \phi + v \times B) \cdot \nabla_v f.$$

Throughout this paper, we use the Lie bracket $[A,B] := AB - BA$ to denote the commutation of two operators. We shall use the notation $f \lesssim g$ as a short-hand for $f \leqslant Cg$, where constant C depends only on the maximum number of commutations.

2.2. Vector fields

For the free transport operator T, there exist vector fields that commute with T, the simplest examples are the translations ∂_t, ∂_{x^i}. In this paper, we will consider the following vector fields as that in [42].

Definition 2.1. We denote by γ the set of all the vector fields, i.e.

$$\gamma := \Big\{ \partial_t,\ \partial_{x^i},\ t\partial_{x^i} + \partial_{v^i},\ \Omega_{ij}^x + \Omega_{ij}^v,\ S^x + S^v,\ t\partial_t + \sum_{i=1}^3 x^i \partial_{x^i} \quad 1 \le i,j \le 3 \Big\},$$

where $S^x = \sum_{i=1}^{3} x^i \partial_{x^i}$, $S^v = \sum_{i=1}^{3} v^i \partial_{v^i}$, $\Omega^x_{ij} = x^i \partial_{x^j} - x^j \partial_{x^i}$, $\Omega^v_{ij} = v^i \partial_{v^j} - v^j \partial_{v^i}$.

Throughout the paper we make the convention that $\Omega^x_{ii} = 0$ and $\Omega^x_{ij} = -\Omega^x_{ji}$, the same is also for v-derivatives. For the sake of simplicity, we denote by Z a generic commuting vector field in γ. For any multi-index $\alpha = (\alpha_1, ..., \alpha_k)$ with $k = |\alpha|$, the operator $Z^\alpha \in \gamma^{|\alpha|}$ is defined by $Z^\alpha = Z^{\alpha_1} Z^{\alpha_2} \cdots Z^{\alpha_k}$.

Definition 2.2. We denote by Γ the macroscopic vector fields corresponding to each element in γ i.e.

$$\Gamma := \left\{ \partial_t,\ \partial_{x^i},\ t\partial_{x^i},\ \Omega^x_{ij},\ S^x,\ t\partial_t + \sum_{i=1}^{3} x^i \partial_{x^i} \quad 1 \le i, j \le 3 \right\}.$$

For functions that only depend on (t, x), such as ϕ, $Z(\phi)$ can be understood as the action of the corresponding macroscopic vector field on ϕ.

First, we recall some important properties of the vector fields.

Lemma 2.1. ([17, 42], Commutation with Δ, ρ) *For any $Z^\alpha \in \gamma^{|\alpha|}$, we have*

$$[Z^\alpha, \Delta] = \sum_{|\beta| \le |\alpha| - 1} c_{\alpha,\beta} Z^\beta \Delta,$$

$$Z^\alpha \rho(f) = \rho(Z^\alpha f) + \sum_{|\beta| \le |\alpha| - 1} c'_{\alpha,\beta} \rho(Z^\beta f).$$

Where $c_{\alpha,\beta}, c'_{\alpha,\beta}$ are globally bounded constants that only depend on $|\alpha|$.

Lemma 2.2. ([17, 42], Commutation within γ) *For any $Z^\alpha \in \gamma^{|\alpha|}$,$Z^{\alpha'} \in \gamma^{|\alpha'|}$ we have*

$$[Z^\alpha, Z^{\alpha'}] = \sum_{|\beta| \le |\alpha| + |\alpha'| - 1} c^{\alpha,\alpha'}_{\beta} Z^\beta.$$

Moreover, if $Z^{\alpha'} = \partial_{x^i}$, we have

$$[Z^\alpha, \partial_{x^i}] = \sum_{j=1}^{n} \sum_{|\beta| \le |\alpha| - 1} c^{\alpha,i}_{\beta,j} \partial_{x^j} Z^\beta.$$

Where $c^{\alpha,\alpha'}_{\beta}, c^{\alpha,i}_{\beta,j}$ are globally bounded constants that depend only on $\max\{|\alpha|, |\alpha'|\}$.

Next, we recall the following conclusions for the Poisson equation $\triangle_x \phi = \rho(f)$ and the screened Poisson equation $\triangle_x \phi - \phi = \rho(f)$.

Lemma 2.3 (Lemma 2.8, [17])**.** *Suppose f is sufficiently regular, ϕ solves the equation $\triangle_x \phi = \rho(f)$, then for any multi-index α and $Z^\alpha \in \gamma^{|\alpha|}$, we have*

$$\triangle_x Z^\alpha \phi = \sum_{|\beta| \le |\alpha|} c^\alpha_\beta \rho(Z^\beta f),$$

where c^α_β are globally bounded constants.

Lemma 2.4 (Lemma 2.9, [17]). *Suppose f is sufficiently regular, ϕ solves the equation $\triangle_x\phi - \phi = \rho(f)$, then for any multi-index α and $Z^\alpha \in \gamma^{|\alpha|}$, we have*

$$\triangle_x Z^\alpha\phi - Z^\alpha\phi = \sum_{k=0}^{|\alpha|}\sum_{|\beta|\le|\alpha|-k} c^\alpha_{k,\beta} G_1 *^{(k)} \rho(Z^\beta f),$$

where $c^\alpha_{k,\beta}$ are globally bounded constants, $G_1 *^{(k)}$ represents the k times convolution of G_1.

2.3. The modified vector fields

In this paper, we will apply the modified vector field method which developed by Smulevici [42] to prove our main results. We first give the definition of the modified vector fields, about the motivation of introducing the modified vector field one can refer to [48] for detailed explanation. Similar to [48], we define the modified vector fields as follows

Definition 2.3. We denote by γ_m the set of the modified vector fields, i.e.,

$$\gamma_m := \Big\{\partial_t,\ \partial_{x^i},\ t\partial_{x^i} + \partial_{v^i} - \sum_{j=1}^3 \Phi^j_i\partial_{x^j},\ \Omega^x_{ij} + \Omega^v_{ij} - \sum_{m=1}^3 \omega^m_{ij}\partial_{x^m},$$
$$S^x + S^v - \sum_{j=1}^3 \xi^j\partial_{x^j},\ t\partial_t + \sum_{i=1}^3 x^i\partial_{x^i} - \sum_{j=1}^3 \tau^j\partial_{x^j},\ \ 1\le i,j\le 3\Big\}.$$

Here the functions Φ^j_i, ω^m_{ij}, ξ^j and τ^j are solutions of the following equations with zero initial data:

$$T_{\phi,B}(\Phi^j_i) = \mu t\partial_{x^j}(t\partial_{x^i}\phi), \tag{2.1}$$
$$T_{\phi,B}(\omega^m_{ij}) = \mu t\partial_{x^m}(\Omega^x_{ij}\phi), \tag{2.2}$$
$$T_{\phi,B}(\xi^j) = \mu t\partial_{x^j}(S^x\phi - 2\phi), \tag{2.3}$$
$$T_{\phi,B}(\tau^j) = \mu t\partial_{x^j}\big((t\partial_t + S^x)\phi\big). \tag{2.4}$$

Here we denote by $\mathcal{M}$ the set of all functions $\Phi^j_i, \omega^m_{ij}, \xi^j$ and τ^j. And we denote by φ a generic function in $\mathcal{M}$. For simplicity of presentation, under the above definitions, we rewritten (2.1)-(2.4) as

$$T_{\phi,B}(\varphi) = \mu t\sum_{i=1}^3\sum_{|\alpha|\le 1} c_{Z,i}\partial_{x^i}Z^\alpha(\phi), \tag{2.5}$$

where $c_{Z,i}$ are some globally bounded constants.

Throughout the paper, we denote by Y a generic modified vector field in γ_m. For any multi-index $\alpha = (\alpha_1, \cdots, \alpha_1)$ with $k = |\alpha|$, the operator $Y^\alpha \in \gamma_m^{|\alpha|}$ is defined by $Y^\alpha = Y^{\alpha^1}Y^{\alpha^2}\cdots Y^{\alpha^k}$. We say that $P(\varphi)$ is a multi-linear form of degree d with signature less than k if $P(\varphi)$ has the following form

$$P(\varphi) = \sum_{\substack{|\alpha^1|+\cdots+|\alpha^d|\le k\\ (\varphi_1,\cdots,\varphi_d)\in\mathcal{M}^d}} C_{\overline{\alpha},\overline{\varphi}}\prod_{j=1}^d Y^{\alpha_j}(\varphi_j),$$

where α^j are all multi-indices and $C_{\overline{\alpha},\overline{\varphi}}$ are constants with $\overline{\alpha} = (\alpha^1, \cdots, \alpha^d)$ and $\overline{\varphi} = (\varphi_1, \cdots, \varphi_d)$.

2.4. Properties of modified vector fields

In this subsection, we will study the main properties of the modified vector field that will be used later.

Lemma 2.5. *(Higher order commutation formula with $T_{\phi,B}$) For any multi-index α, we have*

$$[T_{\phi,B}, Y^\alpha]f = \sum_{d=0}^{|\alpha|+1} \sum_{i=1}^{3} \sum_{\substack{|\gamma|\le|\alpha| \\ |\beta|\le|\alpha|}} P^{\alpha,i}_{d\gamma\beta}(\varphi)[\partial_{x^i} Z^\gamma \phi + Z^\gamma(v\times B) + tZ^\gamma(v\times B)]Y^\beta f.$$

Where $P^{\alpha,i}_{d\gamma\beta}(\varphi)$ are multilinear forms of degree d with signatures less than k such that $k \le |\alpha| - 1$ and $k + |\gamma| + |\beta| \le |\alpha| + 1$.

Proof. Let us first look at the case when $|\alpha| = 1$, the proof is straightforward and is based on the previous equalities. Let us illustrate it with an example, if $Y^i = t\partial_{x^i} + \partial_{v^i} - \sum_{j=1}^{3} \Phi^j_i \partial_{x^j}$, Let us first commute $T_{\phi,B}$ with $Z^i = t\partial_{x^i} + \partial_{v^i}$, we can easily calculate that

$$[T_{\phi,B}, Z^i](f) = -\mu \sum_{j=1}^{3} \partial_{x^j}(Z^i\phi)\partial_{v^j} f - \mu \sum_{j=1}^{3} \big(Z^i(v\times B)^j\big)\partial_{v^j} f.$$

By rewriting ∂_{v^j} as

$$\partial_{v^j} = (t\partial_{x^j} + \partial_{v^j}) - t(\partial_{x^j}),$$

then we have

$$\begin{aligned}
[T_{\phi,B}, Y^i](f) &= [T_{\phi,B}, Z^i](f) - [T_{\phi,B}, \sum_{j=1}^{3} \Phi^j_i \partial_{x^j}](f) \\
&= -\mu \sum_{j=1}^{3} \partial_{x^j}(Z^i\phi)(t\partial_{x^j} + \partial_{v^j})f + \mu \sum_{j=1}^{3} \partial_{x^j}(Z^i\phi)t(\partial_{x^j})f \\
&\quad - \mu \sum_{j=1}^{3} \big(Z^i(v\times B)^j\big)(t\partial_{x^j} + \partial_{v^j})f + \mu \sum_{j=1}^{3} \big(Z^i(v\times B)^j\big)t(\partial_{x^j})f \\
&\quad - \sum_{j=1}^{3} T_{\phi,B}(\Phi^j_i)\partial_{x^j} f + \mu \sum_{j=1}^{3} \Phi^j_i \nabla_x(\partial_{x^j}\phi)\cdot\nabla_v f \\
&\quad + \mu \sum_{j=1}^{3} \Phi^j_i \partial_{x^j}(v\times B)\cdot\nabla_v f.
\end{aligned}$$

By using the equation (2.1) in Definition 2.3, we can cancel the second term and the fifth term on the right-hand side in the above equality, then we have

$$\begin{aligned}
[T_{\phi,B}, Y^i](f) &= \mu \sum_{j=1}^{3} \partial_{x^j}(Z^i\phi)Z^j f \\
&\quad - \mu \sum_{j=1}^{3} \big(Z^i(v\times B)^j\big)Z^j f + \mu \sum_{j=1}^{3} \big(Z^i(v\times B)^j\big)t(\partial_{x^j})f
\end{aligned}$$

$$+\mu\sum_{j=1}^{3}\Phi_i^j\nabla_x(\partial_{x^j}\phi)\cdot\nabla_v f+\mu\sum_{j=1}^{3}\Phi_i^j\partial_{x^j}(v\times B)\cdot\nabla_v f.$$

Now, by just rewriting Z^j as

$$Z^j=Y^j+\sum_{l=1}^{3}\Phi_j^l\partial_{x^l},$$

we have

$$\begin{aligned}[T_{\phi,B},Y^i](f)=&-\mu\sum_{j=1}^{3}\partial_{x^j}(Z^i\phi)Y^jf-\mu\sum_{j,l=1}^{3}\partial_{x^j}(Z^i\phi)\Phi_j^l\partial_{x^l}f\\&-\mu\sum_{j=1}^{3}\big(Z^i(v\times B)^j\big)Y^jf-\mu\sum_{j,l=1}^{3}\big(Z^i(v\times B)^j\big)\Phi_j^l\partial_{x^l}f\\&+\mu\sum_{j=1}^{3}\big(Z^i(v\times B)^j\big)t(\partial_{x^j})f+\mu\sum_{j=1}^{3}\Phi_i^j\nabla_x(\partial_{x^j}\phi)\cdot\nabla_v f\\&+\mu\sum_{j=1}^{3}\Phi_i^j\partial_{x^j}(v\times B)\cdot\nabla_v f.\end{aligned}$$

The first five terms are already in the correct form, note that $\partial_{v^j}=Y^j+\sum_{l=1}^{3}\Phi_j^l\partial_{x^l}-t\partial_{x^j}$ and $\partial_{x^l},\partial_{x^j}\in\gamma_m$, it is easy to know that the sixth term and the seventh term on the right-hand side of the above equality are also in the correct form. Thus the lemma holds for $|\alpha|=1$.

Now assume that the lemma holds for some multi-index α, we want to establish the validity of the inequalities $[T_{\phi,B},Y^\alpha Y]f$. It is clear that

$$[T_{\phi,B},Y^\alpha Y]f=[T_{\phi,B},Y]Y^\alpha f+Y[T_{\phi,B},Y^\alpha]f=I_1f+I_2f,$$

where

$$\begin{aligned}I_1=&[T_{\phi,B},Y]Y^\alpha\\=&\sum_{d=0}^{2}\sum_{i=1}^{3}\sum_{|\gamma|\le1,|\beta|\le1}P_{d\gamma\beta}^{\alpha,i}(\varphi)[\partial_{x^i}Z^\gamma\phi+Z^\gamma(v\times B)+tZ^\gamma(v\times B)]Y^\beta Y^\alpha.\end{aligned}$$

Let $\alpha+\beta$ be a new β, which is of the desired result. Now we consider I_2.

$$\begin{aligned}I_2=&Y[T_{\phi,B},Y^\alpha]\\=&Y\Big(\sum_{d=0}^{|\alpha|+1}\sum_{i=1}^{3}\sum_{|\gamma|\le|\alpha|,|\beta|\le|\alpha|}P_{d\gamma\beta}^{\alpha,i}(\varphi)[\partial_{x^i}Z^\gamma\phi+Z^\gamma(v\times B)+tZ^\gamma(v\times B)]Y^\beta\Big),\end{aligned}$$

which generates three types of terms as described below.

$$\begin{aligned}&Y(P_{d\gamma\beta}^{\alpha,i}(\varphi))[\partial_{x^i}Z^\gamma\phi+Z^\gamma(v\times B)+tZ^\gamma(v\times B)]Y^\beta,\\&P_{d\gamma\beta}^{\alpha,i}(\varphi)Y[\partial_{x^i}Z^\gamma\phi+Z^\gamma(v\times B)+tZ^\gamma(v\times B)]Y^\beta,\end{aligned}$$

$$P^{\alpha,i}_{d\gamma\beta}(\varphi)[\partial_{x^i}Z^\gamma\phi + Z^\gamma(v\times B) + tZ^\gamma(v\times B)]YY^\beta.$$

For the first types of terms, the key point is that

$$Y[Y^{\alpha_1}(\varphi_1)\cdots Y^{\alpha_d}(\varphi_d)] = YY^{\alpha_1}(\varphi_1)\cdots Y^{\alpha_d}(\varphi_d) + \cdots + Y^{\alpha_1}(\varphi_1)\cdots YY^{\alpha_d}(\varphi_d).$$

Its signature increases by 1 at most. For the second types of terms, from $Y = Z + \varphi\partial_x$ we have

$$\begin{aligned}
&Y[\partial_{x^i}Z^\gamma\phi + Z^\gamma(v\times B) + tZ^\gamma(v\times B)]\\
=&(Z+\varphi\partial_x)[\partial_{x^i}Z^\gamma\phi + Z^\gamma(v\times B) + tZ^\gamma(v\times B)]\\
=&\partial_{x^i}ZZ^\gamma\phi + \sum_{j=1}^{n}\partial_{x^j}Z^\gamma\phi + ZZ^\gamma(v\times B) + tZZ^\gamma(v\times B)\\
&+\varphi\partial_x(\partial_{x^i}Z^\gamma\phi) + \varphi\partial_x Z^\gamma(v\times B) + t\varphi\partial_x Z^\gamma(v\times B).
\end{aligned}$$

Note that $\partial_{x^i}\in\gamma_m$, it is easy to know that the second types of terms is also in the correct form. For the last types of terms, it is easy to know that it is also in the correct form. And we complete the proof. □

Lemma 2.6 (Lemma 4.3, [17]). *For any multi-index α, we have*

$$Z^\alpha = \sum_{d=0}^{|\alpha|}\sum_{|\beta|\le|\alpha|} P^\alpha_{d\beta}(\varphi)Y^\beta.$$

Where $P^\alpha_{d\beta}(\varphi)$ are multilinear forms of degree d with signatures less than k such that $k\le|\alpha|-1$ and $k+|\beta|\le|\alpha|$.

Lemma 2.7 (Lemma 4.4, [17]). *Let $f(t,x,v)$ be a sufficiently regular function, then for any multi-index α, we have*

$$\rho(Z^\alpha f) = \sum_{d=0}^{|\alpha|}\sum_{|\beta|\le|\alpha|}\rho(Q^\alpha_{d\beta}(\partial_x\varphi)Y^\beta f) + \sum_{j=0}^{|\alpha|}\sum_{d=1}^{|\alpha|+1}\sum_{|\beta|\le|\alpha|}\frac{1}{(t+1)^j}\rho(P^{\alpha,j}_{d\beta}(\varphi)Y^\beta f).$$

Where $Q^\alpha_{d\beta}(\partial_x\varphi)$ are multilinear forms with respect to $\partial_x\varphi$ of signatures less than k' satisfying $k'\le|\alpha|-1$ and $k'+d+|\beta|\le|\alpha|$, $P^{\alpha,j}_{d\beta}(\varphi)$ are multilinear forms of degree d with signatures less than k satisfying $k\le|\alpha|$ and $k+|\beta|\le|\alpha|$.

Lemma 2.8 (Lemma 4.5, [17]). *We have*

$$Y^\alpha\nabla\phi = Z^\alpha\nabla\phi + \frac{1}{t}\sum_{d=1}^{|\alpha|}\sum_{|\beta|\le|\alpha|}P^\alpha_{d\beta}(\varphi)Z^\beta\nabla\phi.$$

Where $P^\alpha_{d\beta}(\varphi)$ are multilinear forms of degree d with signatures less than k satisfying $k\le|\alpha|-1$ and $k+|\beta|\le|\alpha|$.

3. Proof of the main results

In this section, we will prove our main results by using the modified vector fields method. We divide the proof of Theorem 1.1 into four steps.

Step 1. Bootstrap assumption. We consider the following bootstrap assumptions. Let $T \geq 0$ be the largest time so that, for all $t \in [0,T]$, we have

1.

$$E_N(f) \leq 2\varepsilon. \tag{3.1}$$

2. For any multi-index α with $|\alpha| \leq N-5$ and any $Y^\alpha \in \gamma_m^{|\alpha|}$, we have

$$|Y^\alpha \varphi(t,x,v)| \lesssim \varepsilon^{1/2}(1+\log(t+1)), \qquad \forall\ (x,v) \in \mathbb{R}^3_x \times \mathbb{R}^3_v. \tag{3.2}$$

3. For any multi-index α with $|\alpha| \leq N-6$ and any $Y^\alpha \in \gamma_m^{|\alpha|}$, we have

$$|Y^\alpha \nabla_x \varphi(t,x,v)| \lesssim \varepsilon^{1/2}, \qquad \forall\ (x,v) \in \mathbb{R}^3_x \times \mathbb{R}^3_v. \tag{3.3}$$

4. For any multi-index α with $|\alpha| \leq N-4$ and any $Z^\alpha \in \Gamma^{|\alpha|}$, we have

$$|\nabla_x Z^\alpha \phi(t,x)| \lesssim \frac{\varepsilon^{1/2}}{(t+1)^2}, \qquad \forall\ x \in \mathbb{R}^3_x. \tag{3.4}$$

Step 2. Klainerman-Sobolev inequality for the modified vector field. Based on the bootstrap assumptions, we can get the Klainerman-Sobolev inequality for the modified vector field. One can refer to Proposition 6.1 in [42] for the proof.

Proposition 3.1 (Proposition 6.1, [42])**.** *For any sufficiently regular function $f(x,v)$, we have*

$$\rho(|f|)(t,x) \lesssim \frac{1}{(1+t+|x|)^3} \sum_{|\alpha|\leq 3, Y^\alpha \in \gamma_m^{|\alpha|}} \| Y^\alpha f \|_{L^1(\mathbb{R}^3_x \times \mathbb{R}^3_v)} .$$

And for multi-index $|\alpha| \leq N-3$, we have

$$\begin{aligned}\rho(|Y^\alpha f(t)|) &\lesssim \frac{1}{(1+t+|x|)^3} \sum_{|\beta|\leq |\alpha|+3, Y^\beta \in \gamma_m^{|\beta|}} \| Y^\beta f(t) \|_{L^1(\mathbb{R}^3_x \times \mathbb{R}^3_v)} \\ &\lesssim \frac{1}{(1+t+|x|)^3} \sum_{|\beta|\leq N, Y^\beta \in \gamma_m^{|\beta|}} \| Y^\beta f(t) \|_{L^1(\mathbb{R}^3_x \times \mathbb{R}^3_v)} .\end{aligned}$$

Step 3. The estimates of $\| Y^\alpha(\varphi) Y^\beta(f) \|_{L^1(\mathbb{R}^3_x \times \mathbb{R}^3_v)}$. Let us first estimate the products of type $Y^\alpha(\varphi)Y^\beta(f)$, which is important to improve the bootstrap assumptions in next step. To get the bounds on $Y^\alpha(\varphi)Y^\beta(f)$, we will use the following lemmas.

Lemma 3.1. *Suppose $f(t,x,v)$ is a sufficiently regular function, then for any $t \in [0,T]$, we have*

$$\| f(t) \|_{L^1(\mathbb{R}^3_x \times \mathbb{R}^3_v)} \leq \| f(0) \|_{L^1(\mathbb{R}^3_x \times R^3_v)} + \int_0^t \| T_{\phi,B}(f)(s) \|_{L^1(\mathbb{R}^3_x \times \mathbb{R}^3_v)}\, ds.$$

Proof. From the definition of $T_{\phi,B}$, we have

$$T_{\phi,B}(|f|) = \partial_t |f| + \nabla_x (v|f|) + \mu \nabla_v ((\nabla_x \phi + v \times B)|f|).$$

First, we consider the case when $|f|$ is smooth and has compact support in (x, v) uniformly in $[0, T]$. Integrating the both side of above inequality in (t, x, v), We have

$$\int_0^t \int_x \int_v T_{\phi,B}(|f|) = \int_0^t \int_x \int_v \partial_t |f| + \nabla_x (v|f|) + \mu \nabla_v ((\nabla_x \phi + v \times B)|f|) \\ = \int_x \int_v |f(t)| - \int_x \int_v |f(0)|.$$

Therefore, we have

$$\| f(t) \|_{L^1(\mathbb{R}^3_x \times \mathbb{R}^3_v)} \leq \| f(0) \|_{L^1(\mathbb{R}^3_x \times \mathbb{R}^3_v)} + \int_0^t \| T_{\phi,B}(f)(s) \|_{L^1(\mathbb{R}^3_x \times \mathbb{R}^3_v)} \, ds.$$

For more general function f, we can approximate $|f|$ by functions $\sqrt{f^2 + \eta^2}\chi_\varepsilon(x, v)$, where $\eta > 0$ and $\chi_\varepsilon(x, v)$ is a smooth cut-off function which is 1 in $B_{x,v}(0, \varepsilon^{-1})$ and 0 out side of $B_{x,v}(0, 2\varepsilon^{-1})$. The proof is completed. □

Now we give a different proof of lemma 3.2 in [17], and our method slightly improves the conclusion, which is very important for this paper.

Lemma 3.2. *For any $n \geq 2$, there exist a constant $C > 0$, such that for any $x \in \mathbb{R}^n$*

$$\int_{\mathbb{R}^n} \frac{1}{|y|^{n-1}(1 + |x + y|)^n} dy \leq \frac{C_n}{(1 + |x|)^{n-2}}.$$

Proof. we divide $\mathbb{R}^n$ into two regions: $R_1 = \{|y| \geq 2|x|\}$ and $R_2 = \{|y| < 2|x|\}$. In the region R_1, we have $|x + y| \geq |y| - |x| \geq \frac{|y|}{2}$, then

$$\int_{|y| \geq 2|x|} \frac{1}{|y|^{n-1}(1 + |x + y|)^n} dy \leq \int_{|y| \geq 2|x|} \frac{1}{|y|^{n-1}(1 + \frac{|y|}{2})^n} dy \\ = \frac{n\alpha(n)}{n - 1} \frac{1}{(1 + |x|)^{n-1}} \\ \leq \frac{C}{(1 + |x|)^{n-2}},$$

here $\alpha(n)$ denote the volume of unit ball in $\mathbb{R}^n$. In the region R_2, we have $1+|x+y| \leq 1 + |x| + |y| \leq 3(1 + |x|)$. From [27, p144, exercises 2.4.8(b)], we have for any $n \geq 2$ and real number λ with $0 < \lambda < n$,

$$\int_{R^n} \frac{1}{|x - y|^\lambda (1 + |x|^2)^{n - \frac{\lambda}{2}}} dx = \frac{\pi^{\frac{n}{2}} \Gamma(\frac{n-\lambda}{2})}{\Gamma(n - \frac{\lambda}{2})} \frac{1}{(1 + |y|^2)^{\frac{\lambda}{2}}},$$

here Γ denote the Gamma function. By choosing $\lambda = n - 1$ in above inequality and doing the change of variables, we get

$$\int_{\mathbb{R}^n} \frac{1}{|y|^{n-1}(1 + |x + y|)^{n+1}} dy = \frac{C}{(1 + |x|^2)^{\frac{n-1}{2}}}.$$

Then we have

$$\int_{|y|<2|x|} \frac{1}{|y|^{n-1}(1 + |x + y|)^n} dy \leq \int_{|y|<2|x|} \frac{1 + |x + y|}{|y|^{n-1}(1 + |x + y|)^{n+1}} dy$$

$$\leq 3(1+|x|)\int_{R^n}\frac{1}{|y|^{n-1}(1+|x+y|)^{n+1}}dy$$
$$\leq \frac{C}{(1+|x|)^{n-2}}.$$

□

Lemma 3.3 (Lemma 3.5, [17]). *For any $1 \leq P \leq \infty$ and any function $\Psi \in L^p(\mathbb{R}^3_x)$, we have*

$$\| G_1 * \Psi \|_{L^p(\mathbb{R}^3_x)} \lesssim \| \Psi \|_{L^p(\mathbb{R}^3_x)}, \quad \| \nabla_x G_1 * \Psi \|_{L^p(\mathbb{R}^3_x)} \lesssim \| \Psi \|_{L^p(\mathbb{R}^3_x)} .$$

With the help of Lemma 3.2 and Lemma 3.3, we have the following conclusion.

Lemma 3.4 (Lemma 4.7, [17]). *For any multi-index γ, α with $|\gamma| \leq N$, $|\alpha| \leq N-3$, we have*

$$\| \nabla_x Z^\gamma(\phi) Y^\alpha(f) \|_{L^1(\mathbb{R}^3_x\times\mathbb{R}^3_v)} \lesssim \sum_{|\beta|\leq|\gamma|}\frac{\varepsilon}{(t+1)^2} \| \rho(Z^\beta f) \|_{L^1(\mathbb{R}^3_x)},$$

where (f, ϕ) is a sufficiently smooth solution to the magnetized VP system and the magnetized VY system.

Let us move on to the following important lemmas.

Lemma 3.5. *We denote the velocity support $Q(t)$ by*

$$Q(t) := 1 + \sup\{|v| : f(t,x,v) \neq 0\}.$$

Then for any $t \geq 0$, the velocity support is uniformly bounded i.e., there exist a constant C such that

$$Q(t) \leq C.$$

Proof. The proof of the theorem is based on a careful analysis of the characteristic system

$$\begin{cases} \dot{X}(s) = V(s), \quad X(t,t,x,v) = x, \\ \dot{V}(s) = \mu[\nabla\phi(s,X(s)) + V(s)\times B(s,X(s))], \quad V(t,t,x,v) = v. \end{cases} \tag{3.5}$$

From the characteristic system (3.5), bootstrap assumption (3.4) and the assumption (1.10), we have

$$\begin{aligned} Q(t) &\leq Q(0) + \int_0^t |\nabla\phi(s,X(s)) + V(s)\times B(s,X(s))|ds \\ &\lesssim Q(0) + \int_0^t \frac{\varepsilon^{1/2}}{(s+1)^2}ds + \int_0^t Q(s)\frac{\varepsilon}{(s+1)^3}ds \\ &\lesssim Q(0) + \varepsilon^{1/2} + \int_0^t Q(s)\frac{\varepsilon}{(s+1)^3}ds. \end{aligned}$$

Applying Gronwall's inequality, we get

$$Q(t) \leq (Q(0) + \varepsilon^{1/2})e^{\varepsilon}.$$

Note that the initial data $f_0(x,v)$ has compact support in v, we have $Q(0) \le C$, and we have completed the proof. □

Combining the definitions of vector fields, Lemma 3.5 and the assumption (1.10), we have

Lemma 3.6. *For any multi-index γ with $|\gamma| \le N$, we have*

$$|Z^\gamma(v\times B)| \lesssim (1+|v|)\sum_{\gamma'\le\gamma}|Z^{\gamma'}(B)| \lesssim \frac{\varepsilon}{(t+1)^3}.$$

With the help of Lemma 3.4 and Lemma 3.6, now we can estimate the products of the products of type $Y^\alpha(\varphi)Y^\beta(f)$ as follows.

Lemma 3.7. *For any fixed small number $\sigma > 0$, there exist constants C_σ and ε_σ such that, if $\varepsilon \le \varepsilon_\sigma$, then for all multi-index α, β with $|\alpha| \le N-1$, $|\beta| \le N$ and $|\alpha| + |\beta| \le N+1$, we have*

$$\| Y^\alpha(\varphi)(t)Y^\beta(f)(t) \|_{L^1(\mathbb{R}^3_x\times\mathbb{R}^3_v)} \le C_\sigma(t+1)^\sigma\varepsilon. \tag{3.6}$$

Moreover, for all multi-index α, β with $|\alpha| \le N-2$, $|\beta| \le N$ and $|\alpha|+|\beta| \le N$, and for all $1 \le i \le 3$, we have

$$\| Y^\alpha(\partial_{x^i}\varphi)(t)Y^\beta(f)(t) \|_{L^1(\mathbb{R}^3_x\times\mathbb{R}^3_v)} \le C_\sigma\varepsilon. \tag{3.7}$$

Proof. For the sake of simplicity, we denote

$$\mathcal{F}(t) = \sum_{|\alpha|\le N-1}\ \sum_{\substack{|\beta|\le N\\ |\alpha|+|\beta|\le N+1}} \| Y^\alpha(\varphi)(t)Y^\beta(f)(t) \|_{L^1(\mathbb{R}^3_x\times\mathbb{R}^3_v)},$$

$$\mathcal{G}(t) = \sum_{|\alpha|\le N-2}\sum_{i=1}^{3}\ \sum_{\substack{|\beta|\le N\\ |\alpha|+|\beta|\le N}} \| Y^\alpha(\partial_{x^i}\varphi)(t)Y^\beta(f)(t) \|_{L^1(\mathbb{R}^3_x\times\mathbb{R}^3_v)}\ .$$

We only need to estimate $\mathcal{F}(t)$ and $\mathcal{G}(t)$.

If $|\alpha| \le N-5$, by bootstrap assumption (3.1),(3.2) we have

$$\begin{aligned}\| Y^\alpha(\varphi)(t)Y^\beta(f)(t) \|_{L^1(\mathbb{R}^3_x\times\mathbb{R}^3_v)} &\lesssim \varepsilon^{1/2}(1+\log(t+1)) \| Y^\beta(f)(t) \|_{L^1(\mathbb{R}^3_x\times\mathbb{R}^3_v)}\\ &\lesssim (t+1)^{\sigma_0}\varepsilon^{3/2}.\end{aligned}$$

where $\sigma_0 > 0$ is a very small constant that is to be fixed later. Similarly, for $|\alpha| \le N-6$, by bootstrap assumption (3.3), we have

$$\| Y^\alpha(\partial_{x^i}\varphi)(t)Y^\beta(f)(t) \|_{L^1(\mathbb{R}^3_x\times\mathbb{R}^3_v)} \lesssim \varepsilon^{3/2}.$$

If $|\alpha| > N-5$, since $N \ge 9$, we have $|\beta| \le N-4$. From Lemma 3.1, we can estimate $\| Y^\alpha(\varphi)(t)Y^\beta(f)(t) \|_{L^1(\mathbb{R}^3_x\times\mathbb{R}^3_v)}$ (including the special case $Y^\alpha = Y^{\alpha'}\partial_{x^i}$). Since $\varphi = 0$ at time $t = 0$, we have

$$\| Y^\alpha(\varphi)(t)Y^\beta(f)(t) \|_{L^1(\mathbb{R}^3_x\times\mathbb{R}^3_v)} \le \int_0^t \| T_{\phi,B}(Y^\alpha(\varphi)Y^\beta(f)) \|_{L^1(\mathbb{R}^3_x\times\mathbb{R}^3_v)}\, ds.$$

Where

$$T_{\phi,B}(Y^\alpha(\varphi)Y^\beta(f))$$

$$=Y^{\alpha}(\varphi)T_{\phi,B}(Y^{\beta}(f))+[T_{\phi,B},Y^{\alpha}](\varphi)Y^{\beta}(f)+Y^{\alpha}T_{\phi,B}(\varphi)Y^{\beta}(f)$$
$$=:I_1+I_2+I_3.$$

Estimate for $\| I_1 \|_{L^1(R^3_x\times R^3_v)}$**.** From Lemma 2.5, we have

$$I_1=\sum_{d=0}^{|\beta|+1}\sum_{i=1}^{3}\sum_{|\gamma|\le|\beta|,|\beta'|\le|\beta|}P^{\beta,i}_{d\gamma\beta'}(\varphi)[\partial_{x^i}Z^{\gamma}\phi+Z^{\gamma}(v\times B)+tZ^{\gamma}(v\times B)]Y^{\beta'}(f)Y^{\alpha}(\varphi).$$

Due to $|\beta|\le N-4$, the signature of $P^{\beta,i}_{d\gamma\beta'}(\varphi)$ is less than $|\beta|-1\le N-5$, $d\le|\beta|+1\le N$ and by bootstrap assumption (3.2) we have

$$|P^{\beta,i}_{d\gamma\beta'}(\varphi)|\lesssim(1+\log(t+1))^N\lesssim(t+1)^{\sigma_0},$$

where $\sigma_0\in(0,1)$ is a small number to be fixed later. Meanwhile, as $|\gamma|\le|\beta|\le N-4$, it from bootstrap assumption (3.4) holds

$$|\partial_{x^i}Z^{\gamma}\phi|\lesssim\frac{\varepsilon^{1/2}}{(t+1)^2}.$$

Therefore, we have

$$|P^{\beta,i}_{d\gamma\beta'}(\varphi)\partial_{x^i}Z^{\gamma}\phi|\lesssim\frac{\varepsilon^{1/2}}{(t+1)^{2-\sigma_0}}.$$

From Lemma 3.6 we have $|Z^{\gamma}(v\times B)|\lesssim\frac{\varepsilon}{(t+1)^3}$, then

$$|P^{\beta,i}_{d\gamma\beta'}(\varphi)[Z^{\gamma}(v\times B)+tZ^{\gamma}(v\times B)]|\lesssim\frac{\varepsilon^{1/2}}{(t+1)^{2-\sigma_0}}.$$

In summary,

$$\| I_1 \|_{L^1(\mathbb{R}^3_x\times\mathbb{R}^3_v)}\lesssim\frac{\varepsilon^{1/2}}{(t+1)^{2-\sigma_0}}\mathcal{F}(t).$$

Estimate for $\| I_2 \|_{L^1(\mathbb{R}^3_x\times\mathbb{R}^3_v)}$**.** From Lemma 2.5, we have

$$I_2=\sum_{d=0}^{|\alpha|+1}\sum_{i=1}^{3}\sum_{|\gamma|\le|\alpha|,|\beta'|\le|\alpha|}P^{\alpha,i}_{d\gamma\beta'}(\varphi)[\partial_{x^i}Z^{\gamma}\phi+Z^{\gamma}(v\times B)+tZ^{\gamma}(v\times B)]Y^{\beta'}(\varphi)Y^{\beta}(f).$$

Here the multi-linear form $P^{\alpha,i}_{d\gamma\beta'}(\varphi)$ has signature less than k such that $k\le|\alpha|-1$ and $k+|\gamma|+|\beta'|\le|\alpha|+1\le N$.

(1). If $|\gamma|\le N-4$, then by the bootstrap assumption (3.4), we have

$$|\partial_{x^i}Z^{\gamma}\phi|\lesssim\frac{\varepsilon^{1/2}}{(t+1)^2}.$$

And from Lemma 3.6 we have

$$|Z^{\gamma}(v\times B)+tZ^{\gamma}(v\times B)|\lesssim\frac{\varepsilon^{1/2}}{(t+1)^2}.$$

When $k \leq N-5$, from the bootstrap assumption (3.2), it holds

$$P^{\alpha,i}_{d\gamma\beta'}(\varphi) \lesssim (1+\log(t+1))^N \lesssim (t+1)^{\sigma_0}.$$

Hence, the L^1 norm of $P^{\alpha,i}_{d\gamma\beta'}(\varphi)Y^{\beta'}(\varphi)Y^{\beta}(f)$ is controlled by $(t+1)^{\sigma_0}\mathcal{F}(t)$. When $k > N-5$, we put $P^{\alpha,i}_{d\gamma\beta'}(\varphi)$ and $Y^{\beta'}(\varphi)$ together. Note that the highest order of both of them is no more than $k \leq |\alpha|-1$ and $|\beta'| \leq |\alpha|$. Hence, the L^1 norm of $P^{\alpha,i}_{d\gamma\beta'}(\varphi)Y^{\beta'}(\varphi)Y^{\beta}(f)$ is controlled by $(t+1)^{\sigma_0}\mathcal{F}(t)$. Therefore, we have

$$\| P^{\alpha,i}_{d\gamma\beta'}(\varphi)\partial_{x^i}Z^\gamma\phi Y^{\beta'}(\varphi)Y^{\beta}(f) \|_{L^1(\mathbb{R}^3_x\times\mathbb{R}^3_v)} \lesssim \frac{\varepsilon^{1/2}}{(t+1)^{2-\sigma_0}}\mathcal{F}(t),$$

$$\| P^{\alpha,i}_{d\gamma\beta'}(\varphi)[Z^\gamma(v\times B)+tZ^\gamma(v\times B)]Y^{\beta'}(\varphi)Y^{\beta}(f) \|_{L^1(\mathbb{R}^3_x\times\mathbb{R}^3_v)} \lesssim \frac{\varepsilon^{1/2}}{(t+1)^{2-\sigma_0}}\mathcal{F}(t).$$

So, if $|\gamma| \leq N-4$, we have

$$\| I_2 \|_{L^1(\mathbb{R}^3_x\times\mathbb{R}^3_v)} \lesssim \frac{\varepsilon^{1/2}}{(t+1)^{2-\sigma_0}}\mathcal{F}(t).$$

(2). If $|\gamma| > N-4$, considering the condition $N \geq 9$, we get $k+|\beta'| \leq |\alpha|+1-|\gamma| \leq N-5$, hence by the bootstrap assumption (3.2), we have

$$|P^{\alpha,i}_{d\gamma\beta'}(\varphi)Y^{\beta'}(\varphi)| \lesssim (1+\log(t+1))^N.$$

Since $|\gamma| \leq |\alpha| \leq N-1 \leq N$, $|\beta| \leq N-4 \leq N-3$, by Lemma 3.4, we have

$$\| \partial_{x^i}Z^\gamma(\phi)Y^\beta(f) \|_{L^1(\mathbb{R}^3_x\times\mathbb{R}^3_v)} \lesssim \sum_{|\eta|\leq|\gamma|} \frac{\varepsilon}{(t+1)^2} \| Z^\eta f \|_{L^1(\mathbb{R}^3_x\times\mathbb{R}^3_v)} .$$

By Lemma 2.6 and the bootstrap assumption (3.2), we have

$$\| Z^\eta f \|_{L^1(\mathbb{R}^3_x\times\mathbb{R}^3_v)} \leq \sum_{d'=0}^{|\eta|}\sum_{|\eta'|\leq|\eta|} \| P^{\eta}_{d'\eta'}(\varphi)Y^{\eta'}(f) \|_{L^1(\mathbb{R}^3_x\times\mathbb{R}^3_v)} \lesssim (1+\log(t+1))^N\mathcal{F}(t).$$

So we have

$$\| P^{\alpha,i}_{d\gamma\beta'}(\varphi)\partial_{x^i}Z^\gamma\phi Y^{\beta'}(\varphi)Y^{\beta}(f) \|_{L^1(\mathbb{R}^3_x\times\mathbb{R}^3_v)} \lesssim \frac{\varepsilon^{1/2}}{(t+1)^{2-\sigma_0}}\mathcal{F}(t).$$

From Lemma 3.6 and the bootstrap assumption (3.1), we have

$$|[Z^\gamma(v\times B)+tZ^\gamma(v\times B)]Y^\beta(f)| \lesssim \frac{\varepsilon}{(t+1)^2} \| Y^\beta(f) \|_{L^1(\mathbb{R}^3_x\times\mathbb{R}^3_v)} \lesssim \frac{\varepsilon^2}{(t+1)^2}.$$

So we have

$$\| P^{\alpha,i}_{d\gamma\beta'}(\varphi)[Z^\gamma(v\times B)+tZ^\gamma(v\times B)]Y^{\beta'}(\varphi)Y^{\beta}(f) \|_{L^1(\mathbb{R}^3_x\times\mathbb{R}^3_v)} \lesssim \frac{\varepsilon^{1/2}}{(t+1)^{2-\sigma_0}}.$$

So, if $|\gamma| > N-4$, we also have

$$\| I_2 \|_{L^1(\mathbb{R}^3_x\times\mathbb{R}^3_v)} \lesssim \frac{\varepsilon^{1/2}}{(t+1)^{2-\sigma_0}}\mathcal{F}(t) + \frac{\varepsilon^2}{(t+1)^{2-\sigma_0}}.$$

In summary,

$$\| I_2 \|_{L^1(\mathbb{R}^3_x\times\mathbb{R}^3_v)}\lesssim \frac{\varepsilon^{1/2}}{(t+1)^{2-\sigma_0}}\mathcal{F}(t)+\frac{\varepsilon^2}{(t+1)^{2-\sigma_0}}.$$

Estimate for $\| I_3 \|_{L^1(\mathbb{R}^3_x\times\mathbb{R}^3_v)}$**.** From (2.5), it is known that

$$T_{\phi,B}(\varphi)=t\sum_{i=1}^{3}\sum_{|\eta|\leq 1}c_{Z,i}\partial_{x^i}Z^\eta\phi.$$

Taking account of Lemma 2.8, we have

$$\begin{aligned}
I_3 &= Y^\alpha T_{\phi,B}(\varphi)Y^\beta(f)\\
&= t\sum_{|\eta|\leq|\alpha|+1}c_{\eta,i}\partial_{x^i}Z^\eta(\phi)Y^\beta(f)+\sum_{d=1}^{|\alpha|}\sum_{|\eta|\leq|\alpha|+1}P^\alpha_{d\eta}(\varphi)\partial_{x^i}Z^\eta(\phi)Y^\beta(f),
\end{aligned}$$

where $P^\alpha_{d\eta}(\varphi)$ are multi-linear forms of degree d with signatures less than k satisfying $k\leq|\alpha|\leq N-1$ and $k+|\eta|\leq|\alpha|+1$. As we know from the above inequality, The extra magnetic field has no effect on the treatment of I_3. So the estimate of $\| I_3 \|_{L^1(\mathbb{R}^3_x\times\mathbb{R}^3_v)}$ is the same as the estimate of [17], taking the same way as in [17] we obtain that

$$\begin{aligned}
\| I_3 \|_{L^1(\mathbb{R}^3_x\times\mathbb{R}^3_v)} &\lesssim (t+1)\sum_{|\eta|\leq|\alpha|+1}\| \partial_{x^i}Z^\eta(\phi)Y^\beta(f) \|_{L^1(\mathbb{R}^3_x\times\mathbb{R}^3_v)}+\frac{\varepsilon^{1/2}}{(t+1)^{2-\sigma_0}}\mathcal{F}(t)\\
&\lesssim \frac{\varepsilon^2}{t+1}+\frac{\varepsilon}{t+1}\mathcal{F}(t)+\frac{\varepsilon}{(t+1)^{1-\sigma_0}}\mathcal{G}(t)+\frac{\varepsilon^{1/2}}{(t+1)^{2-\sigma_0}}\mathcal{F}(t).
\end{aligned}$$

If $Y^\alpha=Y^{\alpha'}\partial_{x_j}$, then

$$\| I_3 \|_{L^1(\mathbb{R}^3_x\times\mathbb{R}^3_v)}\lesssim \frac{\varepsilon^2}{(t+1)^2}+\frac{\varepsilon}{(t+1)^{2-\sigma_0}}\mathcal{G}(t)+\frac{\varepsilon^{1/2}}{(t+1)^{2-\sigma_0}}\mathcal{F}(t).$$

In conclusion, from the estimates for $\| I_1 \|_{L^1(\mathbb{R}^3_x\times\mathbb{R}^3_v)}$, $\| I_2 \|_{L^1(\mathbb{R}^3_x\times\mathbb{R}^3_v)}$ and
$\| I_3 \|_{L^1(\mathbb{R}^3_x\times\mathbb{R}^3_v)}$, we have for $|\alpha|>N-5$,

$$\begin{aligned}
&\| T_{\phi,B}(Y^\alpha(\varphi(s))Y^\beta(f(s))) \|_{L^1(\mathbb{R}^3_x\times\mathbb{R}^3_v)}\\
\lesssim&\frac{\varepsilon^2}{t+1}+\frac{\varepsilon}{t+1}\mathcal{F}(t)+\frac{\varepsilon}{(t+1)^{1-\sigma_0}}\mathcal{G}(t)+\frac{\varepsilon^{1/2}}{(t+1)^{2-\sigma_0}}\mathcal{F}(t),
\end{aligned}$$

for $|\alpha|>N-6$, we have

$$\begin{aligned}
&\| T_{\phi,B}(Y^\alpha(\partial_x\varphi(s))Y^\beta(f(s))) \|_{L^1(\mathbb{R}^3_x\times\mathbb{R}^3_v)}\\
\lesssim&\frac{\varepsilon^2}{(t+1)^{2-\sigma_0}}+\frac{\varepsilon}{(t+1)^{2-\sigma_0}}\mathcal{G}(t)+\frac{\varepsilon^{1/2}}{(t+1)^{2-\sigma_0}}\mathcal{F}(t).
\end{aligned}$$

As a consequence, we have

$$\| Y^\alpha(\varphi)(t)Y^\beta(f)(t) \|_{L^1(\mathbb{R}^3_x\times\mathbb{R}^3_v)}$$

$$\leq \int_0^t \| T_{\phi,B}(Y^\alpha(\varphi(s))Y^\beta(f(s))) \|_{L^1(\mathbb{R}_x^3\times\mathbb{R}_v^3)}\, ds$$
$$\lesssim \varepsilon^2 \log(t+1) + \varepsilon^{\frac{1}{2}} \int_0^t \frac{\mathcal{F}(s)}{s+1} ds + \varepsilon \int_0^t \frac{\mathcal{G}(s)}{(s+1)^{1-\sigma_0}} ds,$$

$$\| Y^\alpha(\partial_x\varphi)(t)Y^\beta(f)(t) \|_{L^1(\mathbb{R}_x^3\times\mathbb{R}_v^3)}$$
$$\leq \int_0^t \| T_{\phi,B}(Y^\alpha(\partial_x\varphi(s))Y^\beta(f(s))) \|_{L^1(\mathbb{R}_x^3\times\mathbb{R}_v^3)}\, ds$$
$$\lesssim \varepsilon^2 + \varepsilon^{\frac{1}{2}} \int_0^t \frac{\mathcal{F}(s)}{(s+1)^{2-\sigma_0}} ds + \varepsilon \int_0^t \frac{\mathcal{G}(s)}{(s+1)^{2-\sigma_0}} ds.$$

Therefore

$$\mathcal{F}(t) \lesssim \varepsilon^{\frac{3}{2}}(t+1)^{\sigma_0} + \varepsilon^{\frac{1}{2}} \int_0^t \frac{\mathcal{F}(s)}{s+1} ds + \varepsilon \int_0^t \frac{\mathcal{G}(s)}{(s+1)^{1-\sigma_0}} ds, \tag{3.8}$$

$$\mathcal{G}(t) \lesssim \varepsilon + \varepsilon^{\frac{1}{2}} \int_0^t \frac{\mathcal{F}(s)}{(s+1)^{2-\sigma_0}} ds + \varepsilon \int_0^t \frac{\mathcal{G}(s)}{(s+1)^{2-\sigma_0}} ds. \tag{3.9}$$

Taking the same way as in [17] to deal with (3.8) and (3.8), we can draw the conclusion. □

Step 4. Improving the bootstrap assumption.

Lemma 3.8. *If the initial data f_0 satisfies $E_N[f_0] \leq \varepsilon$, then when ε is small enough, we have that, for all $t \in [0,T]$*

$$E_N[f(t)] \leq \frac{3}{2}\varepsilon.$$

Proof. Taking account of the conservation laws (3.1), we have that, for any multi-index α, $|\alpha| \leq N$

$$E_N[f(t)] \leq E_N[f(0)] + C\int_0^t \| T_{\phi,B}(Y^\alpha f)(s) \|_{L^1(\mathbb{R}_x^3\times\mathbb{R}_v^3)}\, ds.$$

The initial data is already bounded by ε, the main work is to estimate the second term on the right- hand of above inequality. By Lemma 2.5, we have

$$[T_{\phi,B}, Y^\alpha]f$$
$$= \sum_{d=0}^{|\alpha|+1} \sum_{i=1}^{3} \sum_{|\gamma|\leq|\alpha|,|\beta|\leq|\alpha|} P_{d\gamma\beta}^{\alpha,i}(\varphi)[\partial_{x^i} Z^\gamma\phi + Z^\gamma(v\times B) + tZ^\gamma(v\times B)]Y^\beta f,$$

where $P_{d\gamma\beta}^{\alpha,i}$ are multi-linear forms of degree d with signatures less than k such that $k \leq |\alpha|-1$ and $k+|\gamma|+|\beta| \leq |\alpha|+1 \leq N+1$.

(1). When $|\gamma| \leq N-4$, by the bootstrap assumption (3.4), we have

$$|\partial_{x^i} Z^\gamma \phi| \lesssim \frac{\varepsilon^{1/2}}{(t+1)^2}.$$

And from Lemma 3.6, we have

$$|Z^\gamma(v\times B) + tZ^\gamma(v\times B)| \lesssim \frac{\varepsilon}{(t+1)^2}.$$

Since $k+|\beta|\leq N+1$ and $k\leq N-1$, by Lemma 3.7, we have

$$\| P^{\alpha,i}_{d\gamma\beta}(\varphi)Y^{\beta}(f)\|_{L^1(\mathbb{R}^3_x\times\mathbb{R}^3_v)}\lesssim (1+\log(t+1))^{N+1}(t+1)^{\sigma}\varepsilon.$$

By taking σ small enough, we have

$$\| P^{\alpha,i}_{d\gamma\beta}(\varphi)\partial_{x^i}Z^{\gamma}(\phi)Y^{\beta}(f)\|_{L^1(\mathbb{R}^3_x\times\mathbb{R}^3_v)}\lesssim \frac{\varepsilon^{3/2}}{(t+1)^{1+\sigma'}},$$
$$\| P^{\alpha,i}_{d\gamma\beta}[Z^{\gamma}(v\times B)+tZ^{\gamma}(v\times B)]Y^{\beta}f\|_{L^1(\mathbb{R}^3_x\times\mathbb{R}^3_v)}\lesssim \frac{\varepsilon^{3/2}}{(t+1)^{2-\sigma}}\lesssim\frac{\varepsilon^{3/2}}{(t+1)^{1+\sigma'}},$$

where $\sigma'>0$. Therefore, when $|\gamma|\leq N-4$, we have

$$\| P^{\alpha,i}_{d\gamma\beta}(\varphi)[\partial_{x^i}Z^{\gamma}\phi+Z^{\gamma}(v\times B)+tZ^{\gamma}(v\times B)]Y^{\beta}f\|_{L^1(\mathbb{R}^3_x\times\mathbb{R}^3_v)}\lesssim \frac{\varepsilon^{3/2}}{(t+1)^{1+\sigma'}}.$$

(2). When $|\gamma|>N-4$, then $k,|\beta|\leq N-6$ since $N\geq 9$, so from the bootstrap assumption (3.2), we have

$$|P^{\alpha,i}_{d\gamma\beta}(\varphi)|\lesssim (1+\log(t+1))^{N+1}.$$

By Lemma 3.4, we have

$$\| \partial_{x^i}Z^{\gamma}(\phi)Y^{\beta}f\|_{L^1(\mathbb{R}^3_x\times\mathbb{R}^3_v)}\lesssim \frac{\varepsilon}{(t+1)^2}\sum_{|\eta|\leq|\gamma|}\| Z^{\eta}(f)\|_{L^1(\mathbb{R}^3_x\times\mathbb{R}^3_v)}.$$

Thanks to Lemma 2.6, we have

$$Z^{\eta}(f)=\sum_{d=0}^{|\eta|}\sum_{|\eta'|\leq|\eta|}P^{\eta}_{d\eta'}(\varphi)Y^{\eta'}(f),$$

where $P^{\eta}_{d\eta'}(\varphi)$ are multi-linear forms of degree d with signatures less than k such that $k\leq|\eta|-1\leq N-1$ and $k+|\eta'|\leq|\eta|\leq N$. Therefore, by Lemma 3.7, we have

$$\| Z^{\eta}(f)\|_{L^1(\mathbb{R}^3_x\times\mathbb{R}^3_v)}\lesssim (1+\log(t+1))^{N}(t+1)^{\sigma}\varepsilon,$$

which implies that there exist $\sigma'>0$ such that

$$\| P^{\alpha,i}_{d\gamma\beta}(\varphi)\partial_{x^i}Z^{\gamma}(\phi)Y^{\beta}(f)\|_{L^1(\mathbb{R}^3_x\times\mathbb{R}^3_v)}\lesssim \frac{\varepsilon^{2}}{(t+1)^{1+\sigma'}}.$$

From Lemma 3.6 and the bootstrap assumption (3.1), we have

$$\| [Z^{\gamma}(v\times B)+tZ^{\gamma}(v\times B)]Y^{\beta}f\|_{L^1(\mathbb{R}^3_x\times\mathbb{R}^3_v)}\lesssim \frac{\varepsilon}{(t+1)^2}\| Y^{\beta}(f)\|_{L^1(\mathbb{R}^3_x\times\mathbb{R}^3_v)}\lesssim\frac{\varepsilon^2}{(t+1)^2}.$$

Then there exist $\sigma'>0$ such that

$$\| P^{\alpha,i}_{d\gamma\beta}(\varphi)[Z^{\gamma}(v\times B)+tZ^{\gamma}(v\times B)]Y^{\beta}f\|_{L^1(\mathbb{R}^3_x\times\mathbb{R}^3_v)}\lesssim \frac{\varepsilon^{3/2}}{(t+1)^{1+\sigma'}}.$$

Therefore, When $|\gamma|>N-4$, we have

$$\| P^{\alpha,i}_{d\gamma\beta}(\varphi)[\partial_{x^i}Z^{\gamma}\phi+Z^{\gamma}(v\times B)+tZ^{\gamma}(v\times B)]Y^{\beta}f\|_{L^1(\mathbb{R}^3_x\times\mathbb{R}^3_v)}\lesssim \frac{\varepsilon^{3/2}}{(t+1)^{1+\sigma'}}.$$

In summary, we have that, there exists $\sigma' > 0$, such that

$$\| T_{\phi,B}(Y^\alpha f)(s) \|_{L^1(\mathbb{R}^3_x\times\mathbb{R}^3_v)} \lesssim \frac{\varepsilon^{3/2}}{(t+1)^{1+\sigma'}}.$$

Therefore, when ε is small enough, we have

$$\begin{aligned} E_N[f(t)] &\le E_N[f(0)] + C\int_0^t \| T_{\phi,B}(Y^\alpha f)(s) \|_{L^1(\mathbb{R}^3_x\times\mathbb{R}^3_v)}\, ds \\ &\le \varepsilon + C\varepsilon^{3/2}\int_0^t \frac{1}{(s+1)^{1+\sigma'}} ds \\ &\le \frac{3}{2}\varepsilon. \end{aligned}$$

□

Lemma 3.9. *For any multi-index α with $|\alpha| \le N-4$, we have*

$$|\nabla_x Z^\alpha \phi(x)| \lesssim \frac{\varepsilon}{(1+t)(1+t+|x|)},$$

for the magnetized VP system and

$$|\nabla_x Z^\alpha \phi(t,x)| \lesssim \frac{\varepsilon}{(1+t+|x|)^3},$$

for the magnetized VY system.

Proof. For the magnetized VP system, from Lemma 2.3, we have

$$|\nabla_x Z^\alpha \phi(x)| = \sum_{|\beta|\le|\alpha|} \int \frac{1}{|y|^2}|\rho(Z^\beta f)(x-y)|dy.$$

By Lemma 2.7, we have

$$\rho(Z^\beta f) = \sum_{d=0}^{|\beta|}\sum_{|\gamma|\le|\beta|} \rho(Q^\beta_{d\gamma}(\partial_x\varphi)Y^\gamma f) + \sum_{d=0}^{|\beta|}\sum_{d=1}^{|\beta|+1}\sum_{|\gamma|\le|\beta|} \frac{1}{t^j}\rho(P^{\beta,j}_{d\gamma}(\varphi)Y^\gamma f),$$

where $Q^\beta_{d\gamma}$ are multilinear forms with respect to $\partial_x\varphi$ of signatures less than k' such that $k' \le |\beta|-1 \le N-4$ and $k'+d+|\gamma| \le |\beta|$, $P^{\beta,j}_{d\gamma}$ are multilinear forms of degree d with signatures less than k such that $k \le |\beta| \le N-4$ and $k+|\gamma| \le |\beta|$. Now we apply the Klainerman-Sobolev inequality to every term in the above equation, then we have

$$\begin{aligned} &|\rho((Q^\beta_{d\gamma}(\partial_x\varphi))Y^\gamma f)(x-y)| \\ \lesssim &\frac{1}{(1+t+|x-y|)^3}\sum_{|\eta|\le 3} \| Y^\eta[Q^\beta_{d\gamma}(\partial_x\varphi)Y^\gamma f] \|_{L^1(\mathbb{R}^3_x\times\mathbb{R}^3_v)}, \\ &|\rho(P^{\beta,j}_{d\gamma}(\varphi)Y^\gamma f)(x-y)| \\ \lesssim &\frac{1}{(1+t+|x-y|)^3}\sum_{|\eta|\le 3} \| Y^\eta[P^{\beta,j}_{d\gamma}(\varphi)Y^\gamma f] \|_{L^1(\mathbb{R}^3_x\times\mathbb{R}^3_v)} . \end{aligned}$$

Since $N \geq 9$, there is at most one multiplier $Y^{\eta'}(\varphi)$ with $|\eta'| > N-6$, therefore, by the bootstrap assumption (3.2) and Lemma 3.7, we have

$$\begin{aligned}\| \rho(Z^\beta f)(x-y) \|_{L^\infty} \lesssim & \frac{\varepsilon}{(1+t+|x-y|)^3} + \frac{\varepsilon(1+\log(t+1))^N (t+1)\sigma}{(1+t+|x-y|)^3 t} \\ \lesssim & \frac{\varepsilon}{(1+t+|x-y|)^3}.\end{aligned}$$

Now we do the change of variable $y = (t+1)y'$, then

$$\int_{\mathbb{R}^3} \frac{1}{|y|^2} \frac{\varepsilon}{(1+t+|x-y|)^3} dy = \frac{\varepsilon}{(1+t)^2} \int_{\mathbb{R}^3} \frac{1}{|y'|^2} \frac{\varepsilon}{(1+|y' - \frac{x}{1+t}|)^3} dy'.$$

By Lemma 3.2 with $n = 3$, we have

$$|\nabla_x Z^\alpha \phi(x)| \lesssim \frac{\varepsilon}{(1+t)^2 (1+\frac{|x|}{1+t})} = \frac{\varepsilon}{(1+t)(1+t+|x|)}.$$

For the magnetized VY system, we can do the exactly similar estimate by using the exact expression of $\nabla_x Z^\alpha \phi$. By Lemma 2.4, we have that

$$\nabla_x Z^\alpha \phi = \nabla_x G_1 * \Big(\sum_{k=0}^{|\alpha|} \sum_{|\beta| \leq |\alpha| - k} c^\alpha_{k,\beta} G_1 *^{(k)} \rho(Z^\beta f) \Big),$$

where $c^\alpha_{k,\beta}$ are some globally bounded constants. From Lemma 3.3, we have

$$\| \nabla_x Z^\alpha \phi \|_{L^\infty} \lesssim \sum_{|\beta| \leq |\alpha|} \| \rho(Z^\beta f) \|_{L^\infty} .$$

Similarly, we can obtain

$$\| \rho(Z^\beta f)(x) \|_{L^\infty} \lesssim \frac{\varepsilon}{(1+t+|x|)^3}.$$

Therefore, we have

$$\| \nabla_x Z^\alpha \phi \|_{L^\infty} \lesssim \sum_{|\beta| \leq |\alpha|} \| \rho(Z^\beta f) \|_{L^\infty} \lesssim \frac{\varepsilon}{(1+t+|x|)^3}.$$

□

Lemma 3.10. *For any multi-index α with $|\alpha| \leq N-4$, we have*

$$|Y^\alpha \varphi(t,x,v)| \lesssim \varepsilon(1+\log(t+1)).$$

For any multi-index α with $|\alpha| \leq N-6$, we have

$$|Y^\alpha \nabla_x \varphi(t,x,v)| \lesssim \varepsilon.$$

Proof. By method of characteristics, we have that

$$|Y^\alpha \varphi(t,x,v)| \leq \int_0^t \| T_{\phi,B} Y^\alpha \varphi(s) \|_{L^\infty} ds.$$

We just need to estimate $\| T_{\phi,B}Y^\alpha\varphi(s) \|_{L^\infty}$

$$T_{\phi,B}Y^\alpha(\varphi) = Y^\alpha T_{\phi,B}(\varphi) + [T_{\phi,B}, Y^\alpha](\varphi).$$

For the first part $Y^\alpha T_{\phi,B}(\varphi)$, from Definition 2.3, we know that

$$T_{\phi,B}(\varphi) = t\sum_{i=1}^{3}\sum_{|\eta|\le 1} c_{Z,i}\partial_{x^i}Z^\eta\phi.$$

Therefore, by Lemma 2.8 and Lemma 2.2, we have

$$Y^\alpha T_{\phi,B}(\varphi) = t\sum_{|\eta|\le|\alpha|+1} c_{\eta,i}\partial_{x^i}Z^\eta(\phi) + \sum_{d=1}^{|\alpha|}\sum_{|\eta|\le|\alpha|+1} P^\alpha_{d\eta}(\varphi)\partial_{x^i}Z^\eta(\phi),$$

where $P^\alpha_{d\eta}(\varphi)$ are multi-linear forms of degree d with signatures less than k such that $k \le |\alpha| \le N-4$ and $k+|\eta| \le |\alpha|+1 \le N-4$. Then by bootstrap assumption (3.2) and Lemma 3.9, we have

$$|Y^\alpha T_{\phi,B}(\varphi)(t)| \lesssim \frac{\varepsilon[t+(1+\log(t+1))^{N+1}]}{(t+1)^2} \lesssim \frac{\varepsilon}{t+1}.$$

For the second part $[T_{\phi,B}, Y^\alpha](\varphi)$, from Lemma 2.5, we have

$$\begin{aligned}&[T_{\phi,B}, Y^\alpha](\varphi)\\ &= \sum_{d=0}^{|\alpha|+1}\sum_{i=1}^{3}\sum_{\substack{|\gamma|\le|\alpha|\\ |\beta'|\le|\alpha|}} P^{\alpha,i}_{d\gamma\beta'}(\varphi)[\partial_{x^i}Z^\gamma\phi + Z^\gamma(v\times B) + tZ^\gamma(v\times B)]Y^{\beta'}(\varphi),\end{aligned}$$

where $P^{\alpha,i}_{d\gamma\beta'}(\varphi)$ are multilinear forms of degree d with signatures less than k such that $k \le |\alpha|-1 \le N-6$ and $k+|\gamma|+|\beta'| \le |\alpha|+1 \le N-4$. Therefore, by bootstrap assumption (3.2) and Lemma 3.9, we have

$$|P^{\alpha,i}_{d\gamma\beta'}(\varphi)\partial_{x^i}Z^\gamma\phi Y^{\beta'}(\varphi)| \lesssim \frac{\varepsilon(1+\log(t+1))^{N+1}}{(t+1)^2} \lesssim \frac{\varepsilon}{t+1}.$$

From Lemma 3.6 and the bootstrap assumption (3.2), we have

$$|P^{\alpha,i}_{d\gamma\beta'}(\varphi)[Z^\gamma(v\times B) + tZ^\gamma(v\times B)]Y^{\beta'}(\varphi)| \lesssim \frac{\varepsilon(1+\log(t+1))^{N+1}}{(t+1)^2} \lesssim \frac{\varepsilon}{t+1}.$$

Therefore we have

$$|[T_{\phi,B}, Y^\alpha](\varphi)| \lesssim \frac{\varepsilon}{t+1}.$$

In summary, we have

$$\begin{aligned}|Y^\alpha\varphi(t,x,v)| &\le \int_0^t \| T_{\phi,B}Y^\alpha\varphi(s) \|_{L^\infty}\, ds\\ &\lesssim \int_0^t \frac{\varepsilon}{s+1}ds \lesssim \varepsilon(1+\log(t+1)).\end{aligned}$$

Now if $Y^\alpha = Y^\alpha \partial_x$, in all the above estimates, the term $\partial_{x^i} Z^\eta(\phi)$ will in fact be $\partial_{x^i}\partial_{x^j} Z^{\eta'}(\phi)$, which will provide an additional decay power in t since $\partial_{x^i}\partial_{x^j} Z^{\eta'}(\phi) = t^{-1}\partial_{x^i}(t\partial_{x^j}) Z^{\eta'}(\phi)$. So for any fixed small number $0 < \sigma < 1$ we have

$$|Y^\alpha \nabla_x \varphi(t,x,v)| \lesssim \int_0^t \frac{\varepsilon}{(s+1)^{2-\sigma}} ds \lesssim \varepsilon.$$

□

In summary, we have improved the bootstrap assumptions (3.1)-(3.4), which leads to contraction with the largest time T. Therefore end the proof of the estimates (1.11), (1.14) and (1.16) in Theorem 1.1. From (1.11) and Proposition 3.1, we automatically obtain the estimate (1.12). It remains need to prove (1.13), (1.15), (1.17). From Lemma 2.7 in [17], we have

$$(t+|x|)^\alpha \partial_x^\alpha = \sum_{|\beta|\le|\alpha|, Z^\beta \in \gamma^\beta} C_\beta Z^\beta.$$

Applying the above inequality to Proposition 3.1 and Lemma 3.9, it is easy to get the estimates. Therefore end the proof of Theorem 1.1.

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