Journal of Applied Analysis and Computation
Volume 14, Number 3, June 2024, 1674–1697
Website:http://www.jaac-online.com
DOI:10.11948/20230334

CONTINUITY OF THE MULTILINEAR MAXIMAL COMMUTATORS IN SOBOLEV SPACES*

Xixi Jiang[1] and Feng Liu[1,†]

Abstract In the present paper we study the Sobolev continuity of the multilinear maximal commutators and their fractional variants with Lipschitz symbols. More precisely, let $\mathfrak{M}_{\alpha,\vec{b}}$ be the multilinear fractional maximal commutators, where $0 \leq \alpha < mn$ and $\vec{b} = (b_1, \ldots, b_m)$ with each $b_i \in \mathrm{Lip}(\mathbb{R}^n)$. We establish the continuity of $\mathfrak{M}_{\alpha,\vec{b}} : W^{1,p_1}(\mathbb{R}^n) \times \cdots \times W^{1,p_m}(\mathbb{R}^n) \to W^{1,q}(\mathbb{R}^n)$, provided that $1 < p_1, \ldots, p_m < \infty$, $1/q = \sum_{i=1}^m 1/p_i - \alpha/n$ and $1 \leq q < \infty$. The main result we obtain answers a question originally posed by Chen and Liu in 2022. Our main result is new, even in the linear case $m = 1$.

Keywords Multilinear maximal commutator, fractional variants, Sobolev space, continuity.

MSC(2010) 42B25, 46E35.

1. Introduction

This note continues the study of multilinear maximal commutators and their fractional variants. More precisely, let $m \geq 1$, $0 \leq \alpha < mn$ and $\vec{b} = (b_1, \ldots, b_m)$ with each $b_j \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$. For $\vec{f} = (f_1, \ldots, f_m)$ with each $f_j \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, the multilinear fractional maximal commutator with $\vec{b}$ is defined by

$$\mathfrak{M}_{\alpha,\vec{b}}(\vec{f})(x) = \sum_{i=1}^m \mathfrak{M}^i_{\alpha,\vec{b}}(\vec{f})(x),$$

where

$$\mathfrak{M}^i_{\alpha,\vec{b}}(\vec{f})(x) = \sup_{r>0} \frac{1}{|B(x,r)|^{m-\alpha/n}} \int_{B(x,r)^m} |b_i(x) - b_i(y_i)| \prod_{j=1}^m |f_j(y_j)| d\vec{y},$$

where $B(x,r)^m = \overbrace{B(x,r) \times \cdots \times B(x,r)}^{m}$ and $d\vec{y} = dy_1 \cdots dy_m$. When $\alpha = 0$, the operator $\mathfrak{M}_{\alpha,\vec{b}}$ reduces to the usual multilinear maximal commutator $\mathfrak{M}_{\vec{b}}$. Particularly, when $m = 1$, the operator $\mathfrak{M}_{\alpha,\vec{b}}$ becomes the fractional maximal commutator $M_{b,\alpha}$. Meanwhile, the operator $\mathfrak{M}_{\vec{b}}$ is just the maximal commutator M_b.

[†]The corresponding author.
[1]College of Mathematics and System Science, Shandong University of Science and Technology, Qingdao, Shandong 266590, China
[*]This work was supported partly by the National Natural Science Foundation of China (Grant No. 12326371) and the Natural Science Foundation of Shandong Province (Grant No. ZR2023MA022).
Email: jxx15166021527@163.com(X. Jiang), FLiu@sdust.edu.cn(F. Liu)

Recently Chen and Liu [8] established the Sobolev bounds of $\mathfrak{M}_{\alpha,\vec{b}}$ with Lipschitz symbols. The purpose of this note is to establish the Sobolev continuity of $\mathfrak{M}_{\alpha,\vec{b}}$ with Lipschitz symbols. Before stating our main theorem, we introduce some notation and recall relevant results from the literature.

The regularity theory of maximal operators began with Kinnunen's work [22] in which the author observed that the centered Hardy–Littlewood maximal operator

$$Mf(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)|dy,$$

is bounded on the first order Sobolev space $W^{1,p}(\mathbb{R}^n)$ for $1 < p \leq \infty$, where $B(x,r)$ is the open ball in $\mathbb{R}^n$ centered at x with radius r and $|B(x,r)|$ denotes its volume. In [22], Kinnunen used the above bounds to obtain a weak type inequality for the Sobolev capacity, which can be used to study the pointwise behaviour of Sobolev functions by the standard methods (see [12]). Here $W^{1,p}(\mathbb{R}^n)$ is the set of all measurable functions $f\colon \mathbb{R}^n \to \mathbb{R}$ satisfying

$$\|f\|_{W^{1,p}(\mathbb{R}^n)} := \|f\|_{L^p(\mathbb{R}^n)} + \||\nabla f|\|_{L^p(\mathbb{R}^n)} < \infty,$$

where $\nabla f = (D_1 f, \ldots, D_n f)$ is the weak gradient of f. Since then, Kinnunen's work [22] has initiated a new research direction in harmonic analysis. There are many extensions of his work. For example, see [21,23] for the local case, [24] for the fractional case, [6,28] for the multisublinear case. On the other hand, due to the lack of the sublinearity for the weak derivative of maximal functions, the continuity of $M\colon W^{1,p}(\mathbb{R}^n) \to W^{1,p}(\mathbb{R}^n)$ for $1 < p < \infty$ is certainly a nontrivial issue. This was posed by Hajłasz and Onninen in [21] and was addressed in the affirmative by Luiro [32]. Later on, Luiro's result was extended to the local case in [33], to the bilinear case in [6] and to the fractional and multilinear case in [26]. Since the above results do not include the endpoint case $p = 1$, the $W^{1,1}$-regularity of maximal operators has also been studied by many authors. We can consult [2,4,19], among others.

The maximal commutator and its fractional variant have been the subject of many recent articles in harmonic analysis. This topic began with Garcia-Cuerva et al. [17] who introduced the maximal commutator and used its L^p bounds to characterize $\mathrm{BMO}(\mathbb{R}^n)$ functions. It is worth noticing that the maximal commutator plays an important role in the study of commutators of singular integral operators with BMO symbols (see, for instance, [36,37]). In fact, the maximal commutators can also be used to characterize the Lipschitz space (see [43,44]). The investigation on the fractional maximal commutators has attracted the attention of many authors (see [9,10,20]). Other interesting works can be found in [1,3,41,42], among others. For more progresses on commutators of some integrals, we refer to the papers (see [5,7,11,13–16,18,35,38,40]). Commutators in multilinear settings were first studied by Pérez and Torres [34] and were later developed by many authors (see [25] et al.). Particularly, the multilinear maximal commutators associated to cubes were first introduced by Zhang [45] who investigated the multiple weighted estimates for them.

Another extension of the regularity of maximal operators is to investigate the regularity of maximal commutators. In [31], Liu, Xue and Zhang first investigated the boundedness of maximal commutators with Sobolev symbols on the Sobolev spaces. Later on, the above result was extended to the fractional version in [29] and to the local case in [30]. Very recently, Liu and Wang [27] studied the Sobolev boundedness of maximal commutator and its fractional variant with Lipschitz symbols. More precisely, the authors proved that if $1 < p < \infty$, $0 \le \alpha < n/p$ and $1/q = 1/p - \alpha/n$, then $M_{b,\alpha}$ is bounded from $W^{1,p}(\mathbb{R}^n)$ to $W^{1,q}(\mathbb{R}^n)$, provided that b belongs to the inhomogeneous Lipschitz space $\mathrm{Lip}(\mathbb{R}^n)$. Here

$$\mathrm{Lip}(\mathbb{R}^n):\ = \{f : \mathbb{R}^n \to \mathbb{C} \text{ continuous} : \quad \|f\|_{\mathrm{Lip}(\mathbb{R}^n)} < \infty\},$$

where

$$\|f\|_{\mathrm{Lip}(\mathbb{R}^n)} := \|f\|_{L^\infty(\mathbb{R}^n)} + \|f\|_{Lip(\mathbb{R}^n)}$$

and

$$\|f\|_{Lip(\mathbb{R}^n)} := \sup_{x\in\mathbb{R}^n} \sup_{h\in\mathbb{R}^n\setminus\{0\}} \frac{|f(x+h)-f(x)|}{|h|}.$$

The following presents the differentiable properties of the Lipschitz function.

Remark 1.1. Let $b \in \mathrm{Lip}(\mathbb{R}^n)$. It was pointed out in [27] that the weak partial derivatives $D_i b$, $i = 1, \dots, n$, exist almost everywhere. Moreover, we have that $D_i b(x) = \lim_{h\to 0} \frac{b(x+he_i)-b(x)}{h}$ and $|D_i b(x)| \le \|b\|_{Lip(\mathbb{R}^n)}$ for almost every $x \in \mathbb{R}^n$. Here $e_i = (0, \dots, 0, 1, 0, \dots, 0)$ is the canonical i-th base vector in $\mathbb{R}^n$ for $i = 1, \dots, n$.

Very recently, Chen and Liu [8] established the Sobolev continuity of multilinear maximal commutator and its fractional variant.

Theorem A ([8, Theorem 1]). *Let $0 \le \alpha < mn$, $1 \le q < \infty$, $1 < p_1, \dots, p_m < \infty$ and $1/q = \sum_{i=1}^m 1/p_i - \alpha/n$. If $\vec{b} = (b_1, b_2, \cdots, b_m)$ with each $b_i \in \mathrm{Lip}(\mathbb{R}^n)$, then the map*

$$\mathfrak{M}_{\alpha,\vec{b}}\colon W^{1,p_1}(\mathbb{R}^n) \times \cdots \times W^{1,p_m}(\mathbb{R}^n) \to W^{1,q}(\mathbb{R}^n) \tag{1.1}$$

is bounded.

Meanwhile, the authors of [8] posed the following question.

Question 1.1. *It the map* (1.1) *continuous under the conditions of Theorem A?*

This is the main motivation of this note. In the present paper we shall provide a positive answer to the above question.

Theorem 1.1. *Let $0 \le \alpha < mn$, $1 \le q < \infty$, $1 < p_1, \dots, p_m < \infty$ and $1/q = \sum_{i=1}^m 1/p_i - \alpha/n$ and $\vec{b} = (b_1, b_2, \cdots, b_m)$ with each $b_i \in \mathrm{Lip}(\mathbb{R}^n)$. Then the map* (1.1) *is continuous.*

In order to prove Theorem 1.1, the following facts are very useful.

Remark 1.2. Let $0 \le \alpha < mn$, $1 < p_1, \dots, p_m < \infty$, $1/q = 1/p_1 + \cdots + 1/p_m - \alpha/n$ and $1 \le q < \infty$. Let $\vec{b} = (b_1, \dots, b_m)$ with each $b_j \in L^\infty(\mathbb{R}^n)$.

(i) Let us fix $i \in \{1, \dots, m\}$ and $\vec{f} = (f_1, \dots, f_m)$ with each $f_j \in L^{p_j}(\mathbb{R}^n)$. It was pointed out in [8, Remark 2] that

$$\|\mathfrak{M}^i_{\alpha,\vec{b}}(\vec{f})\|_{L^q(\mathbb{R}^n)} \le C_{\alpha,m,n,p_1,\dots,p_m} \|b_i\|_{L^\infty(\mathbb{R}^n)} \prod_{j=1}^m \|f\|_{L^{p_j}(\mathbb{R}^n)}. \tag{1.2}$$

Moreover, the map

$$\mathfrak{M}^i_{\alpha,\vec{b}}\colon L^{p_1}(\mathbb{R}^n) \times \cdots \times L^{p_m}(\mathbb{R}^n) \to L^q(\mathbb{R}^n)$$

is continuous (see [8, Remark 2]). We also point out that

$$|\mathfrak{M}^i_{\alpha,\vec{b}}(\vec{f}_j)-\mathfrak{M}^i_{\alpha,\vec{b}}(\vec{f})|\le\sum_{l=1}^m\mathfrak{M}^i_{\alpha,\vec{b}}(\vec{F}^j_l),\tag{1.3}$$

where $\vec{f}_j=(f_{1,j},\dots,f_{m,j})$ and $\vec{F}^j_l=(f_1,\dots,f_{l-1},f_{l,j}-f_l,f_{l+1,j},\dots,f_{m,j})$.

(ii) For $\vec{f}=(f_1,\dots,f_m)$ with each $f_j\in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, the multilinear fractional maximal operator $\mathfrak{M}_\alpha$ is defined by

$$\mathfrak{M}_\alpha(\vec{f})(x)=\sup_{r>0}\frac{1}{|B(x,r)|^{m-\alpha/n}}\prod_{j=1}^m\int_{\mathbb{R}^n}|f_j(y)|dy.$$

It is well known that

$$\|\mathfrak{M}_\alpha(\vec{f})\|_{L^q(\mathbb{R}^n)}\le C_{\alpha,m,n,p_1,\dots,p_m}\prod_{j=1}^m\|f\|_{L^{p_j}(\mathbb{R}^n)}.\tag{1.4}$$

We now introduce the structure of the paper. In Section 2 we present some preliminary notation and lemmas, which are the main ingredients of proving Theorem 1.1. The proof of Theorem 1.1 will be given in Section 3. We remark that the main ideas employed in the proof of Theorem 1.1 are motivated by [8, 27, 32].

Throughout this paper, the letter C will stand for positive constants not necessarily the same one at each occurrence but is independent of the essential variables. We write $C_{\alpha,\beta}$ for positive constants that depend on the parameters α,β.

2. Preliminaries

In this section we present some preliminary notation and lemmas, which are the main ingredients of proving Theorem 1.1.

2.1. Preliminary notaton

We denote $\mathbb{N}=\{1,2,\dots\}$. Given $A\subset\mathbb{R}^n$, we set $A^c=\mathbb{R}^n\setminus A$. For any suitable function $F(x,y)$ defined on $\mathbb{R}^n\times\mathbb{R}^n$, we let

$$\nabla_xF=(D_{1,x}F,\dots,D_{n,x}F),\quad \nabla_yF=(D_{1,y}F,\dots,D_{n,y}F),$$

were $D_{i,x}F$ (resp., $D_{i,y}F$) is the i-th weak partial derivative of F in x (resp., y). For convenience, for suitable functions b, g, any $x\in\mathbb{R}^n$ and $r>0$, we set

$$u_{r,g}(x)=\int_{B(x,r)}g(y)dy,\qquad u_{b,r,g}(x)=\int_{B(x,r)}|b(x)-b(y)|g(y)dy.$$

In what follows, let $\vec{b}=(b_1,\dots,b_m)$ with each $b_1\in\mathrm{Lip}(\mathbb{R}^n)$ and $\vec{f}=(f_1,\dots,f_m)$ with each $f_j\in L^{p_j}(\mathbb{R}^n)$, where $1<p_1,\dots,p_m<\infty$, $1\le q<\infty$ and $1/q=\sum_{i=1}^m1/p_i-\alpha/n$. For each fixed point $x\in\mathbb{R}^n$, we define an auxiliary function $A^1_{\vec{b},\alpha,x,\vec{f}}\colon[0,\infty)\to\mathbb{R}$ by

$$A^1_{\vec{b},\alpha,x,\vec{f}}(r)=\begin{cases}0, & \text{if } r=0;\\ \dfrac{1}{|B(x,r)|^{m-\alpha/n}}u_{b_1,r,f_1}(x)\displaystyle\prod_{i=2}^m u_{r,f_i}(x), & \text{if } r\in(0,\infty).\end{cases}$$

We define the set $\mathcal{R}_{\vec{b},\alpha}(\vec{f})(x)$ by

$$\mathcal{R}_{\vec{b},\alpha}(\vec{f})(x) := \Big\{r \geq 0 : \ \mathfrak{M}^1_{\alpha,\vec{b}}(\vec{f})(x) = \limsup_{r_k \to r} A^1_{\vec{b},\alpha,x,\vec{f}}(r_k) \text{ for some } r_k > 0\Big\}.$$

It should be pointed out that the function $A^1_{\vec{b},\alpha,x,\vec{f}}(r)$ is continuous on $(0,\infty)$ for all $x \in \mathbb{R}^n$ and at $r = 0$ for almost every $x \in \mathbb{R}^n$. Since

$$A^1_{\vec{b},\alpha,x,\vec{f}}(r) \leq (|b_1(x)| + \|b_1\|_{L^\infty(\mathbb{R}^n)}) \prod_{i=1}^{m} \|f_i\|_{L^{p_i}(\mathbb{R}^n)} |B(x,r)|^{-1/q},$$

we can get that $\lim_{r\to\infty} A^1_{\vec{b},\alpha,x,\vec{f}}(r) = 0$, Note that $\mathcal{R}_{\vec{b},\alpha}(\vec{f})(x)$ is always closed and, from the above, nonempty. Also,

$$\mathfrak{M}^1_{\alpha,\vec{b}}(\vec{f})(x) = A^1_{\vec{b},\alpha,x,\vec{f}}(r) \text{ for every } x \in \mathbb{R}^n \text{ such that } 0 < r \in \mathcal{R}_{\vec{b},\alpha}(\vec{f})(x),$$
$$\mathfrak{M}^1_{\alpha,\vec{b}}(\vec{f})(x) = A^1_{\vec{b},\alpha,x,\vec{f}}(0) \text{ for almost every } x \in \mathbb{R}^n \text{ such that } 0 \in \mathcal{R}_{\vec{b},\alpha}(\vec{f})(x).$$

Let $f \in L^p(\mathbb{R}^n)$ for $1 < p < \infty$. For all $h \in \mathbb{R}$, $|h| > 0$, $y \in \mathbb{R}^n$ and $l \in \{1, 2, \dots, m\}$, we define the functions $f^l_{\tau(h)}$ and f^l_h by

$$f^l_h(x) = \frac{f(x + he_l) - f(x)}{h} \quad \text{and} \quad f^l_{\tau(h)}(x) = f(x + he_l).$$

It is well known that $\|f^l_{\tau(h)} - f\|_{L^p(\mathbb{R}^n)} \to 0$ as $h \to 0$, and if $f \in W^{1,p}(\mathbb{R}^n)$, then $\|f^l_h - D_l f\|_{L^p(\mathbb{R}^n)} \to 0$ as $h \to 0$.

2.2. Derivative formulas of multilinear maximal commutators

In this subsection we establish some derivative formulas for multilinear maximal commutators, which play a key role in the proof of Theorem 1.1. Before that, it is necessary for us to explore some nice properties of $\mathcal{R}_{\vec{b},\alpha}(\vec{f})$. For $R > 0$, we denote by B_R the ball of radius R centered at the origin. For $A \subset \mathbb{R}^n$ and $x \in \mathbb{R}^n$, we set

$$d(x,A) := \inf_{a\in A} |x - a| \quad \text{and} \quad A_{(\lambda)} := \{x \in \mathbb{R}^n; d(x,A) \leq \lambda\} \quad \text{for } \lambda \geq 0.$$

Lemma 2.1. *Let $0 \leq \alpha < mn$, $1 < p_1, \dots, p_m < \infty$, $1 \leq q < \infty$, $1/q = \sum_{i=1}^m 1/p_i - \alpha/n$ and $\vec{b} = (b_1, \dots, b_m)$ with each $b_j \in L^\infty(\mathbb{R}^n)$. Let $\vec{f} = (f_1, \dots, f_m)$ with each $f_i \in L^{p_i}(\mathbb{R}^n)$ and $\vec{f_j} = (f_{1,j}, \dots, f_{m,j})$ with each $f_{i,j} \in L^{p_i}(\mathbb{R}^n)$ for any $j \geq 1$. Assume that $\|f_{i,j} - f_i\|_{L^{p_i}(\mathbb{R}^n)} \to 0$ as $j \to \infty$ for all $i = 1, 2, \dots, m$. Then for all $R > 0$ and $\lambda > 0$, we have*

$$\lim_{j\to\infty} |\{x \in B(0,R); \mathcal{R}_{\vec{b},\alpha}(\vec{f_j})(x) \not\subseteq \mathcal{R}_{\vec{b},\alpha}(\vec{f})(x)_{(\lambda)}\}| = 0. \tag{2.1}$$

Proof. We shall adopt the method of [32, Lemma 2.2] to prove this lemma. In what follows, let us fix $\lambda > 0$, $R > 0$ and $\epsilon \in (0,1)$. An argument similar to those used to derive [32, Lemma 2.2] gives that for any $j \in \mathbb{Z}$, the set $\{x \in \mathbb{R}^n;\ \mathcal{R}_{\vec{b},\alpha}(\vec{f_j})(x) \not\subseteq \mathcal{R}_{\vec{b},\alpha}(\vec{f})(x)_{(\lambda)}\}$ is measurable. In addition, for almost every $x \in B(0,R)$, we can find $\gamma(x) \in \mathbb{N}$ such that

$$A^1_{\vec{b},\alpha,x,\vec{f}}(r) < \mathfrak{M}^1_{\alpha,\vec{b}}(\vec{f})(x) - \frac{1}{\gamma(x)} \quad \text{for all } r \notin \mathcal{R}_{\vec{b},\alpha}(\vec{f})(x)_{(\lambda)}.$$

From the above we can find $\gamma = \gamma(R, \lambda, \epsilon) \in \mathbb{N}$ and a measurable set E with $|E| < \epsilon$ such that

$$\begin{aligned} &B(0,R) \subset \{x \in \mathbb{R}^n : A^1_{\vec{b},\alpha,x,\vec{f}}(r) < \mathfrak{M}^1_{\alpha,\vec{b}}(\vec{f})(x) - \gamma^{-1} \\ &\text{for all } r \notin \mathcal{R}_{\vec{b},\alpha}(\vec{f})(x)_{(\lambda)}\} \cup E. \end{aligned} \tag{2.2}$$

Set

$$\begin{aligned} H_{1,j} &:= \{x \in \mathbb{R}^n : |\mathfrak{M}^1_{\alpha,\vec{b}}(\vec{f}_j)(x) - \mathfrak{M}^1_{\alpha,\vec{b}}(\vec{f})(x)| \geq (4\gamma)^{-1}\}, \\ H_{2,j} &:= \{x \in \mathbb{R}^n : |A^1_{\vec{b},\alpha,x,\vec{f}_j}(r) - A^1_{\vec{b},\alpha,x,\vec{f}}(r)| \geq (2\gamma)^{-1} \ \text{ for all } r \notin \mathcal{R}_{\vec{b},\alpha}(\vec{f})(x)_{(\lambda)}\}, \\ H_{3,j} &:= \{x \in \mathbb{R}^n : A^1_{\vec{b},\alpha,x,\vec{f}_j}(r) < \mathfrak{M}^1_{\alpha,\vec{b}}(\vec{f}_j)(x) - (4\gamma)^{-1} \ \text{ for all } r \notin \mathcal{R}_{\vec{b},\alpha}(\vec{f})(x)_{(\lambda)}\}. \end{aligned}$$

Clearly,

$$\{x \in \mathbb{R}^n : A^1_{\vec{b},\alpha,x,\vec{f}}(r) < \mathfrak{M}^1_{\alpha,\vec{b}}(\vec{f})(x) - \gamma^{-1} \ \text{ for all } r \notin \mathcal{R}_{\vec{b},\alpha}(\vec{f})(x)_{(\lambda)}\} \subset \bigcup_{i=1}^{3} H_{i,j}.$$

Combining the above with (2.2) implies that

$$\{x \in B(0,R); \mathcal{R}_{\vec{b},\alpha}(\vec{f}_j)(x) \not\subseteq \mathcal{R}_{\vec{b},\alpha}(\vec{f})(x)_{(\lambda)}\} \subset H_{1,j} \cup H_{2,j} \cup E \tag{2.3}$$

since $H_{3,j} \subset \{x \in \mathbb{R}^n : \mathcal{R}_{\vec{b},\alpha}(\vec{f}_j)(x) \subset \mathcal{R}_{\vec{b},\alpha}(\vec{f})(x)_{(\lambda)}\}$. In view of (1.3), one has that for any $x \in \mathbb{R}^n$,

$$|\mathfrak{M}^1_{\alpha,\vec{b}}(\vec{f}_j)(x) - \mathfrak{M}^1_{\alpha,\vec{b}}(\vec{f})(x)| \leq \sum_{l=1}^{m} \mathfrak{M}^1_{\alpha,\vec{b}}(\vec{F}^j_l)(x), \tag{2.4}$$

where $\vec{F}^j_l = (f_1, \ldots, f_{l-1}, f_{l,j} - f_l, f_{l+1,j}, \ldots, f_{m,j})$. By our assumption, there exists $N_0 = N_0(\epsilon) \in \mathbb{N}$ such that $\|f_{i,j} - f_i\|_{L^{p_i}(\mathbb{R}^n)} < \frac{\epsilon}{\gamma}$ and $\|f_{i,j}\|_{L^{p_i}(\mathbb{R}^n)} \leq \|f_i\|_{L^{p_i}(\mathbb{R}^n)} + 1$ for any $j \geq N_0$ and $i = 1, 2, \ldots, m$. These facts together with (1.2) and (2.4) imply that

$$\begin{aligned} |H_{1,j}| &\leq \Big|\Big\{x \in \mathbb{R}^n; \sum_{l=1}^{m} \mathfrak{M}^1_{\alpha,\vec{b}}(\vec{F}^j_l)(x) \geq (4\gamma)^{-1}\Big\}\Big| \\ &\leq \sum_{l=1}^{m} |\{x \in \mathbb{R}^n; \mathfrak{M}^1_{\alpha,\vec{b}}(\vec{F}^j_l)(x) \geq (4m\gamma)^{-1}\}| \\ &\leq \sum_{l=1}^{m} (4m\gamma)^q \|\mathfrak{M}^1_{\alpha,\vec{b}}(\vec{F}^j_l)\|^q_{L^q(\mathbb{R}^n)} \\ &\leq C_{\alpha,m,n,p_1,\ldots,p_m} (4m\gamma)^q \|b_1\|^q_{L^\infty(\mathbb{R}^n)} \sum_{l=1}^{m} \|f_{l,j} - f_l\|^q_{L^{p_j}(\mathbb{R}^n)} \\ &\quad \times \prod_{\mu=1}^{l-1} \|f_\mu\|^q_{L^{p_\mu}(\mathbb{R}^n)} \prod_{\nu=l+1}^{m} \|f_\nu\|^q_{L^{p_\nu}(\mathbb{R}^n)} \\ &\leq C_{\alpha,m,n,p_1,\ldots,p_m} \|b_1\|^q_{L^\infty(\mathbb{R}^n)} \prod_{\mu=1}^{m} (1 + \|f_\mu\|_{L^{p_\mu}(\mathbb{R}^n)})^q \epsilon^q, \end{aligned} \tag{2.5}$$

for any $j \geq N_0$. We also note that, as in (1.3),

$$|A^1_{\vec{b},\alpha,x,\vec{f_j}}(r) - A^1_{\vec{b},\alpha,x,\vec{f}}(r)| \leq \sum_{l=1}^{m} \mathfrak{M}^1_{\alpha,\vec{b}}(\vec{F}^j_l)(x),$$

for all $r \in [0,\infty)$. It follows that

$$H_{2,j} \subset \Big\{x \in \mathbb{R}^n : \sum_{l=1}^{m} \mathfrak{M}^1_{\alpha,\vec{b}}(\vec{F}^j_l)(x) \geq (4\gamma)^{-1}\Big\}.$$

An argument similar to (2.5) gives that

$$|H_{2,j}| \leq C_{\alpha,m,n,p_1,\ldots,p_m}\|b_1\|^q_{L^\infty(\mathbb{R}^n)} \prod_{\mu=1}^{m}(1 + \|f_\mu\|_{L^{p_\mu}(\mathbb{R}^n)})^q\epsilon^q, \tag{2.6}$$

for any $j \geq N_0$. Then we get from (2.3), (2.5) and (2.6) that

$$|\{x \in B(0,R); \mathcal{R}_{\vec{b},\alpha}(\vec{f_j})(x) \nsubseteq \mathcal{R}_{\vec{b},\alpha}(\vec{f})(x)_{(\lambda)}\}| \leq C_{\alpha,m,n,p_1,\ldots,p_m,b_1,f_1,\ldots,f_m}\epsilon,$$

for any $j \geq N_0$. This leads to (2.1) and completes the proof. □

Let A, B be subsets of $\mathbb{R}^n$. The Hausdorff distance of A and B is defined by

$$\pi(A,B) := \inf\{\delta > 0 : A \subset B_{(\delta)} \text{ and } B \subset A_{(\delta)}\}.$$

Applying Lemma 2.1 and an argument similar to that in the proof of [32, Corollary 2.3], we have the following.

Lemma 2.2. *Let $1 < p_1,\ldots,p_m < \infty$, $1 \leq q < \infty$, $1/q = \sum_{i=1}^m 1/p_i - \alpha/n$ and $\vec{b} = (b_1,\ldots,b_m)$ with each $b_j \in \mathrm{Lip}(\mathbb{R}^n)$. Let $\vec{f} = (f_1,\ldots,f_m)$ with each $f_j \in L^{p_j}(\mathbb{R}^n)$. Then for all $R > 0$, $\lambda > 0$ and $l \in \{1,2,\ldots,n\}$, we have*

$$|\{x \in B(0,R); \pi(\mathcal{R}_{\vec{b},\alpha}(\vec{f})(x), \mathcal{R}_{\vec{b},\alpha}(\vec{f})(x+he_l)) > \lambda\}| \to 0 \text{ as } h \to 0. \tag{2.7}$$

Proof. Fix $l \in \{1,2,\ldots,n\}$, $\lambda > 0$ and $R > 0$. For (2.7) it is enough to show

$$|\{x \in B(0,R); \mathcal{R}_{\vec{b},\alpha}(\vec{f})(x+he_l) \nsubseteq \mathcal{R}_{\vec{b},\alpha}(\vec{f})(x)_{(\lambda)}\}| \to 0 \text{ as } h \to 0 \tag{2.8}$$

and

$$|\{x \in B(0,R); \mathcal{R}_{\vec{b},\alpha}(\vec{f})(x) \nsubseteq \mathcal{R}_{\vec{b},\alpha}(\vec{f})(x+he_l)_{(\lambda)}\}| \to 0 \text{ as } h \to 0. \tag{2.9}$$

At first we prove (2.8). The proof of (2.8) is similar to that of Lemma 2.1. By a change of variable, one has that $\mathcal{R}_{\vec{b},\alpha}(\vec{f})(x+he_l) = \mathcal{R}_{\vec{b}^l_{\tau(h)},\alpha}(\vec{f}^l_{\tau(h)})(x)$. Here

$$\vec{b}^l_{\tau(h)} = ((b_1)^l_{\tau(h)},\ldots,(b_m)^l_{\tau(h)}), \quad \vec{f}^l_{\tau(h)} = ((f_1)^l_{\tau(h)},\ldots,(f_m)^l_{\tau(h)}).$$

Thus, (2.8) is equivalent to

$$|\{x \in B(0,R); \mathcal{R}_{\vec{b}^l_{\tau(h)},\alpha}(\vec{f}^l_{\tau(h)})(x) \nsubseteq \mathcal{R}_{\vec{b},\alpha}(\vec{f})(x)_{(\lambda)}\}| \to 0 \text{ as } h \to 0. \tag{2.10}$$

Let $\epsilon \in (0,1)$. By the proof of Lemma 2.1, we can conclude that for any $h \in \mathbb{R}$, the set $\{x \in \mathbb{R}^n; \mathcal{R}_{\vec{b}^l_{\tau(h)},\alpha}(\vec{f}^l_{\tau(h)})(x) \not\subseteq \mathcal{R}_{\vec{b},\alpha}(\vec{f})(x)_{(\lambda)}\}$ is measurable. Moreover, there exist $\gamma = \gamma(R,\lambda,\epsilon) \in \mathbb{N}$ and a measurable set E with $|E| < \epsilon$ such that

$$\begin{aligned} &B(0,R) \subset \{x \in \mathbb{R}^n : A^1_{\vec{b},\alpha,x,\vec{f}}(r) < \mathfrak{M}^1_{\vec{b},\alpha}\vec{f}(x) - \gamma^{-1} \\ &\text{for all } r \notin \mathcal{R}_{\vec{b},\alpha}(\vec{f})(x)_{(\lambda)}\} \cup E. \end{aligned} \tag{2.11}$$

Let $h \in \mathbb{R}$ and set

$$\begin{aligned} H_{1,h} &:= \{x \in \mathbb{R}^n : |\mathfrak{M}^1_{\vec{b}^l_{\tau(h)},\alpha}\vec{f}^l_{\tau(h)}(x) - \mathfrak{M}^1_{\vec{b},\alpha}\vec{f}(x)| \geq (4\gamma)^{-1}\}, \\ H_{2,h} &:= \{x \in \mathbb{R}^n : |A^1_{\vec{b}^l_{\tau(h)},\alpha,x,\vec{f}^l_{\tau(h)}}(r) - A^1_{\vec{b},\alpha,x,\vec{f}}(r)| \geq (2\gamma)^{-1} \\ &\qquad \text{for some } r \notin \mathcal{R}_{\vec{b},\alpha}(\vec{f})(x)_{(\lambda)}\}, \\ H_{3,h} &:= \{x \in \mathbb{R}^n : A^1_{\vec{b}^l_{\tau(h)},\alpha,x,\vec{f}^l_{\tau(h)}}(r) < \mathfrak{M}^1_{\vec{b}^l_{\tau(h)},\alpha}\vec{f}^l_{\tau(h)}(x) - (4\gamma)^{-1} \\ &\qquad \text{for all } r \notin \mathcal{R}_{\vec{b},\alpha}(\vec{f})(x)_{(\lambda)}\}. \end{aligned}$$

We note that

$$H_{3,h} \subset \{x \in \mathbb{R}^n : \mathcal{R}_{\vec{b}^l_{\tau(h)},\alpha}(\vec{f}^l_{\tau(h)})(x) \subset \mathcal{R}_{\vec{b},\alpha}(\vec{f})(x)_{(\lambda)}\} \tag{2.12}$$

and

$$\begin{aligned} &\{x \in \mathbb{R}^n : A^1_{\vec{b},\alpha,x,\vec{f}}(r) < \mathfrak{M}^1_{\vec{b},\alpha}\vec{f}(x) - \gamma^{-1} \ \text{ for all } r \notin \mathcal{R}_{\vec{b},\alpha}(\vec{f})(x)_{(\lambda)}\} \\ &\subset \bigcup_{i=1}^{3} H_{i,h}. \end{aligned} \tag{2.13}$$

It follows from (2.11)–(2.13) that

$$\{x \in B(0,R); \mathcal{R}_{\vec{b}^l_{\tau(h)},\alpha}(\vec{f}^l_{\tau(h)})(x) \not\subseteq \mathcal{R}_{\vec{b},\alpha}(\vec{f})(x)_{(\lambda)}\} \subset H_{1,h} \cup H_{2,h} \cup E. \tag{2.14}$$

We notice that

$$\begin{aligned} &|\mathfrak{M}^1_{\alpha,\vec{b}^l_{\tau(h)}}(\vec{f}^l_{\tau(h)})(x) - \mathfrak{M}^1_{\alpha,\vec{b}}(\vec{f})(x)| \\ &\leq \sup_{r>0} \frac{1}{|B(x,r)|^{m-\alpha/n}} \Big| u_{(b_1)^l_{\tau(h)},r,(|f_1|)^l_{\tau(h)}}(x) \prod_{i=2}^{m} u_{r,(|f_i|)^l_{\tau(h)}}(x) \\ &\quad - u_{b_1,r,|f_1|}(x) \prod_{i=2}^{m} u_{r,|f_i|}(x) \Big| \\ &\leq \sup_{r>0} \frac{1}{|B(x,r)|^{m-\alpha/n}} \Big(|u_{(b_1)^l_{\tau(h)},r,(|f_1|)^l_{\tau(h)}}(x) - u_{b_1,r,|f_1|}(x)| \prod_{i=2}^{m} u_{r,(|f_i|)^l_{\tau(h)}}(x) \\ &\quad + u_{b_1,r,|f_1|}(x) \Big| \prod_{i=2}^{m} u_{r,(|f_i|)^l_{\tau(h)}}(x) - \prod_{i=2}^{m} u_{r,|f_i|}(x) \Big| \Big). \end{aligned}$$

Moreover,

$$\begin{aligned} &\Big| \prod_{i=2}^{m} u_{r,(|f_i|)^l_{\tau(h)}}(x) - \prod_{i=2}^{m} u_{r,|f_i|}(x) \Big| \\ &\leq \sum_{i=2}^{m} u_{r,|(f_i)^l_{\tau(h)} - f_i|}(x) \prod_{\nu=i+1}^{m} u_{r,(|f_\nu|)^l_{\tau(h)}}(x) \prod_{\mu=2}^{i-1} u_{r,|f_\mu|}(x), \end{aligned}$$

and

$$|u_{(b_1)^l_{\tau(h)},r,(|f_1|)^l_{\tau(h)}}(x)-u_{b_1,r,|f_1|}(x)| \le u_{(b_1)^l_{\tau(h)},r,|(f_1)^l_{\tau(h)}-f_1|}(x)+u_{(b_1)^l_{\tau(h)}-b_1,r,|f_1|}(x).$$

Hence, we have

$$\begin{aligned}
&\left|\mathfrak{M}^1_{\alpha,\vec{b}^l_{\tau(h)}}(\vec{f}^l_{\tau(h)})(x) - \mathfrak{M}^1_{\alpha,\vec{b}}(\vec{f})(x)\right| \\
&\le \mathfrak{M}^1_{\alpha,\vec{b}^l_{\tau(h)}}(\vec{A}_{h,l})(x) + 2\|b_1\|_{Lip(\mathbb{R}^n)}\mathfrak{M}_\alpha(\vec{B}_{h,l})(x)|h| + \sum_{i=2}^m \mathfrak{M}^1_{\alpha,\vec{b}}(\vec{D}^i_{l,h})(x) =: \Theta_h(x),
\end{aligned}$$

where

$$\begin{aligned}
&\vec{A}_{h,l} = ((f_1)^l_{\tau(h)} - f_1, (f_2)^l_{\tau(h)}, \dots, (f_m)^l_{\tau(h)}), \quad \vec{B}_{h,l} = (f_1, (f_2)^l_{\tau(h)}, \dots, (f_m)^l_{\tau(h)}), \\
&\vec{D}^i_{l,h} = (f_1, \cdots, f_{i-1}, (f_i)^l_{\tau(h)} - f_i, (f_{i+1})^l_{\tau(h)}, \cdots, (f_m)^l_{\tau(h)}).
\end{aligned}$$

Similarly we can conclude that

$$|A^1_{\vec{b}^l_{\tau(h)},\alpha,x,\vec{f}^l_{\tau(h)}}(r) - A^1_{\vec{b},\alpha,x,\vec{f}}(r)| \le \Theta_h(x),$$

for all $r \in [0,\infty)$. Hence, we have

$$\begin{aligned}
&|H_{1,h}| + |H_{2,h}| \\
&\le 2|\{x \in \mathbb{R}^n : \Theta_h(x) \ge (4\gamma)^{-1}\}| \\
&\le 2|\{x \in \mathbb{R}^n : \mathfrak{M}^1_{\alpha,\vec{b}^l_{\tau(h)}}(\vec{A}_{h,l})(x) \ge (4(m+1)\gamma)^{-1}\}| \\
&\quad +2|\{x \in \mathbb{R}^n : 2\|b_1\|_{Lip(\mathbb{R}^n)}\mathfrak{M}_\alpha(\vec{B}_{h,l})(x)|h| \ge (4(m+1)\gamma)^{-1}\}| \\
&\quad +2\sum_{i=2}^m |\{x \in \mathbb{R}^n : \mathfrak{M}^1_{\alpha,\vec{b}}(\vec{D}^i_{l,h})(x) \ge (4(m+1)\gamma)^{-1}\}|.
\end{aligned} \tag{2.15}$$

Note that $\|(f_i)^l_{\tau(h)} - f_i\|_{L^{p_i}(\mathbb{R}^n)} \to 0$ as $h \to 0$ for all $i = 1, 2, \dots, m$. Then there exists $\delta > 0$ such that $\|(f_i)^l_{\tau(h)} - f_i\|_{L^{p_l}(\mathbb{R}^n)} < \frac{\epsilon}{\gamma}$ and $\|(f_i)^l_{\tau(h)}\|_{L^{p_l}(\mathbb{R}^n)} \le \|f_i\|_{L^{p_i}(\mathbb{R}^n)} + 1$ for any $i = 1, 2, \dots, m$. The above facts together with (1.2), (1.4) and (2.15) imply that

$$\begin{aligned}
&|H_{1,h}| + |H_{2,h}| \\
&\le 2(4(m+1)\gamma)^q \|\mathfrak{M}^1_{\alpha,\vec{b}^l_{\tau(h)}}(\vec{A}_{h,l})\|^q_{L^q(\mathbb{R}^n)} + 2(4(m+1)\gamma)^q \sum_{i=2}^m \|\mathfrak{M}^1_{\alpha,\vec{b}}(\vec{D}^i_{l,h})\|^q_{L^q(\mathbb{R}^n)} \\
&\quad +2(8(m+1)\gamma|h|\|b_1\|_{Lip(\mathbb{R}^n)})^q \|\mathfrak{M}_\alpha(\vec{B}_{h,l})\|^q_{L^q(\mathbb{R}^n)} \\
&\le C_{\alpha,m,n,p_1,\dots,p_m} \|b_1\|^q_{L^\infty(\mathbb{R}^n)} \gamma^q \|(f_1)^l_{\tau(h)} - f_1\|^q_{L^{p_1}(\mathbb{R}^n)} \prod_{i=2}^m \|(f_i)^l_{\tau(h)}\|^q_{L^{p_i}(\mathbb{R}^n)} \\
&\quad +C_{\alpha,m,n,p_1,\dots,p_m} \|b_1\|^q_{Lip(\mathbb{R}^n)} \|f_1\|^q_{L^{p_1}(\mathbb{R}^n)} \prod_{i=2}^m \|(f_i)^l_{\tau(h)}\|^q_{L^{p_i}(\mathbb{R}^n)} \epsilon^q \\
&\quad +C_{\alpha,m,n,p_1,\dots,p_m} \gamma^q \sum_{i=2}^m \prod_{\mu=1}^{i-1} \|f_\mu\|^q_{L^{p_\mu}(\mathbb{R}^n)} \|(f_i)^l_{\tau(h)} - f_i\|^q_{L^{p_i}(\mathbb{R}^n)} \\
&\quad \times \prod_{\nu=i+1}^m \|(f_\nu)^l_{\tau(h)}\|^q_{L^{p_\nu}(\mathbb{R}^n)} \\
&\le C_{\alpha,m,n,p_1,\dots,p_m,b_1,f_1,\dots,f_m} \epsilon^q,
\end{aligned}$$

when $|h| < \min\{\delta, \gamma^{-1}\epsilon\}$. Here the above constant $C_{\alpha,m,n,p_1,\dots,p_m,b_1,f_1,\dots,f_m}$ is independent of γ and ϵ. This together with (2.14) gives (2.10).

Next we prove (2.9). It is easy to see that $\mathcal{R}_{\vec{b},\alpha}(\vec{f})(x) = \mathcal{R}_{\vec{b}^l_{\tau(-h)},\alpha}(\vec{f}^l_{\tau(-h)})(x + he_l)$. It follows that if $|h| < 1$, then

$$\begin{aligned}&\{x \in B(0,R); \mathcal{R}_{\vec{b},\alpha}(\vec{f})(x) \not\subseteq \mathcal{R}_{\vec{b},\alpha}(\vec{f})(x + he_l)_{(\lambda)}\}\\ \subset\ &\{x \in B(0,R); \mathcal{R}_{\vec{b}^l_{\tau(-h)},\alpha}(\vec{f}^l_{\tau(-h)})(x + he_l) \not\subseteq \mathcal{R}_{\vec{b},\alpha}(\vec{f})(x + he_l)_{(\lambda)}\}\\ \subset\ &\{x \in B(0,R+1); \mathcal{R}_{\vec{b}^l_{\tau(-h)},\alpha}(\vec{f}^l_{\tau(-h)})(x) \not\subseteq \mathcal{R}_{\vec{b},\alpha}(\vec{f})(x)_{(\lambda)}\}.\end{aligned}$$

This together with (2.8) leads to (2.9). □

In order to prove Theorem 1.1, we need to establish some formulas for the derivatives of maximal commutators, which are the main ingredients of proving Theorem 1.1.

Lemma 2.3. *Let $0 \le \alpha < mn$, $1 < p_1, \dots, p_m < \infty$, $1 \le q < \infty$ and $1/q = \sum_{i=1}^m 1/p_i - \alpha/n$. Let $\vec{f} = (f_1, \dots, f_m)$ with each $f_j \in W^{1,p_j}(\mathbb{R}^n)$ and $\vec{b} = (b_1, \dots, b_m)$ with each $b_j \in \mathrm{Lip}(\mathbb{R}^n)$. Let us fix $l \in \{1, 2, \dots, n\}$. Then*

(i) *For almost every $x \in \mathbb{R}^n$ if $0 < r \in \mathcal{R}_{\vec{b},\alpha}(\vec{f})(x)$, then*

$$\begin{aligned}&D_l\mathfrak{M}^1_{\vec{b},\alpha}(\vec{f})(x)\\ &= \frac{1}{|B(x,r)|^{m-\alpha/n}}\Big(\int_{B(x,r)} D_{l,y}F_{b_1}(x,y)|f_1|(y)dy\\ &+ \int_{B(x,r)} D_{l,x}F_{b_1}(x,y)|f_1|(y)dy + u_{b_1,r,D_l|f_1|}(x)\Big)\prod_{i=2}^m u_{r,|f_i|}(x)\\ &+ \frac{1}{|B(x,r)|^{m-\alpha/n}} u_{b_1,r,|f_1|}(x)\sum_{i=2}^m u_{r,D_l|f_i|}(x)\prod_{\substack{2\le\mu\le m,\\ \mu\ne i}} u_{r,|f_\mu|}(x),\end{aligned} \tag{2.16}$$

where $F_{b_1}(x,y) = b_1(x) - b_1(y)$.

(i) *For almost every $x \in \mathbb{R}^n$ if $0 \in \mathcal{R}_{\vec{b},\alpha}(\vec{f})(x)$, then*

$$D_l\mathfrak{M}^1_{\vec{b},\alpha}(\vec{f})(x) = 0. \tag{2.17}$$

Proof. Note that $\mathfrak{M}^1_{\vec{b},\alpha}(\vec{f}) = \mathfrak{M}^1_{\vec{b},\alpha}(|\vec{f}|)$, where $|\vec{f}| = (|f_1|, \dots, |f_m|)$. It was known that $|u| \in W^{1,p}(\mathbb{R}^n)$ and $|D_l|u||(x) = |D_l u|(x)$ for almost every $x \in \mathbb{R}^n$ and any $l \in \{1, 2, \dots, m\}$ if $u \in W^{1,p}(\mathbb{R}^n)$ for any $1 < p < \infty$. Hence, we may assume without loss of generality that all $f_j \ge 0$.

Let $A_1 := \{x \in \mathbb{R}^n; |b_1(x)| \le \|b_1\|_{L^\infty(\mathbb{R}^n)}\}$ and $A_2 := \{x \in \mathbb{R}^n; Mf_1(x) < \infty\}$. Clearly, $|A_1^c| = 0$. Note that $Mf_1 \in L^{p_1}(\mathbb{R}^n)$ because of $f_1 \in L^{p_1}(\mathbb{R}^n)$. Then $Mf_1(x) < \infty$ for almost every $x \in \mathbb{R}^n$. So $|A_2^c| = 0$. Let A_3 be the set of all $x \in \mathbb{R}^n$ for which x is the Lebesgue point of all $f_1, \dots, f_m$ and $D_l f_1, \dots, D_l f_m$. Clearly, $|A_3^c| = 0$. Let A_4 be the set of all $x \in \mathbb{R}^n$ for which b_1 is differentiable at x. Clearly, $|A_4^c| = 0$. Let $R > 0$. By Lemma 2.1, there exists a sequence $\{s_k\}_{k=1}^\infty$, $s_k > 0$ and $s_k \to 0$ such that $\lim_{k\to\infty} \pi(\mathcal{R}_{\vec{b},\alpha}(\vec{f})(x), \mathcal{R}_{\vec{b},\alpha}(\vec{f})(x + s_k e_l)) = 0$ for almost every $x \in B(0,R)$. Since $f_i \in W^{1,p_i}(\mathbb{R}^n)$, we have $\|(f_i)^l_{\tau(s_k)} - f_i\|_{L^{p_i}(\mathbb{R}^n)} \to 0$

and $\|(f_i)^l_{s_k} - D_l f_i\|_{L^{p_i}(\mathbb{R}^n)} \to 0$ as $k \to \infty$. Hence, we deduce that $\|M((f_i)^l_{\tau(s_k)} - f_i)\|_{L^{p_i}(\mathbb{R}^n)} \to 0$ and $\|M((f_i)^l_{s_k} - D_l f_i)\|_{L^{p_i}(\mathbb{R}^n)} \to 0$ as $k \to \infty$. On the other hand, we get by Theorem A that $\mathfrak{M}^1_{\alpha,\vec{b}}(\vec{f}) \in W^{1,q}(\mathbb{R}^n)$. Thus, we have that $\|(\mathfrak{M}^1_{\alpha,\vec{b}}(\vec{f}))^l_{s_k} - D_l \mathfrak{M}^1_{\alpha,\vec{b}}(\vec{f})\|_{L^q(\mathbb{R}^n)} \to 0$ as $k \to \infty$. Note that $f_1 \in L^{p_1}(\mathbb{R}^n)$ and $|(b_1)^l_{s_k}(x) - D_l b_1(x)| \leq 2\|b_1\|_{Lip(\mathbb{R}^n)}$ for almost every $x \in \mathbb{R}^n$. Applying the dominated convergence theorem, one has that $\||(b_1)^l_{s_k} - D_l b_1|f_1\|_{L^{p_1}(\mathbb{R}^n)} \to 0$ as $k \to \infty$. It follows that $\||M((b_1)^l_{s_k} - D_l b_1|f_1)\|_{L^{p_1}(\mathbb{R}^n)} \to 0$ as $k \to \infty$.

Let $F_{b_1}(x, y) = |b_1(x) - b_1(y)|$. It is clear that $F_{b_1}(x, \cdot) \in Lip(\mathbb{R}^n)$. Moreover, $\|F_{b_1}(x, \cdot)\|_{Lip(\mathbb{R}^n)} \leq \|b_1\|_{Lip(\mathbb{R}^n)}$ for all $x \in \mathbb{R}^n$. By Remark 1.1, for a given $x \in \mathbb{R}^n$ the function $F_{b_1}(x, \cdot)$ is differentiable for almost every $y \in \mathbb{R}^n$. For almost every $y \in \mathbb{R}^n$, we have that $|D_{l,y} F_{b_1}(x, y)| \leq \|b_1\|_{Lip(\mathbb{R}^n)}$. Similarly we see that $F_{b_1}(\cdot, y) \in Lip(\mathbb{R}^n)$ and $\|F_{b_1}(\cdot, y)\|_{Lip(\mathbb{R}^n)} \leq \|b_1\|_{Lip(\mathbb{R}^n)}$ for all $y \in \mathbb{R}^n$. Next, for a fixed $y \in \mathbb{R}^n$ the function $F_{b_1}(\cdot, y)$ is differentiable almost everywhere. Moreover, for almost every $x \in \mathbb{R}^n$, we have that $|D_{l,x} F_{b_1}(x, y)| \leq \|b_1\|_{Lip(\mathbb{R}^n)}$. For convenience, for given $h \in \mathbb{R} \setminus \{0\}$ we define

$$(F_{x,b_1})^l_h(y) = \frac{1}{h}(F_{b_1}(x, y + he_l) - F_{b_1}(x, y)),$$

$$(F_{y,b_1})^l_h(x) = \frac{1}{h}(F_{b_1}(x + he_l, y) - F_{b_1}(x, y)).$$

From the above, we can conclude that there exist a subsequence $\{h_k\}_{k=1}^\infty$ of $\{s_k\}_{k=1}^\infty$ and a measurable set $A_5 \subset B(0, R)$ with $|B(0, R) \setminus A_5| = 0$ such that for any $x \in A_5$, the following hold:

(i) $\lim_{k\to\infty}(\mathfrak{M}^1_{\alpha,\vec{b}}(\vec{f}))^l_{h_k}(x) = D_l \mathfrak{M}^1_{\alpha,\vec{b}}(\vec{f})(x)$; $\lim_{k\to\infty} M((f_i)^l_{\tau(h_k)} - f_i)(x) = 0$, $\lim_{k\to\infty} M((f_i)^l_{h_k} - D_l f_i)(x) = 0$, $\lim_{k\to\infty} M(|(b_1)^l_{h_k} - D_l b_1|f_1)(x) = 0$;

(ii) $\lim_{k\to\infty} \pi(\mathcal{R}_{\vec{b},\alpha}(\vec{f})(x), \mathcal{R}_{\vec{b},\alpha}(\vec{f})(x + h_k e_l)) = 0$;

(iii) $\lim\limits_{k\to\infty} (F_{y,b_1})^l_{h_k}(x) = D_{l,x} F_{b_1}(x, y)$ for any $y \in \mathbb{R}^n$.

Let

$$\begin{aligned}
A_6 &:= \{x \in \mathbb{R}^n : \mathfrak{M}^1_{\alpha,\vec{b}}(\vec{f})(x) = A^1_{\vec{b},\alpha,x,\vec{f}}(0) \text{ if } 0 \in \mathcal{R}_{\vec{b},\alpha}(\vec{f})(x)\},\\
A_7 &:= \bigcap_{k=1}^\infty \{x \in \mathbb{R}^n : \mathfrak{M}^1_{\alpha,\vec{b}}(\vec{f})(x + h_k e_l) = A^1_{\vec{b},\alpha,x+h_k e_l,\vec{f}}(0) \text{ if } 0 \in \mathcal{R}_{\vec{b},\alpha}(\vec{f})(x + h_k e_l)\},\\
A_8 &:= \Big\{x \in \mathbb{R}^n; \lim_{k\to\infty} (F_{y,b_1})^l_{h_k}(x) = D_{l,x} F_{b_1}(x, y) \ \text{ for a.e. } y \in \mathbb{R}^n\Big\}.
\end{aligned}$$

One can easily check that $|A_i^c| = 0$ for any $i \in \{6, 7, 8\}$. Set

$$A_9 := \Big\{x \in \mathbb{R}^n; \lim_{r\to 0^+} \frac{1}{|B(x, r)|} \int_{B(x,r)} |D_l b_1(x) - D_l b_1(y)| f_1(y) dy = 0\Big\}.$$

Note that $|D_l b_1(x) - D_l b_1(y)|f_1(y)| \leq 2\|b_1\|_{Lip(\mathbb{R}^n)} f_1(y)$ for any $x \in A_4$ and almost every $y \in \mathbb{R}^n$. Then by the Lebesgue differentiation theorem we have $|A_9^c| = 0$. So $|(\bigcap_{i=1}^9 A_i)^c| = 0$.

Let $x \in \bigcap_{i=1}^9 A_i$ and $r \in \mathcal{R}_{\vec{b},\alpha}(\vec{f})(x)$. There exists $r_k \in \mathcal{R}_{\vec{b},\alpha}(\vec{f})(x + h_k e_l)$ such that $\lim_{k\to\infty} r_k = r$. We can write

$$D_l \mathfrak{M}^1_{\vec{b},\alpha}(\vec{f})(x) = \lim_{k\to\infty} \frac{1}{h_k}(\mathfrak{M}^1_{\alpha,\vec{b}}(\vec{f})(x + h_k e_l) - \mathfrak{M}^1_{\alpha,\vec{b}}(\vec{f})(x)). \tag{2.18}$$

We consider two cases:

Case (i) ($r>0$). In this case we may assume without loss of generality that $r_k\in(0,2r)$ for all $k\geq 1$. At first we prove that

$$\begin{aligned}
&\lim_{k\to\infty}\frac{1}{h_k}(A^1_{\vec{b},\alpha,x+h_ke_l,\vec{f}}(r_k)-A^1_{\vec{b},\alpha,x,\vec{f}}(r_k))\\
=&\frac{1}{|B(x,r)|^{m-\alpha/n}}\Big(\int_{B(x,r)}D_{l,y}F_{b_1}(x,y)f_1(y)dy\\
&+\int_{B(x,r)}D_{l,x}F_{b_1}(x,y)f_1(y)dy+u_{b_1,r,D_lf_1}(x)\Big)\prod_{i=2}^m u_{r,f_i}(x)\\
&+\frac{1}{|B(x,r)|^{m-\alpha/n}}u_{b_1,r,f_1}(x)\sum_{i=2}^m u_{r,D_lf_i}(x)\prod_{\substack{2\leq\mu\leq m,\\ \mu\neq i}}u_{r,f_\mu}(x).
\end{aligned}\tag{2.19}$$

By a change of variable, it is not difficult to check that

$$A^1_{\vec{b},\alpha,x+h_ke_l,\vec{f}}(r_k)=\frac{1}{|B(x,r_k)|^{m-\alpha/n}}u_{(b_1)^l_{\tau(h_k)},r_k,(f_1)^l_{\tau(h_k)}}(x)\prod_{i=2}^m u_{r_k,(f_i)^l_{\tau(h_k)}}(x).$$

Note that

$$\begin{aligned}
&u_{(b_1)^l_{\tau(h_k)},r_k,(f_1)^l_{\tau(h_k)}}(x)\prod_{i=2}^m u_{r_k,(f_i)^l_{\tau(h_k)}}(x)-u_{b_1,r_k,f_1}(x)\prod_{i=2}^m u_{r_k,f_i}(x)\\
=&(u_{(b_1)^l_{\tau(h_k)},r_k,(f_1)^l_{\tau(h_k)}}(x)-u_{b_1,r_k,f_1}(x))\prod_{i=2}^m u_{r_k,(f_i)^l_{\tau(h_k)}}(x)\\
&+u_{b_1,r_k,f_1}(x)\Big(\prod_{i=2}^m u_{r_k,(f_i)^l_{\tau(h_k)}}(x)-\prod_{i=2}^m u_{r_k,f_i}(x)\Big).
\end{aligned}$$

It follows that

$$\begin{aligned}
&\frac{1}{h_k}(A^1_{\vec{b},\alpha,x+h_ke_l,\vec{f}}(r_k)-A^1_{\vec{b},\alpha,x,\vec{f}}(r_k))\\
\leq&\frac{1}{|B(x,r_k)|^{m-\alpha/n}}\frac{u_{(b_1)^l_{\tau(h_k)},r_k,(f_1)^l_{\tau(h_k)}}(x)-u_{b_1,r_k,f_1}(x)}{h_k}\prod_{i=2}^m u_{r_k,(f_i)^l_{\tau(h_k)}}(x)\\
&+\frac{1}{|B(x,r_k)|^{m-\alpha/n}}u_{b_1,r_k,f_1}(x)\frac{1}{h_k}\Big(\prod_{i=2}^m u_{r_k,(f_i)^l_{\tau(h_k)}}(x)-\prod_{i=2}^m u_{r_k,f_i}(x)\Big).
\end{aligned}\tag{2.20}$$

Fix $i\in\{1,\dots,m\}$. Noting that

$$\begin{aligned}
|u_{r_k,f_i}(x)-u_{r,f_i}(x)|&\leq\int_{\mathbb{R}^n}f_i(y)|\chi_{B(x,r_k)}(y)-\chi_{B(x,r)}(y)|dy\\
&\leq\int_{B(x,2r)}f_i(y)|\chi_{B(x,r_k)}(y)-\chi_{B(x,r)}(y)|dy.
\end{aligned}$$

This together with the fact that $f_i\in L^1(B(x,2r))$ and the dominated convergence theorem implies that

$$\lim_{k\to\infty}u_{r_k,f_i}(x)=u_{r,f_i}(x).\tag{2.21}$$

Similarly we can obtain

$$\lim_{k\to\infty}u_{b_1,r_k,f_1}(x)=u_{b_1,r,f_1}(x).\tag{2.22}$$

Observe that

$$\begin{aligned}&|u_{r_k,(f_i)^l_{\tau(h_k)}}(x)-u_{r,f_i}(x)|\\ \le\ &|u_{r_k,(f_i)^l_{\tau(h_k)}}(x)-u_{r_k,f_i}(x)|+|u_{r_k,f_i}(x)-u_{r,f_i}(x)|\\ \le\ &|B(x,r_k)|M((f_i)^l_{\tau(h_k)}-f_i)(x)+|u_{r_k,f_i}(x)-u_{r,f_i}(x)|.\end{aligned}$$

This together with (2.21) leads to

$$\lim_{k\to\infty}u_{r_k,(f_i)^l_{\tau(h_k)}}(x)=u_{r,f_i}(x). \tag{2.23}$$

Note that $D_lf_i\in L^{p_i}(B(x,2r))$. An argument similar to (2.21) gives that

$$\lim_{k\to\infty}u_{r_k,D_lf_i}(x)=u_{r,D_lf_i}(x). \tag{2.24}$$

In view of (2.24), one has

$$\begin{aligned}&|u_{r_k,(f_i)^l_{h_k}}(x)-u_{r,D_lf_i}(x)|\\ \le\ &|u_{r_k,(f_i)^l_{h_k}}(x)-u_{r_k,D_lf_i}(x)|+|u_{r_k,D_lf_i}(x)-u_{r,D_lf_i}(x)|\\ \le\ &|B(x,r_k)|M((f_i)^l_{h_k}-D_lf_i)(x)+|u_{r_k,D_lf_i}(x)-u_{r,D_lf_i}(x)|\\ \to\ &0\quad\text{as}\quad k\to\infty.\end{aligned} \tag{2.25}$$

Hence, we get from (2.21), (2.23) and (2.25) that

$$\begin{aligned}&\lim_{k\to\infty}\frac{1}{h_k}\Big(\prod_{i=2}^m u_{r_k,(f_i)^l_{\tau(h_k)}}(x)-\prod_{i=2}^m u_{r_k,f_i}(x)\Big)\\ =\ &\lim_{k\to\infty}\sum_{i=2}^m u_{r_k,(f_i)^l_{h_k}}(x)\prod_{\mu=2}^{i-1}u_{r_k,f_\mu}(x)\prod_{\nu=i+1}^m u_{r_k,(f_\nu)^l_{\tau(h_k)}}(x)\\ =\ &\sum_{i=2}^m u_{r,D_lf_i}(x)\prod_{\mu=2}^{i-1}u_{r,f_\mu}(x)\prod_{\nu=i+1}^m u_{r,f_\nu}(x).\end{aligned} \tag{2.26}$$

Next we prove that

$$\begin{aligned}&\lim_{k\to\infty}\frac{u_{(b_1)^l_{\tau(h_k)},r_k,(f_1)^l_{\tau(h_k)}}(x)-u_{b_1,r_k,f_1}(x)}{h_k}\\ =&\int_{B(x,r_k)}D_{l,y}F_{b_1}(x,y)f_1(y)dy+\int_{B(x,r_k)}D_{l,x}F_{b_1}(x,y)f_1(y)dy\\ &+u_{b_1,r_k,(f_1)^l_{h_k}}(x).\end{aligned} \tag{2.27}$$

We have

$$\begin{aligned}&\frac{u_{(b_1)^l_{\tau(h_k)},r_k,(f_1)^l_{\tau(h_k)}}(x)-u_{b_1,r_k,f_1}(x)}{h_k}\\ =&\int_{B(x,r_k)}(F_{x,b_1})^l_{h_k}(y)(f_1)^l_{\tau(h_k)}(y)dy\\ &+\int_{B(x,r_k)}(F_{y+h_ke_l,b_1})^l_{h_k}(x)(f_1)^l_{\tau(h_k)}(y)dy+u_{b_1,r_k,(f_1)^l_{h_k}}(x).\end{aligned} \tag{2.28}$$

Note that

$$\begin{aligned}
&|u_{b_1,r_k,(f_1)^l_{h_k}}(x)-u_{b_1,r,D_lf_1}(x)|\\
&\le u_{b_1,r_k,|(f_1)^l_{h_k}-D_lf_1|}(x)+|u_{b_1,r_k,D_lf_1}(x)-u_{b_1,r,D_lf_1}(x)|\\
&\le 2\|b_1\|_{Lip(\mathbb{R}^n)}|B(x,r_k)|^{1-\alpha/n}M_\alpha((f_1)^l_{h_k}-D_lf_1)(x)\\
&\quad+2\|b_1\|_{Lip(\mathbb{R}^n)}\int_{B(x,2r)}|D_lf_1(y_1)||\chi_{B(x,r_k)}(y_1)-\chi_{B(x,r)}(y_1))|dy_1.
\end{aligned}$$

By the Hölder's inequality, we see that $D_lf_1\in L^1(B(x,2r))$. Applying the dominated convergence theorem we have

$$\lim_{k\to\infty}u_{b_1,r_k,(f_1)^l_{h_k}}(x)=u_{b_1,r,D_lf_1}(x). \tag{2.29}$$

Note that

$$\begin{aligned}
&\Big|\int_{B(x,r_k)}(F_{x,b_1})^l_{h_k}(y)(f_1)^l_{\tau(h_k)}(y)dy-\int_{B(x,r)}D_{l,y}F_{b_1}(x,y)f_1(y)dy\Big|\\
&\le\int_{B(x,r_k)}|(F_{x,b_1})^l_{h_k}(y)||(f_1)^l_{\tau(h_k)}(y)-f_1(y)|dy\\
&\quad+\int_{B(x,r_k)}|(F_{x,b_1})^l_{h_k}(y)-D_{l,y}F_{b_1}(x,y)|f_1(y)dy.
\end{aligned} \tag{2.30}$$

It is clear that

$$\begin{aligned}
&\int_{B(x,r_k)}|(F_{x,b_1})^l_{h_k}(y)||(f_1)^l_{\tau(h_k)}(y)-f_1(y)|dy\\
&\le\|b_1\|_{Lip(\mathbb{R}^n)}|B(x,r_k)|M((f_1)^l_{\tau(h_k)}-f_1)(x).
\end{aligned}$$

This leads to

$$\lim_{k\to\infty}\int_{B(x,r_k)}|(F_{x,b_1})^l_{h_k}(y)||(f_1)^l_{\tau(h_k)}(y)-f_1(y)|dy=0. \tag{2.31}$$

Note that $(F_{x,b_1})^l_{h_k}(y)-D_{l,y}F_{b_1}(x,y)\to 0$ as $k\to\infty$ for almost every $y\in\mathbb{R}^n$. Moreover, we have that $|(F_{x,b_1})^l_{h_k}(y)-D_{l,y}F_{b_1}(x,y)|\le 2\|b_1\|_{Lip(\mathbb{R}^n)}$ for almost every $y\in\mathbb{R}^n$. Since $f_1\in L^1(B(x,2r))$, these facts together with the dominated convergence theorem imply

$$\lim_{k\to\infty}\int_{B(x,r)}|(F_{x,b_1})^l_{h_k}(y)-D_{l,y}F_{b_1}(x,y)|f_1(y)dy=0. \tag{2.32}$$

The same arguments give

$$\begin{aligned}
&\Big|\int_{B(x,r_k)}|(F_{x,b_1})^l_{h_k}(y)-D_{l,y}F_{b_1}(x,y)|f_1(y)dy\\
&\quad-\int_{B(x,r)}|(F_{x,b_1})^l_{h_k}(y)-D_{l,y}F_{b_1}(x,y)|f_1(y)dy\Big|\\
&\le\int_{\mathbb{R}^n}|(F_{x,b_1})^l_{h_k}(y)-D_{l,y}F_{b_1}(x,y)||f_1(y)(\chi_{B(x,r_k)}(y)-\chi_{B(x,r)}(y))|dy\\
&\le 2\|b_1\|_{Lip(\mathbb{R}^n)}\int_{B(x,2r)}|f_1(y)(\chi_{B(x,r_k)}(y)-\chi_{B(x,r)}(y))|dy\to 0\ \text{ as } k\to\infty.
\end{aligned}$$

This together with (2.30)–(2.32) implies that

$$\lim_{k\to\infty}\int_{B(x,r_k)}(F_{x,b_1})^l_{h_k}(y)(f_1)^l_{\tau(h_k)}(y)dy=\int_{B(x,r)}D_{l,y}F_{b_1}(x,y)f_1(y)dy. \tag{2.33}$$

In view of (2.28), (2.29) and (2.33), for (2.27) it suffices to show that

$$\begin{aligned}&\lim_{k\to\infty}\int_{B(x,r_k)}(F_{y+h_ke_l,b_1})^l_{h_k}(x)(f_1)^l_{\tau(h_k)}(y)dy\\&=\int_{B(x,r)}D_{l,x}F_{b_1}(x,y)f_1(y)dy.\end{aligned} \tag{2.34}$$

We get by a change of variable that

$$\int_{B(x,r_k)}(F_{y+h_ke_l,b_1})^l_{h_k}(x)(f_1)^l_{\tau(h_k)}(y)dy=\int_{B(x+h_ke_l,r_k)}(F_{y,b_1})^l_{h_k}(x)f_1(y)dy. \tag{2.35}$$

Note that $\lim_{k\to\infty}(F_{y,b_1})^l_{h_k}(x)=D_{l,x}F_{b_1}(x,y)$ and $|(F_{y,b_1})^l_{h_k}(x)-D_{l,x}F_{b_1}(x,y)|\leq 2\|b_1\|_{Lip(\mathbb{R}^n)}$ for almost every $x\in\mathbb{R}^n$. We also note that $f_1\in L^1(B(x,r))$. Applying the dominated convergence theorem, one has

$$\lim_{k\to\infty}\int_{B(x,r)}(F_{y,b_1})^l_{h_k}(x)f_1(y)dy=\int_{B(x,r)}D_{l,x}F_{b_1}(x,y)f_1(y)dy. \tag{2.36}$$

We may assume without loss of generality that $h_k\leq r$ for all $k\geq 1$. Note that $B(x+h_ke_l,r_k)\subset B(x,3r)$ and $|(F_{y,b_1})^l_{h_k}(x)|\leq\|b_1\|_{Lip(\mathbb{R}^n)}$. An application of the dominated convergence theorem shows that

$$\begin{aligned}&\Big|\int_{B(x+h_ke_l,r_k)}(F_{y,b})^l_{h_k}(x)f_1(y)dy-\int_{B(x,r)}(F_{y,b_1})^l_{h_k}(x)f_1(y)dy\Big|\\&\leq\|b_1\|_{Lip(\mathbb{R}^n)}\int_{B(x,3r)}f_1(y)|\chi_{B(x+h_ke_l,r_k)}(y)-\chi_{B(x,r)}(y)|dy\to 0\quad\text{as}\quad k\to\infty.\end{aligned}$$

This together with (2.35) and (2.36) implies (2.34). Combining (2.20), (2.23), (2.24), (2.25) with (2.26) leads to (2.19).

It follows from (2.18) and (2.19) that

$$\begin{aligned}&D_l\mathfrak{M}^1_{\alpha,\vec{b}}(\vec{f})(x)\\&\leq\frac{1}{|B(x,r)|^{m-\alpha/n}}\Big(\int_{B(x,r)}D_{l,y}F_{b_1}(x,y)f_1(y)dy\\&\quad+\int_{B(x,r)}D_{l,x}F_{b_1}(x,y)f_1(y)dy+u_{b_1,r,D_lf_1}(x)\Big)\prod_{i=2}^m u_{r,f_i}(x)\\&\quad+\frac{1}{|B(x,r)|^{m-\alpha/n}}u_{b_1,r,f_1}(x)\sum_{i=2}^m u_{r,D_lf_i}(x)\prod_{\substack{2\leq\mu\leq m,\\ \mu\neq i}}u_{r,f_\mu}(x).\end{aligned} \tag{2.37}$$

On the other hand, we note that

$$\begin{aligned}
&\frac{1}{h_k}(A^1_{\vec{b},\alpha,x+h_ke_l,\vec{f}}(r)-A^1_{\vec{b},\alpha,x,\vec{f}}(r))\\
&=\frac{1}{h_k}\frac{1}{|B(x,r)|^{m-\alpha/n}}\Big(u_{(b_1)^l_{\tau(h_k)},r,(f_1)^l_{\tau(h_k)}}(x)\prod_{i=2}^m u_{r,(f_i)^l_{\tau(h_k)}}(x)\\
&\quad -u_{b_1,r,f_1}(x)\prod_{i=2}^m u_{r,f_i}(x)\Big)\\
&=\frac{1}{|B(x,r)|^{m-\alpha/n}}\frac{u_{(b_1)^l_{\tau(h_k)},r,(f_1)^l_{\tau(h_k)}}(x)-u_{b_1,r,f_1}(x)}{h_k}\prod_{i=2}^m u_{r,(f_i)^l_{\tau(h_k)}}(x)\\
&\quad +\frac{1}{|B(x,r)|^{m-\alpha/n}}u_{b_1,r,f_1}(x)\frac{1}{h_k}\Big(\prod_{i=2}^m u_{r,(f_i)^l_{\tau(h_k)}}(x)-\prod_{i=2}^m u_{r,f_i}(x)\Big).
\end{aligned}$$

This together with (2.18) and the arguments similar to those used to derive (2.37) implies that

$$\begin{aligned}
&D_l\mathfrak{M}^1_{\alpha,\vec{b}}(\vec{f})(x)\\
&\geq \frac{1}{|B(x,r)|^{m-\alpha/n}}\Big(\int_{B(x,r)}D_{l,y}F_{b_1}(x,y)f_1(y)dy\\
&\quad +\int_{B(x,r)}D_{l,x}F_{b_1}(x,y)f_1(y)dy+u_{b_1,r,D_lf_1}(x)\Big)\prod_{i=2}^m u_{r,f_i}(x)\\
&\quad +\frac{1}{|B(x,r)|^{m-\alpha/n}}u_{b_1,r,f_1}(x)\sum_{i=2}^m u_{r,D_lf_i}(x)\prod_{\substack{2\leq\mu\leq m,\\ \mu\neq i}}u_{r,f_\mu}(x).
\end{aligned}\tag{2.38}$$

In view of (2.37) and (2.38), we have that (2.16) holds for almost every $x\in B_R$.

Case (ii) ($r=0$). In this case we have $\mathfrak{M}^1_{\alpha,\vec{b}}(\vec{f})(x)=A^1_{\vec{b},\alpha,x,\vec{f}}(0)=0$. We get from (2.18) that

$$D_l\mathfrak{M}^1_{\alpha,\vec{b}}(\vec{f})(x)=\lim_{k\to\infty}\frac{1}{h_k}\mathfrak{M}^1_{\alpha,\vec{b}}(\vec{f})(x+h_ke_l)=\lim_{k\to\infty}\frac{1}{h_k}A^1_{\vec{b},\alpha,x+h_ke_l,\vec{f}}(r_k).\tag{2.39}$$

If $r_k=0$ for infinitely many k, then we get from (2.39) that $D_l\mathfrak{M}^1_{\alpha,\vec{b}}(\vec{f})(x)=0$. In what follows, we may assume without loss of generality that $r_k\in(0,1)$ for all $k\geq 1$. By the definition of $\mathfrak{M}^1_{\alpha,\vec{b}}(\vec{f})$, we can conclude that one the the following conditions holds:

(a) there exists $i\in\{2,\dots,m\}$ such that $f_i(y)=0$ for almost every $y\in\mathbb{R}^n$;

(b) $|b_1(x)-b_1(y)|f_1(y)=0$ for almost every $y\in\mathbb{R}^n$.

Now we consider two cases:

(i) If case (a) holds, then we may assume that $f_2(y)=0$ for almost every $y\in\mathbb{R}^n$. Then we have

$$\frac{u_{r_k,(f_2)^l_{\tau(h_k)}}(x)}{h_k}=\frac{1}{h_k}\int_{B(x,r_k)}((f_2)^l_{\tau(h_k)}(y)-f_2(y))dy=u_{r_k,(f_2)^l_{h_k}}(x).$$

Write

$$
\begin{aligned}
&\frac{1}{h_k}A^1_{\vec{b},\alpha,x+h_ke_l,\vec{f}}(r_k)\\
&=\frac{1}{|B(x,r_k)|^{m-\alpha/n}}\frac{1}{h_k}u_{(b_1)^l_{\tau(h_k)},r_k,(f_1)^l_{\tau(h_k)}}(x)\prod_{i=2}^m u_{r_k,(f_i)^l_{\tau(h_k)}}(x)\\
&=|B(x,r_k)|^{\alpha/n}\Big(\frac{1}{|B(x,r_k)|}u_{(b_1)^l_{\tau(h_k)},r_k,(f_1)^l_{\tau(h_k)}}(x)\Big)\\
&\quad\times\Big(\frac{1}{|B(x,r_k)|}u_{r_k,(f_2)^l_{h_k}}(x)\Big)\prod_{i=3}^m\Big(\frac{1}{|B(x,r_k)|}u_{r_k,(f_i)^l_{\tau(h_k)}}(x)\Big).
\end{aligned}\tag{2.40}
$$

Let us fix $i\in\{1,2,\dots,m\}$. We have

$$
\begin{aligned}
&\frac{1}{|B(x,r_k)|}u_{r_k,(f_i)^l_{h_k}}(x)\\
&\le\frac{1}{|B(x,r_k)|}\int_{B(x,r_k)}|(f_i)^l_{h_k}(y)-D_lf_i(y)|dy+\frac{1}{|B(x,r_k)|}\int_{B(x,r_k)}|D_lf_i(y)|dy\\
&\le M((f_i)^l_{h_k}-D_lf_i)(x)+\frac{1}{|B(x,r_k)|}\int_{B(x,r_k)}|D_lf_i(y)|dy.
\end{aligned}
$$

It follows that

$$\lim_{k\to\infty}\frac{1}{|B(x,r_k)|}u_{r_k,(f_i)^l_{h_k}}(x)=|D_lf_i(x)|.\tag{2.41}$$

Similarly it holds that

$$\lim_{k\to\infty}\frac{1}{|B(x,r_k)|}u_{r_k,(f_i)^l_{\tau(h_k)}}(x)=f_i(x).\tag{2.42}$$

Observe that

$$\frac{1}{|B(x,r_k)|}u_{(b_1)^l_{\tau(h_k)},r_k,(f_1)^l_{\tau(h_k)}}(x)\le\|b_1\|_{Lip(\mathbb{R}^n)}r_k\frac{1}{|B(x,r_k)|}u_{r_k,(f_1)^l_{\tau(h_k)}}(x).$$

It follows that

$$\lim_{k\to\infty}\frac{1}{|B(x,r_k)|}u_{(b_1)^l_{\tau(h_k)},r_k,(f_1)^l_{\tau(h_k)}}(x)=0.\tag{2.43}$$

In view of (2.40)-(2.43), one has

$$\lim_{k\to\infty}\frac{1}{h_k}A^1_{\vec{b},\alpha,x+h_ke_l,\vec{f}}(r_k)=0.\tag{2.44}$$

(ii) If case (b) holds, then we have

$$
\begin{aligned}
&\frac{1}{|B(x,r_k)|}\frac{1}{h_k}u_{(b_1)^l_{\tau(h_k)},r_k,(f_1)^l_{\tau(h_k)}}(x)\\
&=\frac{1}{|B(x,r_k)|}\frac{1}{h_k}\int_{B(x,r_k)}|(b_1)^l_{\tau(h_k)}(x)-(b_1)^l_{\tau(h_k)}(y)|(f_1)^l_{\tau(h_k)}(y)dy\\
&=\frac{1}{|B(x,r_k)|}\frac{1}{h_k}\int_{B(x,r_k)}|(b_1)^l_{\tau(h_k)}(x)-(b_1)^l_{\tau(h_k)}(y)|((f_1)^l_{\tau(h_k)}(y)-f_1(y))dy\\
&\quad+\frac{1}{|B(x,r_k)|}\frac{1}{h_k}\int_{B(x,r_k)}|(b_1)^l_{\tau(h_k)}(x)-(b_1)^l_{\tau(h_k)}(y)-(b_1(x)-b_1(y))|f_1(y)dy\\
&\le\|b_1\|_{Lip(\mathbb{R}^n)}|r_k|\frac{1}{|B(x,r_k)|}u_{r_k,(f_1)^l_{h_k}}(x)\\
&\quad+\frac{1}{|B(x,r_k)|}\int_{B(x,r_k)}|(b_1)^l_{h_k}(x)-(b_1)^l_{h_k}(y)|f_1(y)dy.
\end{aligned}
$$

Observe that

$$\begin{aligned}
&\frac{1}{|B(x,r_k)|}\int_{B(x,r_k)}|(b_1)^l_{h_k}(x)-(b_1)^l_{h_k}(y)|f_1(y)dy\\
\leq\ &\frac{1}{|B(x,r_k)|}\int_{B(x,r_k)}|(b_1)^l_{h_k}(x)-(b_1)^l_{h_k}(y)-(D_lb_1(x)-D_lb_1(y))|f_1(y)dy\\
&+\frac{1}{|B(x,r_k)|}\int_{B(x,r_k)}|D_lb_1(x)-D_lb_1(y)|f_1(y)dy\\
\leq\ &|(b_1)^l_{h_k}(x)-D_lb_1(x)|Mf_1(x)+M(((b_1)^l_{h_k}-D_lb_1)f_1)(x)\\
&+\frac{1}{|B(x,r_k)|}\int_{B(x,r_k)}|D_lb_1(x)-D_lb_1(y)|f_1(y)dy\\
\to\ &0\ \text{ as }\ k\to\infty.
\end{aligned}$$

This together with (2.41) leads to

$$\lim_{k\to\infty}\frac{1}{|B(x,r_k)|}\frac{1}{h_k}u_{(b_1)^l_{\tau(h_k)},r_k,(f_1)^l_{\tau(h_k)}}(x)=0. \tag{2.45}$$

It follows from (2.42) and (2.45) that

$$\lim_{k\to\infty}\frac{1}{|B(x,r_k)|^{m-\alpha/n}}\frac{1}{h_k}u_{(b_1)^l_{\tau(h_k)},r_k,(f_1)^l_{\tau(h_k)}}(x)\prod_{i=2}^{m}u_{r_k,(f_i)^l_{\tau(h_k)}}(x)=0.$$

Hence,

$$\lim_{k\to\infty}\frac{1}{h_k}A^1_{\vec{b},\alpha,x+h_ke_l,\vec{f}}(r_k)=0.$$

Combining this with (2.39) leads to

$$D_l\mathfrak{M}^1_{\alpha,\vec{b}}(\vec{f})(x)=0.$$

Since R was arbitrary and $|B(0,R)\setminus(\bigcap_{i=1}^9A_i)|=0$. This proves Lemma 2.3. $\square$

3. Proof of Theorem 1.1

In this section we present the proof of Theorem 1.1.

Proof. Let $\vec{b}=(b_1,\dots,b_m)$ with each $b_j\in\mathrm{Lip}(\mathbb{R}^n)$ and $\vec{f}=(f_1,\dots,f_m)$ with each $f_j\in W^{1,p_j}(\mathbb{R}^n)$, where $0\leq\alpha<mn$, $1<p_1,\dots,p_m<\infty$, $1\leq q<\infty$ and $1/q=\sum_{i=1}^m 1/p_i-\alpha/n$. For any $j\geq1$ let $\vec{f_j}=(f_{1,j},\cdots,f_{m,j})$ be such that $\|f_{i,j}-f_i\|_{W^{1,p_i}(\mathbb{R}^n)}\to0$ as $j\to\infty$. By Remark 1.1 we see that $\|\mathfrak{M}^1_{\alpha,\vec{b}}(\vec{f_j})-\mathfrak{M}^1_{\alpha,\vec{b}}(\vec{f})\|_{L^q(\mathbb{R}^n)}\to0$ as $k\to\infty$. Hence, to prove Theorem 1.1, it is enough to show that

$$\|D_l\mathfrak{M}^1_{\alpha,\vec{b}}(\vec{f_j})-D_l\mathfrak{M}^1_{\alpha,\vec{b}}(\vec{f})\|_{L^q(\mathbb{R}^n)}\to0\ \text{ as } j\to\infty \tag{3.1}$$

for any $l=1,2,\dots,n$.

We only prove (3.1) for $l=1$ since other cases can be proved similarly. We may assume without loss of generality that all $f_{i,j}\geq0$ and $f_i\geq0$.

For convenience, we set $F_{b_1}(x,y)=|b_1(x)-b_1(y)|$. It was pointed out in the proof of Lemma 2.3 that $F_{b_1}(x,\cdot)\in Lip(\mathbb{R}^n)$ and $\|F_{b_1}(x,\cdot)\|_{Lip(\mathbb{R}^n)}\leq\|b_1\|_{Lip(\mathbb{R}^n)}$

for all $x \in \mathbb{R}^n$. Moreover, for almost every $y \in \mathbb{R}^n$, we have that $|D_{1,y}F_{b_1}(x,y)| \leq \|b_1\|_{Lip(\mathbb{R}^n)}$. $F_{b_1}(\cdot,y) \in Lip(\mathbb{R}^n)$ and $\|F_{b_1}(\cdot,y)\|_{Lip(\mathbb{R}^n)} \leq \|b_1\|_{Lip(\mathbb{R}^n)}$ for all $y \in \mathbb{R}^n$. Moreover, for almost every $x \in \mathbb{R}^n$, we have that $|D_{1,x}F_{b_1}(x,y)| \leq \|b_1\|_{Lip(\mathbb{R}^n)}$. For convenience, for a fixed $x \in \mathbb{R}^n$ and $\vec{g} = (g_1, \ldots, g_m)$ with each $g_j \in W^{1,p_j}(\mathbb{R}^n)$, we define the function $\mathcal{B}_{\vec{b},\alpha,x,g}\colon [0,\infty) \to \mathbb{R}$ by $\mathcal{B}_{\vec{b},\alpha,x,\vec{g}}(0) = 0$, and for $r \in (0,\infty)$,

$$\begin{aligned}\mathcal{B}_{\vec{b},\alpha,x,\vec{g}}(r) = {}& \frac{1}{|B(x,r)|^{m-\alpha/n}}\Big(\int_{B(x,r)} D_{l,y}F_{b_1}(x,y)g_1(y)dy \\ &+ \int_{B(x,r)} D_{l,x}F_{b_1}(x,y)g_1(y)dy + u_{b_1,r,D_l|f_1|}(x)\Big)\prod_{i=2}^{m} u_{r,g_i}(x) \\ &+ \frac{1}{|B(x,r)|^{m-\alpha/n}} u_{b_1,r,g_1}(x)\sum_{i=2}^{m} u_{r,D_lg_i}(x) \prod_{\substack{2\leq\mu\leq m,\\ \mu\neq i}} u_{r,g_\mu}(x).\end{aligned}$$

We define the operator T_α by

$$T_\alpha(\vec{g})(x) = 2\|b_1\|_{Lip(\mathbb{R}^n)}\mathfrak{M}_\alpha(\vec{g})(x) + \sum_{i=1}^{m} \mathfrak{M}^1_{\alpha,\vec{b}}(\vec{g}_i)(x),$$

where $\vec{g}_i = (f_1, f_2, \ldots, f_{i-1}, D_lf_i, f_{i+1}, \ldots, f_m)$. By the properties of F_{b_1}, we have that

$$\mathcal{B}_{\vec{b},\alpha,x,\vec{g}}(r) \leq T_\alpha(\vec{g})(x), \tag{3.2}$$

for almost every $x \in \mathbb{R}^n$. By Minkowski's inequality and Remark 1.2,

$$\begin{aligned}&\|T_\alpha(\vec{g})\|_{L^q(\mathbb{R}^n)} \\ &\leq 2\|b_1\|_{Lip(\mathbb{R}^n)}\|\mathfrak{M}_\alpha(\vec{g})\|_{L^q(\mathbb{R}^n)} + \sum_{i=1}^{m}\|\mathfrak{M}^1_{\alpha,\vec{b}}(\vec{g}_i)\|_{L^q(\mathbb{R}^n)} \\ &\leq C_{\alpha,m,n,p_1,\ldots,p_m}\|b_1\|_{Lip(\mathbb{R}^n)}\prod_{j=1}^{m}\|f_j\|_{L^{p_j}(\mathbb{R}^n)} \\ &\quad + C_{\alpha,m,n,p_1,\ldots,p_m}\|b_1\|_{L^\infty(\mathbb{R}^n)}\sum_{i=1}^{m}\|D_lf_i\|_{L^{p_i}(\mathbb{R}^n)}\prod_{\substack{1\leq\mu\leq m,\\ \mu\neq i}}\|f_\mu\|_{L^{p_\mu}(\mathbb{R}^n)} \\ &\leq C_{\alpha,m,n,p_1,\ldots,p_m}\|b_1\|_{\mathrm{Lip}(\mathbb{R}^n)}\prod_{j=1}^{m}\|f_j\|_{W^{1,p_j}(\mathbb{R}^n)}.\end{aligned} \tag{3.3}$$

Let $\epsilon \in (0,1)$. There exists $R > 0$ such that $\|T_\alpha(\vec{f})\|_{L^q((B(0,R))^c)} < \epsilon$. By the absolute continuity of integration, there exists $\eta > 0$ such that for any measurable subset $A \subset B(0,R)$ with $|A| < \eta$ we have $\|T_\alpha(\vec{f})\|_{L^q(A)} < \epsilon$. Observe that for almost every $x \in \mathbb{R}^n$, the function $\mathcal{B}_{\vec{b},\alpha,x,\vec{f}}$ is uniformly continuous on $[0,\infty)$. Hence, for almost every x, we can find $\delta_x > 0$ such that

$$|\mathcal{B}_{\vec{b},\alpha,x,\vec{f}}(r_1) - \mathcal{B}_{\vec{b},\alpha,x,\vec{f}}(r_2)| < |B(0,R)|^{-1/q}\epsilon \ \text{ whenever } |r_1 - r_2| < \delta_x.$$

It follows that

$$B(0,R) := \Big(\bigcup_{j=1}^{\infty}\Big\{x \in B(0,R); \delta_x > \frac{1}{j}\Big\}\Big) \cup E,$$

where $|E| = 0$. Hence, there exists $\delta > 0$ such that

$$\begin{aligned}&|\{x \in B(0,R) : |\mathcal{B}_{\vec{b},\alpha,x,\vec{f}}(r_1) - \mathcal{B}_{\vec{b},\alpha,x,\vec{f}}(r_2)| \\ \geq\ &|B(0,R)|^{-1/q}\epsilon \text{ for some } r_1, r_2 \text{ with } |r_1 - r_2| < \delta\}| =: |G_1| < \frac{\eta}{2}.\end{aligned} \tag{3.4}$$

In view of Lemma 2.1, we can find a positive integer N_1 such that

$$|\{x \in B(0,R); \mathcal{R}_{\vec{b},\alpha}(\vec{f}_j)(x) \not\subseteq \mathcal{R}_{\vec{b},\alpha}(\vec{f})(x)_{(\delta)}\}| =: |H_j| < \frac{\eta}{2}, \quad \forall j \geq N_1.$$

Applying Lemma 2.3, we have that for almost every $x \in \mathbb{R}^n$,

$$\begin{aligned}&|D_1\mathfrak{M}^1_{\alpha,\vec{b}}(\vec{f}_j)(x) - D_1\mathfrak{M}^1_{\alpha,\vec{b}}(\vec{f})(x)| \\ &= |\mathcal{B}_{\vec{b},\alpha,x,\vec{f}_j}(r_1) - \mathcal{B}_{\vec{b},\alpha,x,\vec{f}}(r_2)| \\ &\leq |\mathcal{B}_{\vec{b},\alpha,x,\vec{f}_j}(r_1) - \mathcal{B}_{\vec{b},\alpha,x,\vec{f}}(r_1)| + |\mathcal{B}_{\vec{b},\alpha,x,\vec{f}}(r_1) - \mathcal{B}_{\vec{b},\alpha,x,\vec{f}}(r_2)|\end{aligned} \tag{3.5}$$

for any $r_1 \in \mathcal{R}_{\vec{b},\alpha}(\vec{f}_j)(x)$ and $r_2 \in \mathcal{R}_{\vec{b},\alpha}(\vec{f})(x)$.

For almost every $x \in B(0,R) \setminus (G_1 \cap H_j)$, there exist $r_1 \in \mathcal{R}_{\vec{b},\alpha}(\vec{f}_j)(x)$ and $r_2 \in \mathcal{R}_{\vec{b},\alpha}(\vec{f})(x)$ such that $|r_1 - r_2| < \delta$ and

$$|\mathcal{B}_{\vec{b},\alpha,x,\vec{f}}(r_1) - \mathcal{B}_{\vec{b},\alpha,x,\vec{f}}(r_2)| < |B(0,R)|^{-1/q}\epsilon. \tag{3.6}$$

On the other hand, for any $r \in [0,\infty)$ and almost every $x \in \mathbb{R}^n$,

$$\begin{aligned}&|\mathcal{B}_{\vec{b},\alpha,x,\vec{f}_j}(r) - \mathcal{B}_{\vec{b},\alpha,x,\vec{f}}(r)| \\ \leq\ &2\|b_1\|_{Lip(\mathbb{R}^n)}\sum_{l=1}^m \mathfrak{M}_\alpha(\vec{F}_l^j)(x) + \sum_{i=2}^m \mathfrak{M}^1_{\alpha,\vec{b}}(\vec{G}_i)(x) + \mathfrak{M}^1_{\alpha,\vec{b}}(\vec{P}_i)(x) \\ &+ \sum_{i=2}^m\Big(\sum_{\mu=2}^{i-1}\mathfrak{M}^1_{\alpha,\vec{b}}(\vec{H}_{i,\mu})(x) + \mathfrak{M}^1_{\alpha,\vec{b}}(\vec{I}_i)(x) + \sum_{\mu=2}^{i-1}\mathfrak{M}^1_{\alpha,\vec{b}}(\vec{J}_{i,\mu})(x)\Big) \\ =:\ &\Gamma_j(x),\end{aligned} \tag{3.7}$$

where

$$\begin{aligned}&\vec{F}_l^j = (f_1, \ldots, f_{l-1}, f_{l,j} - f_l, f_{l+1,j}, \ldots, f_{m,j}); \\ &\vec{G}_i = (f_{1,j} - f_1, f_{2,j}, \ldots, f_{i-1,j}, D_l f_{i,j}, f_{i+1,j}, \ldots, f_{m,j}); \\ &\vec{P}_i = (D_l(f_{1,j} - f_1), f_{2,j}, \ldots, f_{m,j}); \\ &\vec{H}_{i,\mu} = (f_1, f_2, \ldots, f_{\mu-1}, f_{\mu,j} - f_\mu, f_{\mu+1,j}, \ldots, f_{i-1,j}, D_l f_{i,j}, f_{i+1,j}, \ldots, f_{m,j}); \\ &\vec{I}_i = (f_1, \ldots, f_{i-1}, D_l(f_{i,j} - f_i), f_{i+1,j}, \ldots, f_{m,j}); \\ &\vec{J}_{i,\mu} = (f_1, \ldots, f_{i-1}, D_l f_i, f_{i+1}, \ldots, f_{\mu-1}, f_{\mu,j} - f_\mu, f_{\mu+1,j}, \ldots, f_{m,j}).\end{aligned}$$

By our assumption, there exists a positive integer N_2 such that $\sup_{1\leq i\leq m}\|f_{i,j} - f_i\|_{W^{1,p_i}(\mathbb{R}^n)} < \epsilon$ and $\sup_{1\leq i\leq m}\|f_{i,j}\|_{W^{1,p_i}(\mathbb{R}^n)} \leq \prod_{i=1}^m(\|f_i\|_{W^{1,p_i}(\mathbb{R}^n)} + 1)$ for any

$j \geq N_2$. By (1.2), (1.4) and Minkowski's inequality, we have

$$
\begin{aligned}
&\|\Gamma_j\|_{L^q(\mathbb{R}^n)} \\
\leq\ & 2\|b_1\|_{Lip(\mathbb{R}^n)} \sum_{l=1}^{m} \|\mathfrak{M}_\alpha(\vec{F}_l^j)\|_{L^q(\mathbb{R}^n)} + \sum_{i=2}^{m} \|\mathfrak{M}^1_{\alpha,\vec{b}}(\vec{G}_i)\|_{L^q(\mathbb{R}^n)} \\
&+\|\mathfrak{M}^1_{\alpha,\vec{b}}(\vec{P}_i)\|_{L^q(\mathbb{R}^n)} + \sum_{i=2}^{m} \Big(\sum_{\mu=2}^{i-1} \|\mathfrak{M}^1_{\alpha,\vec{b}}(\vec{H}_{i,\mu})\|_{L^q(\mathbb{R}^n)} \\
&+\|\mathfrak{M}^1_{\alpha,\vec{b}}(\vec{I}_i)\|_{L^q(\mathbb{R}^n)} + \sum_{\mu=2}^{i-1} \|\mathfrak{M}^1_{\alpha,\vec{b}}(\vec{J}_{i,\mu})\|_{L^q(\mathbb{R}^n)} \Big) \qquad (3.8) \\
\leq\ & C_{\alpha,m,n,p_1,\ldots,p_m} \|b_1\|_{\mathrm{Lip}(\mathbb{R}^n)} \sum_{l=1}^{m} \|f_{l,j} - f_l\|_{W^{1,p_l}(\mathbb{R}^n)} \\
&\times \prod_{\substack{1\leq \mu \leq m, \\ \mu \neq l}} (\|f_{\mu,j}\|_{W^{1,p_\mu}(\mathbb{R}^n)} + \|f_\mu\|_{W^{1,p_\mu}(\mathbb{R}^n)}) \\
\leq\ & C_{\alpha,m,n,p_1,\ldots,p_m,\vec{f}} \|b_1\|_{\mathrm{Lip}(\mathbb{R}^n)} \epsilon,
\end{aligned}
$$

for any $j \geq N_2$. In view of (3.2) and (3.5)–(3.7), we have that for almost every $x \in \mathbb{R}^n$,

$$
\begin{aligned}
&|D_1\mathfrak{M}^1_{\alpha,\vec{b}}(\vec{f}_j)(x) - D_1\mathfrak{M}^1_{\alpha,\vec{b}}(\vec{f})(x)| \\
\leq\ & \Gamma_j(x) + |B(0,R)|^{-1/q} \epsilon \chi_{B(0,R)\setminus(G_1 \cup H_j)}(x) \qquad (3.9) \\
&+2T_\alpha(f)(x)\chi_{G_1\cup H_j \cup (B(0,R))^c}(x).
\end{aligned}
$$

By (3.8), (3.9) and Minkowski's inequality, we have that for any $j \geq \max\{N_1, N_2\}$,

$$
\begin{aligned}
&\|D_1\mathfrak{M}^1_{\alpha,\vec{b}}(\vec{f}_j) - D_1\mathfrak{M}^1_{\alpha,\vec{b}}(\vec{f})\|_{L^q(\mathbb{R}^n)} \\
\leq\ & \|\Gamma_j\|_{L^q(\mathbb{R}^n)} + \||B(0,R)|^{-1/q}\epsilon\|_{L^q(B(0,R))} + 2\|T_\alpha(f)\|_{L^q(G_1\cup H_j\cup(B(0,R))^c)} \\
\leq\ & C_{\alpha,m,n,p_1,\ldots,p_m,\vec{f},b_1} \epsilon.
\end{aligned}
$$

Here we have used the fact that $|G_1 \cup H^j| < \eta$ for $j \geq N_1$. So (3.1) holds for $i = 1$. Theorem 1.1 is now proved. □

References

[1] M. Agcayazi, A. Gogatishvili, K. Koca and R. Mustafayev, *A note on maximal commutators and commutators of maximal functions*, J. Math. Soc. Jpn., 2015, 67(2), 581–593.

[2] J. M. Aldaz and J. Pérez Lázaro, *Functions of bounded variation, the derivative of the one dimensional maximal function, and applications to inequalities*, Trans. Amer. Math. Soc., 2007, 359(5), 2443–2461.

[3] A. M. Alphonse, *An end point estimate for maximal commutators*, J. Fourier Anal. Appl., 2000, 6, 449–456.

[4] D. Beltran and J. Madrid, *Regularity of the centered fractional maximal function on radial functions*, J. Funct. Anal., 2020, 279, 108686.

[5] R. Bu, Z. W. Fu and Y. D. Zhang, *Weighted estimates for bilinear square function with non-smooth kernels and commutators*, Front. Math. China, 2020, 15, 1–20.

[6] E. Carneiro and D. Moreira, *On the regularity of maximal operators*, Proc. Amer. Math. Soc., 2008, 136(12), 4395–4404.

[7] P. Chen, X. T. Duong, J. Li and Q. Y. Wu, *Compactness of Riesz transform commutator on stratified Lie groups*, J. Funct. Anal., 2019, 277, 1639–1676.

[8] T. Chen and F. Liu, *Regularity of commutators of multilinear maximal operators with Lipschitz symbols*, Math. Inequal. Appl., 2022, 25(1), 109–134.

[9] F. Deringoz, V. S. Guliyev and S. G. Hasanov, *Commutators of fractional maximal operator on generalized Orlicz–Morrey spaces*, Positivity, 2018, 22, 141–158.

[10] F. Deringoz, V. S. Guliyev and S. Samko, *Vanishing generalized Orlicz–Morrey spaces and fractional maximal operator*, Publ. Math. Debrecen, 2017, 90(1–2), 125–147.

[11] X. T. Duong, M. Lacey, J. Li, B. D. Wick and Q. Y. Wu, *Commutators of Cauchy-Szego type integrals for domains in Cn with minimal smoothness*, Indiana Univ. Math. J., 2021, 70, 1505–1541.

[12] H. Federer and W. Ziemer, *The Lebesgue set of a function whose distribution derivatives are p-th power summable*, Indiana Univ. Math. J., 1972, 22, 139–158.

[13] Z. W. Fu, R. M. Gong, E. Pozzi and Q. Y. Wu, *Cauchy-Szegö commutators on weighted Morrey spaces*, Math. Nachr., 2023, 296(5), 1859–1885.

[14] Z. W. Fu, S. L. Gong, S. Z. Lu and W. Yuan, *Weighted multilinear Hardy operators and commutators*, Forum Math., 2015, 27, 2825–2852.

[15] Z. W. Fu, L. Grafakos, Y. Lin, Y. Wu and S. H. Yang, *Riesz transform associated with the fractional Fourier transform and applications in image edge detection*, Appl. Comput. Harmon. Anal., 2023, 66, 211–235.

[16] Z. W. Fu, X. M. Hou, M. Y. Lee and J. Li, *A study of one-sided singular integral and function space via reproducing formula*, J. Geom. Anal., 2023, 33. DOI: 10.1007/s12220-023-01340-8.

[17] J. García-Cuerva, E. Harboure, C. Segovia and J. L. Torrea, *Weighted norm inequalities for commutators of strongly singular integrals*, Indiana Univ. Math. J., 1991, 40, 1397–1420.

[18] R. M. Gong, M. N. Vempati, Q. Y. Wu and P. Z. Xie, *Boundedness and compactness of Cauchy-type integral commutator on weighted Morrey spaces*, J. Aust. Math. Soc., 2022, 113(1), 36–56.

[19] C. González-Riquelme, *Continuity for the one-dimensional centered Hardy–Littlewood maximal opeator at the derivative level*, J. Funct. Anal., 2023, 285(9), 110097.

[20] V. S. Guliyev and F. Deringoz, *Some characterizations of Lipschitz spaces via commutators on generalized Orlicz–Morrey spaces*, Mediterr. J. Math., 2018, 15(180), 1–19.

[21] P. Hajłasz and J. Onninen, *On boundedness of maximal functions in Sobolev spaces,* Ann. Acad. Sci. Fenn. Math., 2004, 29(1), 167–176.

[22] J. Kinnunen, *The Hardy–Littlewood maximal function of a Sobolev function*, Israel J. Math., 1997, 100, 117–124.

[23] J. Kinnunen and P. Lindqvist, *The derivative of the maximal function*, J. Reine. Angew. Math., 1998, 503, 161–167.

[24] J. Kinnunen and E. Saksman, *Regularity of the fractional maximal function*, Bull. London Math. Soc., 2003, 35(4), 529–535.

[25] A. K. Lerner, S. Ombrosi, C. Pérez, R. H. Torres and R. Trujillo-González, *New maximal functions and multiple weights for the multilinear Calderón–Zygmund theory*, Adv. Math., 2009, 220, 1222–1264.

[26] F. Liu, *Continuity and approximate differentiability of multisublinear fractional maximal functions*, Math. Inequal. Appl., 2018, 21(1), 25–40.

[27] F. Liu and G. Wang, *Regularity of commutators of maximal operators with Lipschitz symbols*, Taiwan. J. Math., 2021, 25(5), 1007–1039.

[28] F. Liu and H. Wu, *On the regularity of the multisublinear maximal functions*, Canad. Math. Bull., 2015, 58(4), 808–817.

[29] F. Liu and S. Xi, *Sobolev regularity for commutators of the fractional maximal functions*, Banach. J. Math. Anal., 2021, 15(5), 1–36.

[30] F. Liu, Q. Xue and K. Yabuta, *Sobolev boundedness and continuity for commutators of the local Hardy–Littlewood maximal function*, Annales Fennici Mathematici, 2022, 47, 203–235.

[31] F. Liu, Q. Xue and P. Zhang, *Regularity and continuity of commutators of the Hardy–Littlewood maximal function*, Math. Nachr., 2020, 293(3), 491–509.

[32] H. Luiro, *Continuity of the maixmal operator in Sobolev spaces*, Proc. Amer. Math. Soc., 2007, 135(1), 243–251.

[33] H. Luiro, *On the regularity of the Hardy–Littlewood maximal operator on subdomains of* $\mathbb{R}^n$, Proc. Edinburgh Math. Soc., 2010, 53(1), 211–237.

[34] C. Pérez and R. H. Torres, *Sharp maximal function estimates for multilinear singular integrals*, Contemp. Math., 2003, 320, 323–331.

[35] J. M. Ruan, Q. Y. Wu and D. S. Fan, *Weighted Moes for Hausdorff operator and its commutator on the Heisenberg group*, Math. Inequal. Appl., 2019, 22(1), 307–329.

[36] C. Segovia and J. L. Torrea, *Weighted inequalities for commutators of fractional and singular integrals*, Publ. Mat., 1991, 35, 209–235.

[37] C. Segovia and J. L. Torrea, *Higher order commutators for vector-valued Calderón–Zygmund operators*, Trans. Amer. Math. Soc., 1993, 336, 537–556.

[38] S. G. Shi, Z. W. Fu and S. Z. Lu, *On the compactness of commutators of Hardy operators*, Pacific J. Math., 2020, 307, 239–256.

[39] S. G. Shi and S. Z. Lu, *A characterization of Campanato space via commutator of fractional integral*, J. Math. Anal. Appl., 2014, 419, 123–137.

[40] Q. Y. Wu and Z. W. Fu, *Weighted p-adic Hardy operators and their commutators on p-adic central Morrey spaces*, Bull. Malays. Math. Sci. Soc., 2017, 40, 635–654.

[41] M. H. Yang, Z. W. Fu and J. Y. Sun, *Existence and large time behavior to coupled chemotaxis-fluid equations in Besov–Morrey spaces*, J. Differ. Equations, 2019, 266, 5867–5894.

[42] M. H. Yang, Z. W. Fu and J. Y. Sun, *Existence and Gevrey regularity for a two-species chemotaxis system in homogeneous Besov spaces*, Sci. China Math., 2017, 60, 1837–1856.

[43] P. Zhang, *Characterization of Lipschitz spaces via commutators of the Hardy–Littlewood maximal function*, C. R. Acad. Sci. Paris Ser. I, 2017, 355(3), 336–344.

[44] P. Zhang, *Characterization of boundedness of some commutators of maximal functions in terms of Lipschitz spaces*, Anal. Math. Phys., 2019, 9(3), 1411–1427.

[45] P. Zhang, *Multiple weighted estimates for commutators of multilinear maximal function*, Acta Math. Sin. Engl. Ser., 2015, 31(6), 973–994.